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NEUMANN BOUNDARY VALUE PROBLEMS ACROSS RESONANCE*

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Abstract. We obtain an existence-uniqueness result for a second order Neumann boundary value problem including cases where the nonlinearity possibly crosses several points of resonance. Optimal and Schauder fixed points methods are used to prove this kind of results.

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1. Introduction

The aim of this work is to give an existence and uniqueness result for elliptic boundary value problem of second order in resonance, that is, our hypotheses allow that the nonlinearity crosses several eigenvalues of the associated eigenvalue problem. Under suitable conditions, which include the regularity of the function f, the positivity of the function $\frac{\partial f}{\partial x}$ and a uniform upper bound for $\int_0^\pi \frac{\partial}{\partial x} f(t,x)\mathrm{d}t$, the problem

$$\left.\begin{array}{c} -x''(t) - \alpha(t)x'(t) = f(t, x(t)), \text{ in } [0,\pi] \\ A = x'(0);\ B = x'(\pi) \end{array}\right\} \tag{1.1}$$

has a unique solution (Th. 14), where $\alpha : [0,\pi] \to \mathbb{R}$ is a continuous function.

It is well known that in order to study the solutions of a boundary value problem for a second order operator, it is convenient to consider the interaction between the nonlinearity, $f(t,x)$, and the spectrum of the corresponding operator. See [2,3,5].

Our motivation was a paper written by Huaizhong and Yong, [4], in which, for $\alpha \equiv 0$, the Theorem 14 below is obtained. For it, they use Pontryagin's maximum principle and the explicit expression of solutions for the associated linear problem to (1.1). When $\alpha \neq 0$ is a general continuous function, the explicit expression of solutions for the associated linear problem is unknown and so, the use of Pontryagin's maximum principle becomes difficult to handle, in order to obtain the main result (Th. 14), which is obtained in a different way.

In Section 2, we consider the linear problem associated with problem (1.1)

$$\left.\begin{array}{c} -x'' - \alpha(t)x' = \beta(t)x, t \in [0,\pi] \\ x'(0) = x'(\pi) = 0 \end{array}\right\} \tag{1.2}$$

where β is a nonnegative, bounded and measurable function.

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Now we define an optimal control problem for the above linear boundary value problem. We prove the existence of the optimal control and, after doing a qualitative study of the optimal solutions of (1.2), we obtain a lower bound for the minimum value of the functional cost. Moreover, we apply Pontryagin's maximum principle to describe the optimal control. Finally, in Section 3, applying the results of Section 2 and Schauder fixed-point theorem, we obtain our main result (Th. 14). Roughly speaking, we show that even though the function f crosses some eigenvalues, an additional "energy" is necessary to get several solutions for the problem (1.1).

2. Linear problem

We consider the linear boundary problem

$$\left.\begin{array}{c} -x'' - \alpha(t)x' = \beta(t)x, t \in [0,\pi] \\ x'(0) = x'(\pi) = 0 \end{array}\right\}$$

where α is a continuous function on $[0,\pi]$ with $\beta \in L^\infty[0,\pi]$ and $\beta(t) \geq 0$ *a.e.* $t \in [0,\pi]$.

We start by studying the spectral structure of the associated eigenvalue problem.

$$\left.\begin{array}{c} -x'' - \alpha(t)x' = \lambda x, \ t \in [0,\pi] \\ x'(0) = x'(\pi) = 0. \end{array}\right\} \tag{2.3}$$

Lemma 1. *The eigenvalues of the problem* (2.3) *are a sequence of real numbers* $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$, *where* $\lambda_0 = 0$, $\lim \lambda_n = +\infty$ *and all of them are simple.*

Proof. Multiplying equation (2.3) by $\mathrm{e}^{\int_0^t \alpha(s)\,\mathrm{d}s}$ we obtain the equivalent problem

$$\left.\begin{array}{c} -(\mathrm{e}^{\int_0^t \alpha(s)\,\mathrm{d}s}x')' = \lambda \mathrm{e}^{\int_0^t \alpha(s)\,\mathrm{d}s}x, \quad t \in [0,\pi] \\ x'(0) = x'(\pi) = 0. \end{array}\right\} \tag{2.4}$$

From [7] Theorem 27.II, we have that the eigenvalue problem (2.4), and so (2.3), has infinitely many simple real eigenvalues $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ and no other eigenvalues.

Since 0 is clearly an eigenvalue of (2.3), it remains only to see that λ_n can't be negative for all $n \in \mathbb{N}$.

Assume that $\lambda_n < 0$ for some n. Now, by integrating in (2.4) we obtain

$$x'(t) = -\mathrm{e}^{-\int_0^t \alpha(s)\,\mathrm{d}s} \int_0^t \mathrm{e}^{\int_0^s \alpha(\tau)\,\mathrm{d}\tau} \lambda_n x(s)\,\mathrm{d}s, \quad \forall t \in [0,\pi] \tag{2.5}$$

where $x \in C^1[0,\pi]$ is the solution of (2.3) with $\lambda = \lambda_n$ satisfying $x(0) = 1$.

