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NEW CONVEXITY CONDITIONS IN THE CALCULUS OF VARIATIONS AND COMPENSATED COMPACTNESS THEORY *, **

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Abstract. We consider the lower semicontinuous functional of the form $I_f(u) = \int_\Omega f(u)\mathrm{d}x$ where u satisfies a given conservation law defined by differential operator of degree one with constant coefficients. We show that under certain constraints the well known Murat and Tartar's Λ-convexity condition for the integrand f extends to the new geometric conditions satisfied on four dimensional symplexes. Similar conditions on three dimensional symplexes were recently obtained by the second author. New conditions apply to quasiconvex functions.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ be a continuous function and consider the variational functional

$$I_f(u) = \int_\Omega f(\nabla u)\mathrm{d}x, \tag{1.1}$$

where u belongs to the Sobolev space $W^{1,\infty}(\Omega, \mathbb{R}^m)$. It was proved by Morrey in 1952, [50] that the functional $I_f(u)$ is sequentially lower semicontinuous with respect to weak-$*$ convergence in $W^{1,\infty}$ (*i.e.* $\liminf_{\nu\to\infty} I_f(u^\nu) \geq I_f(u)$ provided that $u^\nu \to u$ in $\mathrm{L}^1(\Omega, \mathbb{R}^m)$ and $\nabla u^\nu \rightharpoonup \nabla u$ weakly-$*$ in $\mathrm{L}^\infty(\Omega, \mathbb{R}^{n\times m})$) if and only if f is quasiconvex, which is by definition

$$⨍_Q f(A + \nabla v)\mathrm{d}x \geq f(A) \tag{1.2}$$

for every cube $Q \subseteq \mathbb{R}^n$, every matrix $A \in \mathbb{R}^{n\times m}$ and every function $v \in \mathrm{C}_0^\infty(Q, \mathbb{R}^m)$. The quasiconvexity condition is very hard to verify in practice. It is easier to verify the so-called rank-one convexity condition which

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is a consequence of quasiconvexity. This condition means that every mapping of the form $\mathbb{R} \ni t \mapsto f(A + ta \otimes b)$ is convex for arbitrary $A \in \mathbb{R}^{n\times m}$ and arbitrary rank one matrix $a \otimes b = (a_i b_j)_{i=1,\dots,n; j=1,\dots m} \in \mathbb{R}^{n\times m}$. (see *e.g.* [17, 50, 51, 56, 65]). Those two notions agree if $\min\{n, m\} = 1$, in such cases every quasiconvex function is even convex, or when f is a quadratic form (see *e.g.* [17, 50, 53], Sect. 3, [80], Th. 11).

It has been conjectured by Morrey in 1952 [50] that rank-one convexity does not imply quasiconvexity. This conjecture has been confirmed by Šverák 40 years later in [77] in dimensions $n \geq 2, m \geq 3$ with the example of a polynomial of degree four which is rank-one convex but not quasiconvex. The conjecture is still open in the remaining dimensions $n \geq 2, m = 2$.

However, an alternative to (1.2) algebraic description of quasiconvex functions is known (see [14]), and some numerical approaches to face Morrey's rank-one conjecture are known (see *e.g.* [18,19]), but it is still not possible in general to verify it in practice. There are so far few ways to investigate the quasiconvexity condition directly. It is known that in dimensions $n \geq 2, m \geq 3$ this condition is nonlocal (see [39]). In the same dimensions it is also not invariant with respect to the transposition [40, 58]. For some other related approaches we refer *e.g.* to [1, 5, 10, 25, 33, 34, 48, 53, 64, 68, 69, 73, 75, 76, 78, 83, 84], and references therein. None of the above mentioned properties except the rank-one condition can be described in geometric way.

Recently, the second author has found necessary geometric conditions for quasiconvexity satisfied on certain three dimensional symplexes in $\mathbb{R}^{n\times m}$, [34]. Roughly speaking these conditions, defined as tetrahedral convexity conditions, express the following property: if f agrees with a certain polynomial A on the tetrahedron D from certain class of tetrahedrons in $\mathbb{R}^{n\times m}$ on its vertices and three more other points (where the polynomial and those points are determined by D) then $f \leq A$ inside D. Then the natural question is whether we can expect similar geometric conditions holding on four dimensional symplexes in $\mathbb{R}^{n\times m}$. In particular in the case $n = m = 2$ the dimension of the symplex is the same as the dimension of the domain of f. In this paper we find such conditions. It is worth pointing out that the polynomials in our conditions are of degree no bigger than two. Both mentioned geometric conditions (three and four dimensional) are similar to the familiar convexity conditions, as every convex function which agrees with an affine function in the endpoints of the interval is not bigger than this affine function in its interior. Obviously an interval is a one-dimensional symplex. Note also that our conditions are convenient for a numerical treatment and they are between quasiconvexity and rank-one convexity (three dimensional geometric conditions obtained in [34] have been numerically verified in [74]).

Let us mention that our geometric conditions generalize the so-called Λ-convexity conditions due to Murat an Tartar appearing in the following more general problem. Let $P = (P_1, \dots, P_N) : \mathrm{C}^\infty(\Omega, \mathbb{R}^m) \to \mathrm{C}^\infty(\Omega, \mathbb{R}^N)$ be a differential operator with constant coefficients, given by

$$P_k u = \sum_{i=1,\dots,n, j=1,\dots,m} a^k_{i,j} \frac{\partial u_j}{\partial x_i}, \; k = 1, \dots, N, \tag{1.3}$$

and let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function. Consider instead of (1.1) the functional

$$I_f(u) = \int_\Omega f(u(x))\, \mathrm{d}x, \quad u \in \ker\, P \cap \mathrm{L}^\infty(\Omega, \mathbb{R}^m), \tag{1.4}$$

where ker P is the distributional kernel of the operator P.

In particular, when $P = \operatorname{curl}$ is applied to each column of u (and $u \in \mathbb{R}^k_n$, $k \cdot n = m$) in a simply connected domain, we recover the classical functional of the calculus of variations.

In general the necessary and sufficient conditions for a function f to define a lower semicontinuous functional with respect to sequential weak-$*$-convergence in $\mathrm{L}^\infty(\Omega, \mathbb{R}^m) \cap \ker\, P$ are not known (we refer *e.g.* to, [16], p. 26, [25, 33, 53], Sect. 3, [57, 66, 80], Th. 11, [82] for special cases). The only known general condition is so-called Λ-convexity necessary condition due to Murat and Tartar (see *e.g.* [16], Th. 3.1, [52], Th. 2.1, [53], [65], Th. 10.1, [81], Cor. 9). It reads as follows.

Theorem 1.1. *Define*

$$V = \left\{ (\xi, \lambda) : \xi \in \mathbb{R}^n, \xi \neq 0, \lambda \in \mathbb{R}^m, \sum_{i,j} a^k_{i,j} \xi_i \lambda_j = 0, \text{ for } k = 1, \dots, N \right\},$$
$$\Lambda = \{ \lambda \in \mathbb{R}^m : \text{ there exists } \xi \in \mathbb{R}^n, \xi \neq 0, \text{ such that } (\xi, \lambda) \in V \}.$$

If I_f given by (1.4) is lower semicontinuous (continuous) with respect to L^∞-weak $*$-convergence in $\mathrm{L}^\infty(\Omega, \mathbb{R}^m) \cap$ $\ker P$, then f is Λ-convex (Λ-affine), which means that for every $A \in \mathbb{R}^m$ and every $\lambda \in \Lambda$ the function $\mathbb{R} \ni t \mapsto f(A + t\lambda)$ is convex (affine).

