# AN ISOLATED POINT IN THE HEINIS SPECTRUM 

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#### Abstract

This paper continues the study initiated by Alex Heinis of the set $H$ of pairs ( $\alpha, \beta$ ) obtained as the lower and upper limit of the ratio of complexity and length for an infinite word. Heinis proved that this set contains no point under a certain curve. We extend this result by proving that there are only three points on this curve, namely $(1,1),\left(\frac{3}{2}, \frac{5}{3}\right)$ and $(2,2)$, and moreover the point $\left(\frac{3}{2}, \frac{5}{3}\right)$ is an isolated point in the set $H$. For this, we use Rauzy graphs, generalizing techniques of Ali Aberkane.


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## 1. Introduction

The complexity function $p(n)$, which counts the number of factors of length $n$ in an infinite word $u$ on a finite alphabet $\mathcal{A}$, has been extensively studied. We refer the reader to Chapter 4 in [4] which is devoted to this function. In particular, a vast open question is to know which functions $p$ can be complexity functions; even after taking into account some immediate necessary conditions ( $p$ must be increasing, with $p(m+n) \leq p(m) p(n)$ for all $m, n$ ), it has been known since Morse and Hedlund [11] that some functions, for example $[\sqrt{n}]$, cannot be realized as a complexity function.

More recently, Alex Heinis [7,8] made a fine study of the possible values of the couple $\alpha=\lim \inf _{n \rightarrow \infty} \frac{p(n)}{n}$, $\beta=\lim \sup _{n \rightarrow \infty} \frac{p(n)}{n}$, for $p$ the complexity function of an infinite word $u$ : again, there is one obvious condition, $\alpha \leq \beta$, and a condition from [11] that either $\beta=0$ or $\alpha \geq 1$; but Heinis showed that there are many more constraints: indeed we cannot have $\alpha=\beta \in] 1,2[$, and when we suppose that $1<\alpha<2$, then $\beta$ is bounded from below by the function $\frac{3 \alpha-2}{\alpha}$.

For practical reasons, we restrict ourselves to recurrent infinite words. An infinite word $u$ is recurrent if every factor of $u$ has infinitely many occurrences in $u$. It is likely that the results would not change without this hypothesis, but proofs would be more cumbersome as several additional shapes of Rauzy graphs would have to be considered.

Thus we can define the Heinis spectrum

$$
H=\left\{(\alpha, \beta): u \in \mathcal{A}^{\mathbb{N}}, u \text { recurrent }\right\} .
$$

[^0]By Heinis's result, $H$ is above the curve $C: \beta=\frac{3 \alpha-2}{\alpha}$. Heinis constructed one point in $C \cap H$, namely the point $\left(\frac{3}{2}, \frac{5}{3}\right)$.

The goal of the present paper is to study how close points in $H$ can get to the curve $C$. For this we study Rauzy graphs and their evolution for an infinite word, starting with those of Sturmian words which have only two types of evolution. This was extended by Ali Aberkane in [2] who showed that, when an infinite recurrent word $u$ has a complexity $p(n) \leq \frac{4}{3} n+1$, then we can describe three types of evolutions of Rauzy graphs which are denoted by $O_{1, x}, O_{1, y}$, and $O_{m, x}$.

Then we introduce a new class of infinite words, called words of type $u_{\beta}$, defined by the condition $\beta<\frac{4 \alpha}{2+\alpha}$, we describe the evolution of their Rauzy graphs, adapting the techniques of Aberkane to this new class. A particular case gives the point $\left(\frac{3}{2}, \frac{5}{3}\right)$ in $H$, following Heinis.

In the main part of the paper, we study how properties of evolutions of the Rauzy graphs imply relations between $\alpha$ and $\beta$. We show that if $u$ is an infinite word of type $u_{\beta}$ such that $\beta$ is less than $\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}$, then $\alpha$ is always equal to $\frac{3}{2}$ and $\beta$ is equal to $\frac{5}{3}$. Therefore, we can say that $\left(\frac{3}{2}, \frac{5}{3}\right)$ is the only point of $H$ that lies on $C$ with $1<\alpha<2$, and that it is isolated in $H$.

## 2. Definitions

### 2.1. Some basic notions of combinatorics on words

We first recall some basic definitions in combinatorics on words, and introduce some notations. For more details see [10].
(1) A word is a finite sequence of letters on a finite alphabet $\mathcal{A}$.
(2) We denote by $x[i]$ the i-th letter of a word $x$.
(3) The length of a word $u$ is the number of letters in this word. We denote it by $|u|$.
(4) An infinite word is a sequence indexed by $\mathbb{N}$ with values in $\mathcal{A}$. We denote it by $u=u_{0} u_{1} \ldots$
(5) Let $v$ be a finite word. A word $w$ is a prefix (respectively a suffix) of $v$ if there exists a word $z$ such that $v=w z$ (respectively $v=z w$ ). We denote by $\operatorname{pref}_{i}(v)$ the prefix of length $i$ of $v$ and by $\operatorname{suff}_{i}(v)$ the suffix of length $i$ of $v$. If $w$ is a prefix of length $i$ of $v$, then $w^{-1} v$ is suff $|v|-|w|$,
(6) Let $u$ be an infinite word. The word $u$ is periodic if there exists $T \geq 1$ such that for all $n \geq 0$, we have $u_{n+T}=u_{n}$. We say that $u$ is eventually periodic if there exist $T \geq 1$ and $n_{0} \geq 0$, such that for all $n \geq n_{0}$, $u_{n+T}=u_{n}$. In this case, $u$ can be written $u=v w^{\omega}$ where $v=u_{0} \ldots u_{n_{0}-1}$ and $w=u_{n_{0}} u_{n_{0}+1} \ldots u_{n_{0}+T-1}$.
(7) A word $w$ of length $n$ is a factor of an infinite word $u$ if there exists $n_{0} \in \mathbb{N}$ such that $w=$ $u_{n_{0}} u_{n_{0}+1} \ldots u_{n_{0}+n-1}$. We denote by $L(u)$ the set of all factors of $u$.
(8) Let $u$ be an infinite word. We denote by $L_{n}(u)$ the set of factors of length $n$ of $u$. Then we set $p_{u}(n)=$ $\# L_{n}(u)$ with $p_{u}(0)=1$.
This function $p$ is called the complexity function of $u$.
(9) Let $s(n)=p(n+1)-p(n)$ denote the first difference of the complexity function.
(10) Let $u$ be an infinite word on an alphabet $\mathcal{A}$ and $w$ a factor of $u$.

- The factor $w$ is a right special factor (respectively left special factor) for $u$ if there exist two distinct letters, $a$ and $b$, such that $w a$ and $w b$ (respectively $a w$ and $b w$ ) are in $L(u)$. The set of right special factors (respectively left special factors) of length $n$ is denoted by $R S_{n}$ (respectively $L S_{n}$ ).
- We say that a factor $w$ is a bispecial factor if it is both right special and left special.
(11) Let $u$ be an infinite word on the binary alphabet $\{a, b\}$ and $w$ a bispecial factor of $u$. The factor $w$ is $a$ strong bispecial factor of $u$ if and only if $a w a, b w b, a w b$ and bwa belong to $L(u)$; if $L(u)$ contains only three factors out of these four then $w$ is an ordinary bispecial factor; and finally if it contains only two of them then $w$ is a weak bispecial factor. These definitions can also be used on a larger alphabet when it is known that $w$ has only two extensions to the left and two extensions to the right.
(12) An infinite word $u$ is called Sturmian if its complexity is $p(n)=n+1$ for all $n \in \mathbb{N}$.
(13) An infinite word $u$ is said to be recurrent if every factor of $u$ has infinitely many occurrences in $u$.


### 2.2. The Heinis spectrum

Let $u \in \mathcal{A}^{\omega}$ be a recurrent infinite word. We define $\alpha(u):=\liminf _{n \rightarrow \infty} \frac{p(n)}{n}$ and $\beta(u):=\limsup _{n \rightarrow \infty} \frac{p(n)}{n}$ (usually we will simply write $\alpha(u)=\alpha$ and $\beta(u)=\beta$ where there is no ambiguity).

When $u$ is periodic, then $\alpha=\beta=0$; otherwise, by Morse and Hedlund's Theorem, [11], $\beta \geq \alpha \geq 1$. In particular, if $u$ is a Sturmian word, $\alpha=\beta=1$.