Since $\lambda_n < 0$ and $x(0) > 0$, one can deduce from (2.5) that $x'(t) > 0$, $\forall t \in (0,\pi)$. Then $x(t) > 0$, $\forall t \in [0,\pi]$. Doing $t = \pi$ in (2.5) one has that $\int_0^\pi \mathrm{e}^{\int_0^s \alpha(\tau)\,\mathrm{d}\tau} \lambda_n x(s)\,\mathrm{d}s = 0$ which is a contradiction. □

Choose a suitable admissible set Ω_B as follows:

$$\begin{aligned} \Omega_B = \{\beta \in L^\infty[0,\pi] \setminus \{0\} : 0 \leq \beta(t) \leq B, a.e.\ t \in [0,\pi] \text{ and} \\ (1.2) \ \text{ has nontrivial solution for } \beta = \beta(t)\}, \end{aligned}$$

where $\lambda_1 \leq B$. Our control problem will be to find a function $\beta^* \in \Omega_B$ such that $\beta^*(t)$ minimizes the functional J defined by

$$J(\beta) = \int_0^\pi \beta(t)\,\mathrm{d}t, \ \forall \beta \in \Omega_B,$$

that is,

$$J(\beta^*) = \min_{\Omega_B} J(\beta).$$

In order to prove the existence of a minimum in Ω_B for J we note that every solution of (1.2), for $\beta \in \Omega_B$ has a zero.

Lemma 2. *If $\beta \in \Omega_B$ and x_β is a solution of* (1.2) *for that β, then x_β has a zero.*

Proof.
Arguing as in Lemma 1, we have

$$x'(\pi) = -\mathrm{e}^{-\int_0^\pi \alpha(s)\,\mathrm{d}s} \int_0^\pi \mathrm{e}^{\int_0^s \alpha(\tau)\,\mathrm{d}\tau} \beta(s) x(s)\,\mathrm{d}s = 0,$$

and we conclude the result. □

Lemma 3. *Ω_B is a weakly compact set of $L^1(0,\pi)$.*

Proof. Since the identity map $i : L^2(0,\pi) \to L^1(0,\pi)$ is continuous for the weak topologies and Ω_B is bounded, it is enough to see that Ω_B is a weakly closed subset of $L^2(0,\pi)$.

Pick $\{\beta_n\} \subset \Omega_B$ a sequence and for each n, take x_n a nontrivial solution of problem (1.2) for $\beta = \beta_n$. By linearity we can assume that

$$\|x_n\|_\infty + \|x_n'\|_\infty = 1, \forall n \in \mathbb{N}.$$

Consequently, the sequences $\{x_n\}$ and $\{x_n'\}$ are uniformly bounded and so $\{x_n\}$ is a equicontinuous sequence. From the equation (1.2) for $x = x_n$ we deduce that $\{x_n''\}$ is a uniformly essentially bounded sequence. Hence, $\{x_n'\}$ is also equicontinuous. By the Ascoli-Arzelà theorem, passing to a subsequence if necessary, we can assume that

$$\{x_n\} \to x_0, \quad \{x_n'\} \to y_0 \quad (n \to \infty)$$

uniformly on $[0,\pi]$ for suitable x_0, y_0 continuous functions on $[0,\pi]$. Now, it is clear that $y_0 = x_0'$ and $\|x_0\|_\infty + \|x_0'\|_\infty = 1$, so $x_0 \not\equiv 0$

From the boundness of the sequence $\{\beta_n\}$ in $L^2(0,\pi)$, we can assume, passing to a subsequence if necessary, that $\{\beta_n\}$ converges to $\beta_0 \in L^2(0,\pi)$ for the weakly topology. Passing to the limit in the equation (1.2) for $x = x_n$ and $\beta = \beta_n$ we can deduce that x_0 is a solution of the problem (1.2) for $\beta = \beta_0$, taking into account that $\{x_n\}$ converges "strongly" to x_0 and $\{\beta_n\}$ goes weakly to β_0, both in $L^2(0,\pi)$.

In order to prove that $\beta_0 \in \Omega_B$, it remains only to see that $\beta_0 \not\equiv 0$.

By Lemma 2, for each n there is $t_n \in (0,\pi)$ such that $x_n(t_n) = 0$. We can assume that $t_n \to t^* \in [0,\pi]$. As $x_n \to x_0$ uniformly then $x_0(t^*) = 0$. From the uniqueness for initial value problems for the equation (1.2), we deduce that $t^* \in (0,\pi)$, since $x_0'(0) = x_0'(\pi) = 0$. Therefore x_0 is not a constant function and so $\beta_0 \neq 0$ *a.e.* in $L^2(0,\pi)$. Finally, $\beta_0 \in \Omega_B$. □

Before proving the existence of minimum in Ω_B for J, we recall that a point x of a set C in a vector space is said to be an extreme point of C, if x is not an interior point of any nontrivial segment contained in C. The set of all extreme points of C will be denoted by $extr(C)$.

Theorem 4. *The functional J attains its minimum at an admisible $\beta_0 \in \Omega_B$ such that*

$$\beta_0 \in extr(\overline{co}(\Omega_B)) \quad (\Rightarrow \beta_0 \in extr(\Omega_B)).$$

Moreover, $\min_{\Omega_B} J = \min_{\overline{co}(\Omega_B)} J$.

