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ON THE D0L REPETITION THRESHOLD

Ilya Goldstein[1]

Abstract. The repetition threshold is a measure of the extent to which there need to be consecutive (partial) repetitions of finite words within infinite words over alphabets of various sizes. Dejean's Conjecture, which has been recently proven, provides this threshold for all alphabet sizes. Motivated by a question of Krieger, we deal here with the analogous threshold when the infinite word is restricted to be a D0L word. Our main result is that, asymptotically, this threshold does not exceed the unrestricted threshold by more than a little.

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1. Introduction

For each $n \geq 2$, let Σ_n be an alphabet of size n. Given an infinite word w over Σ_n, we are interested in finite words which appear several times consecutively in w. For $n = 2$ it is obvious that each word, $w \in \Sigma_2^\infty$, has a factor (*i.e.*, subword), u, which is a square (meaning that $u = vv$ for some $v \in \Sigma_2^*$). On the other hand, the Thue-Morse word, $w \in \Sigma_2^\infty$, has the property that, for each letter $a \in \Sigma_2$ and word $v \in \Sigma_2^*$, the word $avava$ is not a factor of w [24]. The *critical exponent, $E(w)$*, of a word $w \in \Sigma_n^\infty$ is given by:

$$E(w) = \sup \left\{ \frac{|u^k v|}{|u|} : k \geq 1, u^k v \,\text{is a factor of}\, w, \,\text{and}\, v \,\text{is a prefix of}\, u \right\}.$$

With this notation, we may say that for each $w \in \Sigma_2^\infty$ we have $E(w) \geq 2$, but there exists a word (the Thue-Morse word) $w \in \Sigma_2^\infty$ such that $E(w) = 2$. One may consider also larger values of n. Define for $n \geq 2$ the *repetition threshold*:

$$RT(n) = \inf \{E(w) : w \in \Sigma_n^\infty\}.$$

[1] Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel; ilyago@bgu.ac.il

The above actually states that $RT\,(2) = 2$, which raises the question of finding $RT\,(n)$ for each n. Thue [23] constructed a word, $w \in \Sigma_3^\infty$, such that $E\,(w) < 2$, which started the study of the exact value of $RT\,(3)$. Later, Dejean [7] proved that $RT\,(3) = 7/4$, $RT\,(4) \geq 7/5$, and trivially $RT\,(n) \geq n/\,(n-1)$ for $n \geq 5$. She stated the now well-known *Dejean's Conjecture*:

$$RT\,(n) = \begin{cases} 2, & n = 2, \\ 7/4, & n = 3, \\ 7/5, & n = 4, \\ n/\,(n-1)\,, & n \geq 5. \end{cases}$$

The conjecture has been completely settled only recently. Pansiot [21] proved its correctness for $n = 4$, then Moulin-Ollagnier [20] generalized his method to prove it for $n \in [5, 11]$, later Mohammad-Noori and Currie [18] generalized it for $n \in [5, 14]$, and a recent work of Carpi [1,2] has proven it for $n \geq 33$. Just this year Currie and Rampersad [4–6] proved its correctness for the remaining cases, i.e., $n \in [15, 32]$.

In the proof of Dejean's Conjecture, the cases $n = 2$ and $n = 3$ are special. In these cases the upper bounds were found using a word which is a fixed point of a morphism $\varphi : \Sigma_n^* \to \Sigma_n^*$, whereas for $n \geq 4$ the upper bounds were established by proving that an appropriate word, $w \in \Sigma_n^\infty$, exists. Fixed points of morphisms on the alphabet Σ_n for some $n \geq 2$ were studied by many researchers, including Ehrenfeucht and Rozenberg [8–12], Cassaigne [3], Frid [13–15], Mossé [19] and Tapsoba [22]. They studied a variety of properties of these words, which have also become to be known as D0L words.

The difference between the cases $n = 2, 3$ and $n \geq 4$ in the proof of Dejean's Conjecture leads to additional questions. Consider the quantity

$$RT_{D0L}\,(n) = \inf \left\{ E\,(w) : w \in \Sigma_n^\infty \text{ such that } w \text{ is a D0L word} \right\},$$

introduced by Krieger [17]. From the proof of Dejean's Conjecture it follows that $RT_{D0L}\,(n) = RT\,(n)$ for $n = 2, 3$. Obviously, $RT_{D0L}\,(n) \geq RT\,(n)$ for all n, but it is not clear whether we actually have here an equality for $n \geq 4$. A construction of Carpi [1] implies that $\lim_{n\to\infty} RT_{D0L}\,(n) = 1$. We would like to compare the speed of convergence with that we have for $RT\,(n)$. To do so, we use a special kind of morphisms: The underlying alphabet consists of the elements of a certain finite abelian group, and the morphism is defined algebraically. (A similar construction was considered in [16], there the subword complexity of the emerging D0L words was studied.) Using such a construction, we are able to show that $RT_{D0L}\,(n) = RT\,(n) + o\,(1/n)$. We note that the morphisms we use to construct these words are uniform, so that the small repetition thresholds are already obtained for a sub-family of that of all D0L words.

We wish to express our gratitude to Dalia Krieger, who introduced us with this subject, and gave a clear explanation about this issue.

