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ON THE EXISTENCE OF VARIATIONS, POSSIBLY WITH POINTWISE GRADIENT CONSTRAINTS

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Abstract. We propose a necessary and sufficient condition about the existence of variations, *i.e.*, of non trivial solutions $\eta \in W_0^{1,\infty}(\Omega)$ to the differential inclusion $\nabla\eta(x) \in -\nabla u(x) + \mathbf{D}$.

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1. A conjecture

Purpose of the present paper is to derive conditions for the existence of (non trivial) solutions $\eta \in W_0^{1,\infty}(\Omega)$ to the differential inclusion

$$\nabla\eta(x) \in -\nabla u(x) + \mathbf{D}$$

where $\mathbf{D}$ is a given set and u is in $W^{1,1}(\Omega)$ and satisfies

$$\nabla u(x) \in co(\mathbf{D});$$

(in the case $\mathbf{D}$ is convex, $\eta = 0$ is always a solution).

The problem of characterizing conditions for the existence of solutions is complex: in $\mathbb{R}^2$, consider the function $v(x_1; x_2) = \sqrt{x_1^2 + x_2^2}$ whose gradient satisfies $\|\nabla v(\cdot)\| = 1$, let $\mathbf{B}$ be the unit ball of $\mathbb{R}^2$ and, on $\Omega \subset \mathbb{R}^2$, consider the inclusion

$$\nabla\eta \in -\nabla v + \mathbf{B}.$$

When Ω is the open disk $x_1^2 + x_2^2 < R^2$, it is easy to see that non trivial solutions η do exist; however, when Ω is the annulus $r^2 < x_1^2 + x_2^2 < R^2$, nontrivial solutions do *not* exist. Hence, the existence or non-existence of nontrivial solutions depends on the geometry of Ω, and cannot be expressed by *local* conditions.

As a motivation for the problem, and for the name of *variations* proposed here for the solutions η, assume we are considering the problem of minimizing

$$\int_\Omega L(\nabla v(x))\,\mathrm{d}x$$

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under given boundary conditions, where L is a convex function, for instance

$$L(\xi) = \begin{cases} 1 - \sqrt{1 - \|\xi\|^2} & \text{for } \|\xi\| \le 1 \\ +\infty & \text{elsewhere;} \end{cases}$$

L is finite for ξ in $\mathbf{B}$, the unit ball of $\mathbb{R}^N$ equipped with the Euclidean norm. Let u be a solution to the minimum problem, and assume that we wish to derive the necessary conditions satisfied by u, hence to compare the values of the integral functional at u and at $u + \eta$. To find these conditions, we have to ask ourselves whether there are nontrivial variations η, such that $\|\nabla u(x) + \nabla \eta(x)\| \le 1$, *i.e.*, solutions to $\nabla \eta(x) \in -\nabla u(x) + \mathbf{B}$. In this case the function u, appearing in the differential inclusion we are investigating, is interpreted as the solution to a variational problem and the set $\mathbf{D}$ as the effective domain of a convex Lagrangean.

We propose the following conjecture, on the existence of non trivial variations. In it, and in the remainder of the paper, by saying that a vector function $p \in L^1_{loc}(\Omega)$ is such that $\operatorname{div}(p) = 0$ we mean that, for every $\eta \in C^\infty_c(\Omega)$, we have

$$\int_\Omega \langle p(x), \nabla \eta(x) \rangle \mathrm{d}x = 0.$$

Conjecture. *Let* $\mathbf{D} \subset \mathbb{R}^N$. *Let* u *be a solution to*

$$\nabla u(x) \in co(\mathbf{D}).$$

Then, the following a) and b) are in alternative:

a) there exists a nontrivial $\eta \in W^{1,\infty}_0(\Omega)$, *solution to*

$$\nabla \eta(x) \in -\nabla u(x) + \mathbf{D} \tag{1}$$

b) there exists a vector function $p \in L^1_{loc}(\Omega)$, $p(x) \neq 0$ *a.e., such that* $\operatorname{div}(p) = 0$, *and*

$$\langle p(x), \nabla u(x) \rangle = \sup_{k \in \mathbf{D}} \langle p(x), k \rangle \tag{2}$$

for almost every $x \in \Omega$.

Examples.

1) In the case $\mathbf{D} = \mathbb{R}^N$, condition b) is never satisfied and variations do always exist.

2) Consider again the function $v(x_1; x_2) = \sqrt{x_1^2 + x_2^2} = \rho$, whose gradient $\nabla v(x_1; x_2) = \frac{1}{\rho}(x_1; x_2)$. When Ω is the annulus $r^2 < x_1^2 + x_2^2 < R^2$, non trivial solutions do not exist, hence a) is never satisfied. Let us show that b) is true: the vector function $p(x_1; x_2) = \frac{1}{x_1^2 + x_2^2}(x_1; x_2)$ has pointwise divergence zero everywhere in Ω; moreover

$$\sup_{k \in B} \langle p(x_1; x_2), k \rangle = \frac{1}{\rho} = \langle p(x_1; x_2), \nabla v(x_1; x_2) \rangle .$$

Hence b) is satisfied.

When Ω is the open disk $x_1^2 + x_2^2 < R^2$, non trivial η exist, so a) is satisfied. The vector p as used before has *not* weak divergence zero in Ω, hence it does *not* prove that b) is satisfied. The fact that b) *cannot* be satisfied will be proved below.

In the present paper we prove the above conjecture under some additional regularity assumption.

2. The case $\nabla u = 0$

In this section we show that the Conjecture is verified in the case $\nabla u = 0$.

