# TILINGS BY $1 \times 1$ AND $2 \times 2$ 

Nicolas Rolin ${ }^{1}$ and Alexandra Ugolnikova ${ }^{1}$


#### Abstract

We consider tilings of a $k \times n$ board by $1 \times 1$ and $2 \times 2$ squares and get combinatorical results on proportions of small squares for $k \leq 10$ in plain case and for $k \leq 8$ in cylindrical case.


Mathematics Subject Classification. 35L05, 35L70.

## 1. Introduction

In the present work, we look at tilings of a $k \times n$ board $(n, k \in \mathbb{N})$ by $1 \times 1$ (small) and $2 \times 2$ (big) squares with no holes or overlaping. The goal is to understand how the average proportion of small squares in all possible tilings of a $k \times n$ rectangle by small and big squares changes when $k, n \rightarrow+\infty$. Another question is to find the number of small squares in tilings of a $k \times n$ rectangle that maximises the number of tilings of a $k \times n$ rectangle. A simpler problem that we study here is to consider that $k$ is fixed and $n \rightarrow+\infty$.

There has been some work done on the subject. When $k=2$, tilings of a $2 \times n$ rectangle by $1 \times 1$ and $2 \times 2$ squares correspond to the Fibonacci sequence. For $k=3$, one can easily show that the number of ways to cover a $3 \times n$ rectangle with $1 \times 1$ and $2 \times 2$ squares is equal to $\frac{1}{3}(-1)^{n}+\frac{1}{3} 2^{n+1}$.

Some results were obtained by Heubach [3, 4]. Namely, explicit formulas for the number of tilings for $k$ up to 5 by using introduced basic blocks and methods of analytic combinatorics for finding poles of generating functions and asymptotics. Bigger cases, however, seem to pose problems, mainly because it becomes difficult due to the number of basic blocks.

This abstract consists of four main sections, introduction and conclusion. In Section 2 we define a set of Bivariate Generating Functions (BGFs) associated with tilings of a $k \times l$ rectangle (supposing that $n=l k$, $l \in \mathbb{N}$ ), present formulas for small cases and calculate distribution of small squares in tilings for $k \leq 10$. In Section 3 we introduce an automaton construction that represents $B G F s$ and their relations. We extract some properties on its structure, present a simplification algorithm that allows to compute BGFs more easily. In Section 4 we present another point of view on this problem related to the matrix representation. In Section 5 we introduce the problem for a cylindrical case. In conlusion, we show combinatorical results on the proportions of small squares for $k \leq 10$ for the plain case and for and $k \leq 8$ for the cylindrical case, and mention some open question.

[^0]
## 2. SEtTings, DEfinitions

### 2.1. Bivariate generating function

In order to study the general case, we introduce BGFs. For the sake of simplicity we shall define them for the case $k=4$ and then generalize the definition. Let

$$
Q_{0000}(z, u)=\sum_{n, p} A_{n, p}^{4} z^{n} u^{p}
$$

be a $B G F$ where the coefficient $A_{n, p}^{4}$ of $z^{n} u^{p}$ is the number of tilings of a $4 \times \frac{n}{4}$ rectangle with exactly $p$ small squares, supposing that $n$ is a multiple of 4 . We want to underline that the rectangle is of area $n$. This choice is due to the simpler way of defining equations on $B G F s$.

Let $Q_{1000}(z, u)$ be a $B G F$ with the coefficient of $z^{n} u^{p}$ being the number of tilings of the initial rectangle with a $1 \times 1$ square cut off from the upper left corner and $Q_{2200}(z, u)$ a $B G F$ with the coefficient of $z^{n} u^{p}$ corresponding to the number of tilings of the initial rectangle with a $2 \times 2$ square cut off from the upper left corner (illustrations are shown in Fig. 1).

From this point on, we will write $B G F s$ without arguments, always meaning that they are $z, u$. A relation on $Q_{0000}, Q_{1000}$ and $Q_{2200}$ can be expressed in the following way:

$$
Q_{0000}=z u Q_{1000}+z^{4} Q_{2200} .
$$

Indeed, in order to obtain $Q_{0000}$, we can either cut off a small square or a big one from the upper left corner of the initial $4 \times \frac{n}{4}$ rectangle. The remaining areas will correspond either to $Q_{1000}$ or to $Q_{2200}$. And because we cut off squares we need to multiply $Q_{1000}$ by $z u$ ( $z$ corresponds to the area occupied by a small square, $u$ - to the one small square) and $Q_{2200}$ by $z^{4}$ respectively.