Proof. Observe that $\Omega_B \neq \emptyset$ because the eigenfunction associated to the eigenvalues λ_1 is a nontrivial solution of problem (1.2) for $\beta(t) \equiv \lambda_1$ and therefore $\beta(t) \equiv \lambda_1$ belongs to Ω_B. Moreover, the closed convex hull of Ω_B, $\overline{co}(\Omega_B)$, is weakly compact in $L^1(0,\pi)$, applying Lemma 3. We can see the functional J defined on $\overline{co}(\Omega_B)$, namely $\widetilde{J}$. Now, by Bauer's Theorem (see for instance [1, Th. V.1]), $\widetilde{J}$ attains its minimum on $\beta_0 \in extr(\overline{co}(\Omega_B))$. By [1, Prop. V.4], we obtain that in fact $\beta_0 \in extr(\Omega_B)$. Then $J(\beta_0) = \widetilde{J}(\beta_0)$ is the minimum value of J on Ω_B. □

The following lemma exhibits some qualitative properties for the solutions of the problem (1.2) when β is a point of minimum for J.

Lemma 5. *Let β_1 be an element in Ω_B. Assume that the problem* (1.2) *for $\beta = \beta_1$ has a nontrivial solution x_1 whose derivative has a zero in $(0,\pi)$. Then there is β_2 in Ω_B such that every nontrivial solution, x_2 of problem* (1.2) *for $\beta = \beta_2$ satisfies $x_2'(t) \neq 0$ in $(0,\pi)$ and $\int_0^\pi \beta_2(t)\,\mathrm{d}t < \int_0^\pi \beta_1(t)\,\mathrm{d}t$.*

Proof. Let $t_1 \in (0,\pi)$ be a zero of x_1'. By Lemma 2 there is $t_2 \in (0,\pi)$ such that $x_1(t_2) = 0$. The uniqueness of initial value problem for equation (1.2) gives $x_1'(t_2) \neq 0$. It is clear that there is $0 \le a < b \le \pi$ such that $t_2 \in (a,b)$, $x_1'(a) = x_1'(b) = 0$ and $x_1'(t) \neq 0, \forall t \in (a,b)$. Clearly $0 < b - a < \pi$.

Consider $F(t) = \int_0^t \frac{\mathrm{d}s}{1 - K\mathrm{e}^{\int_0^s \alpha(\tau)\,\mathrm{d}\tau}} + a,\ t \in (0,\pi)$ where $K < 0$ is chosen such that the equality $F(\pi) = b$ holds (this is possible by virtue of the continuity of F with respect to the parameter K). Note that F defined above is the unique solution of

$$\left.\begin{array}{c} f'' + \alpha(t)f' = \alpha(t)(f')^2 \\ f(0) = a, \quad f(\pi) = b. \end{array}\right\}$$

Now, define $x_2(t) = x_1(F(t)), \forall t \in [0,\pi]$. One can check that x_2 is a solution of problem (1.2) for $\beta = F'(t)^2\beta_1(F(t))$. Moreover, $x_2'(t) = x_1'(F(t))F'(t) \neq 0$, for $t \in (0,\pi)$. Taking $\beta_2 = F'(t)^2\beta_1(F(t))$ one has $\beta_2 \in \Omega_B$, since

$$F'(t) = \frac{1}{1 - K\mathrm{e}^{\int_0^t \alpha(\tau)\,\mathrm{d}\tau}} < 1,\ \forall t \in (0,\pi).$$

Consequently,

$$\int_0^\pi \beta_2 = \int_0^\pi F'(t)^2\beta_1(F(t))\,\mathrm{d}t < \int_0^\pi F'(t)\beta_1(F(t))\,\mathrm{d}t = \int_a^b \beta_1(t)\,\mathrm{d}t \le \int_0^\pi \beta_1(t)\,\mathrm{d}t.$$ □

Corollary 6. *If β is a minimum for J, then every nontrivial solution $x(t)$ of the problem* (1.2) *satisfies $x'(t) \neq 0$ for all $t \in (0,\pi)$. In particular, x has exactly one zero in $[0,\pi]$.*

Note that the minimum of the functional J is strictly positive, so we will estimate this minimum and for this we need to fix some notation.

In the subsequent lemmas we will be in the following setting

(H)
$$\left\{\begin{array}{l} \bullet\ \beta^* \text{ is a minimum of } J \text{ (Th. 4)}, \\ \bullet\ x_1, x_2 \text{ are two solutions (so, linearly dependent) of the problem (1.2) with} \\ x_1(0) = 1, x_2(\pi) = 1, \text{ for } \beta = \beta^*, \text{ and} \\ \bullet\ t^* \in (0,\pi) \text{ is the unique point in } (0,\pi) \text{ satisfying } x_1(t^*) = 0, x_2(t^*) = 0 \\ \text{(Cor. 6)}. \end{array}\right.$$

Lemma 7. $x_1(\pi)x_2(0) = 1$.