2. Definitions and basic properties

2.1. A uniform morphic word

Let k be a positive integer divisible by 4, and m a positive integer divisible by k. Put $n = km$. Let $G = C_m \times C_{4k}$, where C_l is the cyclic group of order l for any positive integer l. We view G also as our alphabet, and put $\Sigma = G$. We want to define a uniform morphism of length m over G. Given any $(q, r) \in G$, put $q = kp + j$ where $p \geq 0$ and $j \in [0, k-1]$, and define

$$\mu\left(q, r\right) = a_0 a_1 \ldots a_{m-1},$$

where for even r:

$$a_i = \begin{cases} \left(rk + i, j\right), & i \equiv 0 \pmod 4, \\ \left(\left(p + r\right)k + i, j\right), & i \equiv 1 \pmod 4, \\ \left(i, j\right), & i \equiv 2 \pmod 4, \\ \left(pk + i, j\right), & i \equiv 3 \pmod 4, \end{cases}$$

and for odd r:

$$a_i = \begin{cases} \left(rk + i, j\right), & i \equiv 0 \pmod 4, \\ \left(\left(p + r\right)k + i, j + (-1)^{\lfloor j/2 \rfloor} \cdot 2\right), & i \equiv 1 \pmod 4, \\ \left(i, j\right), & i \equiv 2 \pmod 4, \\ \left(pk + i, j + (-1)^{\lfloor j/2 \rfloor} \cdot 2\right), & i \equiv 3 \pmod 4. \end{cases}$$

We denote by μ also the extension of μ to a uniform morphism of Σ^*, as well as the mapping induced by μ on $\Sigma^{\mathbb{N}}$. Let $w = w_0 w_1 w_2 \ldots$ be the fixed point of this latter map with $w_0 = (0, 0)$.

3. Main results

Theorem 3.1. *We have*

$$E\left(w\right) = 1 + \frac{1 + 1/\left(m^2 - m\right)}{\left(k-3\right)m + k} < 1 + \frac{k}{\left(k-3\right)n}\cdot$$

Put $\Sigma_i = \{0, 1, \ldots, i-1\}$ for each positive integer i.

Corollary 3.2. *There exists a sequence of uniform morphic words, $(y_i)_{i=1}^{\infty}$, where $y_i \in \Sigma_i^{\mathbb{N}}$, such that*

$$\lim_{i \to \infty} \frac{E\left(y_i\right) - 1}{RT\left(i\right) - 1} = 1.$$

Corollary 3.3.

$$RT_{D0L}\left(n\right) = RT\left(n\right) + o\left(1/n\right).$$

4. Proof of theorem 3.1

A straightforward consequence of the definition of μ is the following lemma.

Lemma 4.1. *Let $\alpha \in G$, with $\mu(\alpha) = a_0 a_1 \ldots a_{m-1}$, where $a_i = (q_i, r_i)$ for $i \in [0, m-1]$. Then $q_i \equiv i \pmod{k}$ for $i \in [0, m-1]$.*

Since $k|m$ and $w = \mu(w_0)\,\mu(w_1)\,\mu(w_2)\ldots$, the lemma yields the following corollary.

Corollary 4.2. *Let $i \geq 0$, and $w_i = (q, r)$. Then $q \equiv i \pmod{k}$.*

For $j \in C_k$, let

$$\begin{aligned} A_j &= \{(i,j) : i \in C_m\}, \\ B_j &= \left\{\left(i, j + (-1)^{\lfloor j/2 \rfloor} + (-1)^{\lfloor j/2 \rfloor + 1 + i}\right) : i \in C_m\right\}. \end{aligned}$$

For example,

$$\begin{aligned} B_1 &= \{(0,1), (1,3), (2,1), (3,3), (4,1), \ldots, (m-1,3)\}, \\ B_3 &= \{(0,3), (1,1), (2,3), (3,1), (4,3), \ldots, (m-1,1)\}. \end{aligned}$$

Lemma 4.3. *Let $\alpha = (q, r) \in G$, where $q = kp + j$ for some $p \geq 0$ and $j \in [0, k-1]$. If $2|r$ then the word $\mu(\alpha)$ is formed of the letters in A_j in some order, while if $2 \nmid r$ then it consists of the letters in B_j in some order.*

Proof. Put $\mu(\alpha) = a_0 a_1 \ldots a_{m-1}$, and suppose that r is even. Let $i \in [0, m-1]$. Then $(i,j) \in A_j$. If $i \equiv 0 \pmod{4}$, then for $t = i - rk \bmod m$ we have $a_t = (i,j)$. If $i \equiv 1 \pmod{4}$, then for $t = i - (r+p)\,k \bmod m$ we have $a_t = (i,j)$. If $i \equiv 2 \pmod{4}$, then $a_i = (i,j)$. Finally, if $i \equiv 3 \pmod{4}$, then for $t = i - kp \bmod m$ we have $a_t = (i,j)$. Thus, there exists some $t \in [0, m-1]$ such that $a_t = (i,j)$, and hence for each $\beta \in A_j$ there exists some $t \in [0, m-1]$ such that $a_t = \beta$. Since the length of $\mu(\alpha)$ is m, just as the size of A_j, it means that $\mu(\alpha)$ is a permutation of the letters in A_j.

Now suppose that r is odd. Let $i \in [0, m-1]$. In case i is even we have $(i,j) \in B_j$, and in case i is odd we have $\left(i, j + (-1)^{\lfloor j/2 \rfloor} \cdot 2\right) \in B_j$. If $i \equiv 0 \pmod{4}$, then for $t = i - rk \bmod m$ we have $a_t = (i,j)$. If $i \equiv 2 \pmod{4}$, then $a_i = (i,j)$. If $i \equiv 1 \pmod{4}$, then for $t = i - (r+p)\,k \bmod m$ we have $a_t = \left(i, j + (-1)^{\lfloor j/2 \rfloor} \cdot 2\right)$. If $i \equiv 3 \pmod{4}$, then for $t = i - kp \bmod m$ we have $a_t = \left(i, j + (-1)^{\lfloor j/2 \rfloor} \cdot 2\right)$. Thus, there exists some $t \in [0, m-1]$ such that $a_t = \left(i, j + (-1)^{\lfloor j/2 \rfloor} + (-1)^{\lfloor j/2 \rfloor + 1 + i}\right)$, and hence for each $\beta \in B_j$ there exists some $t \in [0, m-1]$ such that $a_t = \beta$. Since the length of $\mu(\alpha)$ is m, just as the size of B, it means that $\mu(\alpha)$ is a permutation of the letters in B_j. ☐

The following lemma is a straightforward consequence of the lemmas above.