In the same way we can introduce $Q_{1100}$ and $Q_{1220}$ and we have the relation

$$
Q_{1000}=z u Q_{1100}+z^{4} Q_{1220} .
$$

At each step we change indexes of $Q_{i_{1} i_{2} i_{3} i_{4}}$ by going from left to right in the following way: we permit changing either one 0 to 1 or 00 to 22 , which means changing the left one or two columns of the board that was obtained at the previous step by cutting off either a $1 \times 1$ or a $2 \times 2$ square from the upper left corner of the board. By this rule one can never obtain $Q_{1010}$ or $Q_{1022}$, for example.

As soon as we get to $Q_{i_{1} i_{2} i_{3} i_{4}}$ with all indexes being different from zero, we can use a tetris rule to reduce the indexes of $Q_{i_{1} i_{2} i_{3} i_{4}}$ by one layer with "no charge". With "no charge" here means that, given that our strip is infinite, $Q_{1122}=Q_{0011}, Q_{1111}=Q_{0000}$ and so on.

Using this technique one obtains a finite set of $B G F s Q_{i_{1} i_{2} i_{3} i_{4}}$ and a system of functional equations on them. For $k \geq 5$, the principle of constructing a set of $Q_{i_{1} \ldots i_{k}}$ and a system of functional equations is the same.


Figure 1. $4 \times \frac{n}{4}$ board with cut off corners.

### 2.2. Entropy

In this section, for an easier representation, let us denote as $A_{l}(k)$ the number of configurations of a $k \times l$ rectangle that can be easily obtained from the previous subsection. We define the entropy $\eta$ of our system as follows:

## Definition 2.1.

$$
\eta=\lim _{l, k \rightarrow \infty} A_{l}(k)^{\frac{1}{k l}}
$$

Proposition 2.2. For $n, k, l \in \mathbb{N}$

$$
A_{l}(n+k-1) \leq A_{l}(n) A_{l}(k) \leq A_{l}(n+k)
$$

Proof. The right inequality follows from the fact that when we stick two rectangles of heights $k$ and $n$ together we have a perfect boundary, so the number of configurations is less than in the rectangle of size $k+n$ where there can be big squares on the boundary.

To prove the left inequality let us take a paving of a rectangle of height $k+n-1$, and look at what happens at height $k$. On this level we have two types of big squares, those that come from the $(k-1)$ th level and those that come from the $(k+1)$ th level. This means that we can cut the rectangle in two parts, with each part keeping their big squares from the border ( $k$ th level), and filling the missing spaces with small squares. The result consists of 2 valid pavings of sizes $k$ and $n$ respectively, which can be used to cannonically rebuild the initial rectangle of height $k+n-1$.

Corollary 2.3.

$$
\begin{equation*}
A_{l}(p) \leq A_{l}(p-1) \frac{\phi^{l}}{\sqrt{5}} \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
A_{l}(p) \leq \frac{\phi^{l p}}{\sqrt{5}} \tag{2.2}
\end{equation*}
$$

where $\phi=\frac{\sqrt{5}+1}{2}$.
Proof. Take $p=k+1, n=2$ in the left inequality in Proposition 2.2 and remember that $A_{l}(2.2)$ is the $l$ th Fibinacci number to get (2.1). Then apply (2.1) $p-2$ times to get (2.2).
Corollary 2.4.

$$
A_{l}(n) \geq \frac{\phi^{l\left\lceil\frac{n}{2}\right\rceil}}{\sqrt{5}}
$$

Proof. It suffices to take $k=2$ in the right inequality in Proposition 2.2, apply it $n$ times and use the fact that $A_{l}(2.2)$ is the $l$ th Fibonacci number.
Corollary 2.5. Let

$$
A(n)=\lim _{l \rightarrow \infty} A_{l}(n)
$$

then for $n=k p$

$$
\begin{equation*}
A(k)^{p} \leq A(n) \leq A(k+1)^{p} \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
A(k)^{\frac{1}{k}} \leq \eta \leq A(k+1)^{\frac{1}{k}} \tag{2.4}
\end{equation*}
$$

Proof. From (2.3) we get

$$
A(k)^{\frac{1}{k}} \leq A(n)^{\frac{1}{n}} \leq A(k+1)^{\frac{1}{k}}
$$

If we let $p$ tend to $\infty$ with $k$ being fixed we get (2.4).
For $k$ up till 9 we have bounds (see the Table 1) on $A(k p)$ that become tighter with the increase of $k$. The further we could go with the calculation, the tighter bounds could be obtained.