Theorem 1. *Let $\mathbf{D} \subset \mathbb{R}^N$ and let u be such that $\nabla u = 0 \in co(\mathbf{D})$. Then, the following a) and b) are in alternative:*

a) there exists a nontrivial $\eta \in W_0^{1,\infty}(\Omega)$, solution to $\nabla\eta(x) \in \mathbf{D}$;

b) there exists a vector function $p \in L^1_{loc}(\Omega)$, $p(x) \neq 0$ a.e., such that $\operatorname{div}(p) = 0$, *and for a.e. $x \in \Omega$,*

$$\sup_{k\in\mathbf{D}} \langle p(x), k\rangle = 0.$$

In the proof of Theorem 1 we will need the following lemma, whose proof is a consequence of a result appearing in [4].

Lemma 1. *Let $\Omega \subset \mathbb{R}^N$ an open bounded set, and $\mathbf{D} \subset \mathbb{R}^N$. There exists a nontrivial function $\eta \in W_0^{1,\infty}(\Omega)$ such that $\nabla\eta(x) \in \mathbf{D}$ for a.e. $x \in \Omega$, if and only if $0 \in int(co(\mathbf{D}))$.*

Proof. When $0 \in \operatorname{int}(co(\mathbf{D}))$, by Lemma 1, there exists $\eta \in W_0^{1,\infty}(\Omega)$ such that, *a.e.*, $\nabla\eta(x) \neq 0$, hence η is non trivial and a) is always satisfied. We show that b) cannot be true: in fact, in this case, there must exist a ball $B(0,r) \subset co(\mathbf{D})$ so that, for every non trivial vector function p, we have $\langle p(x), \nabla u(x)\rangle \equiv 0$, while $\sup_{k\in\mathbf{D}} \langle p(x), k\rangle \geq r\|p(x)\|$, that is positive on a set of positive measure.

When $0 \notin \operatorname{int}(co(\mathbf{D}))$, again by Lemma 1, there is no $\eta \in W_0^{1,\infty}(\Omega)$ apart from $\eta = 0$, so that a) is not satisfied. We show that b) is true: in fact, the convex sets 0 and $co(\mathbf{D})$ can be weakly separated, *i.e.* there exists a non zero vector v such that $\langle v, k\rangle \leq 0$ for every $k \in co(\mathbf{D})$, *i.e.*, such that $\sup_{k\in co(\mathbf{D})}\langle v, k\rangle \leq 0$. This constant vector v is the required p : we have $\sup_{k\in co(\mathbf{D})}\langle v, k\rangle \leq \langle v, 0\rangle = 0$ while, since $0 \in co(\mathbf{D})$, $\sup_{k\in co(\mathbf{D})}\langle p, 0\rangle \geq 0$. This ends the proof. □

3. b) implies not a)

We prove that b) implies non a) under the additional assumption that p be locally Lipschitzian in Ω, but no special assumptions on $\mathbf{D}$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be open, $u \in W^{1,1}(\Omega)$ with $\nabla u(x) \in co(\mathbf{D})$ for a.e. $x \in \Omega$. Assume that there exists a vector function $p \in W^{1,\infty}_{loc}(\Omega)$, $p(x) \neq 0$ for $x \in \Omega$, such that* $\operatorname{div}(p) = 0$ *and, for a.e. $x \in \Omega$,*

$$\langle p(x), \nabla u(x)\rangle = \sup_{k\in\mathbf{D}} \langle p(x), k\rangle .$$

Then the only solution $\eta \in W_0^{1,\infty}(\Omega)$ to the differential inclusion

$$\nabla\eta(x) \in -\nabla u(x) + \mathbf{D}$$

is $\eta \equiv 0$.

In the proof we will need the following lemma, a well known result (Liouville's Theorem) for the case of a differentiable p.

Lemma 2. *Let p as in Theorem 2. Let $S(t;x)$ be the solution to the Cauchy problem $\dot{x}(t) = p(x(t))$, $x(0) = x$. Then the map $x \to S(t;x)$ is measure preserving.*

Proof of Lemma 2. Let $\widetilde{\Omega} \subset \Omega$ and $\delta > 0$ be such that solutions issuing from $\widetilde{\Omega}$ are defined on the interval $[0,\delta]$. We wish to prove that for $t \in [0,\delta]$ and $x \in \widetilde{\Omega}$, $J(t;x)$, the Jacobian of the transformation $x \to S(t;x)$, equals

1 *a.e.* By Rademacher's Theorem, for *a.e.* x, (Dp), the matrix of (pointwise) partial derivatives of p exists. By a result of Tsuji [9], for *a.e.* x,

$$J(t;x) = \mathrm{e}^{\int_0^t \operatorname{tr}((Dp)(\tau))\,\mathrm{d}\tau}$$

where the matrix (Dp) is computed along the solution $S(\tau;x)$. We wish to show that for *a.e.* $x \in \Omega$, for *a.e.* $t \in [0,\delta]$, we have $\operatorname{tr}((Dp)(\tau)) = 0$. Let g be any of the components of the vector p; fix $\eta \in C_c^\infty(\Omega)$. The sequence $\frac{g(x+h_n e_i)-g(x)}{h_n}$ converges pointwise *a.e.* to $\frac{\partial g(x)}{\partial x_i}$ and it is (locally) uniformly bounded, so that

$$\frac{1}{h_n}\int [g(x+h_n e_i) - g(x)]\eta(x)\,\mathrm{d}x$$

converges both to $\int \frac{\partial g(\xi)}{\partial \xi_i}\eta(\xi)\,\mathrm{d}\xi$ and ([6], p. 132), to $\int g_i(x)\eta(x)\,\mathrm{d}x$, with g_i the ith Sobolev partial derivative of g. So $\int \left[\frac{\partial g(x)}{\partial x_i} - g_i(x)\right]\eta(x)\,\mathrm{d}x = 0$, hence $\frac{\partial g(x)}{\partial x_i} - g_i(x) = 0$, for all components g and all i, with the exception of a set $E \subset \Omega$ of N dimensional measure zero. In particular, on $\Omega \setminus E$, the pointwise divergence of S with respect to x, $\operatorname{tr}(Dp)$ and the weak divergence $\operatorname{div}(p)$, coincide and are zero.