The Heinis spectrum is the set

$$
H=\left\{(\alpha, \beta): u \in \mathcal{A}^{\mathbb{N}}, u \text { recurrent }\right\} \subseteq(\mathbb{R} \cup\{+\infty\})^{2}
$$

Theorem $2.1([7])$. If $(\alpha, \beta) \in H$ with $\alpha \neq 0$, then $\beta \geq \frac{3 \alpha-2}{\alpha}$.
The following Lemma (implicit in [7]) will be useful to compute $\alpha$ and $\beta$.
Lemma 2.2. Let $u$ be a recurrent infinite word.
(1) If $s(n)$ is eventually constant or if $\lim _{n \rightarrow \infty} s(n)=+\infty$, then $\alpha=\beta=\lim _{n \rightarrow \infty} s(n)$.
(2) Otherwise, let $V^{+}=\{n: s(n)>s(n-1)\}$ and $V^{-}=\{n: s(n)<s(n-1)\}$, then

$$
\alpha=\liminf _{n \in V^{+}} \frac{p(n)}{n} \text { and } \beta=\limsup _{n \in V^{-}} \frac{p(n)}{n} .
$$

Proof.
(1) If $s(n)=d$ for all $n \geq n_{0}$, then $p(n)=p\left(n_{0}\right)+d\left(n-n_{0}\right)$ so that $\lim _{n \rightarrow \infty} \frac{p(n)}{n}=d$. If $\lim _{n \rightarrow \infty} s(n)=+\infty$, then for all $d$ there exists $n_{0}$ such that $s(n) \geq d$ and $p(n) \geq p\left(n_{0}\right)+d\left(n-n_{0}\right)$ for all $n \geq n_{0}$, and thus $\lim _{n \rightarrow \infty} \frac{p(n)}{n}=+\infty$.
(2) If the conditions of (1) are not satisfied, then both $V^{+}$and $V^{-}$are infinite. Let $n_{1}$ and $n_{2}$ be two successive elements of $V^{+}$. Then $s(n) \leq s(n-1)$ for all $n$ such that $n_{1}<n<n_{2}$. It follows that

$$
p(n) \geq p\left(n_{1}\right)+\left(n-n_{1}\right) s(n) \text { and } p(n) \geq p\left(n_{2}\right)-\left(n_{2}-n\right) s(n)
$$

We multiply the first inequation by $\left(n_{2}-n\right)$ and the second by $\left(n-n_{1}\right)$. Summing the resulting inequations, we get

$$
\left(n_{2}-n_{1}\right) p(n) \geq\left(n_{2}-n\right) p\left(n_{1}\right)+\left(n-n_{1}\right) p\left(n_{2}\right)
$$

and thus

$$
\frac{p(n)}{n} \geq \frac{n_{1}\left(n_{2}-n\right)}{n\left(n_{2}-n_{1}\right)} \frac{p\left(n_{1}\right)}{n_{1}}+\frac{n_{2}\left(n-n_{1}\right)}{n\left(n_{2}-n_{1}\right)} \frac{p\left(n_{2}\right)}{n_{2}} \geq \min \left(\frac{p\left(n_{1}\right)}{n_{1}}, \frac{p\left(n_{2}\right)}{n_{2}}\right)
$$

It follows that $\liminf _{n \rightarrow \infty} \frac{p(n)}{n}=\liminf _{n \in V^{+}} \frac{p(n)}{n}$.
The proof for $\beta$ is similar.

## 3. RAUZY GRAPHS AND THEIR EVOLUTION

Let $u$ be an infinite word defined on a finite alphabet $\mathcal{A}$. To describe its structure, we associate with the set of factors of $u$ a family of graphs, called Rauzy graphs. Rauzy graphs were defined in [3], and have proved to be a very useful tool to study families of infinite words with a constrained complexity function, e.g. $p(n)=2 n$ in [12], or $s(n) \leq 2$ in [9].


Figure 1. The first Rauzy graphs associated with the fixed point of the substitution defined by: $\sigma(a)=b b$ et $\sigma(b)=b a$.

Definition 3.1. For all $n \in \mathbb{N}$, the Rauzy graph of order $n$ of $u$ is the labelled directed graph, denoted by $\Gamma_{n}=\Gamma_{n}(u)$, such that:

- Its vertices are the factors of length $n$ of $u$.
- There exists an edge from the vertex $w$ to the vertex $v$ if and only if there exist two elements $a$ and $b$ of $\mathcal{A}$, satisfying $w a=b v$, such that $w a$ is a factor of length $n+1$ of $u$. We then say that the words $w$ and $v$ follow each other in the infinite word $u$. The letter $a$ is the label of the edge from $w$ to $v$.

Remark 3.2. Let $B=\left(w_{0}, w_{1}, w_{2}, \ldots, w_{k}\right)$ be a directed path in the graph $\Gamma_{n}$. For all $i \in[1, k]$, there exist two letters $a_{i}$ and $b_{i}$ in $\mathcal{A}$ such that $w_{i-1} a_{i}=b_{i} w_{i}$. The path $B$ is labelled by the word $a_{1} a_{2} \ldots a_{k}$. The length of this path is $\left|a_{1} a_{2} \ldots a_{k}\right|=k$. Note that several paths may be labelled by the same word.

Definition 3.3. Let $w$ be a factor of $u$ of length $n+k$ for some $k \geq 0$, and let $w_{i}$ be the factor of length $n$ occurring at position $i$ in $w$, for $0 \leq i \leq k$. Then $B=\left(w_{0}, w_{1}, w_{2}, \ldots, w_{k}\right)$ is a directed path in the graph $\Gamma_{n}$, which is called the path associated with $w$. The label of $B$ is a word $w^{\prime}$ such that $w=w_{0} w^{\prime}$.

Definition 3.4. A substitution $\sigma$ is a non-erasing endomorphism of $\mathcal{A}^{*}$.
Example 3.5. Consider the substitution

$$
\sigma:\left\{\begin{array}{l}
a \mapsto b b \\
b \mapsto b a
\end{array}\right.
$$

Then $u=\lim _{n \rightarrow \infty} \sigma^{n}(b)=b a b b b a b a b a b b b a b b b a b b b a b a b a b b b a b \ldots$ is the fixed point of this substitution. The first Rauzy graphs of this infinite word are represented on Figure 1.

### 3.1. Evolution of Rauzy graphs

Informally, we call evolution of a Rauzy graph $\Gamma_{n}$ the sequence of the following Rauzy graphs $\left(\Gamma_{n+1}, \Gamma_{n+2}, \ldots, \Gamma_{n+l}\right)$ for some $l \geq 1$. The question we want to address here is: if we are given only $\Gamma_{n}$, without any knowledge of the infinite word $u$, are we able to predict the evolution of $\Gamma_{n}$, and up to which length $l$ ? More generally, what are the possible evolutions of $\Gamma_{n}$ ?

In order to give a formal definition, we need more notations. Let $\mathcal{G}$ be the set of all possible Rauzy graphs for infinite words on a finite alphabet $\mathcal{A} \subset \mathcal{A}_{0}$ (where $\mathcal{A}_{0}$ is a fixed infinite set, e.g. $\mathcal{A}_{0}=\mathbb{N}$, otherwise $\mathcal{G}$ would not be a properly defined set). Then, we can write:

$$
\mathcal{G}=\left\{\Gamma_{n}(u): n \in \mathbb{N}, \mathcal{A} \text { a finite alphabet, } u \in \mathcal{A}^{\omega}\right\}
$$

Given $G \in \mathcal{G}$, there is an integer $n$, an alphabet $\mathcal{A}$ and an infinite word $u \in \mathcal{A}^{\mathbb{N}}$ such that $G=\Gamma_{n}(u)$. Moreover, $n$ is unique (recall that the vertices of a Rauzy graph are words such that $n$ is their length). An evolution is a partial map $E: \mathcal{G} \rightarrow \mathcal{G}^{*}=\cup_{l \geq 0} \mathcal{G}^{l}$ such that, if $E(G)=\left(G_{1}, G_{2}, \ldots, G_{l}\right)$, there exist $n \in \mathbb{N}$, an alphabet $\mathcal{A}$ and $u \in \mathcal{A}^{\omega}$ such that $\Gamma_{n}(u)=\bar{G}$ and $\Gamma_{n+i}(u)=G_{i}$ for all $i$ from 1 to $l$. Note that usually $E$ will be defined only on a subset of $\mathcal{G}$. We say that $\Gamma_{n}$ undergoes the evolution $E$ (in a given infinite word $u$ ), if there exists $l$ such that $E\left(\Gamma_{n}\right)=\left(\Gamma_{n+1}, \ldots, \Gamma_{n+l}\right)$. Now, we call $\mathcal{E}$ the set of the evolutions. If we take $E_{1}$ and $E_{2}$ from $\mathcal{E}$, let $E_{1} E_{2}$ be the evolution that maps $G$ to $\left(G_{1}, \ldots, G_{l}, G_{1}^{\prime}, \ldots, G_{l^{\prime}}^{\prime}\right)$ where $E_{1}(G)=\left(G_{1}, \ldots, G_{l}\right)$ and $E_{2}\left(G_{l}\right)=\left(G_{1}^{\prime}, \ldots, G_{l^{\prime}}^{\prime}\right)$. This endows $\mathcal{E}$ with a monoid structure.