TABLE 1. $k$ is the height of the region, $z_{0}$ - dominant singularity of the corresponding $B G F$, $\%$ - average percentage of space occupied by small squares.

| $k$ | $z_{0}$ | $\%$ |
| ---: | :---: | :---: |
| 3 | 0.7937 | 55.555 |
| 4 | 0.7721 | 46.954 |
| 5 | 0.7701 | 49.507 |
| 6 | 0.7642 | 47.241 |
| 7 | 0.7621 | 47.759 |
| 8 | 0.7596 | 47.029 |
| 9 | 0.7586 | 47.055 |
| 10 | 0.7656 | 46.764 |

### 2.3. Combinatorical Results

Using traditional combinatorical tools (see, e.g., [2]) we can find formulas for our BGFs and extract some properties. We can solve a system of equations and find $Q_{0 \ldots 0}(z, u)$ for small $k$. It starts getting complex for $k \geq 10$ given that the size of the associated matrix grows exponentially.

For example, for $k=4$

$$
\begin{aligned}
Q_{0000}(z, u) & =\frac{1-z^{4}}{1-z^{4}-z^{4} u^{4}-2 z^{8} u^{4}-z^{8}+z^{12} u^{4}+z^{12}} \\
Q_{0000}(z, 1) & =\frac{1-z^{4}}{1-2 z^{4}-3 z^{8}+2 z^{12}}
\end{aligned}
$$

The coefficients $A_{n}^{k}$ of $Q_{0000}(z, 1)$ correspond to tilings of a $4 \times \frac{n}{4}$ rectangle. They satisfy satisfy the recurrence equation: $a_{n}=2 a_{n-1}+3 a_{n-2}-2 a_{n-3}$ with $a_{0}=a_{1}=1, a_{2}=5$ [A054854] [7].

For $k=5$

$$
Q_{00000}=\frac{u^{2} z^{10}+u z^{5}-1}{1-u^{5} z^{5}-u z^{5}-3 u^{6} z^{10}-4 u^{2} z^{10}-u^{7} z^{15}+3 u^{3} z^{15}-3 u^{4} z^{20}}
$$

The standart technique to calculate the expected value of a certain random variable is to differentiate generating functions and then use the singularity analysis [2]. We are interested in calculating the average proportion of space occupied by small squares. Imagine that we wanted to calculate the expected value of the number of small squares, then we could have used the standart technique that is well detailed in, for example, [5] and look at

$$
\frac{\partial_{u} Q_{0 \ldots 0}(z, 1)}{Q_{0 \ldots 0}(z, 1)}
$$

Since we need the proportion of space occupied by small squares, we need to extract the area $n$ from the denominator. We can do this by differentiating $Q_{0 \ldots 0}(z, u)$ by $z$ and multiplying it by $z$. It gives us the following expression:

$$
\frac{\partial_{u} Q_{0 \ldots 0}(z, 1)}{z \partial_{z} Q_{0 \ldots 0}(z, 1)}
$$

And the singularity analysis will directly give us the average proportions of space occupied by small squares in rectangles. Therefore we calculate the singularities of $Q_{0 \ldots 0}(z, u)$ that are the closest to zero. The list of singularities for $k=2, \ldots, 9$ is the folowing: $0.7861,0.7937,0.7721,0.7701,0.7642,0.7621,0.7961,0.7842$. Naming $z_{0}$ the dominant singularity (closest to zero) we get

$$
\left.\frac{\partial_{u} Q_{0 \ldots 0}(z, u)}{z \partial_{z} Q_{0 \ldots 0}(z, u)}\right|_{\left(1, z_{0}\right)}
$$

Average proportions of space occupied by small squares for $k \leq 10$ are shown in Table 1 .


Figure 2. Automaton for $k=4$.

