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MORPHISMS PRESERVING THE SET OF WORDS CODING THREE INTERVAL EXCHANGE *, **

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Abstract. Any amicable pair φ, ψ of Sturmian morphisms enables a construction of a ternary morphism η which preserves the set of infinite words coding 3-interval exchange. We determine the number of amicable pairs with the same incidence matrix in $\mathrm{SL}^{\pm}(2,\mathbb{N})$ and we study incidence matrices associated with the corresponding ternary morphisms η.

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1. Introduction

Sturmian words are well-described objects in combinatorics on words. They can be defined in several equivalent ways [5], *e.g.* as words coding a two-interval exchange transformation with irrational ratio of lengths of the intervals. Morphisms preserving the set of Sturmian words are called *Sturmian* and they form a monoid generated by three of its elements (see [6, 12]). Let us denote this monoid by $\mathcal{M}_{\text{Sturm}}$.

In this paper, we consider morphisms preserving the set of words coding a three-interval exchange transformation with permutation $(3, 2, 1)$, the so-called *3iet*

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words. We call these morphisms *3iet-preserving*. Monoid of these morphisms, denoted by $\mathcal{M}_{3\text{iet}}$, is not fully described. It is shown (see [10]) that the monoid $\mathcal{M}_{3\text{iet}}$ is not finitely generated. Recently, in [2], pairs of amicable Sturmian morphisms were defined. The authors used this notion to describe morphisms that have as a fixed point a non-degenerate 3iet word, *i.e.* word with complexity $\mathcal{C}(n) = 2n + 1$. Using the operation of "ternarization", we can assign a morphism $\eta = \text{ter}(\varphi, \psi)$ over a ternary alphabet to a pair of amicable Sturmian morphisms. We show that such η is a 3iet-preserving morphism. Moreover, we show that the set

$$\mathcal{M}_{\text{ter}} = \big\{\text{ter}(\varphi, \psi) \big| \varphi, \psi \text{ amicable morphisms}\big\}$$

is a monoid, but it does not cover the whole monoid $\mathcal{M}_{3\text{iet}}$.

We also study the incidence matrices of morphisms $\eta \in \mathcal{M}_{\text{ter}}$. From the definition of amicable Sturmian morphisms φ, ψ we can derive that φ and ψ have the same incidence matrix $\mathbf{A} \in \mathbb{N}^{2\times 2}$, where $\det \mathbf{A} = \pm 1$. As shown in [14], for every matrix $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right)$ with $\det \mathbf{A} = \pm 1$, there exist $p_0 + p_1 + q_0 + q_1 - 1$ Sturmian morphisms. We will show the following theorem concerning the number of pairs of amicable Sturmian morphisms with a given matrix.

Theorem 1.1. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *be a matrix with* $\det \mathbf{A} = \pm 1$*. Then there exist exactly*

$$m\big(\|\mathbf{A}\| - 1\big) + \frac{m}{2}\big(\det \mathbf{A} - m\big) \tag{1.1}$$

pairs of amicable Sturmian morphisms with incidence matrix $\mathbf{A}$*, where* $m = \min\{p_0 + p_1, q_0 + q_1\}$ *and* $\|\mathbf{A}\| = p_0 + p_1 + q_0 + q_1$.

Moreover, for a given matrix $\mathbf{A}$, we will describe all matrices $\mathbf{B} \in \mathbb{N}^{3\times 3}$ such that $\mathbf{B}$ is an incidence matrix of $\eta = \text{ter}(\varphi, \psi)$ for amicable Sturmian morphisms φ, ψ with incidence matrix $\mathbf{A}$.

2. Preliminaries

2.1. Words over finite alphabet

Besides the infinite words, we consider *finite words* over the alphabet $\mathbb{A}$. We write $w = w_0 w_1 \ldots w_{n-1}$, where $w_i \in \mathbb{A}$ for all $i \in \mathbb{N}$, $i < n$. We denote by $|w|$ the length n of the finite word w. We denote by $|w|_a$ the number of occurrences of a letter $a \in \mathbb{A}$ in the word w. The set of all finite words on the alphabet $\mathbb{A}$ including the empty word is denoted by $\mathbb{A}^*$. The set $\mathbb{A}^*$ with the operation of concatenation is a monoid. On the set $\mathbb{A}^*$ we define a relation of *conjugation*: $w \sim w'$, if there exists $v \in \mathbb{A}^*$ such that $wv = vw'$. A *morphism* from $\mathbb{A}^*$ to $\mathbb{B}^*$ is a mapping $\varphi : \mathbb{A}^* \to \mathbb{B}^*$ such that $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathbb{A}^*$. It is clear that a morphism is well defined by images of letters $\varphi(a)$ for all $a \in \mathbb{A}$. If $\mathbb{A} = \mathbb{B}$, then φ is called a *morphism over* $\mathbb{A}$.

The set of *infinite words* over the alphabet $\mathbb{A}$ is denoted by $\mathbb{A}^{\mathbb{N}}$. The action of a morphism can be naturally extended to an infinite word $(u_i)_{i\in\mathbb{N}}$ putting

$\varphi(u) = \varphi(u_0)\varphi(u_1)\varphi(u_2)\dots$ If an infinite word $u \in \mathbb{A}^{\mathbb{N}}$ satisfies $\varphi(u) = u$, we call it a *fixed point* of the morphism φ over $\mathbb{A}$.

To a morphism φ over $\mathbb{A}$ we assign an *incidence matrix* $\mathbf{M}_\varphi$ defined by $(\mathbf{M}_\varphi)_{ab} = |\varphi(a)|_b$ for all $a, b \in \mathbb{A}$. To a finite word $v \in \mathbb{A}^*$ we assign a *Parikh vector* $\Psi(v)$ defined by $\Psi(v)_b = |v|_b$ for all $b \in \mathbb{A}$.

The *language* of an infinite word u is the set of all its factors. Let us recall that a finite word $w \in \mathbb{A}^*$ is a *factor* of $u = (u_i)_{i\in\mathbb{N}}$, if there exist indices $n, j \in \mathbb{N}$ such that $w = u_n u_{n+1} \dots u_{n+j-1}$. The language of an infinite word is denoted by $\mathcal{L}(u)$.

It is known that the language of neither Sturmian nor 3iet word depends on the point $x_0 \in [0, 1)$, the orbit of which the infinite word codes. It depends only on slope ε or parameters α, β.

The *(factor) complexity* of an infinite word u is a mapping $\mathcal{C}_u : \mathbb{N} \to \mathbb{N}$, which returns the number of factors of u of the length n, thus $\mathcal{C}_u(n) = \#\{w \in \mathcal{L}(u) \,\big|\, |w| = n\}$. It is easy to see that a word u is periodic if and only if there exists $n_0 \in \mathbb{N}$ such that $C_u(n_0) \le n_0$.

2.2. Interval exchange

We consider Sturmian words, *i.e.* aperiodic words given by exchange of 2 intervals with permutation $(2, 1)$, and words given by exchange of 3 intervals with permutation $(3, 2, 1)$. Let us recall that general r-interval exchange transformations were introduced already in [11].