We try to contribute to both approaches: the general one and the special variational one. As a result we obtain a general condition stated in Theorem 4.2 and its special case related to the variational approach (Th. 4.3).

Let us mention that the rank-one problem is strongly related to an important and long standing problem in the theory of quasiconformal mappings, as has been recently pointed out by Iwaniec and Astala [4, 29].

The paper is organized as follows. At first we study functionals of the form: $I_f(u) = \int_{[0,1]^2} f(u_1(\tau_1), u_2(\tau_2),$ $u_3(\tau_1 + \tau_2)$, $u_4(\tau_1 - \tau_2) \mathrm{d}\tau_1 \mathrm{d}\tau_2$ and after some preparations made in Sections 2 and 3 we see in Section 4 that if I_f is lower semicontinuous then f satisfies certain conditions on four dimensional symplexes in $\mathbb{R}^4$. Then the general functional given by (1.4) is reduced to that special one after restricting it to subspaces in the kernel of the operator P of the form $\{A + \sum_{i=1}^4 u_i(< x, \xi_i >)\lambda_i\}$, where $(\xi_i, \lambda_i) \in V$ and V is defined in Theorem 1.1. The main result and the discussion are presented in Sections 4 and 5 respectively. The technically complicated part (proof of Lem. 3.2) is given in the Appendix.

The results of this paper and that of [34] are the continuation of the approach started by the second author mainly in [33] where she studied functionals of the general form (1.4) for systems like (1.3) where the kernel of P is the solution of the system of equations like

$$\partial_{V_j} u_j = 0 \text{ for } j = 1, \dots, m, \tag{1.5}$$

$V_1, \dots, V_m$ are linear subspaces of $\mathbb{R}^n$ and the condition $\partial_V u = 0$ means that for every $v \in V$ we have $\partial_v u = 0$. The knowledge about lower semicontinuous functionals on solutions of (1.5) should yield new conditions in the Compensated Compactness Theory and Calculus of Variations. This is because subspaces like (1.5) can be found in kernels of generally defined operators like (1.3), so we can restrict our general functional to such subspaces. The main result of [34] is based on investigation of functional like $I_f(u) =$ $\int_{[0,1]^2} f(u_1(\tau_1), u_2(\tau_2), u_3(\tau_1 + \tau_2)) \mathrm{d}\tau_1 \mathrm{d}\tau_2$, while here the special role is played by the functional $I_f(u) =$ $\int_{[0,1]^2} f(u_1(\tau_1), u_2(\tau_2), u_3(\tau_1 + \tau_2), u_4(\tau_1 - \tau_2) \mathrm{d}\tau_1 \mathrm{d}\tau_2$. Both studied models bring new geometric conditions for lower semicontinuous functionals in the general model (1.4).

Some of the ideas exploited and developed here and in the paper [34] can be tracked back to Murat [54] and Pedregal [66].

We believe that the number of various versions of geometric convexity-like conditions for quasiconvex functions like ours will decrease with the time. They require systematic investigation. Perhaps one of them will lead to the confirmation of Morrey's conjecture in some cases, or perhaps unpossibility to find an example of a rank-one convex function which does not satisfy a geometric condition in the remaining cases of the rank-one conjecture of Morray or will encourage someone to find the proof that quasiconvexity is the same as rank-one convexity.

2. Notation and preliminaries

2.1. **The basic notation**

For a measurable function $u : [0, 1) \to \mathbb{R}$ we denote by $\hat{u}$ its periodic extension outside $[0, 1)$. If $A \subseteq [0, 1]$ is any subset, we write $A^1 = A$ and $A^0 = [0, 1] \setminus A$. Let $[s]_1$ stand for the non integer part of $s \in \mathbb{R}$. In the sequel we will assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded domain. If A is a measurable subset of $\mathbb{R}^n$, by $|A|$ we denote its Lebesgue's measure, while $\mathcal{H}^s$ is the s-dimensional Hausdorff measure. By n-dimensional cube in $\mathbb{R}^n$ we mean an arbitrary set of the form $I_1 \times \cdots \times I_n$ where the I_k's are closed intervals. If X is a topological space, by $\mathrm{C}^0(X)$ we denote the space of continuous functions on X. By $e_1, \ldots, e_n$ we denote the standard basis in $\mathbb{R}^n$, while $< \cdot, \cdot >$ stands for the inner product. If $\xi \in \mathbb{R}^n$ and $a \in \mathbb{R}^m$ by $\xi \otimes a$ we denote the rank one matrix $(\xi_i a_j)_{i=1,\ldots,n, j=1,\ldots,m}$. By χ_B we denote the characteristic function of the set B. Let $j = (j_1, \ldots, j_k) \in \{0,1\}^k$. We describe its length by $|j| = \sum_{i=1}^k j_i$. If H is a given group of transformations of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a mapping, we denote $f_h(x) := f(hx)$. By $\mathbb{R}^{n \times m}$ we denote the space of $n \times m$ matrices.

2.2. Combinatorial objects

By $\mathcal{S}^k$ we define the space of sequences with indices in $\{0,1\}^k$. As $\{0,1\}^k$ consists of 2^k elements, it follows that the space $\mathcal{S}^k$ is isomorphic to $\mathbb{R}^N$ with $N = 2^k$.

Let $i \in \{1, \ldots, l\}$, $\epsilon \in \{0,1\}$, and let us define the transformation of indices $s_i^\epsilon : \{0,1\}^{l-1} \to \{0,1\}^l$ by putting ϵ on the i-th place. Namely, for $j = (j_1, \ldots, j_{l-1}) \in \{0,1\}^{l-1}$ we define

$$
\begin{aligned}
&s_1^\epsilon(j) := (\epsilon, j_1, \ldots, j_{l-1}), \; s_l^\epsilon(j) := (j_1, \ldots, j_{l-1}, \epsilon), \text{ and} \\
&s_i^\epsilon(j) := (j_1, \ldots, j_{i-1}, \epsilon, j_i, \ldots, j_{l-1}) \text{ for } 1 < i < l.
\end{aligned}
$$

However the above transformation depends also on l, but for abbreviation we omit this dependence in the notation.

Then we define the three related operators $\Pi_i^0, \Pi_i^1, \Pi_i : \mathcal{S}^l \to \mathcal{S}^{l-1}$ by

$$
(\Pi_i^\epsilon\{h\})_j = h_{s_i^\epsilon(j)}, \quad \Pi_i\{h\} = \Pi_i^0\{h\} + \Pi_i^1\{h\}.
$$

For example when $i = l$ we have $(\Pi_l\{h\})_j = h_{(j,0)} + h_{(j,1)}$, for every $j \in \{0,1\}^{l-1}$.

