NEIGHBOR ISOLATED TENACITY OF GRAPHS

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Abstract. The tenacity of a graph is a measure of the vulnerability of a graph. In this paper we investigate a refinement that involves the neighbor isolated version of this parameter. The neighbor isolated tenacity of a noncomplete connected graph G is defined to be $NIT(G) = \min\{\frac{|X|+c(G/X)|}{i(G/X)}, i(G/X) \ge 1\}$ where the minimum is taken over all X, the cut strategy of G, i(G/X) is the number of components which are isolated vertices of G/X and c(G/X) is the maximum order of the components of G/X. Next, the relations between neighbor isolated tenacity and other parameters are determined and the neighbor isolated tenacity of some special graphs are obtained. Moreover, some results about the neighbor isolated tenacity of graphs obtained by graph operations are given.

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1. INTRODUCTION

We use Li and Ye [13] and Wei and Zhang [21] for terminology and consider only finite simple connected graphs. Let G = (V, E) be a graph and u a vertex in G. We call $N(u) = \{v : v \in V(G), uv \in E(G)\}$ the open neighborhood of u, and $N[u] = \{u\} \cup N(u)$ its closed neighborhood. We define analogously the open neighborhood $N(S) = \bigcup_{u \in S} N(u)$ for any $S \subseteq V(G)$ and the closed neighborhood $N[S] = \bigcup_{u \in S} N[u]$. A vertex $u \in V(G)$ is subverted when the closed neighborhood N[u] is deleted from G. A vertex subversion strategy $X \subseteq V(G)$ is a set of vertices whose closed neighborhood is deleted from G. The survival subgraph G/X is the subgraph obtained by the subversion strategy X applied to G, *i.e.*, G/X = G - N[X]. X is called a cut-strategy of G if the survival subgraph G/X is disconnected,

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or is a clique, or is an empty set. Let deg(u) denote the degree of the vertex u in G.

It is known that communication systems are often exposed to failures and attacks. So robustness of the network topology is a key aspect in the design of computer networks. The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. In the literature, various measures were defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. Graph vulnerability relates to the study of graph when some of its elements (vertices or edges) are removed. The measures of graph vulnerability are usually invariants that measure how a deletion of one or more network elements changes properties of the network. Parameters used to measure the vulnerability include connectivity, integrity [2], and scattering number [10]. Motivated from Jung's scattering number by replacing $\omega(G-S)$ with i(G-S) in the definition, Wang et al. [19] introduced the isolated scattering number, isc(G), as a parameter to measure the vulnerability of a network. The concept of graph tenacity was introduced by Cozzens et al. in [6], as a measure of network vulnerability and reliability. Graph tenacity has been an active area of research since the concept was introduced in 1992. Cozzens et al. [7] introduced a measure of network vulnerability termed the Mix-tenacity of a graph. Moazzami [16] introduced the concept of edge-tenacity of graphs. However, most of these parameters do not consider the neighborhoods of the affected vertices. On the other hand, in spy networks, if a spy or a station is captured, then adjacent stations are unreliable. Therefore, neighborhoods should be taken into consideration in spy networks. Nevertheless, there are very few parameters concerning neighborhoods such as neighbor connectivity [8], neighbor integrity [4], and neighbor scattering number [22].

The most common vulnerability parameters concerning to spy networks are as follows.

The neighbor connectivity of a graph G is

$$\kappa(G) = \min_{S \subseteq V(G)} \{|S|\},\$$

where S is a subversion strategy of G [8].

The neighbor integrity of a graph G is defined to be

$$NI(G) = \min_{S \subseteq V(G)} \{ |S| + c(G/S) \},\$$

where S is any vertex subversion strategy of G and c(G/S) is the order of the largest connected component of G/S [4].

The vertex neighbor scattering number of a graph G is defined as

$$S(G) = \max\{\omega(G/X) - |X| : X \text{ is a cut} - \text{strategy of } G, \ \omega(G/X) \ge 1\},\$$

where $\omega(G/X)$ denotes the number of connected components in G/X [22].



FIGURE 1. Graph G.

The known parameters concerning neighborhoods do not deal with the number of removed vertices, the number of components which are isolated vertices, and the number of the vertices in the largest component of the remaining graph in a disrupted network simultaneously. In order to fill this void in the literature, the current study proposes a definition of neighbor isolated tenacity, which is a new parameter concerning to these three values.