First, we study the evolutions of length 1 . Let $\Gamma_{n}$ and $\Gamma_{n+1}$ be two consecutive Rauzy graphs of $u$. How are they related? There is a one-to-one correspondence between edges of $\Gamma_{n}$ and vertices of $\Gamma_{n+1}$. Indeed, both are in one-to-one correspondence with factors of length $n+1$ of $u$ : vertices of $\Gamma_{n+1}$ by definition, and edges of $\Gamma_{n}$ when viewed as paths of length 1 associated with factors of length $n+1$. In the sequel, we will identify both edges of $\Gamma_{n}$ and vertices of $\Gamma_{n+1}$ with the associated words. So we can say that the vertices of $\Gamma_{n+1}$ are the edges of $\Gamma_{n}$.
Definition 3.6. Let $G=(V, E)$ be a directed graph. The derived graph of $G$, denoted by $D(G)=\left(V^{\prime}, E^{\prime}\right)$, is the directed graph such that:

- Its vertices are the edges of $G$, i.e. $V^{\prime}=E$.
- It admits an edge from the vertex $x$ to the vertex $y$ when in $G$, the ending vertex of the edge $x$ is the starting vertex of the edge $y$ (we can then say that the edges $x$ and $y$ of $G$ are consecutive).
Remark 3.7 ([1]). The derived graph $D\left(\Gamma_{n}\right)$ of a Rauzy graph $\Gamma_{n}$ is thus the directed graph such that:
- Its vertices are the edges of $\Gamma_{n}$, i.e. the factors of length $n+1$ of u.
- It admits an edge from the vertex $w$ to the vertex $v$ if and only if there exists $a$ and $b$ in $\mathcal{A}$ satisfying $w a=b v$. This edge has label $a$.

We can notice that in the Rauzy graph $\Gamma_{n+1}$, in order to have an edge from $w$ to $v$, wa must belong to $L_{n+1}(u)$. But it is not important in $D\left(\Gamma_{n}\right)$. We know that any edge of $\Gamma_{n+1}$ connects two vertices associated with two factors $w$ and $v$ of $u$ such that $w a=b v$. These two vertices correspond to two consecutive edges of $\Gamma_{n}$ and thus are also connected in $D\left(\Gamma_{n}\right)$. Therefore $\Gamma_{n+1}$ is a subgraph of $D\left(\Gamma_{n}\right)$.

The following proposition shows how $\Gamma_{n}$ and $\Gamma_{n+1}$ are related.
Proposition 3.8 ([5]). Let $u$ be an infinite word and $\left(\Gamma_{n}\right)_{n \in \mathbb{N}}$ be its sequence of Rauzy graphs. We always have $\Gamma_{n+1} \subset D\left(\Gamma_{n}\right)$. Moreover, if $u$ is recurrent and $R S_{n} \cap L S_{n}=\varnothing$ then $\Gamma_{n+1}=D\left(\Gamma_{n}\right)$.
The consequence of this proposition is that, if there is no bispecial factor, we can easily construct the graph $\Gamma_{n+1}$. So, the most important is to study the case where we have a bispecial factor.

To study this case, we introduce the evolution $E_{b} . E_{b}$ is an evolution up to the next graph with a bispecial factor. We can define $E_{b}$ in this way:

$$
\begin{cases}E_{b}\left(G_{0}\right) \text { undefined } & \text { if } G_{0} \text { has no right special factor } \\ E_{b}\left(G_{0}\right)=\varnothing & \text { if } G_{0} \text { has a bispecial factor } \\ E_{b}\left(G_{0}\right)=\left(G_{1}, \ldots, G_{l}\right) & \text { otherwise, where } G_{l} \text { has a bispecial factor, } \\ & G_{0}, \ldots, G_{l-1} \text { have no bispecial factor, } \\ & \text { and } G_{i+1}=D\left(G_{i}\right) \text { for } 0 \leq i \leq l-1\end{cases}
$$

Remark 3.9. If $G_{0}$ has no right special factor, then any $u$ such that $\Gamma_{n}(u)=G_{0}$ is eventually periodic and has no bispecial factor of length more than $n$. Otherwise, there is always $l \geq 1$ such that $\Gamma_{n+l}$ has a bispecial factor.


Figure 2. The two types of Sturmian graph.

### 3.2. The evolution of Rauzy graphs of infinite Sturmian words

First of all, we recall the particular case of infinite Sturmian words.
The evolution of their Rauzy graphs is well-known. We adopt the presentation of Ali Aberkane.

### 3.2.1. Some properties of Sturmian words and their Rauzy graphs

Proposition 3.10 ([5]). Let u be a binary infinite word. The cardinality of the set of right special factors of length $n$, called $R S_{n}$, is equal to $s(n)$. If moreover $u$ is recurrent, the cardinality of the set $L S_{n}$ is also equal to $s(n)$.

Remark 3.11. In the case of an infinite Sturmian word which is always recurrent, we can notice that $s(n)=1$ for all $n \in \mathbb{N}$, i.e. there is only one right special factor and one left special factor for each length.

As every infinite Sturmian word is recurrent, we can say that a Rauzy graph of an infinite Sturmian word $u$ has one of only two possible shapes. We call Sturmian graph a Rauzy graph that has one of these two shapes (even if it is not associated with a Sturmian word). The first is the graph which has only one bispecial factor and the second is the graph which has only one right special factor and one left special factor distinct from each other.

Notation 3.12. We now give a notation for these two types of Sturmian graphs described above and illustrated on Figure 2.
(1) We denote by $S(n, x, y)$ the graph of order $n$ with a bispecial factor and $x, y$ are the words which are the labels of the two loops. We suppose that $|x| \geq|y|$. From $n, x$ and $y$, we can find the vertices. In particular, the bispecial factor $w$ is determined by $n, x$ and $y$.
(2) We denote by $T(n, x, y, z)$ the graph of order $n$ with only one right and one left special factor, $x, y$ and $z$ are the words labelling the branches (see Fig. 2).

## Remark 3.13.

- If $\Gamma_{n}=S(n, x, y)$ and $w$ is the bispecial factor of length $n$, then $x$ and $y$ start with different letters and satisfy $w x=x^{\prime} w$ and $w y=y^{\prime} w$ where $x^{\prime}$ and $y^{\prime}$ are two words that end with different letters.
- If $\Gamma_{n}=T(n, x, y, z)$, if $w_{1}$ is the right special vertex and $w_{2}$ the left special vertex, there exist $x^{\prime}, y^{\prime}$ and $z^{\prime}$ such that $w_{1} x=x^{\prime} w_{2}, w_{1} y=y^{\prime} w_{2}$ and $w_{2} z=z^{\prime} w_{1}$.


### 3.2.2. The derived graph of these two types of graphs

In this part, we study the shape of the derived graph of a Sturmian graph. Suppose first that $\Gamma_{n}$ is a graph of type $T$. Then $\Gamma_{n+1}=D\left(\Gamma_{n}\right)$ by Proposition 3.8. If $\Gamma_{n}=T(n, x, y, z)$ then $\Gamma_{n+1}=D(T(n, x, y, z))=$ :

$$
\begin{cases}S(n+1, x z, y z) & \text { if }|z|=1 \\ T\left(n+1, x \operatorname{pref}_{1}(z), y \operatorname{pref}_{1}(z), \operatorname{suff}_{|z|-1}(z)\right) \text { if }|z|>1\end{cases}
$$



Figure 3. A graph of type $S$ and its derived graph.

Iterating this, we get $E_{b}(T(n, x, y, z))=\left(T\left(n+1, x \operatorname{pref}_{1}(z), y \operatorname{pref}_{1}(z)\right.\right.$, suff $\left._{|z|-1}(z)\right), \ldots, T(n+|z|-$ $\left.\left.1, x \operatorname{pref}_{|z|-1}(z), y \operatorname{pref}_{|z|-1}(z), \operatorname{suff}_{1}(z)\right), S(n+|z|, x z, y z)\right)$.

We suppose now that $\Gamma_{n}$ is a graph of type $S$. We can notice that $D\left(\Gamma_{n}\right)$ is a graph with $n+2$ vertices and $n+4$ edges. Since $p(n+2)=n+3$, then $\Gamma_{n+1}$ is a subgraph of $D\left(\Gamma_{n}\right)$ with $n+3$ edges. Then, to obtain $\Gamma_{n+1}$, we should take off one edge from the graph $D\left(\Gamma_{n}\right)$. Since $\Gamma_{n+1}$ has to be of type $S$ or $T$, the only edges that we can take off are $e_{2}$ and $e_{4}$ (see Fig. 3).

If we take off $e_{2}$ then $\Gamma_{n+1}=T\left(n+1, x \operatorname{pref}_{1}(y), \operatorname{pref}_{1}(y), \operatorname{suff}_{|y|-1}(y)\right)$ (we assume for simplicity that $|y| \geq 2$; if $|y|=1$, we directly get $S(n+1, x y, y)$ ). Then this graph $\Gamma_{n+1}$ undergoes the evolution $E_{b}$ which ends with the graph $\Gamma_{n+|y|}=S(n+|y|, x y, y)$.