Let us identify $\{h\} \in \mathcal{S}^l$ with the mapping from $\{0,1\}^l$ to $\mathbb{R}$. Then operator Π_i^ϵ restricts $\{h\}$ to the subset of those $j \in \{0,1\}^l$ which have ϵ on the i-th place and identifies the new mapping with an element of $\mathcal{S}^{l-1}$. The operator Π_i can be regarded as discrete directional integration of $\{h\}$ in the given i-th direction, with respect to the counting measure.

The following lemma summarizes obvious but useful properties of the operators Π_i and Π_i^ϵ. Its proof is left to the reader as a simple exercise.

Lemma 2.1.

1) *For every $\{h\} \in \mathcal{S}^k$ we have*

$$
\Pi_1 \circ \cdots \circ \Pi_k\{h\} = \sum_{j \in \{0,1\}^k} h_j.
$$

2) *The operator Π_i preserves the sum: if $\{h\} \in \mathcal{S}^k$ sums up to A, then also $\{\Pi_i\{h\}\}$ sums up to A, which is expressed by*

$$
\sum_{j \in \{0,1\}^{k-1}} (\Pi_i\{h\})_j = \sum_{j \in \{0,1\}^k} h_j.
$$

If Ω is the subset of $\mathbb{R}^n$, by $\mathcal{S}^k(\Omega)$ we will denote the space of all measurable functions $\{h\} : \Omega \to \mathcal{S}^k$. If $\{h\} \in \mathcal{S}^k(\Omega)$ is the given function, and $\mathcal{G} : \mathcal{S}^l \mapsto \mathcal{S}^r$ we use the same expression: $\mathcal{G}$ to denote the mapping from $\mathcal{S}^l(\Omega)$ to $\mathcal{S}^r(\Omega)$ induced by $\mathcal{G}$, namely $(\mathcal{G}(\{h\}))(x) = \mathcal{G}(\{h(x)\})$.

2.3. Algebraic and geometric objects

Polynomials and projections

By $\mathcal{A}$ we denote the 11 dimensional subspace of polynomials in $\mathbb{R}^4$ spanned by $\{1, x_i, x_i x_j : i, j \in \{1,2,3,4\}, i \neq j\}$. Let us describe the following operator $P : \mathrm{C}^0(\mathbb{R}^4) \to \mathcal{A}$ expressed in terms of differences of f: $\Delta_i f(x) = f(x + e_i) - f(x)$:

$$\begin{aligned} Pf(x) &= f(0,0,0,0) + \sum_{i=1}^{4} \Delta_i f(0,0,0,0) x_i \\ &\quad + \sum_{(i,j)\in\{(1,3),(1,4),(2,3)\}} \Delta_i \Delta_j f(0,0,0,0) x_i x_j + \Delta_1 \Delta_2 f(0,0,1,0) x_1 x_2 \\ &\quad + \Delta_2 \Delta_4 f(1,0,1,0) x_2 x_4 + \frac{1}{2}\Big(\Delta_3 \Delta_4 f(0,0,0,0) + \Delta_3 \Delta_4 f(1,0,0,0)\Big) x_3 x_4 \\ &:= Qf(x) + \frac{1}{2}(R_1 f(x) + R_2 f(x)), \end{aligned} \tag{2.1}$$

where $x = (x_1, x_2, x_3, x_4)$, $R_1 f(x) = \Delta_3 \Delta_4 f(0,0,0,0) x_3 x_4$, the projectors R_1, R_2 are defined by $R_2 f(x) = \Delta_3 \Delta_4 f(1,0,0,0) x_3 x_4$, and $Qf(x)$ is the remaining term in the expression above.

We have the following lemma. Its simple proof is left to the reader.

Lemma 2.2.

1) *The operator Pf coincides with f in the following* 9 *vertices of the cube* $Q = [0,1]^4$*:* $(0,0,0,0)$*,* $(1,0,0,0)$*,* $(0,1,0,0)$*,* $(0,0,1,0)$*,* $(0,0,0,1)$*,* $(1,0,1,0)$*,* $(1,0,0,1)$*,* $(0,1,1,0)$*,* $(1,1,1,0)$*.*

2) *Pf depends only on values of f in* 12 *vertices of the cube Q: nine vertices described above and:* $(1,0,1,1)$*,* $(1,1,1,1)$*,* $(0,0,1,1)$*.*

3) *If $f \in \mathcal{A}$ we have $Pf = f$, in particular Pf is the projection operator onto the space $\mathcal{A}$.*

Remark 2.1. Note that the class of projection operators from $\mathrm{C}^0(\mathbb{R}^4)$ to $\mathcal{A}$ is large. For example we can define it by taking an arbitrary $f \in \mathrm{C}^0(\mathbb{R}^4)$ and prescribing to it the uniquely defined polynomial which agrees with f in the following 11 vertices of the cube Q: the first one taken arbitrary and the remaining ones linked with the first one by at most two edges of the cube. As convex combination of arbitrary two projection operators is again a projection operator, it follows that the set of projection operators from $\mathrm{C}^0(\mathbb{R}^4)$ to $\mathcal{A}$ is convex.

Remark 2.2. Let us look at the projection operator Pf from Lemma 2.2 more closely. Note that Pf is the convex combination with weights 1/2 of the projection operators: $P_1 f = Qf + R_1 f$, and $P_2 f = Qf + R_2 f$. The operator P_2 agrees with f in the given 11 vertices of the cube Q: 9 vertices described in part 1) of Lemma 2.2 and two more: $(1,0,1,1)$ and $(1,1,1,1)$. The first operator agrees with f in 10 vertices of the cube Q only: the 9 described in part 1) of Lemma 2.2 and in $(0,0,1,1)$.

Special group of invariances

We will consider the group G of linear transformations of $\mathbb{R}^4$ generated by the following ones:

- translations: $y \mapsto b + y$ where $b, y \in \mathbb{R}^4$;
- dilations: $y = (y_1, y_2, y_3, y_4) \mapsto (t_1 y_1, t_2 y_2, t_3 y_3, t_4 y_4)$ where $t_1, \ldots, t_4 \in \mathbb{R}$;
- permutations $\pi_{1,2}$ and $\pi_{3,4}$ defined by $\pi_{1,2}(x_1, x_2, x_3, x_4) = (x_2, x_1, x_3, x_4)$, and $\pi_{3,4}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_4, x_3)$.