The neighbor isolated tenacity of a noncomplete connected graph G is defined to be

$$NIT(G) = \min\left\{\frac{|X| + c(G/X)}{i(G/X)}, \ i(G/X) \ge 1\right\}$$

where the minimum is taken over all X, the cut strategy of G, i(G/X) is the number of components which are isolated vertices of G/X and c(G/X) is the maximum order of the components of G/X. A set $X \subset V(G)$ is said to be the NIT - set of G if $NIT(G) = \frac{|X| + c(G/X)}{i(G/X)}$. In particular, the neighbor isolated tenacity of a complete graph K_n is defined to be n. For example, consider the graph G in Figure 1, where |V(G)| = 12 and |E(G)| = 14. It can be easily seen that |X| = 1, i(G/X) = 5 and c(G/X) = 2. Then, we have $NIT(G) = \frac{3}{5}$.

The following example shows that neighbor isolated tenacity is better than the neighbor connectivity, the neighbor integrity and the neighbor scattering number in measuring the vulnerability of graphs in some situations. Graphs with small neighbor isolated tenacity are more vulnerable.

Example 1.1. It can be easily seen that neighbor connectivity, neighbor integrity and neighbor scattering number of P_{13} , C_{13} and $W_{1,12}$ graphs are equal.

$$\kappa(C_{13}) = \kappa(W_{1,12}) = 2.$$

 $S(C_{13}) = S(W_{1,12}) = 0.$
 $NI(P_{13}) = NI(W_{1,12}) = 4.$



FIGURE 2. Example of *NIT* of graphs.

On the other hand, the neighbor isolated tenacities of P_{13} , C_{13} and $W_{1,12}$ are different, as shown in Figure 2.

$$NIT(P_{13}) = 1, \ NIT(C_{13}) = \frac{5}{3} \text{ and } NIT(W_{1,12}) = \frac{4}{3}.$$

2. Bounds for neighbor tsolated tenacity

Theorem 2.1. If G is a graph of order n, then

$$NIT(G) \ge \frac{\kappa(G) + 1}{n - \kappa(G)}.$$

Proof. Let X be a subversion strategy of G. We know that $\kappa(G) \leq |X| \leq |N[X]|$, $i(G/X) \leq n - |N[X]|$ and $c(G/X) \geq 1$. Thus,

$$\frac{|X| + c(G/X)}{i(G/X)} \ge \frac{|X| + 1}{n - |N[X]|}$$

Therefore, when we take the minimum of both sides,

$$NIT(G) \ge \frac{\kappa(G) + 1}{n - \kappa(G)}$$

The proof is completed.

Theorem 2.2. If G is a graph of order n and minimum vertex degree $\delta(G)$, then

$$NIT(G) \ge \frac{\delta(G) + 2}{n - \kappa(G)(\delta(G) + 1))}.$$

Proof. Let X be a subversion strategy of G. We have $\kappa(G) \leq |X|$ and, for any $v \in V(G)$, $|N[v]| \geq \delta(G) + 1$, so $i(G/X) \leq n - \kappa(G)(\delta(G) + 1)$ and $c(G/X) \geq 1$. By the definition of neighbor isolated tenacity

$$NIT(G) \ge \frac{\delta(G) + 1 + 1}{n - \kappa(G)(\delta(G) + 1))} = \frac{\delta(G) + 2}{n - \kappa(G)(\delta(G) + 1))}.$$

The proof is completed.

Theorem 2.3. If G is a graph of order n and independence number $\alpha(G)$, then

$$NIT(G) \ge \frac{\kappa(G) + 1}{\alpha(G)}$$
.

Proof. Let X be a subversion strategy of G. We know that $\kappa(G) \leq |X|, i(G/X) \leq \alpha(G)$ and $c(G/X) \geq 1$. Hence, we get

$$NIT(G) \ge \frac{\kappa(G) + 1}{\alpha(G)}$$

The proof is completed.

Theorem 2.4. If G is a connected graph of order n, then

$$NIT(G) \ge \frac{2}{n-2}$$

Proof. Let X be a subversion strategy of G. As $|X| \ge 1$ and $|N[X]| \ge 2$, by Theorem 2.1, the result holds.

Theorem 2.5. For any graph G,

$$NIT(G) \ge \frac{NI(G)}{\alpha(G)}$$
.

Proof. Let X be a subversion strategy of G. For any set X of G, we have $|X| + c(G/X) \ge NI(G)$ and $i(G/X) \le \alpha(G)$. Then

$$\frac{|X| + c(G/X)}{i(G/X)} \ge \frac{NI(G)}{i(G/X)}.$$

Thus, we take the minimum of both sides,

$$NIT(G) \ge \frac{NI(G)}{\alpha(G)}$$

The proof is completed.

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3. Neighbor isolated tenacity of several specific classes of graphs

Theorem 3.1. If P_n is a path graph with order $n \ge 6$, then

$$NIT(P_n) = \begin{cases} 1, & n \equiv 1 \pmod{4}; \\ \\ \frac{\lceil \frac{n}{4} \rceil + 1}{\lceil \frac{n}{4} \rceil}, & n \equiv 0, 2, 3 \pmod{4} \end{cases}$$

Proof. Let X be a subversion strategy of P_n and |X| = r. We distinguish two cases.