For t in $[0,\delta]$ and y in $\left\{S(t;x) : x \in \widetilde{\Omega}\right\}$ define the inverse map

$$S^{-1}(t;y) = (t;x).$$

S^{-1} is locally Lipschitzian in its variables and sends the set $[0,\delta]\times E$ into a set $E^* \subset \left([0,\delta]\times\widetilde{\Omega}\right)$ of $N+1$ dimensional measure zero. By Fubini's Theorem, with the exception of a set X_{E^*} of N dimensional measure zero, the segments $\{(t;x) : t \in [0,\delta]\}$ meet the set E^* on a set of 1 dimensional measure zero. This means that for $x \notin X_{E^*}$, for *a.e.* $t \in [0,\delta]$, $S(t;x) \notin ([0,\delta]\times E)$, *i.e.*, that $\operatorname{tr}(Dp(x))$ and $\operatorname{div}(p)$, computed along $S(t;x)$, coincide. □

Proof of Theorem 2.

a) We first notice that condition (2) implies that $\nabla u(x) \in \partial(co(\mathbf{D}))$ for *a.e.* $x \in \Omega$.

In fact, otherwise, we can find a set, of positive measure, $\Omega_* \subset \Omega$ and $\varepsilon > 0$ such that $\nabla u(x) + \varepsilon p(x) \in co(\mathbf{D})$. For $x \in \Omega_*$, we have

$$\sup_{k\in co(\mathbf{D})} \langle p(x), k\rangle \ge \langle p(x), \nabla u(x) + \varepsilon\, p(x)\rangle = \langle p(x), \nabla u(x)\rangle + \varepsilon\|p(x)\|^2 >$$

$$\langle p(x), \nabla u(x)\rangle = \sup_{k\in\mathbf{D}} \langle p(x), k\rangle\,.$$

Recalling that $\sup_{k\in co(\mathbf{D})}\langle p(x), k\rangle = \sup_{k\in\mathbf{D}}\langle p(x), k\rangle$, we obtain a contradiction.

b) To prove the theorem, suppose, by contradiction, that there exists a nontrivial $\eta \in W_0^{1,\infty}(\Omega)$, that verifies condition (1) almost everywhere.

In the case that $\operatorname{int}(co(\mathbf{D})) = \varnothing$, $\mathbf{D}$ is contained in a hyperplane, and condition (1) implies that also $\nabla\eta$ is in a hyperplane, a contradiction to Lemma 1. Hence, in what follows, we consider $\operatorname{int}(co(\mathbf{D})) \neq \varnothing$.

c) *Claim.* For every $x \in \Omega$, there exists c such that $\eta(S(t;x)) = c$ for $t \in (\alpha_x, \beta_x)$, the maximal interval of existence for the solution S.

Proof of this claim. By assumption, for almost every $x \in \Omega$,

$$\langle \nabla u(x), p(x)\rangle = \sup_{k\in\mathbf{D}}\langle k, p(x)\rangle$$

and

$$\langle \nabla \eta(x), p(x) \rangle = \langle -\nabla u(x), p(x) \rangle + \langle \nabla \eta(x) + \nabla u(x), p(x) \rangle \leq$$
$$- \langle \nabla u(x), p(x) \rangle + \sup_{k \in \mathbf{D}} \langle k, p(x) \rangle ,$$

so that

$$\langle \nabla \eta(x), p(x) \rangle \leq 0.$$

Since $\eta \in W_0^{1,\infty}(\Omega)$, the assumption on the divergence of p implies

$$\int_\Omega \langle \nabla \eta(x), p(x) \rangle \, \mathrm{d}x = 0,$$

hence we obtain that, for almost every $x \in \Omega$,

$$\langle \nabla \eta(x), p(x) \rangle = 0.$$

Fix $x^* \in \Omega$. Consider the $N-1$ dimensional affine space

$$V = x^* + p(x^*)^\perp.$$

There exists $\delta > 0$ and $r > 0$, such that a solution $S(t; v)$ to $\dot{x} = p(x)$ and $x(0) = v$ exists for $v \in V \cap B(x^*, r)$ on an interval $(-\delta, \delta)$. The map $(t; v) \to S(t; v)$ is Lipschitzian and invertible. Hence, by the coarea theorem, with the exception of a subset of V of $N-1$ dimensional measure zero, $S(t; v)$ meets the set M, where $\langle \nabla \eta(x), p(x) \rangle \neq 0$, on a subset of $(-\delta, \delta)$ of 1-dimensional measure zero, and, outside of this exceptional set, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \eta(S(t; v)) = \langle \nabla \eta(S(t; v)), p(S(t; v)) \rangle = 0.$$

Hence, there exists a sequence $v_n \to x^*$ such that $\eta(S(t; v_n)) \equiv c_n$ on $(-\delta, \delta)$. Since the limit of solutions is a solution and η is continuous, we have that $\eta(S(t; x^*)) \equiv c$ on $(-\delta, \delta)$. This local reasoning can be extended to the maximal interval of existence, proving the claim.

d) Let $\overline{x}$ be such that $\eta(\overline{x}) > 0$, and define

$$E = \left\{ x \in \Omega : \eta(x) \geq \frac{1}{2} \eta(\overline{x}) \right\} \subset \Omega.$$

The set E is nonempty, compact, $\operatorname{int}(E) \neq \varnothing$ and $d(E, \partial\Omega) > 0$. As a consequence of c), it cannot happen that there exists $x \in E$ such that, for some $t \in (\alpha_x, \beta_x)$, $S(t; x) \notin E$. Hence for every $x \in E$ and every $t \in (\alpha_x, \beta_x)$, $S(t; x) \in E$. By the basic theorems on the prolongability of solutions to ordinary differential equations, it follows then that the solution $S(t; x)$ must be defined for every $t \in \mathbb{R}$, since $d(E, \partial\Omega) > 0$. Hence, for every $t \in \mathbb{R}$, the map $S(t; \cdot)$ is a bijection of E into itself and, in addition, by Lemma 1, it is measure preserving.

e) We wish to apply the following Poincaré recurrence Theorem to the map $S(t; \cdot)$, (see for instance [1] for the proof).