## 3. Automaton representation

For each $k$ let us introduce an automaton. Each $Q_{i_{1} \ldots i_{k}}$ with $i_{j} \in\{0,1,2\}$ for $j=1, \ldots k$ is associated with a state $q=i_{1} \ldots i_{k}$ and each functional equation involving $B G F s$ can be translated into an automaton transition. For example, the relation

$$
Q_{0000}=z u Q_{1000}+z^{4} Q_{2200}
$$

is represented in the following way: an arrow marked by $z u$ goes from the state 1000 to the state 0000 , an arrow marked by $z^{4}$ goes from the state 2200 to the state 0000 . When the tetris rule is applied, we will mark the corresponding arrows by a star.
Commentary: The unusual way, one might say, of directing arrows can be explained by the fact that adding 1 or 22 to the indexes of BGFs corresponds to cutting off corners of the initial rectangle.

For $k=4$ the set of states consists of the states: $0000,1000,2200,1100,1220,2210,2222,1110,1122,1221$, 1111, 0011, 0110, 1011, 2211 and an illustration of the automaton is shown in Figure 2.

We shall refer to the state that consists of all $0 s$ as initial. Calculation of the paths in the automaton that start and end at the initial state will allow us to find formulas for $Q_{0 \ldots 0}$. Our objective is to decrease the computational complexity by reducing the number of states, which basically means reducing the number of functional equations in the system.

### 3.1. EsSENTIAL, NON-ESSENTIAL AND ADDITIONAL STATES

Definition 3.1. A state $q$ of an automaton is called essential if there are at least two arrows coming in and out of $q$ and at least one of the arrows coming out is marked by a star. It is called non-essential otherwise.

Let $E_{k}$ be the set of all essential states for each $k \geq 4$. One can see from Table 1 that $E_{4}=\{1100\}$ and $\left|E_{4}\right|=1$. Let us describe the structure of $E_{k}$ and find $\left|E_{k}\right|$.

Proposition 3.2. A state $q=i_{1} \ldots i_{k}$ of an automaton is essential if and only if $q$ has the following properties:
(1) $q$ consists only of $0 s$ and $1 s$.
(2) All $1 s$ come in consecutive pairs in $q$.
(3) $i_{1}=i_{2}=1$.
(4) There are at least two $0 s$ and the leftmost 0 in $q$ comes in a pair with another 0 .

Proof.
$\Rightarrow$ Let us prove that if $q$ does not have at least one of the four properties, then $q$ is non-essential. If there exists $j \in\{1, \ldots, k\}$ such that $i_{j}=2$ or a block of consecutive $1 s$ whose length is odd in $q$, then no state can be reduced to $q$ by the use of the tetris rule, so no arrow marked by a star comes out of it. If $q$ starts with a zero, then at most one arrow comes out of $q$ (to the state that is reduced to $q$ by the tetris rule). If the leftmost zero in $q$ is isolated, then only one arrow comes in $q$ (from the state with the leftmost 0 being replaced by 1).
$\Leftarrow$ Consider that $q$ has these four properties. Given that the leftmost 0 is not isolated, there are two arrows coming in $q$. And it is clear that there are two arrows that come out of $q$ - one to the state $q^{\prime}=i_{1}+1 \ldots i_{k}+1$ that is reduced to $q$ by the tetris rule and one to a state with a 1 on the left from the leftmost 0 being replaced by a 0 .

Corollary 3.3. With $k \rightarrow \infty$

$$
\left|E_{k}\right| \sim \frac{\phi^{k-3}}{\sqrt{5}}
$$

Proof. Looking at essential states of length $k$ is the same as looking at states of length $k-3$ that are obtained from essential states by deleting the first two $1 s$ and gluing together the first two consecutive $0 s$ in each essential state. The obtaiend states are in bijection with tilings of a strip $1 \times k-3$ by blocks of size $1 \times 1$ and $1 \times 2$.

Proposition 3.4. For $k \geq 4$ the number of essential states $\left|E_{k}\right|$ in the automaton is represented by the following formula:

$$
\left|E_{k}\right|=\sum_{i=1}^{\left\lfloor\frac{k-2}{2}\right\rfloor} \sum_{j=1}^{i}\binom{k-i-j-2}{i-j}
$$

Proof. For every $i$ that corresponds to the number of pairs of 11 we calculate the number of possible essential states. For every $i=1, \ldots,\left\lfloor\frac{k-2}{2}\right\rfloor$ we have a sum on $j=1, \ldots i$ that corresponds to the number of pairs of 11 that precede the first pair of 00 . For $j=1, \ldots i$ the following sum can be obtained:

$$
\binom{k-2-2-(i-1)}{i-1}+\binom{k-4-2-(i-2)}{i-2}+\ldots+\binom{k-2 i-2-(i-i)}{i-i}=\sum_{j=1}^{i}\binom{k-i-j-2}{i-j}
$$

Definition 3.5. A state is called additional if it belongs to a cycle that does not contain any essential or initial states.