Case 1: Assume $n \equiv 1 \pmod{4}$.

If $r \leq \frac{n-1}{4}$, then $i(P_n/X) \leq r+1$ and $c(P_n/X) \geq \lceil \frac{n-3r}{r+1} \rceil$. Thus,

$$\frac{|X| + c(P_n/X)}{i(P_n/X)} \ge \frac{r + \left\lceil \frac{n-3r}{r+1} \right\rceil}{r+1}$$

the function $f(r) = \frac{r + \lceil \frac{n-3r}{r+1} \rceil}{r+1}$ is a decreasing function and it takes its minimum value at $r = \frac{n-1}{4}$ and we have

$$NIT(P_n) \ge \frac{\frac{n-1}{4} + 1}{\frac{n-1}{4} + 1} = 1.$$
(3.1)

It can be easily seen that there is a subversion strategy X^* of P_n such that $|X^*| = \frac{n-1}{4}$, $i(P_n/X^*) = \frac{n-1}{4} + 1$ and $c(P_n/X) = 1$, so

$$NIT(P_n) = 1. \tag{3.2}$$

If $r > \frac{n-1}{4}$, then $i(P_n/X) \le r+1$ and $c(P_n/X) \ge 1$. Thus,

$$\frac{|X| + c(P_n/X)}{i(P_n/X)} \ge \frac{r+1}{r+1}$$

$$NIT(P_n) \ge 1. \tag{3.3}$$

Therefore, by (3.1) - (3.3),

$$NIT(P_n) = 1 \tag{3.4}$$

where $n \equiv 1 \pmod{4}$.

Case 2: Assume
$$n \equiv 0, 2, 3 \pmod{4}$$
.
If $r \leq \lceil \frac{n}{4} \rceil$, then $i(P_n/X) \leq r$ and $c(P_n/X) \geq 1$. Thus,

$$\frac{|X| + c(P_n/X)}{i(P_n/X)} \ge \frac{r+1}{r}$$

the function $f(r) = \frac{r+1}{r}$ is a decreasing function and it takes its minimum value at $r = \lceil \frac{n}{4} \rceil$ and we have

$$NIT(P_n) \ge \frac{\left\lceil \frac{n}{4} \right\rceil + 1}{\left\lceil \frac{n}{4} \right\rceil}.$$
(3.5)

It can be easily seen that there is a subversion strategy X^* of P_n such that $|X^*| = \lceil \frac{n}{4} \rceil$, $i(P_n/X^*) = \lceil \frac{n}{4} \rceil$ and $c(P_n/X) = 1$, so

$$NIT(P_n) = \frac{\left\lceil \frac{n}{4} \right\rceil + 1}{\left\lceil \frac{n}{4} \right\rceil}.$$
(3.6)

If $r \ge \lceil \frac{n}{4} \rceil + 1$, then $i(P_n/X) \le \lceil \frac{n}{4} \rceil$ and $c(P_n/X) \ge 1$. Thus,

$$\frac{|X| + c(P_n/X)}{i(P_n/X)} \ge \frac{r+1}{\lceil \frac{n}{4} \rceil}$$

the function $f(r) = \frac{r+1}{\lceil \frac{n}{4} \rceil}$ is an increasing function and it takes its minimum value at $r = \lceil \frac{n}{4} \rceil + 1$ and we have

$$NIT(P_n) \ge \frac{\left\lceil \frac{n}{4} \right\rceil + 2}{\left\lceil \frac{n}{4} \right\rceil}.$$
(3.7)

Therefore, by (3.5), (3.6) and (3.7),

$$NIT(P_n) = \frac{\left\lceil \frac{n}{4} \right\rceil + 1}{\left\lceil \frac{n}{4} \right\rceil}$$
(3.8)

where $n \equiv 0, 2, 3 \pmod{4}$. By (3.4) and (3.8) we have

$$NIT(P_n) = \begin{cases} 1, & n \equiv 1 \pmod{4}; \\ \frac{\lceil \frac{n}{4} \rceil + 1}{\lceil \frac{n}{4} \rceil}, & n \equiv 0, 2, 3 \pmod{4}. \end{cases}$$

The proof is completed.

Theorem 3.2. If C_n is a cycle graph with order $n \ge 4$, then

$$NIT(C_n) = \begin{cases} \frac{\lceil \frac{n}{4} \rceil + 1}{\lceil \frac{n}{4} \rceil - 1}, n \equiv 0 \ (Mod \ 4); \\ \\ \frac{\lceil \frac{n}{4} \rceil + 1}{\lceil \frac{n}{4} \rceil - 1}, n \equiv 1, 2, 3 \ (Mod \ 4). \end{cases}$$

Proof. The proof is similar to the one of Theorem 3.1.