Lemma 3 (Poincaré)**.** *Let E be a compact, nonempty set such that $\operatorname{int}(E) \neq \varnothing$, and let $\psi : E \to E$ a bijective, measure preserving function. Then, for every $x_0 \in \operatorname{int}(E)$ and every $\varepsilon > 0$, there exists an integer $k > 0$ such that*

$$\psi^k \left(B(x_0, \varepsilon) \right) \cap B(x_0, \varepsilon) \neq \varnothing.$$

Going back to the proof, let $r^0 > 0$ and x^0 be such that $B(x^0, r^0) \subset E$ and let $t_0 > 0$ be such that $S(t_0; x^0) \neq x^0$. Let $V \subset \Omega$ be a neighborhood of the trajectory

$$\left\{S(t; x^0) : t \in [0, t_0]\right\}$$

and let $p^0 > 0$ be such that $\|p(x)\| \geq p^0$ for x in V. Let $r \leq r^0$ be so small that:

$$S(t_0; B(x^0, r)) \cap B(x^0, r) = \varnothing$$

and, for every $\xi \in B(x^0, r)$, the solution $S(t; \xi) \in V$ for $t \in [0, t_0]$. Applying Poincaré's method we obtain that, for every $\rho < r$, there exist $\xi_\rho \in B(x^0, \rho)$ and an integer $\nu_\rho > 1$, such that

$$\left|S(t_0\nu_\rho; \xi_\rho) - x^0\right| \leq \rho.$$

f) Choose $v \in \mathrm{int}(\mathbf{D})$ and let $s > 0$ be such that $B(v, s) \subset \mathbf{D}$. Consider the function u_0 defined by

$$u_0(x) = u(x) - \langle v, x\rangle\,.$$

Condition (2) implies that u_0, computed along $S(t; x)$, for $x \in B(x^0, r)$, is strictly increasing:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} u_0(S(t; x)) = \langle \nabla u(S(t; x)) - v, p(S(t; x))\rangle &= \sup_{k \in D} \langle k - v, p(S(t; x))\rangle \\ &\geq s\,\|p(S(t; x))\| > 0;\end{aligned}$$

in particular, for $\xi \in B(x^0, \rho)$, with $\rho \leq r$, we obtain

$$u_0(S(t_0\nu_\rho; \xi)) - u_0(\xi) \geq t_0\nu_\rho s p^0 \geq t_0 s p^0.$$

This last estimate is independent of ρ.

Apply this estimate to ξ_ρ; we have that both ξ_ρ and $S(t_0\nu_\rho; \xi_\rho)$ are in $B(x^0, \rho)$. By the continuity of u_0 at x^0, the difference $u_0(S(t_0\nu_\rho; \xi_\rho)) - u_0(\xi_\rho)$ can be made arbitrarily small by decreasing ρ, a contradiction. □

The following result completes the discussion of the example in Section 2.

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be the open disk $x_1^2 + x_2^2 < 1$ and $v(x_1; x_2) = \sqrt{x_1^2 + x_2^2}$. There is no vector function $p \in L^1_{loc}(\Omega)$, $p(x) \neq 0$ a.e., such that* $\mathrm{div}\,(p) = 0$, *and*

$$\langle p(x), \nabla v(x)\rangle = \sup_{k \in \mathbf{B}} \langle p(x), k\rangle$$

for almost every $x \in \Omega$.

Proof of Theorem 3. The function $\eta = -\sqrt{x_1^2 + x_2^2} + 1$ is in $W_0^{1,\infty}(\Omega)$ and is a solution to the differential inclusion

$$\nabla\eta(x) \in -\nabla v(x) + \mathbf{B}.$$

Assume that p exists. By assumption we must have, for almost every $x \in \Omega$,

$$\langle p(x), \nabla v(x)\rangle = \|p(x)\|$$

so that $p(x) = \alpha(x)\frac{x}{\|x\|}$, and $\alpha \geq 0$. On the other hand, in c) of the proof of the previous theorem we have obtained that, for almost every $x \in \Omega$,

$$\langle \nabla\eta(x), p(x)\rangle = 0,$$

so that $\alpha(x) = 0$ *a.e.* in Ω. □

4. When $\mathbf{D} = \mathbf{B}$, not a) implies b)

We prove this part of the conjecture in the case $\mathbf{D} = \mathbf{B}$.

Theorem 4. *Let $u \in W^{1,\infty}(\Omega)$ be a solution to $\nabla u(x) \in \mathbf{B}$ and assume that there exist no nontrivial $\eta \in W_0^{1,\infty}(\Omega)$, solution to the differential inclusion*

$$\nabla \eta(x) \in -\nabla u(x) + \mathbf{B}.$$

Then:

i) the solution u belongs to $C^1(\Omega) \cap W^{2,\infty}_{loc}(\Omega)$;

ii) there exists $p \in L^1_{loc}(\Omega)$, $p(x) \neq 0$ for almost every $x \in \Omega$, such that $\operatorname{div}(p) = 0$, *and*

$$\langle p(x), \nabla u(x) \rangle = \sup_{k \in B} \langle p(x), k \rangle$$

for almost every $x \in \Omega$.