The group G described above will be called the *special group of invariances.*

We will consider also its subgroup $\tilde{G}$ consisting of isometries of $[0,1]^4$. It is generated by the following 6 transformations: 4 symmetries s_i: $x_i \mapsto 1 - x_i$ where $i \in \{1, \ldots, 4\}$ and two permutations: π_{12} and π_{34} (note that every its element is of order 2). It is easy to see that the subgroup H_1 of $\tilde{G}$ generated by two symmetries s_1, s_2, and permutation π_{12} is the normal subgroup of $\tilde{G}$. Moreover, the quotient group $\tilde{G}/H_1$ is isomorphic to

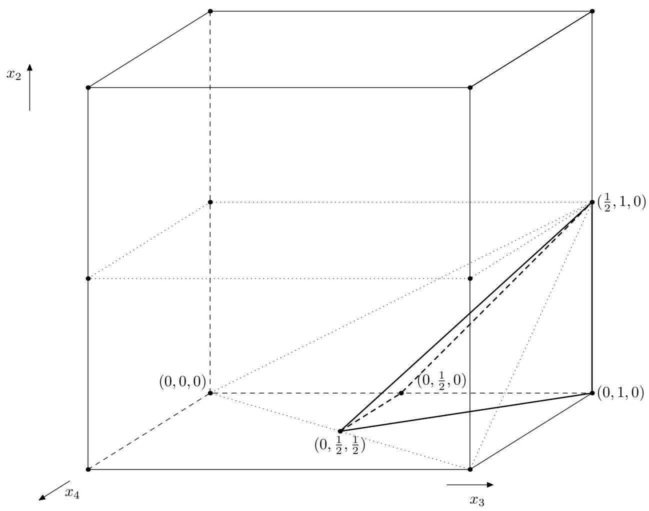

FIGURE 1. $T = R_1 \cap \{x_1 = \frac{1}{2}\} \subset T_3$.

the subgroup H_2 of $\tilde{G}$ generated by symmetries s_3, s_4 and the permutation π_{34}, and both subgroups H_1 and H_2 consist of 8 elements. Then $\tilde{G}$ is isomorphic to $\tilde{G}/H_1 \otimes H_1$, hence it is isomorphic to $H_2 \otimes H_1$ (see *e.g.* [42], Prop. 1 on p. 12). This implies that the group $\tilde{G}$ consists of 64 elements.

The principal symplex and its G-similar cousins

The following symplex in $\mathbb{R}^4$ will play a special role in our development:

$$R_1 = \{x \in Q : x_3 > x_1 + x_2, x_1 \leq 1/2, 2x_1 > x_3 + x_4, x_2 \geq 0, x_4 \geq 0\}. \tag{2.2}$$

The symplex R_1 will be called the *principal* symplex.

Let H be a given group of transformations of $\mathbb{R}^4$ and $D \subseteq \mathbb{R}^4$ be a subset. We will say that the subset $\tilde{D} \subseteq \mathbb{R}^4$ is H-similar to D if there is such $h \in H$ that $\tilde{D} = h(D)$. Let us denote by $Caus(D, H)$ the set of all H-similar to D subsets of $\mathbb{R}^4$. This set will be called the set of H-cousins of D. Let us consider the set $Caus(R_1, G)$ where R_1 is the principal symplex and G is the special group of invariances. One may ask what kind of symlexies will be found there. Obviously, every such symplex is of the form $D \circ Tr(R)$, where R if $\tilde{G}$-cousin of R_1 (note that the set $Caus(R_1, \tilde{G})$ consists of 64 symplexis), Tr is some translation and D is some dilation in $\mathbb{R}^4$. Let us denote vertices of R_1 by: $W_1 = (0,0,0,0)$, $W_2 = (1/2,0,1,0)$, $W_3 = (1/2,0,1/2,0)$, $W_4 = (1/2,0,1/2,1/2)$, and $W_5 = (1/2,1/2,1,0)$ (note that only W_1 is the vertex of the cube $Q = [0,1]^4$). Then R_1 is the convex Minkowski's combination of two sets: the tetrahedron $T \subseteq \mathbb{R}^4$ spanned by W_2, W_3, W_4, W_5, and W_1, which means that $R_1 = \{tx + (1-t)W_1 : t \in [0,1], x \in T\}$.

The tetrahedron T (see Fig. 1) lives on the hyperplane $x_1 = 1/2$ (so $x_1 = const.$) and has three axes perpendicular to each other: W_2W_5 is parallel to e_2, W_2W_3 is parallel to e_3, W_3W_4 is parallel to e_4. Those axes form the polyline in $\mathbb{R}^4$: $W_5W_2W_3W_4$ which uniquely defines the symplex T. Now if we apply the dilation on $\mathbb{R}^4$ we see that the new polyline obtained by this dilation again has three axes parallel to e_2, e_3, e_4 respectively, moreover, the new tetrahedron obtained from T by dilation also lives on the hyperplane $x_1 = const$. Obviously, those mentioned properties: $R_1 = \text{conv}\{W, T\}$, where a) W is one of the vertices of Q and b) T lives on the hyperplane $x_i = const.$ for some i and has three axes perpendicular to each other, remain unchanged under the action of G (where instead of Q we consider its G-cousin).

2.4. **The functional setting**

The special functional

Let $Q = [0,1]^2$ be the unit cube, and $f : \mathbb{R}^4 \to \mathbb{R}$ be continuous. The functional

$$I_f(u) = \int_Q f(u_1(x_1), u_2(x_2), u_3(x_1+x_2), u_4(x_1-x_2)\,\mathrm{d}x_1\mathrm{d}x_2, \tag{2.3}$$

where $u = (u_1, u_2, u_3, u_4)$ and $u_i \in \mathrm{L}^\infty(\mathbb{R})$ for $i = 1, 2, 3, 4$ will play an essential role in our investigations. For this reason such functionals will be called *special.*

We have the following lemma.

Lemma 2.3. *Let $f \in \mathrm{C}^0(\mathbb{R}^4)$, let $\xi_1, \xi_2 \in \mathbb{R}^n$ be two independent vectors and consider the functional*

$$J_{f,\Omega,\xi}(u) = \int_\Omega f(u_1(\langle x, \xi_1\rangle), u_2(\langle x, \xi_2\rangle), u_3(\langle x, \xi_1+\xi_2\rangle), u_4(\langle x, \xi_1-\xi_2\rangle))\,\mathrm{d}x, \tag{2.4}$$

where $\xi = (\xi_1, \xi_2)$, $u = (u_1, u_2, u_3, u_4)$ with $u_i \in \mathrm{L}^\infty(\mathbb{R})$ for $i = 1, \ldots, 4$. The following statements are equivalent:

1) *For an arbitrary bounded domain $\Omega \subseteq \mathbb{R}^n$ with $n \geq 2$ the functional $J_{f,\Omega,\xi}(\cdot)$ given by (2.4) is lower semicontinuous with respect to the weak $*$ convergence of the u_i's in $\mathrm{L}^\infty(\mathbb{R})$.*

2) *The special functional $I_f(\cdot)$ given by (2.3) is lower semicontinuous with respect to the weak $*$ convergence of the u_i's in $\mathrm{L}^\infty(\mathbb{R})$.*

Proof. The implication **1)** $\Rightarrow$ **2)** is obvious. Hence, we prove the implication **2)** $\Rightarrow$ **1)** only. The proof follows by steps: 1) we assume that Ω is a ball. 2) We prove **2)** $\Rightarrow$ **1)** for an arbitrary domain Ω.
Proof of step 1. At first we note that if **2)** holds then Q can be replaced by any cube with edges parallel do the axes. Let us set $y_1 = \langle x, \xi_1\rangle$ and $y_2 = \langle x, \xi_2\rangle$.