Theorem 3.3. If K_{n_1,n_2,\ldots,n_k} is a complete k-partite graph, then

$$NIT(K_{n_1,n_2,...,n_k}) = \frac{2}{\max\{n_1,n_2,...,n_k\} - 1}$$

Proof. As a complete k-partite graph is a join between their stable sets, by Theorem 4.2, one can only choose vertices of a same stable set to be the

subversion strategy X. Clearly, the minimum NIT will be obtained by choosing only one vertex of the maximum stable set n_i to be part of X, with |X| = 1, $c(K_{n_1,n_2,...,n_k}/X) = 1$, and $i(K_{n_1,n_2,...,n_k}/X) = n_i - 1$. Therefore, the result holds.

The following results can be easily obtained from Theorem 3.3. Corollary 3.4. If $K_{1,n}$ is a star graph, then

$$NIT(K_{1,n}) = \frac{2}{n-1}$$

Corollary 3.5. If $K_{a,b}$ is a complete bipartite graph, then

$$NIT(K_{a,b}) = \frac{2}{\max\{a, b\} - 1}$$

Theorem 3.6. Let $T_{k,d}$ be a complete k-ary tree of depth d where $k \ge 2$. Then

$$NIT(T_{k,d}) = \begin{cases} \frac{k^2(k^d - 1) + k^4 - 1}{k^4(k^d - 1) + k^4 - 1}, & d \equiv 0 \pmod{4}; \\ \frac{k^2(k^d - k) + 2k^4 - 2}{k^{d+4} - k^4 - k + 1}, & d \equiv 1 \pmod{4}; \\ \frac{k^{d+2} + k^4 - 2}{k^2(k^{d+2} - 1)}, & d \equiv 2 \pmod{4}; \\ \frac{k(k^{d+1} - 1) + k^4 - 1}{k^3(k^{d+1} - 1)}, & d \equiv 3 \pmod{4}. \end{cases}$$

Proof. Let X be a subversion strategy of $T_{k,d}$ and |X| = r be the number of removing vertices. There are four cases according to the depth of $T_{k,d}$.

Case 1: Let $d \equiv 0 \pmod{4}$.

(i) If
$$1 \le r \le \frac{k(k^d-1)}{k^d-1}$$
, then $i(T_{k,d}/X) \le k^2r + 1$ and $c(T_{k,d}/X) \ge \lfloor \frac{\frac{k^{d+1}-1}{k-1} - k^2r}{k^2r} \rfloor$.
Thus,
 $NIT(T_{k,d}) \ge \min_r \left\{ \frac{r + \lfloor \frac{k^{d+1}-1}{k-1} - k^2r}{k^2r+1} \right\}.$

The function $f(r) = \frac{r + \lfloor \frac{k^{d+1} - 1}{k-1} - k^2 r}{k^2 r + 1} \rfloor$ is a decreasing function and it takes its minimum value at $r = \frac{k(k^d - 1)}{k^4 - 1}$. Then,

$$NIT(T_{k,d}) \ge \frac{\frac{k(k^d - 1)}{k^4 - 1} + \left[\frac{\frac{k^{d+1} - 1}{k-1} - k^2 \frac{k(k^d - 1)}{k^4 - 1}}{k^2 \frac{k(k^d - 1)}{k^4 - 1}}\right]}{k^2 \frac{k(k^d - 1)}{k^4 - 1} + 1}.$$
(3.9)

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(ii) If
$$\frac{k(k^d-1)}{k^d-1} + 1 \le r \le \frac{k^2(k^d-1)}{k^d-1}$$
, then $i(T_{k,d}/X) \le k^2r + 1$ and $c(T_{k,d}/X) \ge 1$.
Then,

$$NIT(T_{k,d}) \ge \min_{r} \left\{ \frac{r+1}{k^2r+1} \right\}$$

The function $f(r) = \frac{r+1}{k^2r+1}$ is a decreasing function and it takes its minimum value at $r = \frac{k^2(k^d-1)}{k^4-1}$. Therefore,

$$NIT(T_{k,d}) \ge \frac{k^2(k^d - 1) + k^4 - 1}{k^4(k^d - 1) + k^4 - 1}.$$
(3.10)

(iii) If $\frac{k^2(k^d-1)}{k^4-1} + 1 \le r$, then we have $i(T_{k,d}/X) \le \frac{k^{d+4}-1}{k^4-1}$ and $c(T_{k,d}/X) \ge 1$. Thus,

$$NIT(T_{k,d}) \ge \min_{r} \left\{ \frac{r+1}{\frac{k^{d+4}-1}{k^4-1}} \right\}$$