Note that only non-essential states can be additional. The interest of looking at additional states is, merely, because in order to properly reduce an automaton and get the explicit formulas for our generating functions, we need to pay attention to all the cycles in the automaton including the cycles that do not pass through the initial state. If not, we might lose some terms in the resulting formulas.

Our objective is to choose additional states in such a way, so that there won't be any cycles left in the automaton that don't include either the initial state, essential states or the chosen additional states. The idea


Figure 3. Cycle with the state $1110 \ldots 0$ for $k \geq 5$.
is to minimize the number of additional states that have to be added to the initial and essential states in order to properly reduce the automaton. We are not going to calculate the number of all additional states. Nor will we minimize this number. Rather, we will define a subset of the set of additional states and try to justify this choice by proving that it provides us with the wanted structure.

Let us take a set that consists of states that have the same structure as the essential states but with an odd block of $1 s$ of size at least 3 on the left from the leftmost 0 . We denote this set by $A_{k}$. It follows from Proposition 2 that

$$
\left|A_{k}\right|=\sum_{i=1}^{\left\lfloor\frac{k-2}{2}\right\rfloor} \sum_{j=1}^{i}\binom{k-2 j-3-i}{i-j}=\left|E_{k-1}\right|
$$

Proposition 3.6. Each state from $A_{k}$ is additional.
Proof. We need to show that every $q \in A_{k}$ belongs to at least one cycle that doesn't contain an essential state or the initial state. For $i=1 q_{1}=1110 \ldots 0$ and the cycle is schematically shown in Figure 2 with $f(z, u)$ and $g(z, u)$ being transition functions between states. There are as many cycles as there are possible ways to get from the state $00011 \ldots \ldots$ to the state $111220 \ldots 0$. It is not difficult to see that there are no essential states between those two states. For other $i$ the structure of cycles that contain $q_{i}$ is analogous.
Corollary 3.7. There are states in $A_{k}$ for $k \geq 6$ that belong to more than cycle with no essential or initial states contained in it.

Now the question is, if we mark the initial state and all the states from $E_{k}$ and $A_{k}$ in the automaton, does it ensure that there are no cycles left that don't contain the marked states? Proving that will justify our choice for keeping these particular states.

Proposition 3.8. Let $q=i_{1} \ldots i_{k}$ be a state that doesn't belong to $A_{k} \cup E_{k} \cup\{0 \ldots 0\}$. Then a cycle (or cycles) that $q$ belongs to, contains states from $A_{k} \cup E_{k} \cup\{0 \ldots 0\}$.

Proof. Let us point out that it is sufficient to prove the statement only for the states that consist of 0 and 1. So let $q=i_{1} \ldots i_{k}$ with $i_{j} \in\{0,1\}, j=1, \ldots k$ (Fig. 3 might help to visualize the cycles.) If the leftmost zero in $q$ is isolated, then there is only one arrow coming in this state from a state from $A_{k} \cup E_{k}$. If the leftmost zero is not isolated, let $i_{1}=\ldots=i_{l}=1, i_{l+1}=i_{l+2}=0,0 \leq l \leq k-2$. If $l=0$, then $q=00 \ldots 011 i_{n+3} \ldots i_{k}$ where $i_{1}, \ldots, i_{n}=0, n \geq 2$. In this case $q$ belongs to a cycle with the state $11 \ldots 100 \ldots 0$ belonging to $A_{k} \cup E_{k}$. The situation is similar if $l=1$ apart from the case when $i_{j}=0$ for all $k+3 \leq j \leq k$. Then $q=10 \ldots 0$ and belongs to a cycle that contains the initial state. For $l \geq 2$ since all $1 s$ after the $l+2$ coordinate in $q$ come in pair, $q$ belongs to a cycle (cycles) with a state from $A_{k} \cup E_{k}$ where the block with leftmost $0 s$ is filled with $1 s$.