Proof. It is clear that there is $c \in \mathbb{R} \setminus \{0\}$ such that $x_1 = c\,x_2$. Hence, $x_1(\pi) = c\,x_2(\pi) = c$, and then $1 = x_1(0) = c\,x_2(0) = x_1(\pi)x_2(0)$. □

Lemma 8. i)

$$\int_0^{t^*} \beta^*(s)\,\mathrm{d}s > \frac{1-x_1(\pi)}{\pi}\mathrm{e}^{-2\pi\|\alpha\|_\infty}.$$

ii)

$$\int_{t^*}^{\pi} \beta^*(s)\,\mathrm{d}s > \frac{1-x_2(0)}{\pi}\mathrm{e}^{-2\pi\|\alpha\|_\infty}.$$

Proof.
i) Integrating in (2.5) for $x = x_1$ and $\lambda_n = \beta^*$ we deduce

$$1 - x_1(\pi) = \int_0^{\pi} \mathrm{e}^{-\int_0^t \alpha(\tau)\,\mathrm{d}\tau} \int_0^t \mathrm{e}^{\int_0^s \alpha(\tau)\,\mathrm{d}\tau} \beta^*(s)x_1(s)\,\mathrm{d}s\,\mathrm{d}t$$

$$< \int_0^{\pi} \mathrm{e}^{t\|\alpha\|_\infty} \int_0^{t^*} \mathrm{e}^{s\|\alpha\|_\infty}\beta^*(s)\,\mathrm{d}s\,\mathrm{d}t \le \pi\mathrm{e}^{2\pi\|\alpha\|_\infty} \int_0^{t^*} \beta^*(s)\,\mathrm{d}s.$$

The second inequality is trivial. For the first inequality we used that the function x_1 is strictly positive in $[0, t^*)$, strictly negative in $(t^*, \pi]$ and $x_1(t) \le 1$ in $[0, \pi]$ (by **(H)** and Cor. 6).

ii) The proof is similar to the proof of i), by using that x_2 is strictly negative in $[0, t^*)$, strictly positive in $(t^*, \pi]$ and $x_2(t) \le 1$ in $[0, \pi]$ (by **(H)** and Cor. 6). □

Now, we present the main tool of the section.

Theorem 9.

$$\min_{\Omega_B} J > \frac{4}{\pi}\mathrm{e}^{-2\pi\|\alpha\|_\infty}.$$

Proof. From the previous lemma we obtain $J(\beta^*) > \frac{\mathrm{e}^{-2\pi\|\alpha\|_\infty}}{\pi}(2 - x_1(\pi) - x_2(0))$. Put $a = -x_1(\pi), b = -x_2(0)$. From Corollary 6 and Lemma 7 it is clear that $a, b > 0$ and $ab = 1$, since $x_1(0) = 1$ and $x_2(\pi) = 1$. Then we conclude $a + b \ge 2$ and so the proof. □

Remark. Observe that with the same computations as in Lemma 8, we obtain a better bound on the $\min_{\Omega_B} J$. Specifically,

$$\min_{\Omega_B} J > \frac{4}{\|\mathrm{e}^{\int_0^t \alpha}\|_\infty \int_0^{\pi} \mathrm{e}^{-\int_0^t \alpha}\,\mathrm{d}t}.$$

However, we will use Theorem 9 for simplicity reasons.

Note that the estimation in Theorem 9 is independent of B, so as a consequence we obtain the following

Corollary 10. *Let $\Omega = \{\beta \in L^\infty(0,\pi) \setminus \{0\} : \beta(t) \ge 0$ a.e. $t \in [0,\pi]$ and the problem (1.2) has nontrivial solution$\}$. Then, $\int_0^{\pi} \beta(t)\,\mathrm{d}t > \frac{4}{\pi}\mathrm{e}^{-2\pi\|\alpha\|_\infty}$ for each $\beta \in \Omega$.*

The above corollary is used now to obtain uniqueness conditions on the problem (1.2), which generalize [4], Theorem 3.

Corollary 11. *Let $\beta \in L^\infty(0,\pi) \setminus \{0\}$ such that $\beta(t) \geq 0$ a.e. $t \in [0,\pi]$ and $\int_0^\pi \beta(t)\,dt \leq \frac{4}{\pi} e^{-2\pi\|\alpha\|_\infty}$. Then for each $g \in L^1(0,\pi), R, S \in \mathbb{R}$ the boundary value problem*

$$\left.\begin{array}{c} x'' + \alpha(t)x' + \beta(t)x = g(t), \quad t \in [0,\pi] \\ x'(0) = R, \ x'(\pi) = S, \end{array}\right\} \tag{2.6}$$

has a unique solution.

Proof. From Corollary 10 the problem (2.6) has at most one solution. Since the equation is linear, the uniqueness implies the existence. □

In order to complete the study of our control problem, we describe now the functions $\beta \in \Omega_B$ where J attains its minimum on Ω_B for $B > \lambda_1$.