The 2-interval exchange transformation S is a mapping $S : [0, 1) \to [0, 1)$. It is determined by its slope $\varepsilon \in [0, 1]$ and is given by

$$Sx = \begin{cases} x + 1 - \varepsilon & \text{if } x \in [0, \varepsilon) \\ x - \varepsilon & \text{if } x \in [\varepsilon, 1). \end{cases}$$

The orbit of a point $x_0 \in [0, 1)$ with respect to the transformation S, *i.e.* the sequence $x_0, Sx_0, S^2x_0, \dots$ can be coded by an infinite word $u = (u_i)_{i=0}^{\infty}$ on the binary alphabet $\{0, 1\}$. The infinite word is given by

$$u_i = \begin{cases} 0 & \text{if } S^i x_0 \in [0, \varepsilon), \\ 1 & \text{if } S^i x_0 \in [\varepsilon, 1). \end{cases} \tag{2.1}$$

It is a well-known fact that for an irrational ε, the word u is Sturmian. Using the same construction on the partition of the interval $(0, 1]$ into $(0, \varepsilon] \cup (\varepsilon, 1]$, we again obtain a Sturmian word. On the other hand, every Sturmian word can be obtained by one of the above two constructions. The set of Sturmian words will be denoted by $\mathcal{W}_{\text{Sturm}}$.

In [12] (the original results can be found in [8, 13]), the authors show that Sturmian words are the aperiodic words with minimal complexity, *i.e.* $\mathcal{C}_u(n) = n+1$ for all $u \in \mathcal{W}_{\text{Sturm}}$ and $n \in \mathbb{N}$. We can see that

$$S^i x_0 = \{x_0 - i\varepsilon\} \quad \text{for all} \quad x_0 \in [0, 1), \tag{2.2}$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the *fractional part* of a number $x \in \mathbb{R}$. Then $u_i = \lfloor x_0 - i\varepsilon \rfloor - \lfloor x_0 - (i+1)\varepsilon \rfloor$, which is exactly the formula how [12] define mechanical words.

We will use another fact about the two-interval exchanges. Let $\varphi \in \mathcal{M}_{\text{Sturm}}$ be a Sturmian morphism. Then the word $v = \varphi(a)$ for $a \in \{0, 1\}$ codes two-interval exchange with the slope $\frac{|v|_0}{|v|}$. We should see this from [12], Lemma 2.1.15. The word a^k is a factor of some Sturmian word, hence the word $\varphi(a)^k$ is balanced for any $k \in \mathbb{N}$, which means that the infinite word $u = \varphi(a)^\omega = \varphi(a)\varphi(a)\varphi(a)\dots$ is balanced and periodic, thus it is rational mechanical. In our terms, this means that it codes a rational 2-interval exchange; it is as well shown there that the slope of the transformation is exactly $\frac{|v|_0}{|v|}$.

The 3-interval exchange transformation T is determined by two parameters $\alpha, \beta \in (0, 1)$ satisfying $\alpha + \beta < 1$. Using parameters α, β and $\gamma = 1 - \alpha - \beta$ we partition the interval $[0, 1)$ into $I_A = [0, \alpha)$, $I_B = [\alpha, \alpha + \beta)$ and $I_C = [\alpha + \beta, 1)$. The mapping T is given by

$$Tx = \begin{cases} x + \beta + \gamma & \text{if } x \in I_A, \\ x - \alpha + \gamma & \text{if } x \in I_B, \\ x - \alpha - \beta & \text{if } x \in I_C. \end{cases}$$

The orbit of a point $x_0 \in [0, 1)$ with respect to the transformation T is coded by a word $u = (u_i)_{i=0}^{\infty}$ over the ternary alphabet $\{A, B, C\}$:

$$u_i = X \quad \text{if} \quad T^i x_0 \in I_X.$$

Similarly to the case of 2-interval exchange transformation, we can define the exchange of 3 intervals using the partition $(0, 1] = (0, \alpha] \cup (\alpha, \alpha + \beta] \cup (\alpha + \beta, 1]$. If $\frac{1-\alpha}{1+\beta}$ is irrational, the infinite word u is aperiodic, and we call it a *3iet word*; the set of these words is denoted by $\mathcal{W}_{3\text{iet}}$. For combinatorial properties of 3iet words, see [9].

Aperiodic words coding 3-interval exchange transformations, called here 3iet words, have the complexity $\mathcal{C}_u(n) \le 2n + 1$ for all $n \in \mathbb{N}$. If a 3iet word $u \in \mathcal{W}_{3\text{iet}}$ satisfies $\mathcal{C}_u(n) = 2n + 1$ for all $n \in \mathbb{N}$, we call it a *non-degenerate* 3iet word; otherwise we call it a *degenerate* 3iet word and it is a quasi-Sturmian word (see [7]).

2.3. Standard pairs and standard morphisms

In [14], the notion of standard pairs is introduced. If we define two operators on pairs of words $L, R : \{0, 1\}^* \times \{0, 1\}^* \to \{0, 1\}^* \times \{0, 1\}^*$ as

$$L(x, y) = (x, xy), \qquad R(x, y) = (yx, y),$$

we say that a pair (x, y) is a *standard pair*, if it can be obtained from the pair $(0, 1)$ by applying the operators L and R finitely many times. For every standard pair (x, y) there exists a word $v \in \{0, 1\}^*$ such that

$$xy = v01 \quad \text{and} \quad yx = v10. \tag{2.3}$$

We say that a binary morphism φ is *standard*, if there exists a standard pair (x, y) such that

$$\begin{aligned} \varphi(0) &= x, \\ \varphi(1) &= y, \end{aligned} \qquad \text{or} \qquad \begin{aligned} \varphi(0) &= y, \\ \varphi(1) &= x. \end{aligned}$$

The authors of [14] show the close connection between the standard morphisms and all the Sturmian morphisms:

(1) Every standard morphism is Sturmian.
(2) For every matrix $\mathbf{A} \in \mathbb{N}^{2\times 2}$ with $\det \mathbf{A} = \pm 1$, there exists exactly one standard morphism φ with incidence matrix $\mathbf{M}_\varphi = \mathbf{A}$.
(3) Every Sturmian morphism $\psi \in \mathcal{M}_{\text{Sturm}}$ is a right conjugate to some standard morphism φ. Let us recall that a morphism ψ over $\mathbb{A}$ is a *right conjugate* to φ, if there exists a finite word $v \in \mathbb{A}^*$ such that

$$\varphi(a)v = v\psi(a) \quad \text{for all letters} \quad a \in \mathbb{A}.$$

2.4. Amicable words and morphisms

In the article [4], authors show the close connection between 3iet and Sturmian words using morphisms $\sigma_{01}, \sigma_{10} : \{A, B, C\}^* \to \{0, 1\}^*$ given by

$$\begin{aligned} \sigma_{01}(A) &= 0, \\ \sigma_{01}(B) &= 01, \\ \sigma_{01}(C) &= 1, \end{aligned} \qquad\qquad \begin{aligned} \sigma_{10}(A) &= 0, \\ \sigma_{10}(B) &= 10, \\ \sigma_{10}(C) &= 1. \end{aligned}$$

In [4], the following theorem is proved.