Remark 1. In the proof of Theorem 4, we will construct a function p that verifies ii). This function p can be interpreted as a mass-transfer vector field, and from condition (2) we see that ∇u determines the optimal direction for p. Hence, we expect p to be of the form $p = \lambda \nabla u$, for a suitable function $\lambda(x)$, and we compute λ by the equation $\operatorname{div}(\lambda \nabla u) = 0$. As appears in [7], this equation is related to the Monge-Kantorovich transport problem. In particular, λ plays the role of a transport density, and is the Lagrange multiplier for the constraint $\nabla u \in \mathbf{B}$.

Proof of Theorem 4. The proof makes use of some results and techniques developed in [5].

a) Fix any point $x^0 \in \Omega$. Using Lemmas 2.2 and 2.3 of [5], from the fact that there is no variation η such that $u(x^0) + \eta(x^0) < u(x^0)$, we infer the existence of at least one unit vector, a direction, d^+, with the property that, for every r such that the ball $B(x^0, r)$ is contained in Ω, we have $u(x^0 + rd^+) - u(x^0) = r$. Such a direction will be called a direction of maximal growth. By the same reasons, since there is no variation η such that $u(x^0) + \eta(x^0) > u(x^0)$, we infer the existence of at least one direction, d^-, such that $u(x^0) - u(x^0 + rd^-) = r$. However we must have that $d^+ = d^-$, in fact, since u is Lipschitzian of constant 1, we have

$$r \left\| d^+ + d^- \right\| \geq \left| u(x^0 + rd^+) - u(x^0 - rd^-) \right| = \left| u(x^0 + rd^+) - u(x^0) - u(x^0 + rd^-) + u(x^0) \right| = 2r,$$

i.e. $\|d^+ + d^-\| = \|d^+\| + \|d^-\|$, that implies $d^+ = d^-$. Notice that this result implies that d^+ and d^- are unique. Hence, from the assumption that there is no variation η, to each $x \in \Omega$ we associate a unique direction $d(x)$ such that $u(x + rd(x)) - u(x) = r$ as long as $x + rd(x) \in \Omega$; *i.e.*, there exists a unique segment $(x + \alpha(x)d(x), x + \beta(x)d(x))$, $\alpha(x) < 0 < \beta(x)$, such that: $x + \alpha(x)d(x) \in \partial\Omega$, $x + \beta(x)d(x) \in \partial\Omega$ and $u(x + \lambda_1 d(x)) - u(x + \lambda_2 d(x)) = \lambda_1 - \lambda_2$ for every $\lambda_1, \lambda_2 \in (\alpha(x), \beta(x))$. The direction d has the following interpretation: at every point x^0 such that $\nabla u(x^0)$ exists, we have that $\nabla u(x^0) = d(x^0)$. In fact, from

$$u(x) - u(x^0) = \left\langle \nabla u(x^0), x - x^0 \right\rangle + \left\| x - x^0 \right\| o\left(\left\| x - x^0 \right\| \right),$$

we obtain

$$r = r \left\langle \nabla u(x^0), d(x^0) \right\rangle + ro(r),$$

that implies $\nabla u(x^0) = d(x^0)$. Moreover, the following property holds: for no $y \in \Omega$ we can have

$$y = x + \lambda \mathrm{d}(x) = x' + \lambda' d(x')$$

unless $d(x') = d(x)$. In fact, otherwise, both $d(x)$ and $d(x')$ would be directions of maximal growth at y, contradicting the uniqueness of $d(y)$.

b) *Claim.* Let ρ be such that $B(x^0,\rho)\subset\Omega$. Then, on $B(x^0,\frac{\rho}{6})$, the map $x\to \mathrm{d}(x)$ is Lipschitzian of constant $\Lambda=\frac{3}{\rho}$.

Proof of this Claim. Let P and P' in $B(x^0,\frac{\rho}{6})$, so that $\|P-P'\|\le\frac{\rho}{3}$. Set $d=d(P)$ and $d'=d(P')$; let O on the line $r=\{P+\lambda d\}$ and O' on the line $r'=\{P'+\lambda d'\}$ be such that $\|O-O'\|=\inf_{Q\in r,Q'\in r'}\|Q-Q'\|$. We have that $(O-O')$ is orthogonal both to r and to r'. Two cases are possible: either, a), $\inf\{\|P-O\|,\|P'-O'\|\}>\frac{\rho}{3}$ or, b), $\inf\{\|P-O\|,\|P'-O'\|\}\le\frac{\rho}{3}$.

Consider case a). Call P the point such that $\|P-O\|\le\|P'-O'\|$. We will need the line $r''=r'+(O-O')$: it is the parallel to r' in the plane containing r and orthogonal to $(O-O')$. Let P'' be the projection of P' on r''. Since $\|P''-O\|=\|P'-O'\|\ge\|P-O\|$, on the segment $[O,P'']$ choose P_i such that $\|P_i-O\|=\|P-O\|$ and consider the isosceles triangle O,P',P_i: we have

$$\frac{\|d-d'\|}{1}=\frac{\|P_i-P\|}{\|P-O\|},$$

so that $\|P-O\|\ge\frac{\rho}{3}$ implies

$$\|d-d'\|\le\frac{3}{\rho}\|P_i-P\|.$$

We claim that $\|P''-P\|\ge\|P_i-P\|$. In fact, the angle P,P_i,P'' is larger than $\frac{\pi}{2}$, being the triangle O,P',P_i isosceles, so that

$$\begin{aligned}\|P''-P\|^2&=\|P-P_i\|^2+\|P_i-P''\|^2+2\langle P-P_i,P_i-P''\rangle\\&\ge\|P-P_i\|^2+\|P_i-P''|^2\ge\|P-P_i\|^2.\end{aligned}$$