Theorem 12. *Assume α differentiable and let $B > \lambda_1$. Then, there is $\beta^* \in \Omega_B$ where J attains its minimum on Ω_B, with the following form:*

$$\beta^*(t) = \begin{cases} B, & 0 \leq t \leq t_1 \\ 0, & t_1 < t < t_2 \\ B, & t_2 \leq t \leq \pi, \end{cases}$$

with $t_1 < t_2 \in (0,\pi)$ satisfying the equation

$$\begin{aligned} 0 = \quad & \exp\left\{-\int_0^{t_2}\alpha\right\} y(t_2)x'(t_1) \\ & - \exp\left\{-\int_0^{t_1}\alpha\right\} x(t_1)y'(t_2) - x'(t_1)y'(t_2)\int_{t_1}^{t_2} \exp\left\{-\int_0^s \alpha\right\} \mathrm{d}s \end{aligned} \tag{2.7}$$

where $\{x, y\}$ are the solutions of the equation (1.2) *for $\beta = B$ satisfying $x(0) = y(\pi) = 1$ and $x'(0) = y'(\pi) = 0$.*

Proof. Choose $B > \lambda_1$ and take $\beta^* \in \Omega_B$ such that J attains its minimum on Ω_B. Let z_0 be the solution of problem (1.2) for $\beta = \beta^*$ satisfying $z_0'(0) = z_0'(\pi) = 0$ and $z_0(0) = 1$. Setting $w_0 = z_0'$ we have that (z_0, w_0) is a solution of the problem

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta^* & -\alpha \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \tag{2.8}$$

with conditions $z(0) = 1$ and $w(0) = 0$. By Theorem 4 and Pontryagin's Maximum Principle (see for instance [6], Th. 4.1), we have $\exists(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$, with $\lambda \geq 0$, and absolutely continuous functions $P, Q : [0,\pi] \to \mathbb{R}$ solutions of the following problem

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = -\begin{pmatrix} 0 & -\beta^* \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \tag{2.9}$$

with conditions $P(\pi) = 0$ and $Q(\pi) = \mu$.

Furthermore,

$$\left\langle \begin{pmatrix} P \\ Q \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -\beta^* & -\alpha \end{pmatrix} \begin{pmatrix} z_0 \\ w_0 \end{pmatrix} \right\rangle + \lambda\beta^* = \tag{2.10}$$

$$\min_{\phi\in[0,B]} \left\langle \begin{pmatrix} P \\ Q \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -\phi & -\alpha \end{pmatrix} \begin{pmatrix} z_0 \\ w_0 \end{pmatrix} \right\rangle + \lambda\phi \ \textit{a.e.}\ [0,\pi]. \tag{2.11}$$

Equivalently,

$$(\lambda - Qz_0)\beta^* = \min_{\phi\in[0,B]} (\lambda - Qz_0)\phi \ \textit{a.e.}\ [0,\pi]. \tag{2.12}$$

From (2.9) we deduce that

$$\begin{array}{c} -Q'' + (\alpha Q)' = \beta^* Q \\ Q(\pi) = \mu, \quad Q'(\pi) = \alpha(\pi)\mu. \end{array} \tag{2.13}$$

Observe that $q(t)=\mathrm{e}^{\int_0^t\alpha}z_0(t)$ satisfies $-q''+(\alpha q)'=\beta^*q$ and $q(\pi)=\mathrm{e}^{\int_0^\pi\alpha}z_0(\pi)$, $q'(\pi)=\alpha(\pi)\mathrm{e}^{\int_0^\pi\alpha}z_0(\pi)$. Then, $Q(t)=\dfrac{\mu\mathrm{e}^{-\int_0^\pi\alpha}}{z_0(\pi)}\mathrm{e}^{\int_0^t\alpha}z_0(t)$.

If $\mu=0$ holds then $\lambda>0$ and $Q\equiv 0$. Now from (2.12) we obtain $\beta^*\equiv 0\notin\Omega_B$. So, $\mu\neq 0$.

If $\lambda=0$ holds then from (2.12) we conclude that $\beta^*\equiv 0\notin\Omega_B$ or $\beta^*\equiv B$ (not possible because $B>\lambda_1$), since the function Qz_0 does not change sing in $[0,\pi]$. Thus, $\lambda>0$.

Again from (2.12)

$$\beta^*(t)=\begin{cases}0, \text{ if } \frac{\mu\mathrm{e}^{-\int_0^\pi\alpha}}{z_0(\pi)}\mathrm{e}^{\int_0^t\alpha}z_0^2(t)<\lambda\\ B, \text{ if } \frac{\mu\mathrm{e}^{-\int_0^\pi\alpha}}{z_0(\pi)}\mathrm{e}^{\int_0^t\alpha}z_0^2(t)>\lambda.\end{cases}\tag{2.14}$$

From Corollary 6, z_0 is strictly monotone, then the number of elements in the set $\mathcal{A}=\{t\in[0,\pi]:\frac{\mu\mathrm{e}^{-\int_0^\pi\alpha}}{z_0(\pi)}\mathrm{e}^{\int_0^t\alpha}z_0^2(t)=\lambda\}$ is either zero, one or two. The cases zero and one give $\beta^*\equiv 0$ in an interval I containing 0 or π and so, z_0 is constant on I, which is a contradiction with Corollary 6. In the case that $\mathcal{A}$ has two elements, the same argument proves that the unique possibility for β^* is

$$\beta^*(t)=\begin{cases}B & \text{if } t\in(0,t_1)\\ 0 & \text{if } t\in(t_1,t_2)\\ B & \text{if } t\in(t_2,\pi).\end{cases}$$