Theorem 2.1. *An infinite ternary word $u \in \{A, B, C\}^{\mathbb{N}}$ is a 3iet word if and only if the words $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are Sturmian.*

This theorem motivated the authors of [3] to introduce the relation of amicability of words.

Definition 2.2. Let $w, w' \in \{0, 1\}^*$, let $b \in \mathbb{N}$. We say that w is *b-amicable* to w', if there exists a factor $v \in \{A, B, C\}^*$ of some 3iet word such that

$$w = \sigma_{01}(v), \qquad w' = \sigma_{10}(v) \quad \text{and} \quad |v|_B = b.$$

We say that w is *amicable* to w', if w is b-amicable to w' for some $b \in \mathbb{N}$, and we denote it by $w \propto w'$.

The ternary word v is called a *ternarization* of w and w', and we write $v = \text{ter}(w, w')$.

It is easy to see that if $w \propto w'$, then they are factors of the same Sturmian word and their Parikh vectors coincide.

The ternarization is given uniquely for a pair w, w'. For, let us see that if ternary words $v^{(1)}$, $v^{(2)}$ differ, then either $\sigma_{01}(v^{(1)}) \neq \sigma_{01}(v^{(2)})$ or $\sigma_{10}(v^{(1)}) \neq \sigma_{10}(v^{(2)})$.

In [3], the notion of amicable words plays a crucial role in the enumeration of words with length n occurring in a 3iet word. In [2], the authors investigate ternary morphisms that have a non-degenerate 3iet fixed point using the following notion of amicability of two Sturmian morphisms.

Definition 2.3. Let φ, ψ be Sturmian morphisms over the alphabet $\{0,1\}$. We say that φ is *amicable* to ψ, if

$$\varphi(0) \propto \psi(0), \quad \varphi(01) \propto \psi(10) \quad \text{and} \quad \varphi(1) \propto \psi(1).$$

We denote this relation by $\varphi \propto \psi$. The morphism η over the ternary alphabet $\{A, B, C\}$, given by

$$\eta(A) = \text{ter}\big(\varphi(0), \psi(0)\big),$$
$$\eta(B) = \text{ter}\big(\varphi(01), \psi(10)\big),$$
$$\eta(C) = \text{ter}\big(\varphi(1), \psi(1)\big),$$

is called the *ternarization* of morphisms φ and ψ, and is denoted by $\eta = \text{ter}(\varphi, \psi)$. The set of these η is denoted by $\mathcal{M}_{\text{ter}}$.

The ternarization of words is given uniquely by the words $u \propto v$, hence the ternarization of morphisms is given uniquely as well.

Example 2.4. Consider Sturmian morphisms φ, ψ given by

$$\varphi(0) = 001, \qquad \varphi(1) = 00101, \qquad \psi(0) = 010, \qquad \psi(1) = 01001.$$

Then $\varphi \propto \psi$ and their ternarization $\eta = \text{ter}(\varphi, \psi)$ satisfies

$$\eta(A) = AB, \qquad \eta(B) = ABABB, \qquad \eta(C) = ABAC.$$

The article [2] states the following theorem:

Theorem 2.5. *Let η be a ternary morphism with non-degenerate 3iet fixed point. Then $\eta \in \mathcal{M}_{\text{ter}}$ or $\eta^2 \in \mathcal{M}_{\text{ter}}$.*

3. Main results

Analogously to the terminology introduced for Sturmian words and morphisms in [6], the ternarization η, having a 3iet fixed point, is *locally 3iet-preserving*, *i.e.* there exists $u \in \mathcal{W}_{\text{3iet}}$ such that $\eta(u) \in \mathcal{W}_{\text{3iet}}$. We now prove a partial result about *(globally) 3iet-preserving* morphisms, *i.e.* ternary morphisms η such that

$$\eta(u) \in \mathcal{W}_{\text{3iet}} \quad \text{for all} \quad u \in \mathcal{W}_{\text{3iet}}.$$

Proposition 3.1. *Let* $\eta = \mathrm{ter}(\varphi, \psi)$ *for amicable Sturmian morphisms* $\varphi \propto \psi$. *Then* η *is a globally 3iet-preserving morphism.*

Proof. Directly from definitions we see that

$$\sigma_{01}\eta(A) = \varphi(0), \qquad \sigma_{01}\eta(B) = \varphi(01), \qquad \sigma_{01}\eta(C) = \varphi(1),$$
$$\sigma_{10}\eta(A) = \psi(0), \qquad \sigma_{10}\eta(B) = \psi(10), \qquad \sigma_{10}\eta(C) = \psi(1).$$

Therefore

$$\sigma_{01}\eta(v) = \varphi\sigma_{01}(v) \qquad \text{and} \qquad \sigma_{10}\eta(v) = \psi\sigma_{10}(v) \tag{3.1}$$

for any factor v of a 3iet word $u \in \mathcal{W}_{3\mathrm{iet}}$. According to Theorem 2.1 we get that $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are Sturmian words, and since φ and ψ are Sturmian morphisms, we obtain that $\sigma_{01}\eta(u)$ and $\sigma_{10}\eta(u)$ are Sturmian words as well, which means, according to the same theorem, that the word $\eta(u)$ is 3iet. □

Proposition 3.2. *Let* $\varphi_i \propto \psi_i$ *be Sturmian morphisms, for* $i = 1, 2$. *Then*

$$\mathrm{ter}(\varphi_1, \psi_1) \circ \mathrm{ter}(\varphi_2, \psi_2) = \mathrm{ter}(\varphi_1 \circ \varphi_2, \psi_1 \circ \psi_2).$$

Proof. It can be shown that the relation of amicability is preserved by composition of morphisms. More precisely $\varphi_1\varphi_2 \propto \psi_1\psi_2$. Denote $\eta_1 = \mathrm{ter}(\varphi_1, \psi_1)$, $\eta_2 = \mathrm{ter}(\varphi_2, \psi_2)$. Using the relation (3.1), we see that for all $v \in \{A, B, C\}^*$

$$\sigma_{01}\eta_1\eta_2(v) = \varphi_1\sigma_{01}\eta_2(v) = \varphi_1\varphi_2\sigma_{01}(v)$$
$$\text{and} \quad \sigma_{10}\eta_1\eta_2(v) = \psi_1\sigma_{10}\eta_2(v) = \psi_1\psi_2\sigma_{10}(v).$$

But this means that $\eta_1\eta_2 = \mathrm{ter}(\varphi_1\varphi_2, \psi_1\psi_2)$. □

As a consequence of previous two propositions, we can state the following theorem.