Remark 3.9. We shall further refer to the states from $A_{k}$ as additional ${ }^{\star}$.






Figure 4. Rules of reduction for an automaton.


Figure 5. Reduced automaton for $k=4$.


Figure 6. Reduced automaton for $k=5$.

### 3.2. Simplified automata

We can simplify an automaton by keeping only the initial, essential and additional* states and reducing all other states. The rules of reduction are shown in Figure 4. We denote by $f_{i j}$ a transition between states $q_{i}$ and $q_{j}$ which is represented by an arrow going from $q_{i}$ to $q_{j}$.

In the case $k=4$, there is one essential state 1100 and no additional states. A reduced automaton for the case $k=4$ is shown in Figure 5 .

For a $5 \times n$ rectangle there is one essential state 11000 and one additional* state 11100 . A reduced automaton is shown in Figure 6. For $k=6,7$ reduced automata are schematically shown in Figure 7.

## 4. Matrix Representation

Another point of view on this problem comes from the problem of non-attacking kings where one looks at the number of ways kings could be put on a rectangular board without having two king attacking each other. For the sake of completeness we mention this approach and results that were obtained previously.


Figure 7. Reduced automata: for $k=6$ (right), for $k=7$ (left).

There is a natural bijection between tilings of a $k \times k$ region by $1 \times 1$ and $2 \times 2$ squares and configurations of non-attacking kings on a $(k-1) \times(k-1)$ board. One simply has to consider that every big square has a king in his left bottom corner, and having no kings attacking each other is equivalent to having no big squares intersecting.

This problem has been studied using the matrix approach [1]. Namely, the use of the adjacency matrix of the graph where nodes are lines of the rectangle, two nodes are connected if the two lines can be put on top of each other (for the use of this approach see also [6]). These adjacency matrices verify the folowing recursion:

$$
A_{0}=(1), A_{1}=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right), \ldots, A_{n}=\left(\begin{array}{cc}
A_{n-1} & A_{n-2} \\
A_{n-2} & 0
\end{array}\right)
$$

This is in relation with the previous method, as the dominant eigenvalue of $A_{i}$ is equal to the inverse of the singularity of $Q_{i}(z, 1)$. In [1] it was used to compute good approximation of the entropy but it can be also used to compute the average proportions of space occupied by small squares for a given $k$. Indeed, one can compute for a fixed $k \times n$ the number of big squares by taking a matrix $M$ which is a diagonal with the number of big squares in the corresponding node, and calculate

$$
b=\mathbf{1}\left(\sum_{p \leq n} A_{k}^{p} M A_{k}^{n-p}\right) \mathbf{1}^{T},
$$

where $\mathbf{1}=(1, \ldots, 1)$.Then the proportion of big squares equals

$$
\frac{4 b}{k c}
$$

where

$$
c=\mathbf{1}\left(\sum_{p \leq n} A_{k}^{p} I A_{k}^{n-p}\right) \mathbf{1}^{T} .
$$

is the number of lines in all the configurations. We can then obtain result when $n \rightarrow \infty$ by putting $A_{k}$ in Jordan form and dividing both $b$ and $c$ by $\lambda_{1}^{n}$ where $\lambda_{1}$ is the dominant eigenvalue of $A_{k}$.

This method allows us to find the result for $k$ up to 9 , the Jordan form beeing the limit factor as it is numerically unstable, hence not available in numerical package, and that exact resolution does not scale well with the size.


Figure 8. Automaton for $k=4$, cylindrical case.


Figure 9. Reduced automaton for $k=4$, cylindrical case.

## 5. Cylindrical case

Let us now consider a $k \times \frac{n}{k}$ rectangle with sewn horizontal borders. We obtain a cylindrical region that we want to tile with $1 \times 1$ and $2 \times 2$ squares. The way of constructing $B G F \mathrm{~s}$ stays the same with the only difference that now we allow having separated 2 s in the first and last positions of the indexes of Generating Functions. The functional equations, therefore, change. For example,

$$
Q_{0000}^{c}=z u Q_{1000}^{c}+z^{4} Q_{2200}^{c}+z^{4} Q_{2002}^{c},
$$

where the index $c$ is used to distinguish between cylindrical and plain cases.
Let us construct an automaton in the same way as before (see Fig. 8).
The notions of essential and additional states stay the same. The set $E_{k}^{c}$ of essential states in the cylindrical case equals $E_{k}$. But the set $A_{k}^{c}$ constists not only of states from $A_{k}$, but also of $E_{k-2}$ and the initial state of size $k-2$ where $1 s$ are added in the first and last positions which might of course create a rightmost block of an odd length but that's due to the fact that there are states with a 2 in the first and the last positions. So $\left|A_{k}^{c}\right|=\left|A_{k}\right|+\left|E_{k-2}\right|+1$.