Take $\{x,y\}$ the solutions of (1.2) with $\beta=B$ satisfying $x(0)=1$, $x'(0)=0$, $y(\pi)=1$ and $y'(\pi)=0$. Let β^* be a function in Ω_B where J attains its minimum value on Ω_B. Then, there are some $t_1,t_2\in(0,\pi)$ such that

$$\beta^*(t)=\begin{cases}B & \text{if } t\in(0,t_1)\\ 0 & \text{if } t\in(t_1,t_2)\\ B & \text{if } t\in(t_2,\pi)\end{cases}$$

and hence, a nontrivial solution x_{β^*} of (1.2) for $\beta=\beta^*$ can be written, when B is not an eigenvalue of (2.3), in the way

$$x_{\beta^*}(t)=\begin{cases}ax(t)+by(t), & \text{if } t\in(0,t_1)\\ c+d\int_0^t\exp\{-\int_0^s\alpha\}\mathrm{d}s, & \text{if } t\in(t_1,t_2)\\ ex(t)+fy(t), & \text{if } t\in(t_2,\pi)\end{cases}\tag{2.15}$$

for suitable $a,\ b,\ c,\ d,\ e,\ f\in\mathbb{R}$, since $\{x,y\}$ is a fundamental system of (1.2) with $\beta=\beta^*$. Now, we impose the continuity and differentiability on x_{β^*} in t_1 and t_2 joint to $x'_{\beta^*}(0)=x'_{\beta^*}(\pi)=0$ and we obtain that the following system

$$\begin{aligned}ax'(0)+by'(0)&=0\\ ax(t_1)+by(t_1)-c-d\int_0^{t_1}\exp\left\{-\int_0^s\alpha\right\}\mathrm{d}s&=0\\ ax'(t_1)+by'(t_1)-d\ \exp\left\{-\int_0^{t_1}\alpha\right\}&=0\\ ex(t_2)+fy(t_2)-c-d\int_0^{t_2}\exp\left\{-\int_0^s\alpha\right\}\mathrm{d}s&=0\\ ex'(t_2)+fy'(t_2)-d\ \exp\left\{-\int_0^{t_2}\alpha\right\}&=0\\ ex'(\pi)+fy'(\pi)&=0\end{aligned}\tag{2.16}$$

has a nontrivial solution $(a,\ b,\ c,\ d,\ e,\ f)$. Equivalently, the determinant D of the following matrix must be 0:

$$0 = D = \det \begin{pmatrix} x'(0) & y'(0) & 0 & 0 & 0 & 0 \\ x(t_1) & y(t_1) & -1 & -\int_0^{t_1} \exp\{-\int_0^s \alpha\}\mathrm{d}s & 0 & 0 \\ x'(t_1) & y'(t_1) & 0 & -\exp\{-\int_0^{t_1} \alpha\} & 0 & 0 \\ 0 & 0 & -1 & -\int_0^{t_2} \exp\{-\int_0^s \alpha\}\mathrm{d}s & x(t_2) & y(t_2) \\ 0 & 0 & 0 & -\exp\{-\int_0^{t_2} \alpha\} & x'(t_2) & y'(t_2) \\ 0 & 0 & 0 & 0 & x'(\pi) & y'(\pi) \end{pmatrix}.$$

Computing the determinant D, one has

$$0 = \exp\left\{-\int_0^{t_2} \alpha\right\} y(t_2)x'(t_1) - \exp\left\{-\int_0^{t_1} \alpha\right\} x(t_1)y'(t_2) - \left(\int_0^{t_2} \exp\left\{-\int_0^s \alpha\right\} \mathrm{d}s - \int_0^{t_1} \exp\left\{-\int_0^s \alpha\right\} \mathrm{d}s\right) x'(t_1)y'(t_2). \quad (2.17)$$

If B is an eigenvalue of (2.3), then the above equation also holds, setting $x = y$. □

Now, we finish this section by showing, as consequence, an improvement of Corollary 11, which generalizes [4], Theorem 2, giving a bound depending of B.