We will apply the coarea formula (see *e.g.* Ths. 3.2.12 and 3.2.22 in [24] for its variants):

$$\int_{\mathbb{R}^m} \left(\int_{\Phi^{-1}(y)} \omega(x)\mathcal{H}^{n-m}(\mathrm{d}x) \right) \mathcal{H}^m(\mathrm{d}x) = \int_\Omega \omega(x)|J_m\Phi(x)|\mathrm{d}x, \tag{2.5}$$

whenever $\Omega \subseteq \mathbb{R}^n$ is an open bounded subset, $\Phi : \Omega \to \mathbb{R}^m$ is a lipshitz transformation of variables (in particular $m \leq n$), $J_m\Phi$ is the $[\binom{n}{m}\binom{n}{m}]$-tuple of $m \times m$ minors of the Jacobi matrix of Φ, and ω is an integrable function on Ω.

Applying this with $m = 2$, $\Phi(x) = (\langle x, \xi_1\rangle, \langle x, \xi_2\rangle)$ and

$$\omega(x) = f(u_1(\langle x, \xi_1\rangle), u_2(\langle x, \xi_2\rangle), u_3(\langle x, \xi_1+\xi_2\rangle), u_4(\langle x, \xi_1-\xi_2\rangle)),$$

we observe that

$$\begin{aligned} c = |J_2\Phi(x)| &\text{ does not depend on } x, \\ \Phi^{-1}(y_1,y_2) &= \{x \in \Omega : y_1 = \langle x, \xi_1\rangle \text{ and } y_2 = \langle x, \xi_2\rangle\} := \Omega_{(y_1,y_2)}, \text{ and} \\ \omega(x) &\equiv f(u_1(y_1), u_2(y_2), u_3(y_1+y_2), u_4(y_1-y_2)) \text{ on } \Phi^{-1}(y_1,y_2). \end{aligned}$$

Let us set $\tilde{\Omega} = \{(y_1,y_2) \in \mathbb{R}^2 : \Omega_{(y_1,y_2)} \neq \emptyset\}$. Then by (2.5) we have

$$\begin{aligned} cJ_{f,\Omega,\xi}(u) &= c\int_\Omega \omega(x)\mathrm{d}x \int_\Omega \omega(x)|J_2\Psi(x)|\mathrm{d}x = \\ &= \int_{\tilde{\Omega}} \mathcal{H}^{n-2}(\Omega_{(y_1,y_2)}) f(u_1(y_1), u_2(y_2), u_3(y_1+y_2), u_4(y_1-y_2))\,\mathrm{d}y_1\mathrm{d}y_2 =: J_f(u). \end{aligned}$$

Hence, the functional $J_{f,\Omega,\xi}$ is lower semicontinuous if and only if the functional $J_f(u)$ is lower semicontinuous. Note that under our assumptions the set $\Omega_{(y_1,y_2)}$ is a $(n-2)$-dimensional ball contained in an affine subspace of codimension 2. It is easy to see that the radius of $\Omega_{(y_1,y_2)}$ moves continuously as (y_1, y_2) ranges through the open set $\tilde{\Omega}$. In particular the mapping $\tilde{\Omega} \ni y \mapsto \mathcal{H}^{n-2}(\Omega_{(y)})$ is continuous. Let us cover the set $\tilde{\Omega}$ by countable family of disjoint cubes $Q_i = Q_i(y^i)$ where $y^i = (y_1^i, y_2^i)$ whose edges are parallel to the axes and consider the functional

$$J_f^N(u) = \sum_{i=1}^{N} \mathcal{H}^{n-2}(\Omega_{(y^i)}) \int_{Q_i} f(u_1(y_1), u_2(y_2), u_3(y_1+y_2), u_4(y_1-y_2))\, \mathrm{d}y_1 \mathrm{d}y_2.$$

If the diameter of each Q_i is small enough, then the continuity of f and continuity of the mapping $y \mapsto \mathcal{H}^{n-2}(\Omega_{(y)})$ yields: for all $\varepsilon > 0$ and N sufficiently large

$$|J_f^N(u) - J_f(u)| < \varepsilon$$

for all functions $u \in \mathrm{L}^\infty$ with $\|u\|_\infty \le R$. This and the lower semicontinuity of J_f^N complete the proof of step 1.

Proof of step 2. Let Ω be an arbitrary open, bounded set. Since we deal with bounded sequences, without loss of generality we may assume that $f \ge 0$. Choose a countable family of disjoint, open balls $\{B_h\}_h$ that cover Ω up to a null set. Then, according to the previous step, we have:

$$J_{f,B_h,\xi}(u) \le \liminf_{k\to\infty} J_{f,B_h,\xi}(u^k),$$

for every h, whenever $u^k \rightharpoonup u$ weakly $*$ in L^∞. Hence and using Fatou's lemma, we have

$$\begin{aligned} J_{f,\Omega,\xi}(u) &= \sum_h J_{f,B_h,\xi}(u) \\ &\le \sum_h \liminf_{k\to\infty} J_{f,B_h,\xi}(u^k) \le \liminf_{k\to\infty} \sum_h J_{f,B_h,\xi}(u^k) = \liminf_{k\to\infty} J_{f,\Omega,\xi}(u^k). \end{aligned}$$

This ends the proof of step 2. □

Now we are going to concentrate on obtaining some basic properties of special functionals.

Let $\mathcal{M}$ be the set of all continuous functions on $\mathbb{R}^4$ which define lower semicontinuous special functional, and $\mathcal{C}$ be the space of those integrands which define weakly continuous special functionals (with respect to the sequential weak-$*$ convergence of the u_i's in L^∞).

One can easily check that the following property holds.

Lemma 2.4. *The set $\mathcal{M}$ is invariant with respect to the action of the special group of invariances G (see Sect. 2.3). This means that if $g \in G$ is taken arbitrary and $f \in \mathcal{M}$ then the mapping $f_g(y) = f(gy)$ also belongs to $\mathcal{M}$.*

Remark 2.3. We will see later that $\mathcal{C} = \mathcal{A}$.

The following lemma describes the restriction of special functional to the set of periodic extensions of characteristic functions of cubes in $\mathbb{R}^4$.

Lemma 2.5. *Let $x = (x_1, x_2, x_3, x_4) \in Q = [0,1]^4$, $u_i(\sigma, x) = \hat{\chi}_{[0,x_i]}(\sigma)$ where $\sigma \in \mathbb{R}$, $i \in \{1,2,3,4\}$ and $u : \mathbb{R}^2 \times Q \to \mathbb{R}^4$ be defined by $u(\tau, x) = (u_1(\tau_1), u_2(\tau_2), u_3(\tau_1+\tau_2), u_4(\tau_1-\tau_2))$ where $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2$. Define*

$$\begin{aligned} h_j(x) = |\{(\tau_1,\tau_2) \in [0,1]^2 \;:\; &\tau_1 \in [0,x_1]^{j_1}, \tau_2 \in [0,x_2]^{j_2}, \\ &[\tau_1+\tau_2]_1 \in [0,x_3]^{j_3}, [\tau_1-\tau_2]_1 \in [0,x_4]^{j_4}\}| \end{aligned}$$

where $j = (j_1, j_2, j_3, j_4) \in \{0,1\}^4$ and $[0,s]^1 = [0,s]$, $[0,s]^0 = (s,1]$.