Lemma 4.4. *Let $\alpha \in G$, and put $\alpha = (q,r)$. Put $\mu(\alpha) = a_0 a_1 \dots a_{m-1}$ and $\mu^2(\alpha) = \pi_0 \pi_1 \dots \pi_{m-1}$, where $\pi_i = \mu(a_i)$ for $i \in [0, m-1]$. If q is even, then for each $i \in [0, m-1]$, π_i is a permutation of the letters in $A_{i \bmod k}$. If q is odd, then π_i is a permutation of the letters in $B_{i \bmod k}$ for each $i \in [0, m-1]$.*

Proof. Let $p \geq 0$ and $j \in [0, k-1]$ be such that $q = pk + j$. For each $i \in [0, m-1]$ put $a_i = (x_i, x'_i)$. The definition of μ guarantees that, in case j is even, for each $i \in [0, m-1]$ the values x'_i are even, while if j is odd then for each $i \in [0, m-1]$ the values x'_i are odd. Moreover, Lemma 4.1 yields that $x_i \equiv i \pmod{k}$. Therefore, Lemma 4.3 implies that in case j is even, then for each $i \in [0, m-1]$ the word π_i is a permutation of the letters in $A_{i \bmod k}$; and in case j is odd then for each $i \in [0, m-1]$, this word is a permutation of the letters in $B_{i \bmod k}$. Since $2|k$, then j is even if and only if q is even, which completes the proof. □

Lemma 4.5. *Let $\alpha \in G$, and let $u, v \in \Sigma^*$ be such that v is a prefix of u, and uv is a factor of the word $\mu^2(\alpha)$. Then $|v| \leq 1$, and if $|v| = 1$, then $|u| \geq k(m-1)$.*

Proof. Put $\alpha = (q,r)$, $\mu(\alpha) = a_0 a_1 \dots a_{m-1}$, $\mu^2(\alpha) = b_0 b_1 \dots b_{m^2-1}$, and $a_i = (x_i, x'_i)$ for $i \in [0, m-1]$. Lemma 4.4 yields that $\mu^2(\alpha) = \pi_0 \pi_1 \dots \pi_{m-1}$, where either $\pi = \mu(a_i)$ is a permutation of the letters in $A_{i \bmod k}$ for each $i \in [0, m-1]$, or $\pi_i = \mu(a_i)$ is a permutation of the letters in $B_{i \bmod k}$ for each $i \in [0, m-1]$. Note that the sets A_j, $j \in C_k$, are pairwise disjoint, as are the sets B_j, $j \in C_k$. Since uv is a factor of $\mu^2(\alpha)$, there exist a $t \in [0, m-1]$ and $l \geq 0$ such that $t + l \leq m-1$, uv is a factor of $\pi_t \pi_{t+1} \pi_{t+2} \dots \pi_{t+l}$, and uv is neither a factor of $\pi_{t+1} \pi_{t+2} \dots \pi_{t+l}$ nor a factor of $\pi_t \pi_{t+1} \pi_{t+2} \dots \pi_{t+l-1}$. Since π_t is a permutation and v a prefix of u, the word uv is not a factor of π_t, and hence $l > 0$.

Suppose that $|v| = 2$. Put $v = \beta_1 \beta_2$, and $\beta_i = (y_i, y'_i)$ for $i = 1, 2$. The previous paragraph yields that β_2 is a factor of π_{t+l}. Since v is a prefix of u, β_2 is either a factor of π_t or the first letter of π_{t+1}. In the latter case $y'_2 - y'_1$ is odd, and hence the definition of μ yields that β_2 is also the first letter of π_{t+l}. Therefore, β_1 is the last letter of π_t and the last letter of π_{t+l-1}, which yields by the previous paragraph that $t \equiv t + l - 1 \pmod{k}$. Hence we may put $x_i = k z_i + z'_i$ for $i \in [0, m-1]$, where $z_i \geq 0$ and $z'_i \in [0, k-1]$, and we have both $y_1 = z_t k + m - 1$ and $y_1 = z_{t+l-1} k + m - 1$. Therefore $k z_t = k z_{t+l-1}$, and since $t \equiv t + l - 1 \pmod{k}$ Lemma 4.1 yields $z'_t = z'_{t+l-1}$. Thus, $x_t = x_{t+l-1}$, which contradicts Lemma 4.3, and hence β_2 is not the first letter of π_{t+1}. Symmetrically means β_2 is not the first letter of π_{t+l} , and hence v is a factor of π_{t+l} and a factor of π_t.

Since v is a factor of π_t and π_{t+l}, the permutations π_t and π_{t+l} have a common letter, and hence $t \equiv t + l \pmod{k}$. Since $4|k$, it means that l is even, and hence the definition of μ yields $x'_t = x'_{t+l}$. If y_1 is even, then the definition of μ implies that $y_2 - y_1 - 1 \equiv z_t k \pmod{m}$, and also $y_2 - y_1 - 1 \equiv z_{t+l} k \pmod{m}$. Thus, $z_{t+l} k = z_t k$. Since $t \equiv t + l \pmod{k}$, Lemma 4.1 implies $z'_{t+l} = z'_t$, and hence $x_t = x_{t+l}$. Therefore, $\alpha_t = \alpha_{t+l}$, which contradicts Lemma 4.3. Similarly, in case $y_1 \equiv 1 \pmod{4}$, we have $y_2 - y_1 - 1 \equiv -x'_t k - z_t k \pmod{m}$ and $y_2 - y_1 - 1 \equiv -x'_{t+l} k - z_{t+l} k \pmod{m}$. Hence, since $x'_t = x'_{t+l}$, we also have $z_t k = z_{t+l} k$, which implies $x_t = x_{t+l}$. Hence, in case $y_1 \equiv 1 \pmod{4}$ we also have $\alpha_t = \alpha_{t+l}$, which is a contradiction. In the last case, $y_1 = 3 \pmod 4$, we have $y_2 - y_1 - 1 \equiv x'_t k - z_t k \pmod m$ and $y_2 - y_1 - 1 \equiv x'_{t+l} k - z_{t+l} k \pmod m$. Again, since $x'_t = x'_{t+l}$, we also have $x_t = x_{t+l}$, which yields $\alpha_t = \alpha_{t+l}$, and again we have a contradiction. Thus, $|v| = 2$ yields a contradiction, and therefore $|v| \leq 1$.