If we take off $e_{4}$ then $\Gamma_{n+1}=T\left(n+1, y \operatorname{pref}_{1}(x), \operatorname{pref}_{1}(x), \operatorname{suff}|x|-1(x)\right)$. Then $\Gamma_{n+1}$ undergoes the evolution $E_{b}$ which ends with $\Gamma_{n+|x|}=S(n+|x|, y x, x)$.
Definition 3.14. We define two types of evolutions between two graphs of type $S$ :

- Let $O_{1, y}$ be the evolution defined on graphs of type $S$ by:
$O_{1, y}(S(n, x, y))=\left(G_{1}, G_{2}, \ldots, G_{|y|}\right)$ where $G_{i}=T\left(n+i, x \operatorname{pref}_{i}(y)\right.$, $\left.\operatorname{pref}_{i}(y), \operatorname{suff}_{|y|-i}(y)\right)$ for $1 \leq i<|y|$, and $G_{|y|}=S(n+|y|, x y, y)$.
- Let $O_{1, x}$ be the evolution defined on graphs of type $S$ by:
$O_{1, x}(S(n, x, y))=\left(G_{1}, G_{2}, \ldots, G_{|x|}\right)$ where $G_{i}=T\left(n+i, y \operatorname{pref}_{i}(x)\right.$, $\left.\operatorname{pref}_{i}(x), \operatorname{suff}_{|x|-i}(x)\right)$ for $1 \leq i<|x|$, and $G_{|x|}=S(n+|x|, y x, x)$.
We conclude that the sequence of Rauzy graphs of a Sturmian word $u$ is therefore characterized by the sequence of evolutions $\left(E_{k}\right)_{k \geq 0} \in\left\{O_{1, x}, O_{1, y}\right\}^{\mathbb{N}}$. Indeed, starting from $n_{0}=0, x_{0}=a, y_{0}=b$, it defines by induction a sequence $\left(n_{k}, x_{k}, y_{k}\right) \in \mathbb{N} \times \mathcal{A}^{*} \times \mathcal{A}^{*}$ such that $\Gamma_{n_{k}}=S\left(n_{k}, x_{k}, y_{k}\right)$ and $E_{k}\left(\Gamma_{n_{k}}\right)$ ends with $\Gamma_{n_{k+1}}$.


## 4. The infinite words of TyPe $u_{\beta}$

### 4.1. Definition

In general, the Rauzy graphs of an infinite word may have many different shapes, with complicated evolutions. However, if the complexity is assumed to be small enough, then the range of possible shapes and evolutions becomes smaller and resembles that of Sturmian words. The purpose of this section is to define the right class of words for our study, large enough to include all words whose contribution to the Heinis spectrum lies in a neighborhood of $C$, and small enough to minimize the number of different evolutions.

We first observe that, if $\alpha<2$, then infinitely many Rauzy graphs have a shape already encountered with Sturmian words.

Lemma 4.1. Let $u$ be a recurrent infinite word which is not eventually periodic. If $\alpha<2$ then there exists an infinite number of $n$ such that $\Gamma_{n}$ is of type $S$.

Proof. If we suppose that $s(n) \geq 2$ for all $n$ large enough, then $p(n) \geq 2 n+c$, which implies that $\alpha \geq 2$ which is excluded. From this, we can say that $s(n)=1$ for an infinite number of $n$, since $s(n) \geq 1$ as $u$ is not eventually periodic. Therefore, for such an $n, \Gamma_{n}$ is one of the Sturmian graphs. If it is not already of type $S$, then it will evolve to a graph $\Gamma_{n^{\prime}}$ of type $S$ with $n^{\prime}>n\left(n^{\prime}-n\right.$ being the length of one branch of $\left.\Gamma_{n}\right)$.

In [2], Aberkane considered the class of infinite words with complexity $p(n) \leq \frac{4}{3} n+1$. This is not enough for us, as the point $\left(\frac{3}{2}, \frac{5}{3}\right)$ cannot be reached by such words, so we need to extend Aberkane's results to a larger class. One of the key argument in [2] is that, when a Rauzy graph is of type $S$, say $\Gamma_{n}=S(n, x, y)$, some of the a priori possible evolutions require that $s(n+k) \geq 2$ for all $k$ such that $1 \leq k \leq|x|$, which is impossible when $p(n) \leq \frac{4}{3} n+1$. In the next lemma, we study more precisely when this happens.

Lemma 4.2. Let $u$ be an infinite word. We assume that there exist infinitely many $n$ such that $\Gamma_{n}$ is of type $S$, and $s(n+k) \geq 2$ for all $k$ such that $1 \leq k \leq|x|$, where $\Gamma_{n}=S(n, x, y)$. Then

$$
\alpha(4-\beta) \leq 2 \beta
$$

Proof. If $\beta \geq 2$, clearly

$$
\alpha(4-\beta) \leq 2 \alpha \leq 2 \beta
$$

We assume now that $\beta<2$.
Let $\varepsilon$ be such that $0<\varepsilon \leq 2-\beta$. Since $\alpha=\liminf _{n \rightarrow \infty} \frac{p(n+1)}{n}$ and $\beta=\limsup _{n \rightarrow \infty} \frac{p(n+1)}{n}$, there exists $n_{0} \in \mathbb{N}$ such that, for all $n \geq n_{0}, \alpha-\varepsilon \leq \frac{p(n+1)}{n} \leq \beta+\varepsilon$. Let $n \geq n_{0}$ be such that $\Gamma_{n}=S(n, x, y)$ and $s(n+k) \geq 2$ for $1 \leq k \leq|x|$.

We can notice that

$$
p(n+|x|+1)=p(n+1)+\sum_{k=1}^{|x|} s(n+k) \geq p(n+1)+2|x|
$$

since

$$
s(n+k) \geq 2
$$

We know that $p(n+1)=|x|+|y|$. Then, we have

$$
\left\{\begin{array}{l}
p(n+1)=|x|+|y| \geq(\alpha-\varepsilon) n  \tag{4.1}\\
3|x|+|y| \leq p(n+|x|+1) \leq(\beta+\varepsilon)(n+|x|) \\
|x| \geq|y|
\end{array}\right.
$$

Equations (4.1) and (4.3) give

$$
\begin{equation*}
|x| \geq \frac{1}{2}(\alpha-\varepsilon) n . \tag{4.4}
\end{equation*}
$$

Equation (4.2) gives

$$
(|x|+|y|)+(2-\beta-\varepsilon)|x| \leq(\beta+\varepsilon) n
$$

According to (4.1) and (4.4), we have

$$
(\alpha-\varepsilon) n+\frac{1}{2}(2-\beta-\varepsilon)(\alpha-\varepsilon) n \leq(\beta+\varepsilon) n
$$

which implies that

$$
(\alpha-\varepsilon)\left(2-\frac{\beta}{2}-\frac{\varepsilon}{2}\right) \leq \beta+\varepsilon
$$

By letting $\varepsilon$ tend to 0 , we get

$$
\alpha(4-\beta) \leq 2 \beta
$$



Figure 4. Graphs of type $R$ and $Q$.
We are now ready to define our class of words, that we named $u_{\beta}$. It is exactly the class of recurrent words for which the conclusion of Lemma 4.2 does not hold.

Definition 4.3. Let $u$ be a recurrent infinite word. We say that $u$ is an infinite word of type $u_{\beta}$ if

$$
\beta<\frac{4 \alpha}{2+\alpha}
$$

Remark 4.4. If $u$ is of type $u_{\beta}$, then $\alpha \leq \beta<\frac{4 \alpha}{2+\alpha}$, therefore $0<\alpha<2$, and $\beta<2$. In particular, since $\alpha \neq 0$, $u$ is not eventually periodic, so in fact $1 \leq \alpha \leq \beta<2$. By Lemma 4.1, infinitely many Rauzy graphs are of type $S$.
Remark 4.5. Let $u$ be a recurrent infinite word of complexity $p(n) \leq \frac{4}{3} n+1$, which is the class of infinite words defined by Aberkane in [2], then $\beta \leq \frac{4}{3}$ which implies that this infinite word is of type $u_{\beta}$ (except in the particular case $\alpha=1, \beta=\frac{4}{3}$ ). We are going to show that the techniques of Aberkane allow to extend some of his results to the class of infinite words of type $u_{\beta}$.

### 4.2. Evolution of Rauzy graphs of infinite words of type $u_{\beta}$

We consider now an infinite word of type $u_{\beta}$. As with Sturmian words, our aim is to look at the evolution of graphs between two successive graphs of type $S$.
In this part, we see the difference between the evolution of graphs of Sturmian words and of infinite words of type $u_{\beta}$. In fact, together with the two cases previously represented, where one edge is removed from $D\left(\Gamma_{n}\right)$ to get $\Gamma_{n+1}$, we can also have $\Gamma_{n+1}=D\left(\Gamma_{n}\right)$. However, it is not permitted to take off both $e_{2}$ and $e_{4}$ (see Fig. 3), as this would yield an infinite periodic word, which is excluded as $\alpha>0$.
Definition 4.6. We define the graphs of types $R$ and $Q$ :

- Let $R\left(n, x, y, z_{1}, z_{2}, z_{3}, z_{4}\right)$ be the graph which contains two right and two left special factors with six branches labelled by $x, y, z_{1}, z_{2}, z_{3}, z_{4}$ as indicated in Figure 4.
- Let $Q\left(n, x, y, z_{1}, z_{2}, z_{3}\right)$ be the graph which contains one right special factor, one left special factor and one bispecial factor with five branches labelled by $x, y, z_{1}, z_{2}, z_{3}$ as indicated in Figure 4.
Definition 4.7. We define the evolutions $O_{m, x}$ for all $m \geq 2$, where we don't take off the edges $e_{2}$ and $e_{4}$. Let $O_{m, x}$ be the evolution defined on graphs of type $S$ by: $O_{m, x}(S(n, x, y))=\left(G_{1}, \ldots, G_{|x|}\right)$ when $|x|>(m-1)|y|$ (otherwise not defined), where:
- $G_{j|y|+i}=R\left(n+j|y|+i, \operatorname{suff}_{|x|-j|y|-i}(x), \operatorname{suff}_{|y|-i}(y), \operatorname{pref}_{j|y|+i}(x)\right.$, $\left.\operatorname{pref}_{j|y|+i}(x), y^{j} \operatorname{pref}_{i}(y), \operatorname{pref}_{i}(y)\right)$ for $1 \leq i<|y|$ and $0 \leq j \leq m-2$.
- $G_{j|y|}=Q\left(n+j|y|, \operatorname{suff}_{|x|-j|y|}(x), y, \operatorname{pref}_{j|y|}(x), \operatorname{pref}_{j|y|}(x), y^{j}\right)$ for $1 \leq j \leq m-1$.
- $G_{(m-1)|y|+i}=T\left(n+(m-1)|y|+i, y^{m} \operatorname{pref}_{(m-1)|y|+i}(x), \operatorname{pref}_{(m-1)|y|+i}(x)\right.$, suff $\left._{|x|-(m-1)|y|-i}(x)\right)$ for $1 \leq i<|x|-(m-1)|y|$
- $G_{|x|}=S\left(n+|x|, y^{m} x, x\right)$.