Theorem 3.3. *The set* $\mathcal{M}_{\mathrm{ter}}$ *of all ternarizations of amicable Sturmian morphisms with the operation of composition of morphisms is a sub-monoid of the monoid* $\mathcal{M}_{3\mathrm{iet}}$ *of all globally 3iet-preserving morphisms.*

Unfortunately, $\mathcal{M}_{\mathrm{ter}} \subsetneq \mathcal{M}_{3\mathrm{iet}}$. Consider for example the morphism

$$\eta(A) = B, \qquad \eta(B) = CAC, \qquad \eta(C) = C. \tag{3.2}$$

As shown in [10], this morphism is 3iet-preserving, but it can be easily verified that it is not a ternarization of any pair of Sturmian morphisms, using the following statement.

Proposition 3.4. *A ternary morphism* η *is a ternarization, i.e.* $\eta \in \mathcal{M}_{\mathrm{ter}}$, *if and only if it satisfies*

$$\sigma_{01}\eta(B) = \sigma_{01}\eta(AC) \quad \textit{and} \quad \sigma_{10}\eta(B) = \sigma_{10}\eta(CA).$$

Proof. The implication (⇒). Suppose $\eta = \mathrm{ter}(\varphi, \psi)$. According to (3.1) we get

$$\sigma_{01}\eta(B) = \varphi\sigma_{01}(B) = \varphi(01) = \varphi\sigma_{01}(AC) = \sigma_{01}\eta(AC),$$
$$\sigma_{10}\eta(B) = \psi\sigma_{10}(B) = \psi(10) = \psi\sigma_{10}(CA) = \sigma_{10}\eta(CA).$$

The implication (⇐). Define morphisms φ, ψ as

$$\varphi(0) = \sigma_{01}\eta(A), \qquad \psi(0) = \sigma_{10}\eta(A),$$
$$\varphi(1) = \sigma_{01}\eta(C), \qquad \psi(1) = \sigma_{10}\eta(C).$$

Immediately we get $\mathrm{ter}\big(\varphi(0), \psi(0)\big) = \eta(A)$ and $\mathrm{ter}\big(\varphi(1), \psi(1)\big) = \eta(C)$. The words $\varphi(01)$ and $\psi(10)$ satisfy

$$\varphi(01) = \sigma_{01}\eta(AC) = \sigma_{01}\eta(B) \quad \text{and} \quad \psi(10) = \sigma_{10}\eta(CA) = \sigma_{10}\eta(B),$$

which means that $\mathrm{ter}\big(\varphi(01), \psi(10)\big) = \eta(B)$. □

For the morphism (3.2), we get $\sigma_{01}\eta(B) = 010 \neq 011 = \sigma_{01}\eta(AC)$. Another even simpler example of a 3iet-preserving morphism that is not a ternarization is the morphism interchanging the letters A and C.

Now, our goal will be to determine the number of amicable pairs of morphisms with incidence matrix $\mathbf{A}$ of $\det \mathbf{A} = \pm 1$. We will use the notion of b-amicable morphisms.

Definition 3.5. Let φ and ψ be binary morphisms and let $b \in \mathbb{N}$. We say that φ is *b-amicable* to ψ, if φ is amicable to ψ and the number of occurrences of B in $\mathrm{ter}\big(\varphi(01), \psi(10)\big)$ is b.

We now determine the numbers of pairs of b-amicable Sturmian morphisms.

Proposition 3.6. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *be a matrix with* $\det \mathbf{A} = \pm 1$ *and* $b \in \mathbb{N}$. *Put* $p = p_0 + p_1$, $q = q_0 + q_1$. *Then the number* $c_{\mathbf{A}}(b)$ *of pairs of* b*-amicable morphisms with matrix* $\mathbf{A}$ *is equal to*

$$c_{\mathbf{A}}(b) = \begin{cases} \|\mathbf{A}\| - b & \text{if } \det \mathbf{A} = +1 \text{ and } 1 \leq b \leq \min\{p, q\}, \\ \|\mathbf{A}\| - b - 2 & \text{if } \det \mathbf{A} = -1 \text{ and } 0 \leq b \leq \min\{p, q\} - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\|\mathbf{A}\| = p + q$.

First, let us state the following lemma.

Lemma 3.7. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *be a matrix with* $\det \mathbf{A} = \pm 1$ *and* $b \in \mathbb{N}$. *Put* $p = p_0 + p_1$, $q = q_0 + q_1$ *and* $N = \|\mathbf{A}\| = p + q$. *Let* S *be a two-interval exchange with the slope* p/N. *Let* $w^{(k)}$ *be a word of the length* N *that codes* S *with the start point* k/N, *for* $k \in \{0, \ldots, N-1\}$.

Then $w^{(k)}$ *is* b*-amicable to* $w^{(\bar{k})}$ *if and only if* $0 \leq b \leq \min\{p, q\}$ *and* $\bar{k} - k = b$.

Proof. Using (2.2), we see that $S^i(k/N) \equiv (k-ip)/N \pmod 1$, which is equivalent to $NS^i(k/N) \equiv k-ip \pmod N$. We know that the numbers p and N are co-prime, thus the mapping $f_k : \{0,\dots,N-1\} \to \{0,\dots,N-1\}$ given by the congruence $f_k(i) \equiv k - ip \pmod N$ is a bijection. As well, $f_{\bar{k}}(i) - f_k(i) \equiv \bar{k} - k \pmod N$.

Denote $m = \min\{p,q\}$ and $b = \bar{k} - k$. Consider the following cases:

- Case $b < 0$. We shall see that $w^{(k)}$ is lexicographically larger than $w^{(\bar{k})}$, *i.e.* if $i \in \mathbb{N}$ is the first position such that $w_i^{(k)} \neq w_i^{(\bar{k})}$, then $w_i^{(k)} = 1$ and $w_i^{(\bar{k})} = 0$. Directly from the definition of amicability, if $w^{(k)} \propto w^{(\bar{k})}$ and $w^{(k)} \neq w^{(\bar{k})}$, then $w^{(k)}$ is lexicographically smaller than $w^{(\bar{k})}$. These two facts make a contradiction.
- Case $b \in \{0,\dots,m\}$. Let $\mathcal{I}_a \subset \{0,\dots,N-1\}$ be a set of indices i such that $w_i^{(k)} = a$ and $w_i^{(\bar{k})} \neq a$, for both $a = 0,1$. To show that $w^{(k)}$ is b-amicable to $w^{(\bar{k})}$, we need to show that $i \in \mathcal{I}_0$ implies $i+1 \in \mathcal{I}_1$ and $\#\mathcal{I}_0 = \#\mathcal{I}_1 = b$. The fact that $\left|w^{(k)}\right|_0 = \left|w^{(\bar{k})}\right|_0$ follows to $\#\mathcal{I}_0 = \#\mathcal{I}_1$.
  Let i be an index such that $f_k(i) \in [p-b,p)$, thus $w_i^{(k)} = 0$. Then $f_{\bar{k}}(i) \in [p,p+b)$, thus $w_i^{(\bar{k})} = 1$. This means $i \in \mathcal{I}_0$. For these i, we have $f_k(i+1) \in [N-b,N)$ and $f_{\bar{k}}(i+1) \in [0,b)$, which means $i \in \mathcal{I}_1$. There are exactly b such indices i.
  It remains to show that we covered the whole set $\mathcal{I}_0$. Suppose $f_k(i) < p-b$, then $f_{\bar{k}}(i) < p$ and $w_i^{(\bar{k})} = 0$, which means $i \notin \mathcal{I}_0$. Suppose $f_k(i) \geq p$, then $w_i^{(k)} = 1$, which means $i \notin \mathcal{I}_0$.
- Case $b \in \{m+1,\dots,N-m-1\}$. Let i be such index that $f_k(i) = p-1$. Then $f_k(i+1) = N-1$.
  If $p \leq q$, then $f_{\bar{k}}(i) = b+p-1$ and $f_{\bar{k}}(i+1) = b-1$, which means that $w_i^{(k)}w_{i+1}^{(k)} = 01$ and $w_i^{(\bar{k})}w_{i+1}^{(\bar{k})} = 11$.
  If $p > q$, then $f_{\bar{k}}(i) = b-q-1$ and $f_{\bar{k}}(i+1) = b-1$, which means that $w_i^{(k)}w_{i+1}^{(k)} = 01$ and $w_i^{(\bar{k})}w_{i+1}^{(\bar{k})} = 00$.
  Both these are in contradiction with $w^{(k)} \propto w^{(\bar{k})}$.
- Case $b \in \{N-m,\dots,N-1\}$.
  Suppose $p < q$. Then $j = 2p$ solves the inequalities