Now, suppose that $|v| = 1$, and put $v = \beta = (y, y')$. Since β is a factor of π_t and π_{t+l}, we have $t \equiv t + l \pmod k$. Therefore, $k|l$, and since $l > 0$, it implies that $l \geq k$. Hence, in case $l > k$ we have $l \geq 2k$, and therefore $|u| \geq (2k-2)\, m \geq k\,(m-1)$. Now, suppose that $l = k$. Since $\mu^2(\alpha) = b_0 b_1 \dots b_{m^2-1}$, we have $\pi_t = b_{tm} b_{tm+1} b_{tm+2} \dots b_{tm+m-1}$, and $\pi_{t+k} = b_{(t+k)m} b_{(t+k)m+1} \dots b_{(t+k+1)m-1}$. Since β is a factor of π_t, there exists an $i \in [0, m-1]$ such that $\beta = b_{tm+i}$. Since $4|k$, we have $x'_t = x'_{t+k}$, and therefore the definition of μ implies that in case i is even we have $b_{(t+k)m+i} = \beta$. Since π_{t+k} is a permutation, it implies that $|u| = km > (k-1)\, m$. Since $4|k$, the definition of μ yields that $x_{t+k} = x_t + k$. Therefore, in case i is odd we have $b_{(t+k)m+i'} = \beta$, where $i' = i - k \bmod m$. Thus, if i is odd, then either $|u| = km - k$ or $|u| = (k+1)\, m - k$. Both cases imply $|u| \geq (k-1)\, m$, which completes the proof. ☐

Lemma 4.6. *Let $(q, r), (q', r') \in \Sigma$ be such that $q' \equiv q + 1 \pmod k$, and put $\alpha = (q, r)$ and $\beta = (q', r')$. Let $u, v \in \Sigma^*$ be such that v is a prefix of u, and uv is a factor of the word $\mu^2(\alpha\beta)$. Then $|v| \leq 1$, and if $|v| = 1$, then $|u| \geq (k-3)\, m + k$.*

Proof. Put $\mu(\alpha\beta) = a_0 a_1 \dots a_{2m-1}$, $\mu^2(\alpha\beta) = b_0 b_1 \dots b_{2m^2-1}$, and $a_i = (x_i, x'_i)$ for each $i \in [0, 2m-1]$. Since $2|k$ and $q' \equiv q + 1 \pmod k$, Lemma 4.4 yields that $\mu^2(\alpha) = \pi_0 \pi_1 \dots \pi_{m-1}$, and $\mu^2(\beta) = \pi_m \pi_{m+1} \dots \pi_{2m-1}$, where either for each $i \in [0, m-1]$ the word π_i is a permutation of the letters in $A_{i \bmod k}$ and for each $i \in [m, 2m-1]$ it is a permutation of the letters in $B_{i \bmod k}$, or for each $i \in [0, m-1]$ it is a permutation of the letters in $B_{i \bmod k}$ and for each $i \in [m, 2m-1]$ it is a permutation of the letters in $A_{i \bmod k}$.

Since uv is a factor of $\mu^2(\alpha\beta)$, there exist a $t \geq 0$ and an $l > 0$, such that $t + l \leq 2m^2$, and $uv = b_t b_{t+1} b_{t+2} \dots b_{t+l-1}$. In case $t + l \leq m^2$, the word uv is a factor of $\mu^2(\alpha)$, and in case $t \geq m^2$, it is a factor of $\mu^2(\beta)$. In both cases, Lemma 4.5 ensures that $|v| \leq 1$, and if $|v| = 1$ then $|u| \geq k\,(m-1) \geq (k-2)\, m + k$. Therefore, in those cases the lemma is true. Hence, from now on, suppose that $t < m^2$ and $t + l > m^2$.

Suppose that $|v| = 2$, let $v = \gamma_1 \gamma_2$, and put $\gamma_i = (y_i, y'_i)$ for $i = 1, 2$. As in the proof of Lemma 4.5, there is no way for γ_1 to be the last letter of $\pi_{\lfloor t/m \rfloor}$ and π_j for some $j \in [0, m-1] \setminus \{\lfloor t/m \rfloor\}$, and therefore $t + l \neq m^2 + 1$. Thus, $t + l > m^2 + 1$, and therefore v is a factor of $\mu^2(\beta)$. If, for each $i \in [0, m-1]$, the word π_i is a permutation of the letters in $B_{i \bmod k}$, then since v is a factor of $\mu^2(\beta)$ we have either $\gamma'_2 - \gamma'_1 = 0$ or $\gamma'_2 - \gamma'_1 = 1$, and since v is a factor of $\mu^2(\alpha)\, b_{m^2+1}$, then we have either $|\gamma'_2 - \gamma'_1| = 2$, or $\gamma'_2 - \gamma'_1 = 3$, or $\gamma'_2 - \gamma'_1 = -1$. Thus, since $k \geq 4$, we have a contradiction. Therefore, π_i is a permutation of the letters in $A_{i \bmod k}$ for each $i \in [0, m-1]$. Hence, since v is a factor of $\mu^2(\beta)$, we have either $|\gamma'_2 - \gamma'_1| = 2$, or $\gamma'_2 - \gamma'_1 = 3$, or $\gamma'_2 - \gamma'_1 = -1$, and since v is a factor of $\mu^2(\alpha)\, b_{m^2}$ we have either $\gamma'_2 - \gamma'_1 = 0$ or $\gamma'_2 - \gamma'_1 = 1$. Again, since $k \geq 4$, we have a contradiction, which means that $|v| \neq 2$, and hence $|v| = 1$.