Then the following properties hold.

i) *For every continuous function* $f : \mathbb{R}^4 \to \mathbb{R}$ *we have*

$$\int_{[0,1]^2} f(u(\tau, x))\, \mathrm{d}\tau = \sum_{j\in\{0,1\}^4} h_j(x) f(j).$$

ii) *Functions* $\{h_j(x)\}$ *define the distribution of the probability measure concentrated in 16 points* $j \in \{0,1\}^4$, *in particular all the* h_j*'s are nonnegative.*

iii) *For every* $j \in \{0,1\}^4$ *the mapping* $x \mapsto h_j(x)$ *is continuous.*

Some further properties of functions $\{h_j(x)\}$ and the computation of their values in selected subregions of the cube Q will be presented in Section 3 and in the Appendix. From now functions $\{h_j(x)\}$ and their shiftings $\{h^i_j(x)\} = \{\Pi_i h(x)\}_j$ will be called *special distributions of measures* (note that the shiftings also define distributions of probability measures).

The key point in our argumentation will be the following lemma.

Lemma 2.6. *Assume that* $f \in \mathrm{C}^0(\mathbb{R}^4)$ *and* f *defines the special functional, which is lower semicontinuous with respect to the weak-∗ convergence of the* u_i*'s in* $\mathrm{L}^\infty(\Omega)$*. Then if* u_1, u_2, u_3, u_4 *are bounded and periodic of period* 1*, we have*

$$\int_{[0,1]^2} f(u_1(\tau_1), u_2(\tau_2), u_3(\tau_1+\tau_2), u_4(\tau_1-\tau_2))\mathrm{d}\tau_1\mathrm{d}\tau_2 \geq f\left(\int_0^1 u_1(\tau)\mathrm{d}\tau, \ldots, \int_0^1 u_4(\tau)\mathrm{d}\tau\right).$$

Proof. This follows from the Riemann–Lebesgue's Theorem applied to functions $u^\nu(x) = u(\nu x)$, where $u = (u_1, u_2, u_3, u_4)$ and $f^\nu(x) = f(u^\nu(x))$ (see *e.g.* Lem. 1.2 on p. 8 in [16]). It only remains to check that $\int_{[0,1]^2} u_3(\tau_1+\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 = \int_0^1 u_3(\tau)\mathrm{d}\tau$ and $\int_{[0,1]^2} u_4(\tau_1-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 = \int_0^1 u_4(\tau)\mathrm{d}\tau$. □

3. Special distributions of measures

3.1. Invariances

Through this section we assume that $Q = [0,1]^4$ is the unit cube and h_j's are the same as in Lemma 2.5. Let $\{h^i_j(x)\}_{j\in\{0,1\}^3} = \{\Pi_i h(x)\}_{j\in\{0,1\}^3}$. Note that whereas formally $\{h\}$ as well as $\{\Pi_i h\}$ depend on all four coordinates $x_1, \ldots, x_4$, but every component h^i_j of $\{\Pi_i h\}$ depends on at most three coordinates among $x_1, \ldots, x_4$, namely those x_k with $k \neq i$. Therefore we will treat the mapping $x \mapsto \{h^i_j(x)\}_{j\in\{0,1\}^3}$ as the function defined on $[0,1]^3$. On the other hand in some places we will denote coordinates of $x \in [0,1]^3$ in $h^i_j(x)$ in the following way: in $h^1_j(x)$ by (x_2, x_3, x_4), in $h^2_j(x)$ by (x_1, x_3, x_4), in $h^3_j(x)$ by (x_1, x_2, x_4), in $h^4_j(x)$ by (x_1, x_2, x_3) to indicate that in fact h^i_j's are functions of four variables, but we forget about the given one. The choice of the notation will be obvious from the context.

Let $k \in \mathbb{N}$ and $A : \mathbb{R}^k \to \mathbb{R}^k$ be an arbitrary affine transformation such that A restricted to $\{0,1\}^k$ is a bijection. Then A preserves the whole cube $Q = [0,1]^k$ and A defines the mapping $\mathcal{S}^k(Q) \to \mathcal{S}^k(Q)$ by expression

$$A\{h_j(x)\}_{j\in\{0,1\}^k} = \{h_{Aj}(Ax)\}_{j\in\{0,1\}^k}. \tag{3.1}$$

Our goal is to look for those affine transformations of $\mathbb{R}^3$ and $\mathbb{R}^4$ which leave the distributions $\{h^i_j(x)\}_{j\in\{0,1\}^3}$ and $\{h_j(x)\}_{j\in\{0,1\}^4}$ invariant. We start with the following lemma.

Lemma 3.1 (Lemma about invariances). *The distributions $\{h^i_j(x)\}_{j\in\{0,1\}^3}$'s and $\{h_j(x)\}_{j\in\{0,1\}^4}$ are invariant with respect to the following isometries of $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively.*

1) $\{h^4_j(x)\}_{j\in\{0,1\}^3}$ *is invariant under the permutation of axes* $\pi_{12}(x_1,x_2,x_3)=(x_2,x_1,x_3)$, *and symmetry with respect to the point* $(1/2,1/2,1/2)$, $S(x_1,x_2,x_3)=(1-x_1,1-x_2,1-x_3)$.

2) $\{h^1_j(x)\}_{j\in\{0,1\}^3}$ *is invariant with respect to isometries of* $\mathbb{R}^3$, $B_1,B_2:\mathbb{R}^3\to\mathbb{R}^3$ *given by* $B_1(x_1,x_2,x_3)=(x_1,1-x_3,1-x_2)$, $B_2(x_1,x_2,x_3)=(1-x_1,x_3,x_2)$.

3) *Let* $C_3:\mathbb{R}^3\to\mathbb{R}^3$ *be the affine isometry given by* $C_3(x_1,x_2,x_3)=(x_1,1-x_2,x_3)$. *Then for every* $x\in[0,1]^3$ *and every* $j\in\{0,1\}^3$ *we have*

$$h^3_j(x)=h^4_{C_3j}(C_3x).$$

In particular $\{h^3_j(x)\}_{j\in\{0,1\}^3}$ *is invariant with respect to mappings*

$$D_1(x_1,x_2,x_3)=C_3\pi_{12}C_3(x_1,x_2,x_3)=(1-x_2,1-x_1,x_3)\text{ and }S.$$

4) *Let* $C_1:\mathbb{R}^3\to\mathbb{R}^3$ *be an affine isometry given by* $C_1(x_1,x_2,x_3)=(x_1,x_2,1-x_3)$. *Then for every* $x\in[0,1]^3$ *and every* $j\in\{0,1\}^3$ *we have*

$$h^2_j(x)=h^1_{C_1j}(C_1x).$$

In particular $\{h^2_j(x)\}_{j\in\{0,1\}^3}$ *is invariant with respect to the following permutation of axes* $\pi_{23}(x_1,x_2,x_3)=(x_1,x_3,x_2)$ *and the mapping* $S\circ\pi_{23}$.

5) $\{h_j(x)\}_{j\in\{0,1\}^4}$ *is invariant with respect to orientation preserving isometries of* $\mathbb{R}^4$ *given by*

$$A_1(x)=(1-x_1,x_2,1-x_4,1-x_3),\quad A_2(x)=(x_2,x_1,x_3,1-x_4),$$

where $x=(x_1,x_2,x_3,x_4)$.