The function $f(r) = \frac{r+1}{\frac{k^{d+4}-1}{k^4-1}}$ is an increasing function and it takes its minimum value at $r = \frac{k^2(k^d-1)}{k^4-1} + 1$. Then,

$$NIT(T_{k,d}) \ge \frac{k^2(k^d - 1) + 2k^4 - 2}{k^{d+4} - 1}.$$
(3.11)

It can be easily seen that there is a subversion strategy X^* of $T_{k,d}$ such that $|X^*| = \frac{k^2(k^d-1)}{k^4-1}$ where X^* contains all the vertices on the $\{2nd, 6th, 10th, 12th, \ldots, (d-2)th\}$ levels. Then, $i(T_{k,d}/X^*) = \frac{k^{d+4}-1}{k^4-1}$ and $c(T_{k,d}/X) = 1$. Thus,

$$NIT(T_{k,d}) = \frac{k^2(k^d - 1) + k^4 - 1}{k^4(k^d - 1) + k^4 - 1}.$$
(3.12)

The proof is completed by (3.9)-(3.12).

Case 2: Let $d \equiv 1 \pmod{4}$.

(i) If $1 \le r \le \frac{k^2(k^d - k)}{k^4 - 1}$, then $i(T_{k,d}/X) \le k^2r + 1$ and $c(T_{k,d}/X) \ge \lceil \frac{k^{d+1} - 1}{k - 1} - k^2r \rceil$. Then,

$$NIT(T_{k,d}) \ge \min_{r} \left\{ \frac{r + \lceil \frac{k^{d+1} - 1}{k-1} - k^2 r}{k^2 r} \rceil}{k^2 r + 1} \right\}.$$

The function $f(r) = \frac{r + \left\lceil \frac{k^{d+1} - 1 - k^2 r}{k^2 r} \right\rceil}{k^2 r + 1}$ is a decreasing function and it takes its minimum value at $r = \frac{k^2 (k^d - k)}{k^4 - 1}$. Therefore,

$$NIT(T_{k,d}) \ge \frac{\frac{k^2(k^d - k)}{k^4 - 1} + \left\lceil \frac{\frac{k^{d+1} - 1}{k - 1} - k^2 \frac{k^2(k^d - k)}{k^4 - 1}}{\frac{k^2 \frac{k^2(k^d - k)}{k^4 - 1}}{\rceil} \right\rceil}{k^2 \frac{k^2(k^d - k)}{k^4 - 1} + 1} \cdot$$
(3.13)

(ii) If $r = \frac{k^2(k^d-k)}{k^4-1} + 1$, then $i(T_{k,d}/X) \le \frac{k^{d+4}-k^4-k+1}{k^4-1}$ and $c(T_{k,d}/X) \ge 1$. Hence,

$$NIT(T_{k,d}) \ge \frac{\frac{k^2(k^d-k)}{k^4-1} + 2}{\frac{k^{d+4}-k^4-k+1}{k^4-1}} = \frac{k^2(k^d-k) + 2k^4 - 2}{k^{d+4}-k^4-k+1}.$$
 (3.14)

(iii) If $\frac{k^2(k^d-k)}{k^4-1} + 1 < r$, then $i(T_{k,d}/X) \le \frac{k^{d+4}-k^4-k+1}{k^4-1}$ and $c(T_{k,d}/X) \ge 1$. Then, $NIT(T_{k,d}) \ge \min_r \left\{ \frac{r+1}{\frac{k^{d+4}-k^4-k+1}{k^4-1}} \right\}$.

The function $f(r) = \frac{r+1}{\frac{k^{d+4}-k^4-k+1}{k^4-1}}$ is an increasing function and it takes its minimum value at $r = \frac{k^2(k^d-1)}{k^4-1} + 2$. Therefore,

$$NIT(T_{k,d}) \ge \frac{\frac{k^2(k^d-1)}{k^4-1} + 3}{\frac{k^{d+4}-k^4-k+1}{k^4-1}} = \frac{k^2(k^d-k) + 3k^4 - 3}{k^{d+4}-k^4-k+1}.$$
 (3.15)

It is obvious that there is a subversion strategy X^* of $T_{k,d}$ such that $|X^*| = \frac{k^2(k^d-k)}{k^4-1} + 1$ where X^* contains all the vertices on the {3rd, 7th, 11th, ..., (d-2)th} levels and one of the vertices on the first level. Then $i(T_{k,d}/X^*) = \frac{k^{d+4}-k^4-k+1}{k^4-1}$ and $c(T_{k,d}/X) = 1$. Hence, we get

$$NIT(T_{k,d}) = \frac{k^2(k^d - k) + 2k^4 - 2}{k^{d+4} - k^4 - k + 1}.$$
(3.16)

The proof is completed by (3.13)-(3.16).

Case 3: Let $d \equiv 2 \pmod{4}$.