We have shown that

$$\|d-d'\|\le\frac{3}{\rho}\|P_i-P\|\le\frac{3}{\rho}\|P''-P\|\le\frac{3}{\rho}\|P'-P\|.$$

Consider case b). Consider the two points O and O'; since $\|O-O'\|\le\|P-P'\|$, we obtain that both points O and O' are in $B(x^0,\rho)$, so that u is defined at O and O'. For case b), we assign names to the points P and P' by assuming that $u(O')\ge u(O)$. With this choice of names, consider again the lines r, r' and set again $r''=r'+(O-O')$. On r consider the segment $[A,D]$, centered at O, such that $\|A-O\|=\|D-O\|=\frac{\rho}{3}$; on r', the segment $[B',C']$, centered at O', such that $\|B'-O'\|=\|C'-O'\|=\frac{\rho}{3}$; orientations are chosen so that $A=O+\frac{\rho}{3}d$ and $B'=O'+\frac{\rho}{3}d'$. Call B and C the projections of B' and C' on the line r''. We obtain

$$\begin{aligned}\|B'-D\|&\ge u(B')-u(D)=u(B')-u(O')+u(O')-u(O)+u(O)-u(D)\\&\ge u(B')-u(O')+u(O)-u(D)=\frac{\rho}{3}+\frac{\rho}{3}.\end{aligned}$$

Set $H=\frac{1}{2}A+\frac{1}{2}B$. We have:

$$\begin{aligned}\|H-O\|^2&=\left\|\frac{1}{2}(A-O)+\frac{1}{2}(B-O)\right\|^2=\left\|\frac{1}{2}(O-D)+\frac{1}{2}(B-O)\right\|^2\\&=\left\|\frac{1}{2}(B-D)\right\|^2=\frac{1}{4}\left(\|B'-D\|^2-\|O-O'\|^2\right),\end{aligned}$$

the last equality deriving from the Pytagorean Theorem applied to the triangle D, B, B'. Hence we have:

$$\begin{aligned}\frac{1}{4}\|A-B\|^2 &= \|B-H\|^2 = \|B-O\|^2 - \|H-O\|^2\\ &= \left(\frac{\rho}{3}\right)^2 - \frac{1}{4}\left(\|B'-D\|^2 - \|O-O'\|^2\right) \le \left(\frac{\rho}{3}\right)^2 - \frac{1}{4}\left(\frac{2\rho}{3}\right)^2 + \frac{1}{4}\|O-O'\|^2\\ &= \frac{1}{4}\|O-O'\|^2.\end{aligned}$$

We obtain

$$\|A-B\| = \left\|O+\frac{\rho}{3}d - \left(O+\frac{\rho}{3}d'\right)\right\| \le \|O-O'\| \le \|P-P'\|.$$

We conclude that, for case b) as well, we have

$$\|d-d'\| \le \frac{3}{\rho}\|P-P'\|$$

proving the claim.

c) We claim that, as a consequence of the Lipschitzianity of d, we have that $u \in C^1(\Omega) \cap W^{2,\infty}_{loc}(\Omega)$. The directions of the coordinate axis are denoted by e_i.

Fix x; let $B(x,r) \subset \Omega$ and let Λ be a Lipschitz constant for d in $B(x,r)$. We first notice that if it happens that on the intersection of the line $\{x+te_i : t \in \mathbb{R}\}$ with $B(x,r)$, u is differentiable at $x+te_i$ for almost every t, then we must have

$$|u(x+he_i) - u(x) - h\langle d(x), e_i\rangle| \le h^2\Lambda.$$

In fact, the Lipschitzian map $t \to u(x+te_i)$ is the integral of its derivative, that coincides, for *a.e.* t, with $\langle d(x+te_i), e_i\rangle$, so that

$$|u(x+he_i) - u(x) - h\langle d(x), e_i\rangle| = \left|h\int_0^1 \langle d(x+she_i) - d(x), e_i\rangle \,\mathrm{d}s\right| \le h^2\Lambda.$$

Notice next that, since $\nabla u(x)$ exists for *a.e.* $x \in \Omega$, there must exists a sequence $x_n \to x$ such that, on the intersection of the line $\{x_n + te_i : t \in \mathbb{R}\}$ with $B(x,r)$, $\nabla u(x_n + te_i)$ exists for *a.e.* t. Then we have:

$$\begin{aligned}|u(x+he_i) - u(x) - h\langle d(x), e_i\rangle| = &\,|u(x_n+he_i) - u(x_n) - h\langle d(x_n), e_i\rangle + h\langle d(x_n) - d(x), e_i\rangle\\ &+ u(x+he_i) - u(x_n+he_i) + u(x_n) - u(x)|\\ &\le h^2\Lambda + h\Lambda|x_n - x| + 2|x_n - x|.\end{aligned}$$

Letting $n \to \infty$ we obtain that $\frac{\partial u}{\partial x_i}$ exists at x and equals $\langle d(x), e_i\rangle$. Since the gradient is continuous, we obtain that u is differentiable and that $u \in C^1(\Omega)$.

Fix $\eta \in C^\infty_c(\Omega)$. Then on $\operatorname{supp}(\eta)$, $\nabla u(x) = d(x)$ is uniformly Lipschitzian: hence, see [8], for each component d^i and each j there is g^i_j such that

$$\int_\Omega g^i_j \eta \,\mathrm{d}x = -\int_\Omega d^i \frac{\partial \eta}{\partial x_j}\,\mathrm{d}x.$$

This proves i).

d) As established in the Remark, the map p, as required in ii), will be of the form $\lambda(x)d(x)$. To find λ amounts to finding a weak solution to the equation $\operatorname{div}(\lambda(x)d(x)) = 0$, where $\operatorname{div}(d(x)) \in L^\infty_{loc}(\Omega)$.