Put $v = \gamma = (y, y')$. The definition of t and l implies that $b_t = \gamma$ and $b_{t+l-1} = \gamma$. Therefore, γ is a factor of both π_j and $\pi_{j'}$ for $j = \lfloor t/m \rfloor$ and $j' = \lfloor (t+l-1)/m \rfloor$. Thus, π_j and $\pi_{j'}$ have a common letter, and since either π_j is a permutation of the letters in $A_{j \bmod k}$ and $\pi_{j'}$ is a permutation of the letters in $B_{j' \bmod k}$, or π_j is a permutation of the letters in $B_{j \bmod k}$ and $\pi_{j'}$ a permutation of the letters in $A_{j' \bmod k}$, then either $j' \equiv j \pmod{k}$ or $j' \equiv j \pm 2 \pmod{k}$. Thus, $j' - j \geq k - 2$, and hence $l - 1 > (k-3)\, m$. Lemma 4.1 also implies $t \equiv t + l - 1 \pmod{k}$, and hence $k|m$ implies that $l - 1 \geq (k-3)\, m + k$. This means that $|u| \geq (k-3)\, m + k$, and completes the proof. □

Corollary 4.7. *Let $u, v \in \Sigma^*$ be such that v is a prefix of u, and uv is a factor of w. If $|v| = 1$ then $|u| \geq (k-3)\, m + k$, and if $|v| = 2$ then $|u| \geq m^2$.*

The corollary deals with repetitions of single letters and of words of length 2 in w. From now on, put $w_i = (x_i, x'_i)$ for $i \geq 0$. We turn to study repetitions of factors of length 3 and 4 in w, *i.e.* the existence of $i, j \geq 0$ with $i < j$ such that

$$w_i = w_j, \qquad w_{i+1} = w_{j+1}, \qquad w_{i+2} = w_{j+2}, \tag{4.1}$$

and on occasion also $w_{i+3} = w_{j+3}$.

Lemma 4.8. *Let $i, j \geq 0$ be such that (4.1) is satisfied. Then $i \equiv j \pmod{m}$.*

Proof. Since $4|k$ and $w_i = w_j$, Corollary 4.2 guarantees that $i \equiv j \pmod{4}$. Let $t \in [0, 3]$ be such that $i + t \equiv 2 \pmod{4}$. If $t \neq 3$, then by the definition of μ we have $x_{i+t} \equiv i + t \pmod{m}$ as well as $x_{j+t} \equiv j + t \pmod{m}$. Since $w_{i+t} = w_{j+t}$, we have $i + t \equiv j + t \pmod{m}$, which yields $i \equiv j \pmod{m}$.

It remains to deal with the case $t = 3$. We have $i \equiv 3 \pmod{4}$. If $x'_{i+1} - x'_i$ is odd then $\lfloor i/m \rfloor < \lfloor (i+1)/m \rfloor$, and therefore $i \equiv m - 1 \pmod{m}$. Since $w_j = w_i$ and $w_{j+1} = w_{i+1}$, we analogly obtain $j \equiv m - 1 \pmod{m}$, and hence $i \equiv j \pmod{m}$. Now, suppose that $x'_{i+1} - x'_i$ is even, and therefore $i \equiv 3 \pmod{4}$ yields $\lfloor i/m \rfloor = \lfloor (i+2)/m \rfloor$ as well as $\lfloor j/m \rfloor = \lfloor (j+2)/m \rfloor$. Therefore, in such a case the definition of μ implies that $x_{i+2} - x_{i+1} - x_i - 1 \equiv -i \pmod{m}$ as well as $x_{j+2} - x_{j+1} - x_j - 1 \equiv -j \pmod{m}$. Since the equalities in (4.1) hold, we deduce that $i \equiv j \pmod{m}$, which completes the proof. □

Lemma 4.9. *Let $i, j \geq 0$ be such that (4.1) is satisfied. If either $i \equiv 0 \pmod{4}$ or $i \equiv 1 \pmod{4}$, then $w_{\lfloor i/m \rfloor} = w_{\lfloor j/m \rfloor}$; if $i \equiv 3 \pmod{4}$, then $w_{\lfloor (i+1)/m \rfloor} = w_{\lfloor (j+1)/m \rfloor}$.*

Proof. Since i and j satisfy the equalities in (4.1), Lemma 4.8 implies that $i \equiv j \pmod{m}$, and in particular $i \equiv j \pmod{4}$. In case $i \equiv 0 \pmod{4}$, the definition of μ gives $x_{i+1} - x_i - 1 + x'_i = x_{\lfloor i/m \rfloor}$, and $x_i - x_{i+2} + 2 \equiv k x'_{\lfloor i/m \rfloor} \pmod{m}$. Therefore, in case $i \equiv 0 \pmod{4}$ the equalities in (4.1) imply $x_{\lfloor i/m \rfloor} = x_{\lfloor j/m \rfloor}$, and $k x'_{\lfloor i/m \rfloor} \equiv k x'_{\lfloor j/m \rfloor} \pmod{m}$. Since $k|m$ and $m \geq k^2$, it follows that $x'_{\lfloor i/m \rfloor} = x'_{\lfloor j/m \rfloor}$, and therefore in case $i \equiv 0 \pmod{4}$ we have $w_{\lfloor i/m \rfloor} = w_{\lfloor j/m \rfloor}$.

In case $i \equiv 1 \pmod{4}$, the definition of μ shows that $x_{i+2} - x_{i+1} - 1 + x'_{i+1} = x_{\lfloor i/m \rfloor}$, and $x_i - x_{i+2} + 2 \equiv k x'_{\lfloor i/m \rfloor} \pmod{m}$. Therefore, if $i \equiv 1 \pmod{4}$ then the equalities in (4.1) imply $x_{\lfloor i/m \rfloor} = x_{\lfloor j/m \rfloor}$, $kx'_{\lfloor i/m \rfloor} \equiv kx'_{\lfloor j/m \rfloor} \pmod{m}$, and similarly to the previous case $w_{\lfloor i/m \rfloor} = w_{\lfloor j/m \rfloor}$ as well.