We can notice that this evolution is not a simple evolution such as $E_{b}$ which is from a graph which contains a bispecial factor to the next graph which has a bispecial factor since the graphs of type $Q$ also contain a bispecial


Figure 5. The possible evolutions of a graph of type $S$ with a strong bispecial factor.
factor. The graph $G_{0}=S(n, x, y)$ contains a strong bispecial factor $w$. For $1 \leq j \leq m-2, G_{j|y|}$ contains an ordinary bispecial factor $w y^{j}$ and $G_{(m-1)|y|}$ has a weak bispecial factor which is $w y^{m-1}$.

The following proposition generalizes Lemma 4 in [1].
Proposition 4.8. Let $u$ be an infinite word of type $u_{\beta}$, then for all $n$ large enough such that $\Gamma_{n}$ is of type $S$, the evolution between $\Gamma_{n}$ and the next graph of type $S$ is one of $O_{m, x}, O_{1, x}$, or $O_{1, y}$.

Proof. Let $\Gamma_{n}=S(n, x, y)$, and define $w, x^{\prime}$ and $y^{\prime}$ as in Remark 5. If $\Gamma_{n+1} \neq D\left(\Gamma_{n}\right)$, then as for a Sturmian word we get evolution $O_{1, x}$ or $O_{1, y}$.

Assume now that $\Gamma_{n+1}=D\left(\Gamma_{n}\right)$, so that $s(n+1)=2$. Let $n_{1}=\min \{k>n: s(k)=1\}-1$, which is well-defined since $s(k)$ is infinitely often 1 . By Lemma 4.2 , since $\beta<\frac{4 \alpha}{2+\alpha}$, we have $n_{1}<n+|x|$ except for finitely many $n$. We thus assume that $n$ is large enough, so that $n<n_{1}<n+|x|$. Then $s(k) \geq 2$ for all $k \in\left[n+1, n_{1}\right]$, and $s\left(n_{1}+1\right)=1$.

Since $s\left(n_{1}+1\right)-s\left(n_{1}\right)<0$, there is a weak bispecial factor $w^{\prime}$ of length $n_{1}$. Let $B$ be the path in $\Gamma_{n}$ associated with $w^{\prime}$. Since $w^{\prime}$ is bispecial, the path $B$ must start in a left special factor and end in a right special factor, that is start in $w$ and end in $w$. Therefore $B$ is labelled by a word in $\{x, y\}^{*}$, and thus $w^{\prime} \in w\{x, y\}^{*}$. Then $n_{1}=n+i|x|+j|y|$, and $n_{1}<n+|x|$ implies $i=0$. So we have $n_{1}=n+j|y|$, for some $j \geq 1$. This implies in particular that $|y|<|x|$, and that

$$
w^{\prime}=w y^{j} .
$$

The graph $\Gamma_{n+1}$ evolves by $E_{b}$ to $\Gamma_{n+|y|}$, which has exactly one bispecial factor $w y$, one left special factor and one right special factor. The evolution of $\Gamma_{n+|y|}$ gives four cases for the graph $\Gamma_{n+|y|+1}$. These cases are illustrated on Figure 5:

- Case 1: $w y$ is a strong bispecial factor. Then $\Gamma_{n+|y|+1}=G_{1}^{\prime}$.
- Case 2: $w y$ is an ordinary bispecial factor, with edge $e_{6}$ missing. Then $\Gamma_{n+|y|+1}=G_{2}^{\prime}$.
- Case 3: wy is an ordinary bispecial factor, with edge $e_{5}$ missing. Then $\Gamma_{n+|y|+1}=G_{3}^{\prime}$.
- Case 4: wy is a weak bispecial factor. Then $\Gamma_{n+|y|+1}=G_{4}^{\prime}$.

Now, we must discuss according to the nature of bispecial factors of length between $n$ and $n_{1}$, i.e. the nature of $w y^{l}, 1 \leq l \leq j-1$.

Assume that one of the factors $w y^{l}$ is a strong bispecial factor. Then $x^{\prime} w y^{l} x=w x y^{l} x$ and $x^{\prime} w y^{l} y=w x y^{l} y$ are factors. Also $w x x$ and $w x y$ are factors since $w$ is a strong bispecial factor. Hence, the suffixes of length $n_{1}+1$ of $w x y^{l}$ and $w x$ are both right special factors. They are distinct since their suffixes of length $n+1$ are the two distinct extensions of $w$ to the left, i.e., since we know that $w x y^{l}=x^{\prime} y^{\prime l} w$ and $w x=x^{\prime} w$ and the suffixes of $x^{\prime} y^{\prime l} w$ and $x^{\prime} w$ of length $n+1$ are different then we can say that $w x y^{l}$ and $w x$ have distinct suffixes of length $n_{1}+1$. Then $s\left(n_{1}+1\right) \geq 2$. This is a contradiction.

Therefore, for all $l \in[1, j-1]$, the word $w y^{l}$ is an ordinary bispecial factor since $w x y^{l} y$, wyy $y^{l} x$, wyy $y^{l} y$ necessarily are factors. In particular, $G_{1}^{\prime}$ never occurs.

For all $k \in\left[n+1, n_{1}-1\right]$, we have $s(k+1)=s(k)$, because all bispecial factor of those lengths are ordinary. Then $s(k)=2$ for all $k \in\left[n+1, n_{1}\right]$.

- If $j=1$, there is a weak bispecial factor $w^{\prime}$ of length $n+|y|$ so $\Gamma_{n+|y|+1}$ is graph $G_{4}^{\prime}$ in Figure 5. Then $\Gamma_{n+|x|}$ is of type $S$ and the evolution between $\Gamma_{n}$ and $\Gamma_{n+|x|}$ is $O_{2, x}$.
- If $j \geq 2, \Gamma_{n+|y|+1}$ is graph $G_{3}^{\prime}$ (not $G_{2}^{\prime}$ since $w y^{3}$ is a factor), which is of type $R$ and evolves by $E_{b}$ to a graph of type $Q, \Gamma_{n+2|y|}$. If $j>2$, this process repeats until we reach $\Gamma_{n+j|y|}$, which is of type $Q$ and has a weak bispecial factor. Then $\Gamma_{n+j|y|+1}$ is a graph of type $T$ similar to $G_{4}^{\prime}$, which evolves by $E_{b}$ to a graph of type $S, \Gamma_{n+|x|}$. We recognize that the evolution between $\Gamma_{n}$ and $\Gamma_{n+|x|}$ is $O_{m, x}$ with $m=j+1$.

Definition 4.9. Let $u$ be an infinite word of type $u_{\beta}$, and $n_{0} \in \mathbb{N}$ be the smallest integer such that $\Gamma_{n_{0}}$ is of type $S$ and for every $n \geq n_{0}$, if $\Gamma_{n}$ is of type $S$ then $\Gamma_{n}$ undergoes one of the evolutions $O_{1, x}, O_{1, y}$, or $O_{m, x}$ $(m \geq 2)$. The sequence of evolutions of $u$ is the sequence $\left(E_{i}\right) \in\left\{O_{1, x}, O_{1, y}, O_{m, x}: m \geq 2\right\}^{\mathbb{N}}$ such that $\Gamma_{n_{i}}$ undergoes evolution $E_{i}$, where $\Gamma_{n_{i}}$ is the $i$ th graph of type $S$ after $\Gamma_{n_{0}}$. We say that $E \in \mathcal{E}$ occurs an infinite number of times (in the sequence of evolutions of $u$ ) if there exist $l \in \mathbb{N}$ and infinitely many $i \in \mathbb{N}$ such that $E=E_{i} E_{i+1} \ldots E_{i+l-1}$. Otherwise, we say that $E$ does not occur from a certain rank.