(i) If
$$1 \leq r \leq \frac{k(k^d - k^2)}{k^4 - 1} + 1$$
, then $i(T_{k,d}/X) \leq k^2 r + 1$ and $c(T_{k,d}/X) \geq \left\lceil \frac{k^{d+1} - 1}{k^2 r} \right\rceil$. Thus,
 $\left\lceil \frac{k^{d+1} - 1}{k^2 r} \right\rceil$. Thus,

$$NIT(T_{k,d}) \ge \min_{r} \left\{ \frac{r + \left\lceil \frac{k^{u+1}-1}{k-1} - k^2 r \right\rceil}{k^2 r + 1} \right\}$$

The function $f(r) = \frac{r + \lceil \frac{k^{d+1} - 1 - k^2 r}{k^2 r} \rceil}{k^2 r + 1}$ is a decreasing function and it takes its minimum value at $r = \frac{k(k^d - k^2)}{k^4 - 1} + 1$. Then,

$$NIT(T_{k,d}) \ge \frac{\left(\frac{k(k^d - k^2)}{k^4 - 1} + 1\right) + \left[\frac{\frac{k^{d+1} - 1}{k - 1} - k^2 \left(\frac{k(k^d - k^2)}{k^4 - 1} + 1\right)}{k^2 \left(\frac{k(k^d - k^2)}{k^4 - 1} + 1\right) + 1}\right]}{k^2 \left(\frac{k(k^d - k^2)}{k^4 - 1} + 1\right) + 1} \cdot (3.17)$$

(ii) If $\frac{k(k^d - k^2)}{k^4 - 1} + 1 < r \le \frac{k^{d+2} - 1}{k^4 - 1}$, then $i(T_{k,d}/X) \le \frac{k^{d+2} - 1}{k^4 - 1} + (k^2 - 1)r$ and $c(T_{k,d}/X) \ge 1$. Thus,

$$NIT(T_{k,d}) \ge \min_{r} \left\{ \frac{r+1}{\frac{k^{d+2}-1}{k^4-1} + (k^2-1)r} \right\}$$

The function $f(r) = \frac{r+1}{\frac{k^{d+2}-1}{k^4-1} + (k^2-1)r}$ is a decreasing function and it takes its minimum value at $r = \frac{k^{d+2}-1}{k^4-1}$. Then,

$$NIT(T_{k,d}) \ge \frac{\frac{k^{d+2}-1}{k^4-1} + 1}{\frac{k^{d+2}-1}{k^4-1} + (k^2-1)(\frac{k^{d+2}-1}{k^4-1})} = \frac{k^{d+2}+k^4-2}{k^2(k^{d+2}-1)}.$$
 (3.18)

(iii) If $\frac{k^{d+2}-1}{k^4-1} < r$, then $i(T_{k,d}/X) \le \frac{k^2(k^{d+2}-1)}{k^4-1}$ and $c(T_{k,d}/X) \ge 1$. Then, $NIT(T_{k,d}) \ge \min_r \left\{ \frac{r+1}{\frac{k^2(k^{d+2}-1)}{k^4-1}} \right\}$.

The function $f(r) = \frac{r+1}{\frac{k^2(k^{d+2}-1)}{k^4-1}}$ is an increasing function and it takes its minimum value at $r = \frac{k^{d+2}-1}{k^4-1} + 1$. Therefore,

$$NIT(T_{k,d}) \ge \frac{\frac{k^{d+2}-1}{k^4-1}+2}{\frac{k^2(k^{d+2}-1)}{k^4-1}} = \frac{k^{d+2}+2k^4-3}{k^2(k^{d+2}-1)}.$$
(3.19)

It is obvious that there is a subversion strategy X^* of $T_{k,d}$ such that $|X^*| = \frac{k^{d+2}-1}{k^4-1}$ where X^* contains all the vertices on the {0th, 4th, 8th, ..., (d-2)th} levels. Then $i(T_{k,d}/X^*) = \frac{k^2(k^{d+2}-1)}{k^4-1}$ and $c(T_{k,d}/X) = 1$. Hence, we get

$$NIT(T_{k,d}) = \frac{\frac{k^{d+2}-1}{k^4-1} + 1}{\frac{k^2(k^{d+2}-1)}{k^4-1}} = \frac{k^{d+2}+k^4-2}{k^2(k^{d+2}-1)}.$$
(3.20)

The proof is completed by (3.17)-(3.20).

Case 4: Let $d \equiv 3 \pmod{4}$.