Now suppose that $i \equiv 3 \pmod 4$. If $x'_{i+1} - x'_i$ is odd, then the definition of μ yields $i+1 \equiv 0 \pmod m$ as well as $j+1 \equiv 0 \pmod m$. Hence, is such a case we have $x_{i+1} = kx'_{\lfloor (i+1)/m \rfloor}$ and $x_{i+2} - x_{i+1} - 1 + x'_{i+1} = x_{\lfloor (i+1)/m \rfloor}$. Therefore in such a case the equalities in (4.1) give $x_{\lfloor (i+1)/m \rfloor} = x_{\lfloor (j+1)/m \rfloor}$ and $kx'_{\lfloor i/m \rfloor} \equiv kx'_{\lfloor j/m \rfloor} \pmod m$, which similarly to the previous cases implies $w_{\lfloor (i+1)/m \rfloor} = w_{\lfloor (j+1)/m \rfloor}$. On the other hand, in case $x'_{i+1} - x'_i$ is even, the definition of μ yields $\lfloor (i+1)/m \rfloor = \lfloor i/m \rfloor$, and hence $x_{i+2} - x_{i+1} - 1 + x'_{i+1} = x_{\lfloor (i+1)/m \rfloor}$ and $x_{i+2} - x_i - 2 = kx'_{\lfloor (i+1)/m \rfloor}$. Therefore, in such a case the equalities in (4.1) give $x_{\lfloor (i+1)/m \rfloor} = x_{\lfloor (j+1)/m \rfloor}$, and $kx'_{\lfloor (i+1)/m \rfloor} \equiv kx'_{\lfloor (j+1)/m \rfloor} \pmod m$, which yields $w_{\lfloor (i+1)/m \rfloor} = w_{\lfloor (j+1)/m \rfloor}$ and completes the proof. □

The previous lemma dealt with repetitions of three-lettered factors, and it also gives the following corollary, which completes the study of three- and four-lettered factors in w.

Corollary 4.10. *Let $i, j \geq 0$ be such that (4.1) is satisfied. If $i \equiv 2 \pmod 4$ and $w_{i+3} = w_{j+3}$, then $w_{\lfloor (i+2)/m \rfloor} = w_{\lfloor (j+2)/m \rfloor}$.*

The last lemma and corollary show that each repetition of four- or more letters is formed through a repetition of a single lettered word, a two-lettered word, or a three-lettered word. Lemmas 4.4, 4.5, and Corollary 4.7 studied these repetitions. The following lemmas are tools which will allow us to complete the study of the critical exponent.

Lemma 4.11. *Let $i, i' \geq 0$ be such that $w_i \neq w_{i'}$ and $i \equiv i' \pmod k$. If $x'_i = x'_{i'}$ then for each $t \geq 1$ we have $w_{im^t+j} = w_{i'm^t+j}$ for $j \in [0, (m^t-1)/(m-1)-1]$ and $w_{im^t+j} \neq w_{i'm^t+j}$ for $j = (m^t-1)/(m-1)$. If $x'_i \neq x'_{i'}$ then for each $t \geq 1$ we have $w_{im^t+j} = w_{i'm^t+j}$ for $j \in \left[0, \left(m^{t-1}-1\right)/(m-1)-1\right]$ and $w_{im^t+j} \neq w_{i'm^t+j}$ for $j = \left(m^{t-1}-1\right)/(m-1)$.*

Proof. First, suppose that $x'_i = x'_{i'}$. We claim that for $t \geq 1$ we have $w_{im^t+j} = w_{i'm^t+j}$ for $j \in [0, (m^t-1)/(m-1)-1]$, but $w_{im^t+j} \neq w_{i'm^t+j}$ and $x'_{im^t+j} = x'_{i'm^t+j}$ for $j = (m^t-1)/(m-1)$. Since $i \equiv i' \pmod k$, Corollary 4.2 implies $x_i \equiv x_{i'} \pmod k$, and hence $x'_i = x'_{i'}$ implies $w_{im} = w_{i'm}$. On the other hand, since $w_i \neq w_{i'}$ we have $x_i \neq x_{i'}$, and therefore $x_i \equiv x_{i'} \pmod k$ implies $w_{im+1} \neq w_{i'm+1}$, although $x'_i = x'_{i'}$ yields $x'_{im+1} = x'_{i'm+1}$. Hence, the statement is true for $t = 1$.

Now, suppose the statement is true for some $t \geq 1$. Therefore, $w_{im^{t+1}+j} = w_{i'm^{t+1}+j}$ for each $j \in [0, m(m^t-1)/(m-1)-1]$, and since $x'_{im^t+j} = x'_{i'm^t+j}$ for $j = (m^t-1)/(m-1)$, by the same token as in the previous paragraph we have $w_{im^{t+1}+j} = w_{i'm^{t+1}+j}$, $w_{im^{t+1}+j+1} \neq w_{i'm^{t+1}+j+1}$ and $x'_{im^{t+1}+j+1} = x'_{i'm^{t+1}+j+1}$ for $j = m(m^t-1)/(m-1)$. Since $m(m^t-1)/(m-1) = \left(m^{t+1}-1\right)/(m-1) - 1$, the statement turns out to be true for $t+1$ as well, which proves the lemma for the case $x'_i = x'_{i'}$.