## 5. Infinite words where only $O_{2, x}$ OCCURS

The following lemma implies that $\left(\frac{3}{2}, \frac{5}{3}\right)$ belongs to $H$, which was stated in [7] with a different proof. However Lemma 5.1 is stronger as it applies to a whole class of words, characterized by the evolution of its Rauzy graphs, and this will be needed at the end of the proof of Theorem 6.5.

Lemma 5.1. Let $u$ be an infinite word with infinitely many graphs of type $S$. Let $n_{0} \in \mathbb{N}$, and suppose that for every $n \geq n_{0}$, if $\Gamma_{n}$ is a graph of type $S$, then it undergoes the evolution $O_{2, x}$. Then $(\alpha, \beta)=\left(\frac{3}{2}, \frac{5}{3}\right)$.

Proof. We may assume that $\Gamma_{n_{0}}$ is of type $S$. Let $\left(n_{i}\right)_{i \in \mathbb{N}}$ be the increasing sequence of orders $n \geq n_{0}$ for which $\Gamma_{n}$ is of type $S$, and let $\Gamma_{n_{i}}=S\left(n_{i}, x_{i}, y_{i}\right)$. Then the next graph of type $S$ is the graph $\Gamma_{n_{i+1}}=S\left(n_{i}+\left|x_{i}\right|, y_{i}^{2} x_{i}, x_{i}\right)$, so that

$$
\left\{\begin{array}{l}
n_{i+1}=n_{i}+\left|x_{i}\right| \\
\left|x_{i+1}\right|=\left|x_{i}\right|+2\left|y_{i}\right| \\
\left|y_{i+1}\right|=\left|x_{i}\right|
\end{array}\right.
$$

Then we can write this system as a product of a matrix and a vector as follows:

$$
\left(\begin{array}{c}
n_{i+1} \\
\left|x_{i+1}\right| \\
\left|y_{i+1}\right|
\end{array}\right)=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 2 \\
0 & 1 & 0
\end{array}\right)\left(\begin{array}{c}
n_{i} \\
\left|x_{i}\right| \\
\left|y_{i}\right|
\end{array}\right) .
$$

We note $A=\left(\begin{array}{lll}1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0\end{array}\right)$. Then we can notice that $\left(\begin{array}{c}n_{i} \\ \left|x_{i}\right| \\ \left|y_{i}\right|\end{array}\right)=A^{i}\left(\begin{array}{c}n_{0} \\ \left|x_{0}\right| \\ \left|y_{0}\right|\end{array}\right)$.
To find $A^{i}$, we should write $A$ as a product of a diagonal matrix $D$ by a transfer matrix $P$ and its inverse, i.e. $A=P D P^{-1}$. Then $A^{i}=P D^{i} P^{-1}$.

As the eigenvalues of $A$ are $\{-1,1,2\}$,
we have $D=\left(\begin{array}{ccc}2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{array}\right)$ and then $D^{i}=\left(\begin{array}{ccc}2^{i} & 0 & 0 \\ 0 & (-1)^{i} & 0 \\ 0 & 0 & 1\end{array}\right)$. Moreover, we may take the matrix $P=\left(\begin{array}{ccc}2 & 1 & 1 \\ 2 & -2 & 0 \\ 1 & 2 & 0\end{array}\right)$ and then $P^{-1}=\left(\begin{array}{ccc}0 & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{6} & \frac{1}{3} \\ 1 & -\frac{1}{2} & -1\end{array}\right)$.

Now, we can say that $\left(\begin{array}{c}n_{i} \\ \left|x_{i}\right| \\ \left|y_{i}\right|\end{array}\right)=P D^{i} P^{-1}\left(\begin{array}{c}n_{0} \\ \left|x_{0}\right| \\ \left|y_{0}\right|\end{array}\right)$.
Hence $\left(\begin{array}{c}n_{i} \\ \left|x_{i}\right| \\ \left|y_{i}\right|\end{array}\right)=\left(\begin{array}{c}\frac{2^{i+1}}{3}\left(\left|x_{0}\right|+\left|y_{0}\right|\right)+\frac{(-1)^{i}}{3}\left(-\frac{1}{2}\left|x_{0}\right|+\left|y_{0}\right|\right)-\frac{1}{2}\left|x_{0}\right|-\left|y_{0}\right|+n_{0} \\ \frac{2^{i+1}}{3}\left(\left|x_{0}\right|+\left|y_{0}\right|\right)-\frac{2}{3}(-1)^{i}\left(-\frac{1}{2}\left|x_{0}\right|+\left|y_{0}\right|\right) \\ \frac{2^{i}}{3}\left(\left|x_{0}\right|+\left|y_{0}\right|\right)+\frac{2}{3}(-1)^{i}\left(-\frac{1}{2}\left|x_{0}\right|+\left|y_{0}\right|\right)\end{array}\right)$.
Observe that $s(n)=2$ when $n_{i}+1 \leq n \leq n_{i}+\left|y_{i}\right|$ and $s(n)=1$ when $n_{i}+\left|y_{i}\right|+1 \leq n \leq n_{i+1}$. By Lemma 2.2, with $V^{+}=\left\{n_{i}+1: i \geq 0\right\}$ and $V^{-}=\left\{n_{i}+\left|y_{i}\right|+1: i \geq 0\right\}$, we deduce that

$$
\alpha=\liminf _{i \rightarrow \infty} \frac{p\left(n_{i}+1\right)}{n_{i}}
$$

and

$$
\beta=\limsup _{i \rightarrow \infty} \frac{p\left(n_{i}+\left|y_{i}\right|+1\right)}{n_{i}+\left|y_{i}\right|} .
$$

We can write that

$$
p\left(n_{i}+1\right)=\left|x_{i}\right|+\left|y_{i}\right|
$$

and

$$
p\left(n_{i}+\left|y_{i}\right|+1\right)=p\left(n_{i}+1\right)+\sum_{k=1}^{\left|y_{i}\right|} s\left(n_{i}+k\right)=p\left(n_{i}+1\right)+2\left|y_{i}\right|
$$

(since $\Gamma_{n_{i}}$ undergoes evolution $O_{2, x}$, we have $s\left(n_{i}+k\right)=2$ for $\left.1 \leq k \leq\left|y_{i}\right|\right)$.
Then

$$
\frac{p\left(n_{i}+\left|y_{i}\right|+1\right)}{n_{i}+\left|y_{i}\right|}=\frac{\left|x_{i}\right|+3\left|y_{i}\right|}{n_{i}+\left|y_{i}\right|}
$$

By replacing $n_{i},\left|x_{i}\right|$ and $\left|y_{i}\right|$ with their values, we will have:

$$
\begin{aligned}
\frac{\left|x_{i}\right|+\left|y_{i}\right|}{n_{i}} & =\frac{2^{i}\left(\left|x_{0}\right|+\left|y_{0}\right|\right)}{\frac{2^{i+1}}{3}\left(\left|x_{0}\right|+\left|y_{0}\right|\right)+\frac{(-1)^{i}}{6}\left(2\left|y_{0}\right|-\left|x_{0}\right|\right)-\frac{1}{2}\left|x_{0}\right|-\left|y_{0}\right|+n_{0}} \\
& =\frac{\left|x_{0}\right|+\left|y_{0}\right|}{\frac{2}{3}\left(\left|x_{0}\right|+\left|y_{0}\right|\right)+\frac{(-1)^{i}}{2^{i} 6}\left(2\left|y_{0}\right|-\left|x_{0}\right|\right)-\frac{\frac{1}{2}\left|x_{0}\right|+\left|y_{0}\right|-n_{0}}{2^{i}}}
\end{aligned}
$$

Letting $i \rightarrow \infty$, we obtain that

$$
\begin{equation*}
\frac{\left|x_{i}\right|+\left|y_{i}\right|}{n_{i}} \rightarrow \frac{3}{2} . \tag{5.1}
\end{equation*}
$$

Similarly,

$$
\frac{\left|x_{i}\right|+3\left|y_{i}\right|}{n_{i}+\left|y_{i}\right|}=\frac{2^{i} \frac{5}{3}\left(\left|x_{0}\right|+\left|y_{0}\right|\right)+\frac{2}{3}(-1)^{i}\left(2\left|y_{0}\right|-\left|x_{0}\right|\right)}{2^{i}\left(\left|x_{0}\right|+\left|y_{0}\right|\right)+\frac{(-1)^{i}}{2}\left(2\left|y_{0}\right|-\left|x_{0}\right|\right)-\frac{1}{2}\left|x_{0}\right|-\left|y_{0}\right|+n_{0}} .
$$

Letting $i \rightarrow \infty$, we obtain that

$$
\begin{equation*}
\frac{\left|x_{i}\right|+3\left|y_{i}\right|}{n_{i}+\left|y_{i}\right|} \rightarrow \frac{5}{3} . \tag{5.2}
\end{equation*}
$$

Finally, (5.1) and (5.2) give that $(\alpha, \beta)=\left(\frac{3}{2}, \frac{5}{3}\right)$.
Example 5.2. Consider the substitution

$$
\sigma:\left\{\begin{array}{l}
a \mapsto b b \\
b \mapsto b a .
\end{array}\right.
$$

Then $u=\lim _{n \rightarrow \infty} \sigma^{n}(b)=b a b b b a b a b a b b b a b b b a b b b a b a b a b b b a b \ldots$ is the fixed point of this substitution. It is called period-doubling word [6]. All Rauzy graphs of $u$ of type $S$, starting from $\Gamma_{1}$, undergo the evolution $O_{2, x}$. Therefore Lemma 5.1 applies and $(\alpha, \beta)=\left(\frac{3}{2}, \frac{5}{3}\right)$.

