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DEJEAN'S CONJECTURE HOLDS FOR $N \geq 27$ *, **

James Currie[1] and Narad Rampersad[1]

Abstract. We show that Dejean's conjecture holds for $n \geq 27$. This brings the final resolution of the conjecture by the approach of Moulin Ollagnier within range of the computationally feasible.

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Repetitions in words have been studied since the beginning of the previous century [15,16]. Recently, there has been much interest in repetitions with fractional exponent [1,3,6–8,10]. For rational r with $1 < r \leq 2$, a **fractional r-power** is a non-empty word $w = xx'$ such that x' is the prefix of x of length $(r-1)|x|$. For example, 010 is a 3/2-power. A basic problem is that of identifying the repetitive threshold for each alphabet size $n > 1$:

> What is the infimum of r such that an infinite sequence on n letters exists, not containing any factor of exponent greater than r?

The infimum is called the **repetitive threshold** of an n-letter alphabet, denoted by $RT(n)$. Dejean's conjecture [6] is that

$$RT(n) = \begin{cases} 7/4, & n = 3 \\ 7/5, & n = 4 \\ n/(n-1) & n \neq 3, 4. \end{cases}$$

Thue[16], Dejean [6] and Pansiot[13], respectively established the values $RT(2)$, $RT(3)$, $RT(4)$. Moulin Ollagnier [12] verified Dejean's conjecture for $5 \leq n \leq 11$, and Mohammad-Noori and Currie [11] proved the conjecture for $12 \leq n \leq 14$.

Recently, Carpi [3] showed that Dejean's conjecture holds for $n \geq 33$. Carpi's result is computation-free, and resolving Dejean's conjecture is thus reduced to filling a finite gap. Conceptually, one would hope that the gap could now be filled

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[1] Department of Mathematics and Statistics, University of Winnipeg, 515 Portage Avenue, Winnipeg, Manitoba R3B 2E9, Canada; `j.currie@uwinnipeg.ca`

from below, using the methods of [11,12]. Since these approaches are computationally intensive, optimizing Carpi's result is important. The present authors improved part of Carpi's constructions to show that Dejean's conjecture holds for $n \geq 30$ (see [4]). In the present note we show that in fact Dejean's conjecture holds for $n \geq 27$.

Remark 1. Some months after the first draft of this paper, its goal has been vindicated: the final resolution of the conjecture via methods of Moulin Ollagnier becomes computationally feasible; in a recent paper the present authors proved Dejean's conjecture by resolving computationally the cases $n \leq 26$. Dejean's conjecture is correct! (See [5] and also [14] for another independent proof.)

The following definitions are from [3]: for any non-negative integer r let $A_r = \{1, 2, \ldots, r\}$. Fix $n \geq 27$. Let $m = \lfloor (n-3)/6 \rfloor$. If a is a letter, then let $|v|_a$ denote the number of occurrences of a in the word v. Let $\ker \psi = \{v \in A_m^* \mid \forall a \in A_m,$ 4 *divides* $|v|_a\}$. (We use this as a definition; it is in fact the assertion of Carpi's Lem. 9.1.) A word $v \in A_m^+$ is a **ψ-kernel repetition** if it has period q and a prefix v' of length q such that $v' \in \ker \psi$ and $(n-1)(|v|+1) \geq nq - 3$. In [4] we introduced the following definition: if v has period q and its prefix v' of length q is in $\ker \psi$, we say that q is a **kernel period** of v.

Let $B = \{0, 1\}$ and let S_n be the permutation group on n elements. Consider the morphism $\phi : B^* \to S_n$ generated by

$$\begin{aligned} \phi(0) &= (1 \quad 2 \quad 3 \quad \cdots \quad (n-1)) \\ \phi(1) &= (1 \quad 2 \quad 3 \quad \cdots \quad (n-1) \quad n). \end{aligned}$$

This map is due to Pansiot [13]. A word $u \in B^*$ is a **k-stabilizing word** if $\phi(u)$ fixes $\{1, 2, 3, \ldots, k\}$. The set of k-stabilizing words (for fixed n) is denoted by **$\mathrm{Stab}_n(k)$**. Note that if $i < j$ then $\mathrm{Stab}_n(j) \subseteq \mathrm{Stab}_n(i)$.

A map $\gamma_n : B^* \to A_n^*$ is defined by

$$\gamma_n(b_1 b_2 \ldots b_\ell) = a_1 a_2 \ldots a_\ell$$

where $a_i \phi(b_1 b_2 \ldots b_\ell) = 1$ for $1 \leq i \leq \ell$.

Carpi introduces a morphism $f : A_m^* \to B^*$ generated by

$$\begin{aligned} f(1) &= y^p x (101)^{2m} \\ f(a) &= y^p x (101)^{2m-2a} 010 (101)^{2a-1} \end{aligned}$$

where $2 \leq a \leq m$, $p = \lfloor n/2 \rfloor$, y is the suffix of $(01)^n$ of length $n-1$ and x is the suffix of y of length $|y| - 6m$.