(i) If
$$1 \le r < \frac{k^{d+1}-1}{k^4-1}$$
, then $i(T_{k,d}/X) \le k^2r + 1$ and $c(T_{k,d}/X) \ge \lceil \frac{\frac{k^{d+1}-1}{k-1} - k^2r}{k^2r} \rceil$.
Thus,
 $NIT(T_{k,d}) \ge \min_r \left\{ \frac{r + \left\lceil \frac{k^{d+1}-1}{k^2r} - k^2r \right\rceil}{k^2r + 1} \right\}.$

The function $f(r) = \frac{r + \lceil \frac{k^{d+1}-1}{k-1} - k^2 r \rceil}{k^2 r + 1}$ is a decreasing function and it takes its minimum value at $r = \frac{k^{d+1}-1}{k^4-1} - 1$. Then,

$$NIT(T_{k,d}) \ge \frac{\left(\frac{k^{d+1}-1}{k^4-1}-1\right) + \left[\frac{\frac{k^{d+1}-1}{k-1}-k^2\left(\frac{k^{d+1}-1}{k^4-1}-1\right)}{k^2\left(\frac{k^{d+1}-1}{k^4-1}-1\right)}\right]}{k^2\left(\frac{k^{d+1}-1}{k^4-1}-1\right) + 1} \cdot$$
(3.21)

(ii) If $\frac{k^{d+1}-1}{k^4-1} \leq r \leq \frac{k(k^{d+1}-1)}{k^4-1}$, then $i(T_{k,d}/X) \leq \frac{k(k^{d+1}-1)}{k^4-1} + (k^2-1)r$ and $c(T_{k,d}/X) \geq 1$. Then,

$$NIT(T_{k,d}) \ge \min_{r} \left\{ \frac{r+1}{\frac{k(k^{d+1}-1)}{k^4-1} + (k^2-1)r} \right\}.$$

The function $f(r) = \frac{r+1}{\frac{k(k^{d+1}-1)}{k^4-1} + (k^2-1)r}$ is a decreasing function and it takes its minimum value at $r = \frac{k(k^{d+1}-1)}{k^4-1}$. Therefore,

$$NIT(T_{k,d}) \geq \frac{\frac{k(k^{d+1}-1)}{k^4-1} + 1}{\frac{k(k^{d+1}-1)}{k^4-1} + (k^2 - 1)(\frac{k(k^{d+1}-1)}{k^4-1})} = \frac{k(k^{d+1}-1) + k^4 - 1}{k^3(k^{d+1}-1)}.$$
(3.22)

(iii) If $\frac{k(k^{d+1}-1)}{k^4-1} < r \le \frac{k^{d+1}-1}{k-1} - 2$, then $i(T_{k,d}/X) \le \frac{k^3(k^{d+1}-1)}{k^4-1}$ and $c(T_{k,d}/X) \ge 1$. Then,

$$NIT(T_{k,d}) \ge \min_{r} \left\{ \frac{r+1}{\frac{k^3(k^{d+1}-1)}{k^4-1}} \right\}$$

The function $f(r) = \frac{r+1}{\frac{k^3(k^{d+1}-1)}{k^4-1}}$ is an increasing function and it takes its minimum value at $r = \frac{k(k^{d+1}-1)}{k^4-1} + 1$. Therefore,

$$NIT(T_{k,d}) \ge \frac{\frac{k(k^{d+1}-1)}{k^4-1} + 2}{\frac{k^3(k^{d+1}-1)}{k^4-1}} = \frac{k(k^{d+1}-1) + 2k^4 - 2}{k^3(k^{d+1}-1)}.$$
 (3.23)

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It is clear that there is a subversion strategy X^* of $T_{k,d}$ such that $|X^*| = \frac{k(k^{d+1}-1)}{k^4-1}$ where X^* contains all the vertices on the {1st, 5th, 9th, ..., (d-2)th} levels. Then $i(T_{k,d}/X^*) = \frac{k^3(k^{d+1}-1)}{k^4-1}$ and $c(T_{k,d}/X) = 1$. Hence, we get

$$NIT(T_{k,d}) = \frac{\frac{k(k^{d+1}-1)}{k^4-1} + 1}{\frac{k^3(k^{d+1}-1)}{k^4-1}} = \frac{k(k^{d+1}-1) + k^4 - 1}{k^3(k^{d+1}-1)}.$$
 (3.24)

The proof is completed by (3.21) - (3.24).

Definition 3.7. [3] The gear graph is a wheel graph with a vertex added between each pair adjacent vertices of the outer cycle. The gear graph Ge_k has 2k + 1vertices and 3k edges.

Theorem 3.8. If Ge_k is a gear graph, then

$$NIT(Ge_k) = \frac{2}{k} \cdot$$

Proof. Let X be a subversion strategy of Ge_k , |X| = r and $\deg(u) = k$. If $r \ge 1$, then we have $i(Ge_k/X) \leq k$ and $c(Ge_k/X) \geq 1$. So,

$$\frac{|X| + c(Ge_k/X)}{i(Ge_k/X)} \ge \frac{r+1}{k}$$

the function $f(r) = \frac{r+1}{k}$ is an increasing function and it takes its minimum value at r = 1 and we have

$$NIT(Ge_k) \ge \frac{2}{k} \cdot$$

It is obvious that there is a subversion strategy X^* of Ge_k such that $X^* = \{u\}$, then we have $i(Ge_k/X) = k$ and $c(Ge_k/X) = 1$. Hence,

$$NIT(Ge_k) = \frac{2}{k}$$

The proof is completed.