Now, suppose that $x'_i \neq x'_{i'}$. As mentioned above, we have $x_i \equiv x_{i'} \pmod{k}$. Therefore, we have $x_{im} \neq x_{i'm}$, although $x'_{im} = x'_{i'm}$. On the one hand, we have $w_{im} \neq w_{i'm}$ which implies that the lemma is true also for the case where $x'_i \neq x'_{i'}$ and $t = 1$. On the other hand, we have already proved the lemma for the case $x'_i = x'_{i'}$, and hence $x'_{im} = x'_{i'm}$ and $x_{im} \neq x_{i'm}$ show that for each $t \geq 1$ we have $w_{im^t+j} = w_{i'm^t+j}$ for each $j \in \left[0, \left(m^{t-1} - 1\right) / (m-1) - 1\right]$ and $w_{im^t+j} \neq w_{i'm^t+j}$ for $j = \left(m^{t-1} - 1\right) / (m-1)$. □

Lemma 4.12. *Let $i, i' \geq 0$ be such that $x_i \neq x_{i'}$. Then $x_{(i+1)m^t-1} \neq x_{(i'+1)m^t-1}$ for each $t \geq 1$.*

Proof. The statement is obviously true for $t = 0$. Now, suppose that the statement is true for some $t \geq 0$. Since $x_{(i+1)m^t-1} \neq x_{(i'+1)m^t-1}$ and $(i+1)\,m^t - 1 \equiv (i'+1)\,m^t - 1 \pmod{k}$, we also have $x_{(i+1)m^t-1} \equiv x_{(i'+1)m^t-1} \pmod{k}$, and therefore the inequality $x_{(i+1)m^t-1} \neq x_{(i'+1)m^t-1}$ and the definition of μ give $x_{(i+1)m^{t+1}-1} \neq x_{(i'+1)m^{t+1}-1}$, which completes the induction and thus the proof of the lemma. □

Proof of Theorem 3.1. According to Lemma 4.9 and Corollary 4.10, any repetition of a word of length of four or more stems from a repetition of a word of length at most three. Therefore, we may study repetitions of long blocks by studying repetitions of short (*i.e.*, up to length three) blocks.

A repetition of a word of length two is given by an index, $i \geq 0$, and some $l > 0$, such that $w_i = w_{i+l}$, $w_{i+1} = w_{i+l+1}$, $w_{i-1} \neq w_{i+l-1}$ and $w_{i+2} \neq w_{i+l+2}$. Note that Corollary 4.7 ensures that $l \geq m^2$. Moreover, Corollary 4.2 implies $k|l$. Obviously, for $t \geq 1$, we have $w_{im^t+j} = w_{(i+l)m^t+j}$ for $j \in [0, 2m^t - 1]$, and a repetition formed out of this double lettered repetition is defined by means of an index $i' \in [(i-1)\,m^t + 1, im^t]$ and an $s \geq 2m^t$, such that $w_{i'+j} = w_{i'+lm^t+j}$ for $j \in [0, s-1]$. Since $w_{i-1} \neq w_{i+l-1}$, we have either $w_{im-1} \neq w_{im+lm-1}$, or $w_{im-2} \neq w_{im+lm-2}$, or $x_{im-3} \neq x_{im+lm-3}$. In the first two cases, Lemma 4.9 and Corollary 4.7 yield that $i' > im^t - 2m^{t-1}$, and in the latter case Lemma 4.12 yields $i' \geq im^t - 2m^{t-1}$. Thus, $i' \geq im^t - 2m^{t-1}$, and therefore Lemma 4.11 and $w_{i+2} \neq w_{i+l+2}$ yield that $s \leq 2m^{t-1} + 2m^t + (m^t - 1)/(m-1)$. Hence, for a factor uv of w, where $u, v \in \Sigma^*$ and v is a prefix of u, which is formed out of a double lettered repetition, we have

$$\frac{|uv|}{|u|} = \frac{m^t \cdot l + s}{m^t \cdot l} \leq 1 + \frac{2/m + 2 + (1 - 1/m^t)/(m-1)}{m^2} < 1 + \frac{1 + 1/\left(m^2 - m\right)}{(k-3)\,m + k},$$

where the last inequality is due to the inequality $m \geq k^2$.

A similar calculation can be applied to the repetitions formed out of the three-lettered repetitions, *i.e.*, cases where there is an index $i \geq 0$ and an $l > 0$, such that $w_i = w_{i+l}$, $w_{i+1} = w_{i+l+1}$, $w_{i+2} = w_{i+l+2}$, $w_{i-1} \neq w_{i+l-1}$ and $w_{i+3} \neq w_{i+l+3}$. Since Corollary 4.7 ensures that $l \geq m^2$, in a similar way we see that for a factor uv of w, where $u, v \in \Sigma^*$ and v is a prefix of u, which is formed out of a three-lettered repetition, we have

$$\frac{|uv|}{|u|} \leq 1 + \frac{2/m + 3 + (1 - 1/m^t)/(m-1)}{m^2} < 1 + \frac{1 + 1/\left(m^2 - m\right)}{(k-3)\,m + k},$$

for some $t \geq 1$.

The last and most interesting case, namely of repetitions formed out of a single-lettered repetition, *i.e.*, a case where there is an index $i \geq 0$ and an $l > 0$, such that $w_i = w_{i+l}$, $w_{i-1} \neq w_{i+l-1}$ and $w_{i+1} \neq w_{i+l+1}$. First, Corollary 4.2 implies that $k|l$, and Corollary 4.7 ensures that $l \geq (k-3)\,m + k$. For the cases where $l \geq (k-3)\,m + 4k$ we may apply a similar calculation to the one used above. Hence, we conclude that for a factor uv of w, where $u, v \in \Sigma^*$ and v is a prefix of u, which is formed out of such a repetition, we have

$$\frac{|uv|}{|u|} \leq 1 + \frac{2/m + 1 + (1 - 1/m^t)/(m-1)}{(k-3)\,m + 4k} < 1 + \frac{1 + 1/\left(m^2 - m\right)}{(k-3)\,m + k},$$