The concepts of so-called **short repetitions** and **kernel repetitions** were introduced by Moulin Ollagnier [12]. His work is complicated by the fact that his short repetitions are words over A_n, while his kernel repetitions are words over B (although they code words over A_n via Pansiot's map). Without going into the details, we recall that he reduced the construction of an infinite word over n letters attaining threshold $n/(n-1)$ to avoiding both short repetitions and kernel repetitions. Moulin Ollagnier's binary words were fixed points of morphisms. In [11], a technique was introduced for dealing separately with short repetitions and kernel repetitions; the binary words given there can be viewed as being produced by HD0L's: they have the form $g(h^\omega(0))$ where all words coded by $g(B^*)$ avoid short repetitions, and each h is chosen to eliminate kernel repetitions.

Carpi's work follows essentially this strategy. The lemmas of his paper show that $f(B^*)$ avoids short repetitions if $n \geq 30$. For $m = 5$ (corresponding to $n \geq 33$) he produces an infinite word w_5 over A_m such that $f(w_5)$ avoids kernel repetitions. The exact statement of this division of work into short vs. kernel repetitions is the following:

Proposition 2 ([3], Prop. 3.2)**.** *Let $v \in B^*$. If a factor of $\gamma_n(v)$ has exponent larger than $n/(n-1)$, then v has a factor u satisfying one of the following conditions:*

(i) *$u \in Stab_n(k)$ and $0 < |u| < k(n-1)$ for some $k \leq n-1$;*
(ii) *u is a kernel repetition of order n.*

In our previous note, we improved only the second part of Carpi's construction; he had shown that for $n \geq 30$, no factor u of $f(A_m^*)$ satisfied condition (i) above. As Carpi therefore states at the beginning of Section 9 of [3]:

> By the results of the previous sections, at least in the case $n \geq 30$, in order to construct an infinite word on n letters avoiding factors of any exponent larger than $n/(n-1)$, it is sufficient to find an infinite word w on the alphabet A_m avoiding ψ-kernel repetitions.

For $m = 5$, Carpi was able to produce such an infinite word, based on a paperfolding construction. He thus established Dejean's conjecture for $n \geq 33$. The present authors refined this by constructing an infinite word w_4 on the alphabet A_4 avoiding ψ-kernel repetitions. This established Dejean's conjecture for $n \geq 30$. We remark that for $30 \leq n \leq 32$ the word on A_n verifying Dejean's conjecture for n is $\gamma_n(v)$, where $v = f(w_4)$.

In the present note, we improve on the first aspect of Carpi's attack, by showing that for $27 \leq n \leq 29$, no factor u of $v = f(w_4)$ satisfies (i) above. This implies that Dejean's conjecture holds for $n \geq 27$. Since f is r-uniform where $r = (p+1)(n-1)$, to show that (i) holds for v it suffices to check that no factor $u \in f(B^3)$ satisfies (i). In principle, this involves considering all factors of $f(B^3)$ of length less than $(n-1)^2$. However, we shorten this computation considerably by combining several of Carpi's lemmas.

Lemma 3. *Suppose $n \geq 18$. Suppose that $u \in f(A_m^*) \cap Stab_n(k)$ and $|u| < k(n-1)$ for some $k \in \{1, 2, \ldots, n-1\}$. Then $|u| = r(n-1)$ for some r, $p+1 \leq r < k \leq 16$.*

Proof. Propositions and lemmas referenced in this proof are in [3]. By Proposition 5.1, $k \geq 4$ so that $u \in \mathrm{Stab}_n(4)$. It then follows from Proposition 6.3 that $|u| \geq (p+1)(n-1)$. Since $|u| < k(n-1)$, we deduce that $k > p+1$. From $n \geq 18$ this means that $k > 10$, so that surely $u \in \mathrm{Stab}_n(7)$. Applying Lemma 7.1, we see that $|u|$ is divisible by $n-1$. We may thus write $|u| = r(n-1)$, $p+1 \leq r < k$. By the contrapositive of Proposition 7.2, $u \notin \mathrm{Stab}_n(17)$. It follows that $k \leq 16$. □

We verify that Dejean's conjecture holds for $n = 27, 28, 29$ by exhaustively examining factors u of $f(B^3)$ of length $r(n-1)$ for $p+1 \leq r \leq 15$, and verifying that such u are not in $\mathrm{Stab}_n(k)$ for any k, $r < k \leq 16$. For $n = 28, 29$, the check only involves $r = 15$, $k = 16$. For $n = 27$, we also must consider $r = 14$. Code written in SAGE running on a PC performed the necessary verifications in about half an hour. The code is available at www.uwinnipeg.ca/~currie/kstab.sage.

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