Corollary 13. *Fix $B > \lambda_1$. Assume that α is differentiable and let $\beta \in L_\infty(0, \pi) \setminus \{0\}$ such that $0 \le \beta(t) \le B$ a.e. in $[0, \pi]$. Let $\{x, y\}$ solutions of equation (1.2) for $\beta = B$ satisfying $x(0) = y(\pi) = 1$ and $x'(0) = y'(\pi) = 0$. If $\int_0^\pi \beta(t)\mathrm{d}t \le B(t_1 + \pi - t_2)$, where $0 < t_1 < t_2 < \pi$ is a solution of the algebraic equation*

$$0 = \exp\left\{-\int_0^{t_2} \alpha\right\} y(t_2)x'(t_1) - \exp\left\{-\int_0^{t_1} \alpha\right\} x(t_1)y'(t_2) - \left(\int_0^{t_2} \exp\left\{-\int_0^r \alpha\right\} \mathrm{d}r - \int_0^{t_1} \exp\left\{-\int_0^r \alpha\right\} \mathrm{d}r\right) x'(t_1)y'(t_2) \quad (2.18)$$

which minimizes the expression $t_1 - t_2$, then for each $f \in L_1(0, \pi)$, $C,\ D \in \mathbb{R}$ the boundary value problem

$$\left.\begin{array}{c} x'' + \alpha(t)x' + \beta(t)x = f(t), \quad t \in [0, \pi] \\ x'(0) = C,\ x'(\pi) = D, \end{array}\right\} \quad (2.19)$$

has a unique solution.

3. Nonlinear problem

The results of the previous section will become essential to get the main theorem concerning the existence and uniqueness of solution for the following nonlinear boundary value problem.

$$\left.\begin{array}{c} -x'' - \alpha(t)x' = f(t, x), \quad t \in [0, \pi] \\ x'(0) = A,\ x'(\pi) = B, \end{array}\right\}$$

where $A, B \in \mathbb{R}$, $\alpha : [0,\pi] \to \mathbb{R}$ is a continuous function and the nonlinearity $f : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ satisfies that f, f_x are continuous on $[0,\pi] \times \mathbb{R}$.

Theorem 14. *Assume that the following requirements are fulfilled:*

i) $0 \le f_x(t,x) \le \beta(t)$ *on* $[0,\pi] \times \mathbb{R}$, *where* $\beta \in L^\infty(0,\pi)$ *and satisfies* $\int_0^\pi \beta(t)\,\mathrm{d}t \le \frac{4\exp\{-2\pi\|\alpha\|_\infty\}}{\pi}$.

ii) *For each* $x \in C[0,\pi]$ *one has* $f_x(t,x(t)) \ne 0$ *a.e. on* $[0,\pi]$ *and* $\int_0^\pi \exp\{\int_0^s \alpha(\tau)\,\mathrm{d}\tau\} f(s,0)\,\mathrm{d}s = 0$.

Then, the problem (1.1) *has a unique solution.*

Proof. We first prove the uniqueness. Without loss of generality we can assume that $A = B = 0$. Pick x_1, x_2 solutions of the problem (1.1), then $x(t) = x_1(t) - x_2(t)$ is a solution of the problem

$$\left.\begin{array}{c} -x'' - \alpha(t)x' = x\displaystyle\int_0^1 f_x(t, x_2 + \theta x)\,\mathrm{d}\theta, t \in [0,\pi] \\ x'(0) = x'(\pi) = 0 \end{array}\right\}. \tag{3.20}$$

Take $\beta_0(t) = \int_0^1 f_x(t, x_2 + \theta x)\,\mathrm{d}\theta$. Now, x is a solution of the problem (1.2) for $\beta = \beta_0$. From our requirements $\beta_0 \in L^\infty(0,\pi)$, $0 \le \beta_0(t)$ *a.e.* on $[0,\pi]$ and $\int_0^\pi \beta_0 \le \frac{4\exp\{-2\pi\|\alpha\|_\infty\}}{\pi}$. Then, applying Corollary 10, $x \equiv 0$.

The existence of solution will be obtained by an argument of Schauder fix point type. To do it, we write the nonlinear problem (1.1) in the equivalent form

$$\left.\begin{array}{c} -x'' - \alpha(t)x' = b(t,x)x + f(t,0),\ t \in [0,\pi] \\ x'(0) = x'(\pi) = 0 \end{array}\right\} \tag{3.21}$$

where $b(t,x) = \int_0^1 f_x(t,\theta x)\,\mathrm{d}\theta$. Let $X = \{x \in C^1[0,\pi] : x'(0) = x'(\pi) = 0\}$ provided with the norm $\|x\|_X = \sup_{[0,\pi]} |x(t)| + \sup_{[0,\pi]} |x'(t)|$. It is known that $(X, \|\cdot\|_X)$ is a Banach space. Define the operator $T : X \to X$ as $x \to Tx = y_x$, where y_x is the solution of the linear problem

$$\left.\begin{array}{c} -y'' - \alpha(t)y' = b(t,x)y + f(t,0), \quad t \in [0,\pi] \\ y'(0) = y'(\pi) = 0. \end{array}\right\} \tag{3.22}$$

Observe that by Corollary 11, the above problem has unique solution and then the operator T is well defined. Let us see that T is bounded, *i.e.*, $\exists M > 0$ such that $\|Tx\|_X \le M$ for each $x \in X$. Otherwise, there exists a sequence $\{x_n\} \in X$ satisfying $\|y_{x_n}\|_X \to +\infty$. From the estimate

$$0 \le b(t,x_n) \le \beta(t), \quad t \in [0,\pi],$$

we deduce the existence of a sequence, noted again by $\{b(t,x_n)\}$, such that $\{b(t,x_n)\} \rightharpoonup \beta_1$, where the weak limit function satisfies

$$0 \le \beta_1(t) \le \beta(t), t \in [0,\pi].$$

We know that $y_n := y_{x_n}$ is the solution defined by problem (3.22) and then

$$\int_0^\pi y_n'h' - \int_0^\pi \alpha y_n'h = \int_0^\pi b(t,x_n)y_nh + \int_0^\pi f(t,0)h,\ \forall h \in H^1(0,\pi), \forall n \in \mathbb{N}. \tag{3.23}$$