$$\begin{aligned} &p \leq j < N, && p \leq j+b-N < N,\\ &p \leq j-p < N, && 0 \leq j+b-p-N < p. \end{aligned}$$

  Let i be an index such that $f_k(i) = j$. Then the previous inequalities give $w_i^{(k)}w_{i+1}^{(k)} = 11$ and $w_i^{(\bar{k})}w_{i+1}^{(\bar{k})} = 10$, which is in a contradiction with $w^{(k)} \propto w^{(\bar{k})}$.
  Suppose $p > q$. Then $j = 2p-b-1$ solves the inequalities

$$\begin{aligned} &0 \leq j < p, && 0 \leq j+b-N < p,\\ &p \leq j-p+N < N, && 0 \leq j+b-p < p. \end{aligned}$$

  Let i be an index such that $f_k(i) = j$. Then the previous inequalities give $w_i^{(k)}w_{i+1}^{(k)} = 01$ and $w_i^{(\bar{k})}w_{i+1}^{(\bar{k})} = 00$, which is a contradiction with $w^{(k)} \propto w^{(\bar{k})}$. □

Proof of Proposition 3.6. Let S be a 2-interval exchange transformation with the slope $\varepsilon = p/N$. Let $k \in \mathbb{Z}$ and denote $w^{(k)}$ the word of the length $N = \|\mathbf{A}\|$ that codes the orbit of the point $\{k/N\}$ with respect to S. From [14] we know that for every Sturmian morphism φ with $\mathbf{M}_\varphi = \mathbf{A}$, there exists $k \in \{0, \dots, N-1\}$ such that $\varphi(01) = w^{(k)}$, we will denote this morphism $\varphi^{(k)}$.

Let φ_{std} be a standard morphism with $\mathbf{M}_{\varphi_{\text{std}}} = \mathbf{A}$. Every Sturmian morphism $\varphi^{(k)}$ is a right conjugate to φ_{std}, which means that there exist words $v, v' \in \{0,1\}*$ such that

$$\varphi^{(k)}(aa') = v01v' \quad \text{and} \quad \varphi^{(k)}(a'a) = v10v',$$

where letters a, a' satisfy $aa' = 01$ for $\det \mathbf{A} = +1$ and $aa' = 10$ for $\det \mathbf{A} = -1$. This gives that $\varphi(aa')$ is 1-amicable to $\varphi(a'a)$.

Morphism $\varphi^{(k)}$ is b-amicable to $\varphi^{(\bar{k})}$ if and only if the following conditions are satisfied:

1. $\varphi^{(k)}(01)$ is b-amicable to $\varphi^{(\bar{k})}(10)$;
2. $\varphi^{(k)}(01)$ is amicable to $\varphi^{(\bar{k})}(01)$;
3. Parikh vectors satisfy $\Psi(\varphi^{(k)}(0)) = \Psi(\varphi^{(\bar{k})}(0))$.

The 2nd and 3rd conditions assures that $\varphi^{(k)}(0) \propto \varphi^{(\bar{k})}(0)$ and $\varphi^{(k)}(1) \propto \varphi^{(\bar{k})}(1)$.

Let us discuss the cases $\det \mathbf{A} = +1$ and $\det \mathbf{A} = -1$.

- Case $\det \mathbf{A} = +1$. We know that $\varphi^{(k)}(01)$ is 1-amicable to $\varphi^{(k)}(10)$, implying by Lemma 3.7 that $\varphi^{(k)}(10) = w^{(k+1)}$. This excludes $k = N-1$.
  The 3rd condition is immediately satisfied by $\mathbf{M}_{\varphi^{(k)}} = \mathbf{M}_{\varphi^{(\bar{k})}}$. To satisfy the 1st condition, we need $(\bar{k}+1) - k = b$. To satisfy the 2nd condition, we need $0 \le \bar{k} - k \le \min\{p, q\}$. These facts gives $0 \le k \le \bar{k} \le N-2$ and $1 \le b \le \min\{p, q\}$, because the value $b = \min\{p, q\} + 1$ is denied by Lemma 3.7. For each admissible b, we have exactly $N - b$ pairs of indices $(k, \bar{k})$.
- Case $\det \mathbf{A} = -1$. We know that $\varphi^{(k)}(10)$ is 1-amicable to $\varphi^{(k)}(01)$, implying by Lemma 3.7 that $\varphi^{(k)}(10) = w^{(k-1)}$. This excludes $k = 0$.
  The 3rd condition is immediately satisfied by $\mathbf{M}_{\varphi^{(k)}} = \mathbf{M}_{\varphi^{(\bar{k})}}$. To satisfy the 1st condition, we need $(\bar{k}-1) - k = b$. To satisfy the 2nd condition, we need $0 \le \bar{k} - k \le \min\{p, q\}$. These facts gives $1 \le k \le \bar{k} \le N-1$ and $0 \le b \le \min\{p, q\} - 1$, because the value $b = -1$ is denied by Lemma 3.7. For each admissible b, we have exactly $N - b - 2$ pairs of indices $(k, \bar{k})$. □

Remark 3.8. The proof shows an interesting fact: suppose that

$$\text{the word } \varphi^{(k)}(01) \text{ is } (b - \Delta)\text{-amicable to } \varphi^{(\bar{k})}(01) \tag{3.3}$$

and $c_{\mathbf{A}}(b) \neq 0$. Then the morphism $\varphi^{(k)}$ is b-amicable to $\varphi^{(\bar{k})}$. The reason is as follows: In the proof we considered all pairs of $(k, \bar{k})$ and to satisfy (3.3) there is no other choice but $\bar{k} - k = b - \Delta$. The condition $c_{\mathbf{A}}(b) \neq 0$ is what we needed in the proof to show that $\varphi^{(k)}(01)$ is b-amicable to $\varphi^{(\bar{k})}(10)$. Thus the conditions 1, 2 from the proof are true; the condition 3 is straightforward.