Fix $x^* \in \Omega$ and consider the corresponding level set for the function u, *i.e.* $\{x : u(x) = u(x^*)\}$. We claim that we can parametrize locally this set by a differentiable and invertible map ϕ_{x^*} from an open set V_{x^*} in

a $N-1$ space, to Ω, *i.e.* that there exists V_{x^*}, ϕ_{x^*}, r^* such that $u\left(\phi_{x^*}(\xi)\right) \equiv u(x^*)$, for every $\xi \in V_{x^*}$ and $\phi\left(V_{x^*}\right) = \{u(x) = u(x^*)\} \cap B(x^*, r^*)$.
Proof of this Claim. Consider the $N-1$ dimensional space $d(x^*)^{\perp}$, defined by the equation $\langle d(x^*), x\rangle = \langle d(x^*), x^*\rangle$; let $d_i(x^*) \neq 0$, set the $N-1$ vector ξ be $\xi_j = x_j$, $j \neq i$, and set ξ^* be $\xi_j^* = x_j^*$, $j \neq i$, so that $d(x^*)^{\perp}$ is the image of the affine map ℓ, given by $\ell(\xi)_j = x_j$, $j \neq i$, and

$$\ell(\xi)_i = \frac{\langle d(x^*), x^*\rangle - \sum_{j\neq i} d_j(x^*)\xi_j}{d_i(x^*)}.$$

The map ℓ is one to one from $\mathbb{R}^{N-1}$ to $\mathbb{R}^N$. For ξ in a sufficiently small neighborhood V_{x^*} of ξ^*, so that the maps are defined, we have that $u(\ell(\xi) + td(\ell(\xi))) = u(\ell(\xi)) + t$ and $u(\ell(\xi) + td(\ell(\xi)))$ assumes the value $u(x^*)$ for $u(x^*) - u(\ell(\xi))$. The required parametrization is given by the (differentiable) map $\phi_{x^*}(\xi) = \ell(\xi) + (u(x^*) - u(\ell(\xi)))d(\ell(\xi))$. The map ϕ_{x^*} is invertible: assume that $\ell(\xi) + t(\xi)d(\ell(\xi)) = \ell(\xi') + t(\xi')d(\ell(\xi')) = P$; then $u(P) - u(x^*) = t(\xi)$, $u(P) - u(x^*) = t(\xi')$ and, by the results of a), $d(\ell(\xi)) = d(\ell(\xi'))$, so that $\ell(\xi) = \ell(\xi')$ and $\xi = \xi'$.

e) Consider the flow $S(t;x) = x + td(x)$: it is a solution to the Cauchy problem

$$\frac{d}{\mathrm{d}t} S(t;x) = d(S(t;x)), \quad S(0;x) = x.$$

In particular, consider the map $(t;\xi) \to S(t;\phi_{x^*}(\xi))$: by the basic theorems on uniqueness for ordinary differential equations, and by the invertibility of ϕ_{x^*}, it is an invertible map.

We will denote by D the square matrix of partial derivatives of the vector field $d(x)$ and by $M_x(t)$ the square matrix of partial derivatives of $S(t;x)$ with respect to the space variables, computed at x, *i.e.* $M_x(t) = I + tD(x)$. Since the vector field d is autonomous, we have the basic identity

$$M_x(t)d(x) = d(S(t;x)).$$

In addition, Lindelöf's Theorem on differentiability with respect to initial conditions implies that

$$\det(M_x(t)) = \mathrm{e}^{\int_0^t \operatorname{tr} D(s)\, \mathrm{d}s}$$

where the trace of D appearing at the right hand side is computed along $S(s;x)$. As a consequence of the uniform Lipschitzianity of d on compact subsets of Ω, we have that on a compact set, there exists k such that $\det(M_x(t)) \geq k > 0$. Denote by Φ_ξ the $N \times (N-1)$ matrix of partial derivatives of ϕ with respect to ξ. We obtain that

$$D_{(t;\xi)}(S(t;\phi(\xi))) = \left(d(S(t;\phi(\xi))); M_{\phi(\xi)}(t)\Phi_\xi\right)$$

and, recalling that $\mathrm{d}(S(t;\phi(\xi))) = d(\phi(\xi)) = M_{\phi(\xi)}(t)d(\phi(\xi))$, we obtain

$$\det\left(D_{(t;\xi)}(S(t;\phi(\xi)))\right) = \det(M_{\phi(\xi)}(t)) \det(d(\phi(\xi)); \phi_{\xi_1}; \ldots; \phi_{\xi_{N-1}}).$$

f) An easy contradiction argument shows that the set

$$O_{x^*} = \{(t;\xi) : \alpha(\phi_{x^*}(\xi)) < t < \beta(\phi_{x^*}(\xi)); \xi \in V_{x^*}\}$$

is an open subset of $\mathbb{R} \times \mathbb{R}^{N-1}$ and, being the continuous map $S(t;\phi_{x^*}(\xi))$ one to one, its image S_{x^*} is an open subset of Ω.

Consider a countable covering of Ω by sets S_{x_n}, $n = 1,\dots$ (for brevity we will set $\mathrm{S}_{x_n} = \mathrm{S}_n$, $V_{x_n} = V_n$ and $\phi_{x_n} = \phi_n$). Fix $x \in \mathrm{S}_n$; let t and ξ be such that $x = S(t;\phi_n(\xi))$ and set

$$\lambda_n(x) = \frac{1}{\det M_{\phi_n(\xi)}(t)}\cdot$$

This definition sets (arbitrarily) λ_n to be 1 on the level set $\{x : u(x) = u(x_n)\} \cap \mathrm{S}_n$. Set $E_1 = \Omega \cap \mathrm{S}_1$; $E_{n+1} = \Omega \cap [\mathrm{S}_{n+1} \setminus E_n]$, so that $\Omega = \bigcup E_n$, and the E_n are disjoint.