## 6. Relations between $\beta$ and $\alpha$ according to the evolutions that occur

Lemma 6.1. Let $u$ be an infinite word of type $u_{\beta}$. If the evolutions $O_{m, x}$ occur an infinite number of times with $m \geq 3$, then we have

$$
\beta \geq \frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}
$$

Proof. If $\alpha=1$, then the result is trivial since $\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}=1$. We assume in the rest of the proof that $\alpha \neq 1$. By Remark 7, we then have $1<\alpha \leq \beta<2$. Let $\varepsilon>0$ and $n$ be such that $\Gamma_{n}=S(n, x, y)$ undergoes the evolution $O_{m, x}$, for some $m \geq 3$.

As in the proof of Lemma 4.2, we know that for all $n$ large enough,

$$
(\alpha-\varepsilon) n \leq p(n+1)=|x|+|y| .
$$

Since $s(n+k)=2$, for $1 \leq k \leq(m-1)|y|$, then

$$
p(n+(m-1)|y|+1)=p(n+1)+\sum_{k=1}^{(m-1)|y|} s(n+k)=|x|+(2 m-1)|y| .
$$

Since, for all $n$ large enough,

$$
p(n+(m-1)|y|+1) \leq(\beta+\varepsilon)(n+(m-1)|y|)
$$

then

$$
|x|+(2 m-1)|y| \leq(\beta+\varepsilon)(n+(m-1)|y|) .
$$

We have also, for all $n$ large enough,

$$
p(n+|x|+1)=2|x|+m|y| \geq(\alpha-\varepsilon)(n+|x|) .
$$

Let $\xi=\frac{|x|}{n}$ and $\eta=\frac{|y|}{n}$. We have

$$
\left\{\begin{array}{l}
(\beta+\varepsilon)[1+(m-1) \eta] \geq \xi+(2 m-1) \eta \\
\xi+\eta \geq \alpha-\varepsilon \\
2 \xi+m \eta \geq(\alpha-\varepsilon)(1+\xi)
\end{array}\right.
$$

and thus

$$
\left\{\begin{array}{l}
-\xi-[(2-(\beta+\varepsilon))(m-1)+1] \eta \geq-(\beta+\varepsilon)  \tag{6.1}\\
\xi+\eta \geq(\alpha-\varepsilon) \\
(2-(\alpha-\varepsilon)) \xi+m \eta \geq(\alpha-\varepsilon)
\end{array}\right.
$$

We multiply (6.2) by $\lambda$ and (6.3) by $\mu$. Then in order to cancel $\xi$ and $\eta$ in (6.1) $+\lambda(6.2)+\mu(6.3)$, we should find the values of $\lambda$ and $\mu$ in this system:

$$
\left\{\begin{array}{l}
-1+\lambda+(2-(\alpha-\varepsilon)) \mu=0  \tag{6.4}\\
-[(2-(\beta+\varepsilon))(m-1)+1]+\lambda+m \mu=0
\end{array}\right.
$$

Now, we subtract (6.5) from (6.4), hence we have

$$
(2-(\beta+\varepsilon))(m-1)+(2-(\alpha-\varepsilon)-m) \mu=0
$$

then

$$
\mu=\frac{(2-(\beta+\varepsilon))(m-1)}{m+(\alpha-\varepsilon)-2}
$$

From (6.4), we can say that $\lambda=1-(2-(\alpha-\varepsilon)) \mu$. It is clear that $\mu \geq 0$ if $\varepsilon$ is small enough. We can also see that $\lambda$ is positive, in fact:

$$
\begin{aligned}
\lambda(m+\alpha-\varepsilon-2) & =m+\alpha-\varepsilon-2-(2-\alpha+\varepsilon)(2-\beta-\varepsilon)(m-1) \\
& =m(1-(2-\alpha+\varepsilon)(2-\beta-\varepsilon))-(2-\alpha+\varepsilon)(\beta-1+\varepsilon) \\
& =(m-1)(2-\alpha+\varepsilon)(\beta-1+\varepsilon)+m(\alpha-1-\varepsilon)
\end{aligned}
$$

The last expression is positive if $\varepsilon$ is small enough because $1<\alpha<2$ and $1<\beta<2$.
Moreover, by doing $(6.1)+\lambda(6.2)+\mu(6.3)$, we obtain $(\beta+\varepsilon) \geq(\alpha-\varepsilon)(\lambda+\mu)$. Then we have

$$
(\beta+\varepsilon) \geq(\alpha-\varepsilon)[1+\mu(\alpha-\varepsilon-1)]
$$

which implies that

$$
(m+\alpha-\varepsilon-2)(\beta+\varepsilon) \geq(m+\alpha-\varepsilon-2)(\alpha-\varepsilon)+(\alpha-\varepsilon)(\alpha-\varepsilon-1)(2-(\beta+\varepsilon))(m-1)
$$

and then

$$
(\beta+\varepsilon)\left[(m-1)(\alpha-\varepsilon)^{2}+(2-m)(\alpha-\varepsilon)+(m-2)\right] \geq(\alpha-\varepsilon)^{2}(2 m-1)-m(\alpha-\varepsilon)
$$

By letting $\varepsilon$ tend to 0 , we obtain

$$
\beta\left[(m-1) \alpha^{2}+(2-m) \alpha+(m-2)\right] \geq \alpha^{2}(2 m-1)-m \alpha
$$

Now, we should look at the sign of $(m-1) \alpha^{2}+(2-m) \alpha+(m-2)$, so we calculate $\Delta=(2-m)^{2}-4(m-1)(m-2)=$ $(m-2)(2-3 m)<0$, since $m \geq 3$.

Then

$$
(m-1) \alpha^{2}+(2-m) \alpha+(m-2)>0
$$

so we have

$$
\beta \geq \frac{\alpha^{2}(2 m-1)-m \alpha}{(m-1) \alpha^{2}+(2-m) \alpha+(m-2)}
$$

We know that

$$
\frac{\alpha^{2}(2 m-1)-m \alpha}{(m-1) \alpha^{2}+(2-m) \alpha+(m-2)}=\frac{m\left(2 \alpha^{2}-\alpha\right)-\alpha^{2}}{m\left(\alpha^{2}-\alpha+1\right)+\left(-\alpha^{2}+2 \alpha-2\right)}
$$

which is an increasing function of $m$, then

$$
\beta \geq \min \left\{\frac{m\left(2 \alpha^{2}-\alpha\right)-\alpha^{2}}{m\left(\alpha^{2}-\alpha+1\right)+\left(-\alpha^{2}+2 \alpha-2\right)}: m \geq 3\right\}=\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1} .
$$

Lemma 6.2. Let $u$ be an infinite word of type $u_{\beta}$. If the evolution $O_{2, x} O_{1, y}$ occurs an infinite number of times then we have

$$
\beta \geq \frac{(6 \alpha-5) \alpha}{2 \alpha^{2}-1}
$$

Proof. The result trivially holds if $\alpha=1$, so we assume that $1<\alpha \leq \beta<2$.
Let $\varepsilon>0$ and $n$ be such that $\Gamma_{n}=S(n, x, y)$ undergoes the evolution $O_{2, x} O_{1, y}$.
We know that, for all $n$ large enough,

$$
(\alpha-\varepsilon) n \leq p(n+1)=|x|+|y| \leq(\beta+\varepsilon) n
$$

Since $\Gamma_{n+|y|}=Q\left(n+|y|\right.$, suff $\left.{ }_{|x|-|y|}(x), y, \operatorname{pref}_{|y|}(x), \operatorname{pref}_{|y|}(x), y\right), \Gamma_{n+|x|}=S\left(n+|x|, y^{2} x, x\right)$ and $\Gamma_{n+2|x|}=$ $S\left(n+2|x|, y^{2} x^{2}, x\right)$, we can say that, for all $n$ large enough,

$$
p(n+2|x|+1)=3|x|+2|y| \geq(\alpha-\varepsilon)(n+2|x|)
$$

and

$$
p(n+|y|+1)=|x|+3|y| \leq(\beta+\varepsilon)(n+|y|) .
$$

Let $\xi=\frac{|x|}{n}$ and $\eta=\frac{|y|}{n}$, so that

$$
\left\{\begin{array}{l}
(\alpha-\varepsilon)(1+2 \xi) \leq 3 \xi+2 \eta \\
\xi+3 \eta \leq(\beta+\varepsilon)(1+\eta) \\
(\alpha-\varepsilon) \leq \xi+\eta
\end{array}\right.
$$

Hence, we have

$$
\left\{\begin{array}{l}
(3-2(\alpha-\varepsilon)) \xi+2 \eta \geq(\alpha-\varepsilon)  \tag{6.6}\\
-\xi+(\beta+\varepsilon-3) \eta \geq-(\beta+\varepsilon) \\
\xi+\eta \geq(\alpha-\varepsilon)
\end{array}\right.
$$

We multiply (6.7) by $\lambda$ and (6.8) by $\mu$. Then, in order to cancel $\xi$ and $\eta$ in (6.6) $+\lambda(6.7)+\mu(6.8)$, we should find the values of $\lambda$ and $\mu$ in the following system:

$$
\left\{\begin{array}{l}
(3-2(\alpha-\varepsilon))-\lambda+\mu=0  \tag{6.9}\\
2+\lambda(\beta+\varepsilon-3)+\mu=0
\end{array}\right.
$$

Now, we subtract (6.10) from (6.9), hence we have $\lambda(\beta+\varepsilon-2)-(1-2(\alpha-\varepsilon))=0$. Then $\lambda=\frac{2(\alpha-\varepsilon)-1}{2-(\beta+\varepsilon)}$. From (6.9), we can say that $\mu=\lambda-(3-2(\alpha-\varepsilon))$.