Proof. This follows from the following changes of variables in the calculation of $\{h_j\}$'s: 1) $\overline{\tau_1}=\tau_2$, $\overline{\tau_2}=\tau_1$, and $\overline{\tau_1}=1-\tau_1$, $\overline{\tau_2}=1-\tau_2$; 2) $\overline{\tau}_1=1-\tau_1$, $\overline{\tau}_2=\tau_2$, and $\overline{\tau}_1=\tau_1$, $\overline{\tau}_2=1-\tau_2$; 3) $\overline{\tau}_1=\tau_1$, $\overline{\tau}_2=1-\tau_2$; 4) $\overline{\tau}_1=\tau_2$, $\overline{\tau}_2=\tau_1$; 5) $\overline{\tau}_1=1-\tau_1$, $\overline{\tau}_2=\tau_2$ and $\overline{\tau}_1=\tau_2$, $\overline{\tau}_2=\tau_1$. $\square$

Let us introduce the following definition.

Definition 3.1. We will say that the set $D\subseteq[0,1]^4$ is $\mathcal{A}$-regular if there is a decomposition $D=\cup_{i=1}^r D_i$ for some $r\in\mathbb{N}$, where for every $i\in\{1,\dots,r\}$ the set D_i is connected and the mapping $D_i\ni x\mapsto h_j(x)$ is represented by an element of $\mathcal{A}$ for every $j\in\{0,1\}^4$. Sets D_i will be called regular components of D.

We will say that $D\subseteq[0,1]^4$ is maximal $\mathcal{A}$-regular subset of $[0,1]^4$ if D is $\mathcal{A}$-regular, and an arbitrary $\mathcal{A}$-regular subset of $[0,1]^4$ is contained in D.

We are in the position to formulate the following result.

Lemma 3.2. *The maximal $\mathcal{A}$-regular subset of $Q=[0,1]^4$ equals $R=\cup_{i=1}^8 R_i$, where symplexes R_i are regular components of R, R_1 is the principal symplex given by (2.2) $R_2,\dots,R_8$ are obtained by the condition $R_j=E_j(R_1)$ where E_j's are isometries given by $E_1=id$, $E_2=A_1$, $E_3=A_2$, $E_4=A_2\circ A_1$, $E_5=(A_1\circ A_2)^2$,*

$E_6 = (A_1 \circ A_2)^2 \circ A_1$, $E_7 = A_1 \circ A_2 \circ A_1$, $E_8 = A_1 \circ A_2$, and A_1 and A_2 are the same as in Lemma 3.1. The functions $h_j(x)$ on the set R_1 are described in the table below.

$(0,0,0,0)$	$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,1,0)$
$1-(x_1+x_2+x_3$ $+x_4)+x_1x_3+x_1x_4$ $+x_2x_3+1/2x_3x_4$	$x_1-x_1x_3$ $-x_1x_4+\frac{1}{2}x_3x_4$	$x_2(1-x_3)$	$x_3-x_1x_3-x_2x_3$ $-\frac{1}{2}x_3x_4+x_1x_2$
$(0,0,0,1)$	$(1,1,0,0)$	$(1,0,1,0)$	$(1,0,0,1)$
$x_4-x_1x_4$ $-\frac{1}{2}x_3x_4$	0	$x_1x_3-x_1x_2$ $-\frac{1}{2}x_3x_4+x_2x_4$	$x_1x_4-\frac{1}{2}x_3x_4$
$(0,1,1,0)$	$(0,1,0,1)$	$(0,0,1,1)$	$(1,1,1,0)$
$x_2x_3-x_1x_2$	0	$\frac{1}{2}x_3x_4$	$x_1x_2-x_2x_4$
$(0,1,1,1)$	$(1,0,1,1)$	$(1,1,0,1)$	$(1,1,1,1)$
0	$\frac{1}{2}x_3x_4-x_2x_4$	0	x_2x_4

Functions h_j defined on R_1.

The functions $h_j(x)$ for $x \in R_i$ and $i \in \{2,\dots,8\}$ are obtained from those on R_1 according to the rule given in Lemma 3.1: $h_j(x) = h_{E_i j}(E_i x)$.

Proof of the above lemma is given in the Appendix.

4. The main result

We start with the following lemma.

Lemma 4.1. *Let us define the following operator*

$$\mathrm{C}^0(\mathbb{R}^4) \ni f \mapsto \sum_{j\in\{0,1\}^4} \overline{h}_j(x) f(j) \in \mathrm{C}^0(\mathbb{R}^4), \tag{4.1}$$

where $\overline{h}_j = h_j|_{R_1}$, h_j's are described by Lemma 3.2, and R_1 is the principal symplex given by (2.2). Then the operator described above is the projection operator onto the space $\mathcal{A}$, and agrees with the operator Pf described by (2.1).

Proof. The fact that the range of the operator given by (4.1) is contained in the space $\mathcal{A}$ follows directly from Lemma 3.2. Now it suffices to describe the operator given by (4.1) in the basis of space $\mathcal{A}$. □

Remark 4.1. Note that for $j = (1,1,1,1)$ we have $\overline{h}_{1111}(1,1,1,1) = 1$ but not all other coefficients $\overline{h}_j(1,1,1,1)$ are equal to zero. In particular $Pf(1,1,1,1) \neq f(1,1,1,1)$.

The following lemma will be crucial to obtain our main results.

Lemma 4.2. *Let G be the special group of invariances, R_1 be the principal symplex (see Sect. 2.3), and assume that the symplex $R \subseteq \mathbb{R}^4$ is G-similar to R_1, that is $R = g(R_1)$ for some $g \in G$. Suppose that f defines a special functional which is lower semicontinuous with respect to the weak-$*$ convergence of the u_i's in L^∞. Then for every $x \in R$ we have*

$$f(x) \leq (g^* Pf)(x),$$

where $g^*Pf(x) = \sum_{j\in\{0,1\}^4} \overline{h}_j(g^{-1}x) f(gj)$, P is the same as in (2.1), $\overline{h}_j = h_j|_{R_1}$, h_j's are described by Lemma 3.2.

Proof. The proof follows by steps: 1) we obtain the result for $R = R_1$ and 2) we complete the proof of the theorem.

Proof of step 1. Let $x \in R_1$ and $u(\tau, x)$ be the same as in Lemma 2.5. According to Lemma 2.6 we have

$$f(x) \leq \int_{[0,1]^2} f(u(\tau, x))\mathrm{d}\tau.$$

Using Lemmas 2.5 and 3.2 we see that the right hand side equals $\sum_{j\in\{0,1\}^4} \overline{h}_j(x)f(j)$.