for some $t \geq 1$, where the last inequality is due to the inequality $3/(m-1) \leq 3k/((k-3)\,m+k)$. Now, we deal with the case $l < (k-3)\,m + 4k$. Since $k|l$ we have either $l = (k-3)\,m + 3k$, or $l = (k-3)\,m + 2k$, or $l = (k-3)\,m + k$. Therefore, Lemma 4.5 implies that $\lfloor i/m^2 \rfloor \neq \lfloor (i+l)/m^2 \rfloor$, and since $l < m^2$ we have $\lfloor (i+l)/m^2 \rfloor = \lfloor i/m^2 \rfloor + 1$. Due to Corollary 4.2 we have $x_{\lfloor (i+l)/m^2 \rfloor} \equiv x_{\lfloor i/m^2 \rfloor} + 1 \pmod{k}$, and hence Lemma 4.4 and equality $w_i = w_{i+l}$ imply that $\lfloor (i+l)/m \rfloor = \lfloor i/m \rfloor + k - 2$. Therefore, the definition of μ yields that i is odd, and hence $x'_{i+1} \neq x'_{i+l+1}$. Let a factor uv of w, where $u, v \in \Sigma^*$ and v is a prefix of u, which is formed out such a repetition. Therefore, there exist a $t \geq 1$, an index $i' \in [(i-1)\,m^t + 1, im^t]$, and an $s \geq m^t$, such that $w_{i'+j} = w_{i'+lm^t+j}$ for $j \in [0, s-1]$, and $uv = w_{i'}w_{i'+1}\ldots w_{i'+lm^t+s}$. Similarly as the previous cases, we have $i' \geq im^t - 2m^{t-1}$, and since $x'_{i+1} \neq x'_{i+l+1}$ Lemma 4.11 yields that $s \leq 2m^{t-1} + 2m^t + \left(m^{t-1} - 1\right)/(m-1)$. Thus, we have

$$\frac{|uv|}{|u|} \leq 1 + \frac{2/m + 1 + \left(1 - 1/m^{t-1}\right)/m\,(m-1)}{(k-3)\,m + 3k} < 1 + \frac{1 + 1/\left(m^2 - m\right)}{(k-3)\,m + k},$$

where the last inequality is due to the inequality $2/(m-1) \leq 2k/((k-3)\,m+k)$.

Now to the last two cases, $l = (k-3)\,m + 2k$ and $l = (k-3)\,m + k$. Since $\lfloor (i+l)/m \rfloor = \lfloor i/m \rfloor + k - 2$ and $k > 4$, there is no way that $x'_{\lfloor i/m \rfloor}k - 2k = x'_{\lfloor i/m \rfloor + 2k}k$ or $x'_{\lfloor i/m \rfloor}k - k = x'_{\lfloor i/m \rfloor + 2k}k$. Therefore, since either $i \equiv 1 \pmod{4}$ or $i \equiv 3 \pmod{4}$, and $m \nmid l$, we have $x_{i-1} \neq x_{i+l-1}$. Hence, Lemma 4.12 implies that for some $t \geq 1$ we have $x_{im^t-1} \neq x_{(i+l)m^t-1}$, and as we have already shown, we also have $w_{(i+1)m^t+j} \neq w_{(i+l+1)m^t+j}$ for $j = \left(m^{t-1} - 1\right)/(m-1)$. It follows that, a repetition formed out of these cases, *i.e.*, an index $i' \in [(i-1)\,m^t + 1, im^t]$ and an $s \geq m^t$, such that $w_{i'+j} = w_{i'+lm^t+j}$ for $j \in [0, s-1]$, satisfies $i' = im^t$ and $s \leq m^t + \left(m^{t-1} - 1\right)/(m-1)$. Thus, for a factor uv of w, where $u, v \in \Sigma^*$

and v is a prefix of u, which is formed out of these two cases, we have

$$\frac{|uv|}{|u|} = \frac{m^t \cdot l + s}{m^t \cdot l} \leq 1 + \frac{1 + \left(1 - 1/m^{t-1}\right) / \left(m^2 - m\right)}{(k-3)\, m + k} < 1 + \frac{1 + 1/\left(m^2 - m\right)}{(k-3)\, m + k}\cdot$$

Therefore, the bottom line is that for any factor uv of w, where $u, v \in \Sigma^*$ and v is a prefix of u, we have

$$\frac{|uv|}{|u|} < 1 + \frac{1 + 1/\left(m^2 - m\right)}{(k-3)\, m + k},$$

which proves that

$$E(w) \leq 1 + \frac{1 + 1/\left(m^2 - m\right)}{(k-3)\, m + k}\cdot \tag{4.2}$$

On the other hand, since $w_0 = (0,0)$, we have $w_{m-2k} = (m-2k, 0)$, which yields $w_{m^2-2km} = (0,0)$ and $w_{m^2-2km+1} = (m-2k+1, 0)$. Consequently,

$$w_{m^3-2km^2+m-k+3} = (m-k+3, 0)\,, \qquad w_{m^3-2km^3+m+1} = (m-2k+1, 1)\,.$$

Hence, for the indexes

$$i = m^4 - 2km^3 + m^2 - km + 3m + m - k + 3, \qquad j = m^4 - 2km^3 + m^2 + m + 3,$$

we have $w_i = (m-2k+3, 3)$ and $w_j = (m-2k+3, 3)$, which is a single lettered repetition. Note that $l = j - i = (k-3)\, m + k$. For each $t \geq 1$ put $u_t = w_{im^t} w_{im^t+1} \ldots w_{jm^t-1}$, and $v_t = w_{jm^t} w_{jm^t+1} \ldots w_{(j+1)m^t + (m^{t-1}-1)/(m-1)-1}$. Obviously, for each t the word $u_t v_t$ is a factor of w, and on the other hand, the equality $w_i = w_j$ and Lemma 4.11 ensure that v_t is a prefix of u_t. Thus, for each $t \geq 1$ we have

$$E(w) \geq \frac{|u_t v_t|}{|u_t|} = 1 + \frac{1 + \left(1 - 1/m^{t-1}\right) / \left(m^2 - m\right)}{(k-3)\, m + k},$$

and therefore

$$E(w) \geq 1 + \frac{1 + 1/\left(m^2 - m\right)}{(k-3)\, m + k}\cdot$$

The inequality above, together with the inequality in (4.2), completes the proof of the theorem. ☐

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