If $\varepsilon$ is small enough, it is clear that $\lambda \geq 0$. We can easily prove that $\mu \geq 0$, in fact:

$$
\begin{aligned}
\mu(2-\beta-\varepsilon) & =(2(\alpha-\varepsilon)-1)-(3-2(\alpha-\varepsilon))(2-\beta-\varepsilon) \\
& =2(\alpha-\varepsilon-1)(3-\beta-\varepsilon)+(\beta+\varepsilon-1)
\end{aligned}
$$

The last expression is positive if $\varepsilon$ is small enough since $1<\alpha<2$ and $1<\beta<2$. Moreover, by doing $(6.6)+\lambda(6.7)+\mu(6.8)$, we have $\lambda(\beta+\varepsilon) \geq(\alpha-\varepsilon)(1+\mu)$. Then we have,

$$
(2(\alpha-\varepsilon)-1)(\beta+\varepsilon) \geq(\alpha-\varepsilon)[2-(\beta+\varepsilon)+2(\alpha-\varepsilon)-1-(3-2(\alpha-\varepsilon))(2-(\beta+\varepsilon))] .
$$

Letting $\varepsilon$ tend to 0 , we get

$$
(2 \alpha-1) \beta \geq \alpha(6 \alpha+2 \beta-2 \alpha \beta-5)
$$

and then

$$
\left(2 \alpha^{2}-1\right) \beta \geq \alpha(6 \alpha-5)
$$

Since $2 \alpha^{2}-1>0$ we have $\beta \geq \frac{\alpha(6 \alpha-5)}{2 \alpha^{2}-1}$.
Lemma 6.3. Let $u$ be an infinite word of type $u_{\beta}$. If the evolution $O_{2, x} O_{1, x}$ occurs an infinite number of times then we have

$$
\beta \geq \frac{4 \alpha^{2}-3 \alpha}{2 \alpha^{2}-2 \alpha+1} .
$$

Proof. Let $\varepsilon$ be small enough and $n$ large enough such that $\Gamma_{n}=S(n, x, y)$ undergoes the evolution $O_{2, x} O_{1, x}$.
With similar arguments as in the previous proof, we have:

$$
\left\{\begin{array}{l}
3|x|+4|y| \geq(\alpha-\varepsilon)(n+2|x|+2|y|) \\
|x|+3|y| \leq(\beta+\varepsilon)(n+|y|) \\
|x|+|y| \geq(\alpha-\varepsilon) n
\end{array}\right.
$$

Now, let $\xi=\frac{|x|}{n}$ and $\eta=\frac{|y|}{n}$, then we have this new system:

$$
\left\{\begin{array}{l}
(3-2(\alpha-\varepsilon)) \xi+2(2-(\alpha-\varepsilon)) \eta \geq(\alpha-\varepsilon)  \tag{6.11}\\
(\beta+\varepsilon-3) \eta-\xi \geq-(\beta+\varepsilon) \\
\xi+\eta \geq(\alpha-\varepsilon)
\end{array}\right.
$$

By doing suitable combinations between these equations, we find the following result

$$
\beta \geq \frac{4 \alpha^{2}-3 \alpha}{2 \alpha^{2}-2 \alpha+1}
$$

Lemma 6.4. If $\beta<\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}$, then we have

$$
\beta<\frac{(6 \alpha-5) \alpha}{2 \alpha^{2}-1} \quad \text { and } \quad \beta<\frac{4 \alpha^{2}-3 \alpha}{2 \alpha^{2}-2 \alpha+1}
$$

Proof. Observe first that $\beta<\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}$ implies that $1<\alpha<2$. We have to study the sign of the difference between the quotients $\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}$ and $\frac{(6 \alpha-5) \alpha}{2 \alpha^{2}-1}$. We can write

$$
\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}-\frac{(6 \alpha-5) \alpha}{2 \alpha^{2}-1}=\frac{-2(\alpha-1)(2-\alpha)^{2} \alpha}{\left(2 \alpha^{2}-1\right)\left(2 \alpha^{2}-\alpha+1\right)},
$$



Figure 6. Graphic representation.
which is negative. So

$$
\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}<\frac{(6 \alpha-5) \alpha}{2 \alpha^{2}-1}
$$

which implies that $\beta<\frac{(6 \alpha-5) \alpha}{2 \alpha^{2}-1}$, since $\beta<\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}$.
Similarly, we calculate the following difference

$$
\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}-\frac{4 \alpha^{2}-3 \alpha}{2 \alpha^{2}-2 \alpha+1}=\frac{-2 \alpha^{2}(\alpha-1)(2-\alpha)}{\left(2 \alpha^{2}-2 \alpha+1\right)\left(2 \alpha^{2}-\alpha+1\right)}<0 .
$$

Then we can say that

$$
\beta<\frac{4 \alpha^{2}-3 \alpha}{2 \alpha^{2}-2 \alpha+1} .
$$

Theorem 6.5. Let $u$ be a recurrent infinite word such that $\beta<\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}$ and $\beta<\frac{4 \alpha}{2+\alpha}$, then $(\alpha, \beta)=\left(\frac{3}{2}, \frac{5}{3}\right)$.
Remark 6.6. We can reformulate Theorem 6.5 in another way:

- If $\alpha \leq \frac{5}{3}$, then either $\beta \geq \frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}$ or ( $\left.\alpha, \beta\right)=\left(\frac{3}{2}, \frac{5}{3}\right)$.
- If $\alpha \geq \frac{5}{3}$ then $\beta \geq \frac{4 \alpha}{2+\alpha}$.

The different functions involved in the proof of Theorem 6.5 are represented on Figure 6.
Now, we prove Theorem 6.5.
Proof. Let $u$ be an infinite word satisfying both conditions. Since $\beta<\frac{4 \alpha}{2+\alpha}$, it is of type $u_{\beta}$. Since $\beta<\frac{5 \alpha^{2}-3 \alpha}{2 \alpha^{2}-\alpha+1}$, we can say by using Lemma 6.1 that, from a certain rank, there is no evolution $O_{m, x}$, with $m \geq 3$. Therefore we have only evolutions $O_{1, x}, O_{2, x}$ and $O_{1, y}$.

By using Lemma 6.4, we have $\beta<\frac{(6 \alpha-5) \alpha}{2 \alpha^{2}-1}$ and $\beta<\frac{4 \alpha^{2}-3 \alpha}{2 \alpha^{2}-2 \alpha+1}$, then from a certain rank, evolution $O_{2, x} O_{1, y}$ does not occur by Lemma 6.2, nor evolution $O_{2, x} O_{1, x}$ by Lemma 6.3. These two results imply that from a certain rank, either $O_{2, x}$ does not occur at all, but this would imply that $\alpha=1$, which is excluded, or only $O_{2, x}$ occurs. Therefore, by using Lemma 5.1, we conclude that $(\alpha, \beta)=\left(\frac{3}{2}, \frac{5}{3}\right)$.

Corollary 6.7. Let $H=\left\{(\alpha, \beta): u \in \mathcal{A}^{\mathbb{N}}\right.$, u recurrent $\}$ and $C=\left\{(\alpha, \beta): \beta=\frac{3 \alpha-2}{\alpha}\right\}$. Then

$$
H \cap C=\left\{(1,1),\left(\frac{3}{2}, \frac{5}{3}\right),(2,2)\right\}
$$

and $\left(\frac{3}{2}, \frac{5}{3}\right)$ is an isolated point in $H$.
Proof. Let $(\alpha, \beta) \in H \cap C$.

- If $\alpha<1$ or $\alpha>2$ then $\beta=\frac{3 \alpha-2}{\alpha}<\alpha$, which is impossible.
- If $1<\alpha<2$ then the conditions of Theorem 6.5 are satisfied, so that $(\alpha, \beta)=\left(\frac{3}{2}, \frac{5}{3}\right)$.

Therefore, $H \cap C \subset\left\{(1,1),\left(\frac{3}{2}, \frac{5}{3}\right),(2,2)\right\}$. These three values are obtained since: $(\alpha, \beta)=(1,1)$ for Sturmian words, $(\alpha, \beta)=\left(\frac{3}{2}, \frac{5}{3}\right)$ for Example 5.2 and $(\alpha, \beta)=(2,2)$ for words of complexity $2 n+1$, see [3].

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