In the case $k=4$ there is one essential state 1100 as in plain case and one additional* state 1001 . The rules of reduction stay the same and a reduced automaton is shown in Figure 9.


Figure 10. Reduced automata in cylindrical case: $k=5$ (left), $k=6$ (right).
TABLE 2. $k$ is the height of the region, \% plain - average percentage of space occupied by small squares in the plain case, $\%$ cylindrical - in the cylindrical case.

| $k$ | \% plain | \% cylindrical |
| ---: | :---: | :---: |
| 3 | 55.555 | 51.823 |
| 4 | 46.954 | 42.606 |
| 5 | 49.507 | 46.605 |
| 6 | 47.241 | 44.680 |
| 7 | 47.759 | 45.594 |
| 8 | 47.029 | 45.147 |
| 9 | 47.055 |  |
| 10 | 46.764 |  |

In this case we have

$$
Q_{0000}^{c}=\frac{1-z^{4}}{1-3 u^{4} z^{8}+2 z^{1} 2-u^{4} z^{4}-2 z^{8}-z^{4}},
$$

and small squares occupy, in average, 0.466 of the space which is smaller than in the plain case. It is rather understandable - because of the sewn boarders there are less constraints on the way big squares can be placed.

Reduced automata for $k=5,6$ are schematically shown in Figure 10.

## 6. Conclusion

Average proportions of space occupied by small squares for $k \leq 10$ for the plain case and $k \leq 8$ for the cylindrical case are shown in Table 2.

Representation by automata allows us to reduce the computational complexity and obtain combinatorical results for larger $k$. Although it remains unclear what to do when $k$ grows given that even after reduction complexity stays exponential. Thefore, the main questions are still open: does the sequence of average proportions converge in the plain/cylindrical case? If so, what is its limit? Is there a relation between automata of sizes $k$ and $k+1$ in the plain/cylindrical case? What can we say about the average proportion of small squares if both $k$ and $n$ tend to infinity? And finally, what is the number of small squares that maximises the number of tilings of a $k \times n$ rectangle? These questions probably need different approach.

Acknowledgements. We would like to thank Olivier Bodini, Thomas Fernique and Michael Rao for fruitful discussions and a referee for substantial remarks.

## References

[1] N.J. Calkin, K. James, Sh. Purvis, Sh. Race, K. Schneider and M. Yancey, Counting kings: as easy as $\lambda_{1}, \lambda_{2}, \lambda_{3} \ldots$ Congr. Numer. 183 (2006) 83-95.
[2] P. Flajolet and R. Sedgewick, Analytical Combinatorics. Cambridge University Press (2009).
[3] S. Heubach, Tiling an m-by-n Area with Squares of Size up to k-by-k with m $\leq 5$. Congr. Numerantium 140 (1999) $43-64$.
[4] S. Heubach and P. Chinn, Patterns Arising From Tiling Rectangles with $1 \times 1$ and $2 \times 2$ Squares. Congr. Numerantium 150 (2001) 173-192.
[5] M.-L. Lackner and M. Wallner, An invitation to analytic combinatorics and lattice path counting. Lecture note of the 2015 ALEA in Europe Young Researchers' Workshop (2015).
[6] R.J. Mathar, Tilings of Rectangular Regions by Rectangular Tiles: Counts Derived from Transfer Matrices. Preprint arXiv:1406. 7788 (2014).
[7] N.J.A. Sloane and S. Plouffe, The Encyclopedia of Integer Sequences. Academic Press, San Diego (1995).
Communicated by D. Jamet.
Received March 24, 2016. Accepted April 6, 2016.


[^0]:    Keywords and phrases. Tiling, square tiles, generating functions, automaton, strip tilings, tiling graph.
    1 LIPN, Paris 13, France. nicolas.rolin@lipn.univ-paris13.fr