In general, define $\lambda(x) = \sum \lambda_n(x)\chi_{E_n}$. On a compact subset of Ω, we have that $\lambda_n(x) \le h$ where h does not depend on n, so that $\lambda \in L^\infty_{loc}(\Omega)$. We claim that, for every $\eta \in C^\infty_c(\Omega)$, we have

$$\int_\Omega \lambda(x)\,\langle d(x), \nabla\eta(x)\rangle \;\mathrm{d}x = \sum_n \int_{E_n} \lambda_n(x)\,\langle d(x), \nabla\eta(x)\rangle \;\mathrm{d}x = 0$$

i.e. that the map $p(x) = \lambda(x)d(x)$ has divergence zero.

On E_n consider the change of variables given by $x = S(t;\phi_n(\xi))$, with Jacobian $J_n(t;\xi) = \left|\det D_{(t;\xi)}(S(t;\phi(\xi)))\right|$. We have

$$\begin{aligned}\lambda_n(S(t;\phi_n(\xi)))J_n(t;\xi) &= \frac{1}{\det M_{\phi_n(\xi)}(t)}\left|\det M_{\phi(\xi)}(t)\det\left(d(\phi(\xi)); \phi_{\xi_1}; \dots; \phi_{\xi_{n-1}}\right)\right| \\ &= \left|\det\left(d(\phi(\xi)); \phi_{\xi_1}; \dots; \phi_{\xi_{n-1}}\right)\right|,\end{aligned}$$

so that

$$\begin{aligned}\int_{E_n} \lambda_n(x)\,\langle d(x), \nabla\eta(x)\rangle \;\mathrm{d}x &= \int_{E_n} \lambda_n(S(t;\phi_n(\xi)))\,\langle d(S(t;\phi_n(\xi))), \nabla\eta(S(t;\phi_n(\xi)))\rangle\, J_n(t;\xi)\,d(t;\xi) \\ &= \int\left(\int_{\alpha(\phi_n(\xi))}^{\beta(\phi_n(\xi))} \lambda_n(S(t;\phi_n(\xi)))\,\langle d(S(t;\phi_n(\xi))), \nabla\eta(S(t;\phi_n(\xi)))\rangle\, J_n(t;\xi)\mathrm{d}t\right)\mathrm{d}\xi \\ &= \int\left(\int_{\alpha(\phi_n(\xi))}^{\beta(\phi_n(\xi))} \frac{\mathrm{d}}{\mathrm{d}t}\eta(S(t;\phi_n(\xi)))\mathrm{d}t\right)\left|\det\left(d(\phi(\xi)); \phi_{\xi_1}; \dots; \phi_{\xi_{n-1}}\right)\right|\,\mathrm{d}\xi.\end{aligned}$$

Since, for every ξ, $S(\alpha(\phi_n(\xi));\phi_n(\xi))$ and $S(\beta(\phi_n(\xi));\phi(\xi))$ belong to $\partial\Omega$, we obtain that $\eta(S(\alpha(\phi(\xi));\phi(\xi))) = \eta(S(\beta(\phi(\xi));\phi(\xi))) = 0$ for every ξ, so that

$$\int_\Omega \lambda(x)\,\langle d(x), \nabla\eta(x)\rangle \;\mathrm{d}x = 0.$$

g) We have

$$\langle p(x), \nabla u(x)\rangle = \langle \lambda(x)d(x), \nabla u(x)\rangle = \lambda(x) = \sup_{k\in B}\langle p(x), k\rangle\,,$$

concluding the proof. □

Remark 2. The vector $p(\cdot)$ admits a divergence in the integral sense, but need not belong to $W^{1,1}_{loc}(\Omega)$.

In fact, in $\mathbb{R}^2$ consider

$$\Omega = \left\{(x;y) : x^2 + y^2 < 1, x \le 0, y > 0\right\} \cup \left\{(x;y) : x^2 + (y-1)^2 < 1, x \ge 0, y < 1\right\}.$$

On Ω set $P = (x;y)$ and

$$u(P) = \begin{cases} \sqrt{x^2 + y^2} & \text{if}\quad x \le 0 \\ 1 - \sqrt{x^2 + (y-1)^2} & \text{otherwise.}\end{cases}$$

Then

$$\nabla u(P) = \begin{cases} \frac{P}{\|P\|} & \text{if} \quad x \le 0 \\ \frac{(0;1)-P}{\|P-(0;1)\|} & \text{otherwise} \end{cases}$$

and

$$\Delta u(P) = \begin{cases} \frac{1}{\|P\|} & \text{if} \quad x \le 0 \\ \frac{-1}{\|P-(0;1)\|} & \text{otherwise.} \end{cases}$$

One verifies that the differential equation for λ

$$\langle \nabla\lambda(P), \nabla u(P)\rangle + \lambda(P)\Delta u(P) = 0$$

admits the solution

$$\lambda(P) = \begin{cases} \frac{1}{\|P\|} & \text{if} \quad x \le 0 \\ \frac{1}{\|P-(0;1)\|} & \text{otherwise.} \end{cases}$$

Hence

$$p(P) = \begin{cases} \frac{P}{\|P\|^2} & \text{if} \quad x \le 0 \\ \frac{(0;1)-P}{\|P-(0;1)\|^2} & \text{otherwise,} \end{cases}$$

that has a jump discontinuity through the line $x = 0$. Hence p cannot belong to $W^{1,1}(\Omega)$.

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