4. Graph operations and neighbor isolated tenacity

In this section we consider results on the neighbor isolated tenacity of the join and corona of two graphs.

4.1. JOIN

In this subsection, we consider some results on the neighbor isolated tenacity of the join of two graphs.

Definition 4.1. [9] The join $G = G_1 + G_2$ has graph set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \text{ and } v \in V(G_2)\}.$

Theorem 4.2. Let G and H be two connected graphs, then

$$NIT(G+H) = \min\{NIT(G), NIT(H)\}.$$

Proof. Let X be a subversion strategy of G + H and $i((G + H)/X) \ge 1$. Since every vertex of G is adjacent to all vertices of H and conversely, $X \subseteq V(G)$ and $X \cap V(H) = \emptyset$ or $X \subseteq V(H)$ and $X \cap V(G) = \emptyset$. There are two cases according to the elements of X.

Case 1: Let $X_1 \subseteq V(G)$ be the NIT - set of G such that $NIT(G) = \frac{|X_1| + c(G/X_1)}{i(G/X_1)}$ and $X = X_1$. Since every vertex of G is adjacent to all vertices of H, we have

$$\frac{|X| + c((G+H)/X)}{i((G+H)/X)} = \frac{|X_1| + c(G/X_1)}{i(G/X_1)} = NIT(G).$$
(4.1)

Case 2: Let $X_2 \subseteq V(H)$ be the NIT-set of H such that $NIT(H) = \frac{|X_2|+c(G/X_2)}{i(G/X_2)}$ and $X = X_2$. The proof Case 2 is similar to that of Case 1. Therefore, the result holds.

The following results can be easily obtained from Theorem 4.2.

Corollary 4.3. If G is a noncomplete graph, then

$$NIT(K_m + G) = NIT(G).$$

Corollary 4.4. Let $m \ge 1$ and $n \ge 4$ be positive integers. Then

$$NIT(K_m + C_n) = \begin{cases} \frac{\lceil \frac{n}{4} \rceil + 1}{\lceil \frac{n}{4} \rceil - 1}, n \equiv 0 \pmod{4}; \\ \frac{\lceil \frac{n}{4} \rceil + 1}{\lceil \frac{n}{4} \rceil - 1}, n \equiv 1, 2, 3 \pmod{4}. \end{cases}$$

Corollary 4.5. Let $m \ge 3$ and $n \ge 4$ be positive integers. Then

$$NIT(P_m + C_n) = \begin{cases} 1, & m \equiv 1 \pmod{4}; \\ \frac{\lceil \frac{n}{4} \rceil + 1}{\lceil \frac{n}{4} \rceil}, & m \equiv 0, 2, 3 \pmod{4}. \end{cases}$$

Corollary 4.6. Let $m \ge 3$ and $n \ge 3$ be positive integers. Then

$$NIT(P_m + K_{1,n}) = \frac{2}{n-1} \cdot$$

4.2. Corona

We begin with the definition of the corona of two graphs.

Definition 4.7. [9] The corona $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies H_i of H, and then joining the *i*th vertex of G to every vertex of H_i .

Theorem 4.8. Let G and H be two connected graphs of order m and n, respectively. Then

$$NIT(G \circ H) \leq NIT(H)$$

Proof. Let X be a subversion strategy of $G \circ H$ and X_1 be the NIT - set of H such that $NIT(H) = \frac{|X_1| + c(H/X_1)}{i(H/X_1)}$.

If $|X| = m \cdot |X_1|$, then $i((G \circ H)/X) = m \cdot i(H/X_1)$ and $c((G \circ H)/X) \leq m \cdot c(H/X_1)$. Thus,

$$\frac{|X| + c((G \circ H)/X)}{i((G \circ H)/X)} \le \frac{m.|X_1| + m.c(H/X_1)}{m.i(H/X_1)} \cdot$$

So we have

$$NIT(G \circ H) \le NIT(H).$$

The proof is completed.

5. CONCLUSION

Reliability and efficiency are important criteria in the design of networks. When we want to design a network, we wish that it is as stable as possible. Any network can be modelled as a connected graph. In this study, a new graph theoretical parameter namely the neighbor isolated tenacity has been presented for the network vulnerability. If we want to choose the stabler graph among the graphs which have the same order and the same size, one way is to choose the graph with maximum neighbor isolated tenacity.

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