Proof of Theorem 1.1. The formula (1.1) can be obtained by summation of numbers $c_{\mathbf{A}}(b)$ from the previous proposition. □

To each pair of amicable Sturmian morphisms, an incidence matrix of its ternarization is assigned. We now fully describe which matrices from $\mathbb{N}^{3\times 3}$ are matrices of ternarizations.

Theorem 3.9. *A matrix* $\mathbf{B} \in \mathbb{N}^{3\times 3}$ *is the incidence matrix of the ternarization of a pair of amicable Sturmian morphisms if and only if there exists a matrix* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *with* $\det \mathbf{A} = \Delta = \pm 1$ *and numbers* $b_0, b_1 \in \mathbb{N}$ *such that*

(*a*) $\left|\frac{b_0(p_1+q_1)-b_1(p_0+q_0)}{p_0+q_0+p_1+q_1}\right| < 1$;

(*b*) $\frac{1-\Delta}{2} \leq b_0 + b_1 \leq \min\{p_0+p_1, q_0+q_1\} - \frac{\Delta+1}{2}$;

(*c*) $\mathbf{B} = \mathbf{P}\begin{pmatrix} \mathbf{A} & \begin{smallmatrix} b_0 \\ b_1 \end{smallmatrix} \\ 0\ 0 & \Delta \end{pmatrix}\mathbf{P}^{-1}$, *where* $\mathbf{P} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$.

Proof of the implication ($\Rightarrow$). Let us denote $p = p_0 + p_1$, $q = q_0 + q_1$, $N = p + q$ and $b = b_0 + b_1 + \Delta$. Then we can see that condition (c) gives

$$\mathbf{B} = \begin{pmatrix} p_0 - b_0 & b_0 & q_0 - b_0 \\ p - b & b & q - b \\ p_1 - b_1 & b_1 & q_1 - b_1 \end{pmatrix}. \tag{3.4}$$

The fact that (c) is necessary for $\mathbf{B}$ to be an incidence matrix of a ternarization is shown in [1], Remark 13. Condition (b) is necessary according to Proposition 3.6, so we only need to show that (a) is satisfied for the matrix of the ternarization $\eta = \mathrm{ter}(\varphi, \psi)$ of a pair of amicable Sturmian morphisms $\varphi \propto \psi$.

We can see that $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right)$ is necessarily an incidence matrix of both φ and ψ. Let S be a 2-interval exchange transformation with a rational slope $\varepsilon = p/N$. Then there exist numbers $k, \bar{k} \in \{0, \ldots, N-2\}$ such that $\varphi(01)$, $\psi(01)$ code transformation S with start points $x_0 = k/N$, $\bar{x}_0 = \bar{k}/N$, respectively; moreover, $\bar{k} - k = b - \Delta$. We need to determine the value of $b_0 = \left|\mathrm{ter}\big(\varphi(0), \psi(0)\big)\right|_B$. The number b_0 is equal to the number of indices $i \in \{0, 1, \ldots, p_0 + q_0 - 1\}$ such that $S^i x_0 \in \big[(p - b + \Delta)/N, p/N\big)$, because for exactly these i, we have $S^i x_0 < p/N \leq S^i \bar{x}_0$.

Let $X = \big\{\{x_0 - ip/N\} \big| i \in \mathbb{N}, 0 \leq i < p_0 + q_0\big\}$. Put $p' = p + \Delta/(p_0 + q_0)$, and let $Y = \big\{\{x_0 - ip'/N\} \big| i \in \mathbb{N}, 0 \leq i < p_0 + q_0\big\}$. We can see that $0 \leq \Delta\big((x_0 - ip/N) - (x_0 - ip'/N)\big) = i/(p_0+q_0)N < 1/N$. Thus $x_0 - ip/N \in \big[\frac{p-b+\Delta}{N}, \frac{p}{N}\big)$ if and only if

$$x_0 - ip'/N \in \begin{cases} \big(\frac{p-b}{N}, \frac{p-1}{N}\big] & \text{in the case } \Delta = +1, \\ \big[\frac{p-b-1}{N}, \frac{p}{N}\big) & \text{in the case } \Delta = -1. \end{cases} \tag{3.5}$$

In both cases, the length of the interval is $\frac{b-\Delta}{N}$. From $\Delta = \det \mathbf{A} = \det \begin{pmatrix} p_0 & p_0+q_0 \\ p & N \end{pmatrix}$, it is easy to see that

$$\frac{p'}{N} = \frac{p + \Delta/(p_0+q_0)}{N} = \frac{p}{N} + \frac{p_0 N - p(p_0+q_0)}{N(p_0+q_0)} = \frac{p_0}{p_0+q_0}.$$

Because p_0 is co-prime to p_0+q_0, we get $\big\{\{ip_0/(p_0+q_0)\}\big|i \in \mathbb{N}, 0 \le i < p_0+q_0\big\} = \big\{i/(p_0+q_0)\big|i \in \mathbb{N}, 0 \le i < p_0+q_0\big\}$. But this means that the set Y is uniformly distributed on the interval $[0,1)$, therefore

$$b_0 = \#\Big(X \cap \big[\tfrac{p-b+\Delta}{N}, \tfrac{p}{N}\big)\Big) \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\},$$

where $\beta = (p_0+q_0)\frac{b-\Delta}{N}$ is number of elements of Y multiplied by the length of the interval (3.5). Together we get

$$|\beta - b_0| < 1, \tag{3.6}$$

which is equivalent to condition (a). □

The proof of the other implication is divided into several lemmas.

Lemma 3.10. *Let* $\mathbf{A} = \left(\begin{smallmatrix} p_0 & q_0 \\ p_1 & q_1 \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ *with* $\det \mathbf{A} = \Delta = \pm 1$*, let* $b \in \mathbb{N}$ *with* $\frac{1+\Delta}{2} \le b \le \min\{p_0+p_1, q_0+q_1\} - \frac{1-\Delta}{2}$.

Denote $N = \|\mathbf{A}\|$*,* $p = p_0+p_1$ *and* $q = q_0+q_1$ *integers,* $I = \big[\frac{p-b+\Delta}{N}, \frac{p}{N}\big)$ *an interval,* $X_k = \big\{\{k/N\}, S\{k/N\}, S^2\{k/N\}, \ldots, S^{p_0+q_0-1}\{k/N\}\big\}$ *a set of numbers for any* $k \in \mathbb{Z}$*, where* S *is the 2-interval exchange with the slope* $\varepsilon = p/N$*, and denote* $\beta = \frac{p_0+q_0}{N}(b-\Delta)$.