By normalizing $z_n(t) := \frac{y_n(t)}{\|y_n\|_X}$, we obtain

$$\int_0^\pi z_n'h' - \int_0^\pi \alpha z_n'h = \int_0^\pi b(t,x_n)z_nh + \frac{1}{\|y_n\|_X}\int_0^\pi f(t,0)h,\ \forall h \in H^1(0,\pi), \forall n \in \mathbb{N}.$$

From the previous equation one deduces that z_n'' are uniformly bounded, then by an Ascoli-Arzela argument $\{z_n\} \to z$ and $\{z_n'\} \to z'$ uniformly on $[0,\pi]$. Taking limit in the corresponding expressions, $z \neq 0$ (recall that $\|z\|_X = 1$) is a weak solution of

$$\int_0^\pi z'h' - \int_0^\pi \alpha z'h = \int_0^\pi \beta_1 zh,\ \forall h \in H^1(0,\pi). \tag{3.24}$$

Now, if one takes $h(t) = \mathrm{e}^{\int_0^t \alpha(\theta)\,\mathrm{d}\theta}$ in (3.23), one obtains

$$0 = \int_0^\pi \mathrm{e}^{\int_0^s \alpha(\theta)\,\mathrm{d}\theta}[b(s,x_n)y_n + f(s,0)]\,\mathrm{d}s,$$

and hence, by hypothesis ii)

$$\int_0^\pi \mathrm{e}^{\int_0^s \alpha(\theta)\,\mathrm{d}\theta} b(s,x_n)y_n = 0.$$

This implies that the function y_n has a zero in $[0,\pi]$ and consequently also z. As z is nontrivial, the zero is actually in $(0,\pi)$ and so, z is a nonconstant function. Thus, it follows from (3.24) that β_1 is in $L^\infty(0,\pi)\backslash\{0\}$. On the other hand, hypothesis *i)* of theorem and Corollary 11 imply that $z \equiv 0$ which is a contradiction. Therefore, the operator T is bounded.

Let us see now that the operator $T : X \to X$ is continuous. Take a convergent sequence $\{x_n\} \to x_0 \in X$. Then, we need to show that $y_n \to y_0$ in X, where y_n and y_0 are the corresponding solutions of problem (3.22) for $x = x_n$ and $x = x_0$, respectively. If the sequence is not convergent, then $\exists \eta > 0$ and a subsequence $y_{n'} \notin B_X(y_0,\eta)$, $\forall n \in \mathbb{N}$. On the other hand, we know that $y_{n'}''$ is uniformly bounded in X (taking into account the equation (3.22) and the boundness of operator T), thus there exists a new subsequence $y_{n''} \to y$, which converges to some $y \in X$. Passing to the limit in the equation satisfied by $y_{n''}$, the uniqueness of solution for problem (3.22) implies that $y \equiv y_0$ which is a contradiction. Therefore, $y_n \to y_0$ and then operator T is continuous.

To finish the proof, consider $T : B_X(0,M) \subset X \to X$ and arguing as above it is possible to show that T is a compact operator. Then, Schauder's fix point theorem does the rest. □

Remark. Note that the second condition in i) of Theorem 14 is only a L^1-norm condition for the function β, and so, a such β can take any positive value. For example, fix $\gamma > 0$ and take $\beta = \gamma \boldsymbol{\chi}_I$, where I is a subinterval of $[0,\pi]$ with length small enough, and $\boldsymbol{\chi}_I$ is the characteristic function of I. Therefore, the Theorem 14 include cases where there is interaction between the nonlinearity f_x and any positive eigenvalue of the spectrum. This interaction also exists from the right in zero. This phenomena is known in the literature as resonant phenomena.

To finish, we show a nonlinear version of Corollary 13 which generalizes [4], Theorem A.

Corollary 15. *Fix $B > \lambda_1$ and suppose α differentiable. Assume that the following requirements are fulfilled:*

i) $0 \le f_x(t,x) \le \beta(t) \le B$ *on* $[0,\pi] \times \mathbb{R}$, *where* $\beta \in L^\infty(0,\pi)$ *and satisfies* $\int_0^\pi \beta(t)\,\mathrm{d}t \le B(t_1 + \pi - t_2)$. *Here* $0 < t_1 < t_2 < \pi$ *are defined as in Corollary* 13.

ii) *For each* $x \in C[0,\pi]$ *one has* $f_x(t,x(t)) \neq 0$ *a.e. on* $[0,\pi]$ *and* $\int_0^\pi \exp\{\int_0^s \alpha(\tau)\,\mathrm{d}\tau\} f(s,0)\,\mathrm{d}s = 0$.

Then, the problem (1.1) *has a unique solution.*

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