Then for all $b_0 \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\}$ *such that*

$$b_0 \le \min\{p_0, q_0\} \quad \textit{and} \quad b-\Delta-b_0 \le \min\{p_1, q_1\}, \tag{3.7}$$

there exist $k', k'' \in \{0, \ldots, N-1\}$*,* $k' \ne k''$ *such that*

$$\#(X_{k'} \cap I) = \#(X_{k''} \cap I) = b_0. \tag{3.8}$$

Proof. Denote $r(k) = \#(X_k \cap I)$ for $k \in \mathbb{Z}$. We can see that $\sum_{k=0}^{N-1} r(k) = (b-\Delta)(p_0+q_0)$. According to (3.6), we know that $r(k) \in \big\{\lfloor\beta\rfloor, \lceil\beta\rceil\big\}$ for all $k \in \mathbb{Z}$. Let

$$C_L = \#\big\{k \in \{0,\ldots,N-1\}\big|r(k) = \lfloor\beta\rfloor\big\},$$
$$C_U = \#\big\{k \in \{0,\ldots,N-1\}\big|r(k) = \lceil\beta\rceil\big\}.$$

These numbers satisfy the equations

$$C_L\lfloor\beta\rfloor + C_U\lceil\beta\rceil = N\beta$$
$$\text{and} \qquad C_L + C_U = N. \tag{3.9}$$

If $C_L = 0$ or $C_U = 0$, necessarily $\beta \in \mathbb{N}$ and (3.8) is satisfied for all $k \in \mathbb{Z}$.

If $C_L \ge 2$, we have two different $k \in \mathbb{Z}$ satisfying (3.8) for $b_0 = \lfloor\beta\rfloor$. Similarly if $C_U \ge 2$, we have two different $k \in \mathbb{Z}$ satisfying (3.8) for $b_0 = \lceil\beta\rceil$.

We will show that $C_L = 1$ implies $\lfloor\beta\rfloor$ not to satisfy the condition (3.7), and similarly for C_U and $\lceil\beta\rceil$.

If C_U and C_L are non-zero then there is a unique solution

$$C_L = N\{-\beta\} \quad \text{and} \quad C_U = N\{\beta\}.$$

Using relation $p_0N - (p_0+p_1)(p_0+q_0) = \Delta$, we get

$$\begin{aligned} C_U &\equiv (p_0+q_0)(b-\Delta) \pmod N \\ b-\Delta &\equiv -\Delta(p_0+p_1)C_U \pmod N. \end{aligned} \tag{3.10}$$

Let us suppose $C_U = 1$ or $C_L = 1$, *i.e.* $C_U \equiv \pm 1 \pmod N$ due to (3.9). Then (3.9) and (3.10) lead to $b = (p_0+p_1)+\Delta$ or $b = (q_0+q_1)+\Delta$. For $\Delta = +1$, this is in contradiction with the conditions. For $\Delta = -1$, discuss the following two cases.

- Case $b = (p_0+p_1)+\Delta$. This happens when $C_U = 1$. But it means that $b_0 = \lceil \beta \rceil$ is equal to $\lceil \frac{p_0N-\Delta}{N} \rceil = p_0+1$ and this case is excluded by the condition (3.7).
- Case $b = (q_0+q_1)+\Delta$. This happens when $C_L = 1$. But it means that $b_0 = \lfloor \beta \rfloor$ is equal to $q_0 - 1$ hence $b - \Delta - b_0 = q_1 + 1$, which is excluded by (3.7). □

Lemma 3.11. *Let us have the same hypothesis as in Lemma 3.10.*

Define morphisms φ_k for $k \in \mathbb{Z}$ in the following way:

- *the word $\varphi_k(0)$ codes $\{k/N\}, S\{k/N\}, \dots, S^{p_0+q_0-1}\{k/N\}$;*
- *the word $\varphi_k(1)$ codes $S^{p_0+q_0}\{k/N\}, \dots, S^{N-1}\{k/N\}$.*

Let $k_0 \in \mathbb{Z}$ be such integer that $\#(X_{k_0} \cap I) = \#(X_{k_0-p} \cap I)$. Then

$$\varphi_{k_0} \propto \varphi_{k_0+b-\Delta} \quad \text{or} \quad \varphi_{k_0-p} \propto \varphi_{k_0-p+b-\Delta},$$

and the number of B's in the ternarization of the images of the letter 0 is $\#(X_{k_0} \cap I)$.

Proof. Let $k \in \mathbb{Z}$ and let us consider the orbit

$$\{k/N\}, S\{k/N\}, \dots, S^{p_0+q_0-1}\{k/N\}. \tag{3.11}$$

Let $t^{(k)}$ be a word of the length p_0+q_0 that codes (3.11) to the alphabet $\{0, 0', 1, 1'\}$ with the following code:

$$t_i^{(k)} = \begin{cases} 0 & \text{if } S^i\{k/N\} \in \left[0, \frac{p-b+\Delta}{N}\right), \\ 0' & \text{if } S^i\{k/N\} \in \left[\frac{p-b+\Delta}{N}, \frac{p}{N}\right) = I, \\ 1 & \text{if } S^i\{k/N\} \in \left[\frac{p}{N}, \frac{N-b+\Delta}{N}\right), \\ 1' & \text{if } S^i\{k/N\} \in \left[\frac{N-b+\Delta}{N}, 1\right). \end{cases} \tag{3.12}$$

From definition of S, we see that $t_i^{(k)} = 0' \Leftrightarrow t_{i+1}^{(k)} = 1'$. Define two morphisms $\tau, \tau' : \{0, 0', 1, 1'\}^* \to \{0, 1\}^*$ as

$$\begin{aligned} &\tau(0) = 0, && \tau(0') = 0, && \tau(1) = 1, && \tau(1') = 1, \\ &\tau'(0) = 0, && \tau'(0') = 1, && \tau'(1) = 1, && \tau(1') = 0. \end{aligned}$$

If $t^{(k)}$ does not start with $1'$ and does not end with $0'$, then the word $\varphi_k(0) = \tau(t^{(k)})$ is $|t^{(k)}|_{0'}$-amicable to $\tau'(t^{(k)}) = \varphi_{k+b-\Delta}(0)$. Moreover, $|t^{(k)}|_{0'} = \#(X_k \cap I)$. To show this, notice that $S\{k_0/N\} = \{(k_0-p)/N\}$, which means that there exist letters $a, a' \in \{0, 0', 1, 1'\}$ such that $t^{(k_0)}a = a't^{(k_0-p)}$ and $a = 0' \Leftrightarrow a' = 0'$, because the numbers of letters $0'$ in the words $t^{(k_0)}$ and $t^{(k_0-p)}$ coincide.

Consider these two cases:

- If $a = 0'$ then the last letter of $t^{(k_0)}$ is not $0'$ since this implies $a' = 1'$. This yields $\varphi_k(0) \propto \varphi_{k+b-\Delta}(0)$ for $k = k_0$.
- If $a \neq 0'$ then $t^{(k_0-p)}$ does not start with $1'$ and does not end with $0'$. This yields $\varphi_k(0) \propto \varphi_{k+b-\Delta}(0)$ for $k = k_0 - p$.

Similar reasoning leads to the amicability of the images of the letter 1. Thus by concatenation $\varphi_k(01) \propto \varphi_{k+b-\Delta}(01)$. The condition on b is the same as in Proposition 3.6, hence Remark 3.8 applies. □

Lemma 3.12. *Let us have the same hypothesis as in Lemma 3.10.*

Let $k_0 \in \mathbb{Z}$ be a number such that if $\Delta = -1$ and $b = \min\{p, q\} - 1$ then

$$k_0 \not\equiv \begin{cases} -1 \pmod N & \text{in the case } p > q, \\ p - b - 1 \pmod N & \text{in the case } p < q. \end{cases} \tag{3.13}$$

Then

$$\#(X_{k_0} \cap I) = \#(X_{k_0+p} \cap I) \quad \text{or} \quad \#(X_{k_0} \cap I) = \#(X_{k_0-p} \cap I).$$

Proof. Define the words $t^{(k)}$ by (3.12) in the same way as in the previous proof. Denote $\ell = p_0 + q_0$. Then we know that there exist letters $a_0, \dots, a_{\ell+1} \in \{0, 0', 1, 1'\}$ such that

$$\begin{aligned} t^{(k_0+p)} &= a_0 a_1 a_2 \dots a_{\ell-1}, \\ t^{(k_0)} &= \quad a_1 a_2 \dots a_{\ell-1} a_\ell, \\ t^{(k_0-p)} &= \qquad a_2 \dots a_{\ell-1} a_\ell a_{\ell+1}. \end{aligned}$$

Let us remind that $\#(X_k \cap I) = |t^{(k)}|_{0'}$. The proof will be done by contradiction. Suppose that $|t^{(k_0+p)}|_{0'} \neq |t^{(k_0)}|_{0'} \neq |t^{(k_0-p)}|_{0'}$. There are only two possible values of these numbers, thus $|t^{(k_0+p)}|_{0'} = |t^{(k_0-p)}|_{0'}$. This together gives either $a_0 = a_{\ell+1} = 0'$ or $a_1 = a_\ell = 0'$. It means that there exist $\xi \in I = \left[\frac{p-b+\Delta}{N}, \frac{p}{N}\right)$ and $\omega \in \{+1, -1\}$ such that $S^{\ell+\omega}\xi \in I$. Without the loss of generality $\xi \in \frac{1}{N}\mathbb{Z}$. Since $\ell p = p_0 N - \Delta$, we have

$$S^{\ell+\omega}\xi \equiv \xi - \frac{(\ell+\omega)p}{N} \equiv \xi + \frac{\Delta - \omega p}{N} \pmod 1.$$

Because $|S^{\ell+\omega}\xi - \xi| < 1$ we have

$$\begin{aligned} S^{\ell+\omega}\xi - \xi &= \frac{\Delta - \omega p}{N} \\ \text{or} \quad S^{\ell+\omega}\xi - \xi &= \frac{\Delta - \omega p}{N} + \omega = \frac{\Delta + \omega q}{N}, \end{aligned}$$

since $1-p/N=q/N$. This enforces $b-1-\Delta \geq \min\{p,q\}-1$ for the interval I to be large enough to contain both ξ and $S^{\ell+\omega}\xi$.

For $\Delta=+1$, this is in contradiction with $b \leq \min\{p,q\}$.

For $\Delta=-1$ we get only one admissible $b=\min\{p,q\}-1$. The case $p=\min\{p,q\}$ means $\omega=-1$ and $\xi=\frac{p-b-1}{N}$, which implies $k_0 \equiv p-b-1 \pmod N$. The case $q=\min\{p,q\}$ means $\omega=+1$ and $\xi=\frac{p-1}{N}$, which implies $k_0 \equiv -1 \pmod N$. Both these cases are excluded by (3.13). □

Proof of the implication ($\Leftarrow$). From [1], Remark 13, the incidence matrix of the ternarization $\operatorname{ter}(\varphi,\psi)$ is fully described by the matrix $\mathbf{A}$ and numbers b_0 and $b=b_0+b_1+\Delta$. The condition (a) is equivalent to (3.6) and it gives at most two values of b_0. If $\beta \in \mathbb{N}$, there is nothing to do as we have at least one pair of b-amicable morphisms $\varphi \propto \psi$ for $\mathbf{A}$, and its incidence matrix satisfies all three conditions.

For $\beta \notin \mathbb{N}$, we want to show that for both $b_0 \in \{\lfloor\beta\rfloor, \lceil\beta\rceil\}$ there exist $\varphi \propto \psi$ with $\left|\operatorname{ter}\big(\varphi(0),\psi(0)\big)\right|_B = b_0$. Because the elements of the matrix $\mathbf{B}$ are non-negative, the condition (3.7) of Lemma 3.10 is satisfied and we have two different k', k''. At least one of them satisfies (3.13). Lemma 3.12 then provides k_0 satisfying the conditions of Lemma 3.11 that gives a pair of amicable Sturmian morphisms, ternarization of which has the incidence matrix $\mathbf{B}$. □

4. Conclusions and open problems

Matrices of 3iet-preserving morphisms were studied in [1]. The authors give a necessary condition on $\mathbf{B} \in \mathbb{N}^{3\times 3}$ to be an incidence matrix of a 3iet-preserving morphism:

$$\mathbf{B}\mathbf{E}\mathbf{B}^{\mathsf{T}} = \pm\mathbf{E}, \quad \text{where} \quad \mathbf{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

However, this condition is not sufficient. In our contribution, we study 3iet-preserving morphisms $\eta = \operatorname{ter}(\varphi,\psi)$ arising from pairs of amicable Sturmian morphisms $\varphi \propto \psi$. Our Theorem 3.9 gives sufficient and necessary condition for any matrix $\mathbf{B} \in \mathbb{N}^{3\times 3}$ to satisfy $\mathbf{B} = \mathbf{M}_\eta$ for some ternarization $\eta = \operatorname{ter}(\varphi,\psi)$.

It remains to answer the question about the role of the monoid

$$\mathcal{M}_{\text{ter}} = \big\{\operatorname{ter}(\varphi,\psi) \big| \varphi,\psi \text{ amicable morphisms}\big\}$$

in the whole monoid $\mathcal{M}_{3\text{iet}}$ of all 3iet-preserving morphisms. It seems that using similar proof as for Theorem 2.5 (see [2]) we can prove the following statement.

Corollary 4.1. *Let $\eta \in \mathcal{M}_{3\text{iet}}$. Then one of η, $\eta\circ\xi_1$, $\eta\circ\xi_2$ or $\eta\circ\xi_1\circ\xi_2$ is in $\mathcal{M}_{\text{ter}}$, where*

$$\xi_1(A)=C, \qquad \xi_1(B)=B, \qquad \xi_1(C)=A,$$
$$\xi_2(A)=B, \qquad \xi_2(B)=ACA, \qquad \xi_2(C)=A.$$

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