Proof of step 2. Let $f_g(y) = f(gy)$. By Lemma 2.4 and by step 1 applied to f_g we get $f_g(y) \leq Pf_g(y)$ for every $y \in R_1$. Now it suffices to substitute $x = gy \in R$. $\square$

Remark 4.2. Let $\tilde{G} \subseteq G$ be the subgroup of G of isometries of $[0,1]^4$ introduced in Section 2.3. It consists of 64 elements, let us denote them by $g_1, g_2, \ldots, g_{64}$, where $g_k = E_k$, for $k = 1, \ldots, 8$, E_k's are the same as in Lemma 3.2, and $g_9, \ldots, g_{64}$ are the remaining elements of $\tilde{G}$ written in an arbitrary order. Define $R_i = g_i(R_1)$ for $i = 1, \ldots, 64$. According to Lemma 4.2 the inequality $f(x) \leq (g_i^* Pf)(x)$ holds for every $x \in R_i$ and for every $i \in \{1, \ldots, 64\}$. Note that Lemma 3.2 guarantees such inequality only for $i = 1, \ldots, 8$, as coefficients $h_j(x)$ belong to the space $\mathcal{A}$ only for x being in symplexis $R_1, \ldots, R_8$. In particular the application of Lemma 2.4 gives stronger result than that which follows directly from Lemmas 3.2 and 2.5.

As a corollary we obtain the following result.

Corollary 4.1. *If we assume additionally in Lemma 4.2 that $f \in \mathcal{C}$ then we have $f(x) = (Pf)(x)$ for every $x \in \mathbb{R}^4$, and consequently the space $\mathcal{C}$ agrees with $\mathcal{A}$.*

First proof of Corollary 4.1. At first we note that for arbitrary open and bounded set $U \subseteq \mathbb{R}^4$ we will find $g \in G$ such that $U \subset g(R_1)$. Thus according to Lemma 4.2 applied to f and $-f$ we see that f agrees in U with a certain polynomial belonging to the space $\mathcal{A}$. As two polynomials which agree on an open subset of $\mathbb{R}^n$ must be the same, we see that $f \in \mathcal{A}$. This implies $\mathcal{C} \subseteq \mathcal{A}$. The reverse inclusion is obtained by direct computation. $\square$

The above result could also be obtained as the consequence of the following known result due to Murat and Tartar (see *e.g.* [16], Th. 3.3 on p. 27, [53], Th. 5.1, [80], Th. 18).

Theorem 4.1. *Assume that $f : \mathbb{R}^m \to \mathbb{R}$ defines a weakly continuous functional I_f given by (1.4). Then f is a polynomial of degree* $\min\{n, m\}$*, moreover, the following property is satisfied:*

$$\begin{cases} \text{Given arbitrary } r \in \mathbb{N} \text{ and } (\xi_1, \lambda_1), \ldots, (\xi_r, \lambda_r) \in V \text{ such that} \\ \quad \operatorname{rank}\{\xi_1, \ldots, \xi_r\} \leq r - 1, \text{ we have for arbitrary } s \in \mathbb{R}^m \\ \quad\quad f^{(r)}(s)(\lambda_1, \ldots, \lambda_r) = 0. \end{cases} \tag{4.2}$$

Second proof of Corollary 4.1. It reminds to note that in our case we have $\Lambda = \cup_{i=1}^4 \mathrm{span}\{e_i\} \subseteq \mathbb{R}^4$, and $V \subseteq \mathbb{R}^2 \times \mathbb{R}^4$. Hence $\mathcal{C} \subseteq \mathcal{A}$, while the reverse inclusion is obtained by direct computation. $\square$

Let us introduce the following definition.

Definition 4.1. Let $P : \mathrm{C}^0(\mathbb{R}^4) \to \mathcal{A}$ be the projection operator defined by (2.1), R_1 be the principal symplex given by (2.2) and G be the special group of invariances. We will say that $f \in \mathrm{C}^0(\mathbb{R}^4)$ is a *sub* $\mathcal{A}$-function with respect to the projection operator P on G-similar sets to R_1 (in short *sub*$(\mathcal{A}, P, R_1, G)$) if the inequality

$$f(x) \leq (g^* Pf)(x)$$

holds for every $g \in G$ and every $x \in g(R_1)$ where $(g^* Pf)(x) = \sum_{j\in\{0,1\}^4} h_j(g^{-1}x)f(gj)$.

Remark 4.3. In other words the above definition means that the inequality $f(x) \leq (Pf)(x)$ is satisfied for every $x \in R_1$ and the same inequality is satisfied for f_g, for every $g \in G$.

Remark 4.4. Note that the above definition is related to the classical definition of convexity. Let us take instead of $\mathcal{A}$ the space of affine functions on $\mathbb{R}$ and denote it by A. Take $R = [0, 1]$, and let P be the projection operator from $\mathrm{C}^0(\mathbb{R})$ to A, given by

$$Pf(x) = (1 - x)f(0) + xf(1) := h_1(x)f(0) + h_2(x)f(1),$$

and consider the group G_1 of transformations of $\mathbb{R}$ generated by translations $x \mapsto a + x$ and dilations $x \mapsto bx$. Then one could define in analogous way the class of *sub* A-functions with respect to the projection operator P on G_1-similar sets to R by saying that $f \in sub(A, P, R, G_1)$ if the inequality

$$f(x) \leq (h_1(g^{-1}(x))f(g(0)) + h_2(g^{-1}(x))f(g(1))$$

holds for every $g \in G_1$ and $x \in g(R)$ where $g \in G_1$. As an arbitrary $g \in G_1$ is of the form: $g(t) = a + bt$, the above inequality is equivalent to the inequality:

$$f((1 - t)a + t(a + b)) \leq (1 - t)f(a) + tf(a + b),$$

which holds for arbitrary $a, b \in \mathbb{R}$, and $t \in [0, 1]$. But this is nothing else than convexity.

Now we are in the position to state our main result.

Theorem 4.2. *Assume that $f \in \mathrm{C}^0(\mathbb{R}^m)$ defines a weakly lower semicontinuous functional I_f given by (1.4), V and Λ are the same as in Theorem 1.1. Then f satisfies the following condition:*
Let $(\xi_1, \lambda_1), (\xi_2, \lambda_2), (\xi_3, \lambda_3), (\xi_4, \lambda_4) \in V$ are such that λ_i's are linearly independent, ξ_1 and ξ_2 are linearly independent, $\xi_3 = \xi_1 + \xi_2$ and $\xi_4 = \xi_1 - \xi_2$. Then for arbitrary $A \in \mathbb{R}^m$ the mapping

$$\mathbb{R}^4 \ni x = (x_1, x_2, x_3, x_4) \mapsto \tilde{f}(x) = f\left(A + \sum_{i=1}^{4} x_i \lambda_i\right)$$

is a $sub(\mathcal{A}, P, R_1, G)$ function.

In the case when f defines a weakly continuous functional I_f the described mapping above belongs to the space $\mathcal{A}$.

Proof. It suffices to note that for arbitrary $u_i \in \mathrm{L}^\infty(\mathbb{R})$ $(i = 1, 2, 3, 4)$ the set $\{A + \sum_{i=1}^4 u_i(< x, \xi_i >)\lambda_i\}$ is the subset of $\ker P$. In particular $\tilde{f}$ defines special functional which is lower semicontinuous with respect to the weak-$*$ convergence of the u_i's in L^∞. $\square$

Remark 4.5. Obviously, the second statement can be deduced directly from Theorem 4.1. A similar result for $\tilde{f}$ depending on three variables only is given in Theorem 3.2 in [34].

Our next results apply directly to the variational case. As the consequence of Theorem 4.2 we obtain the following theorem.

Theorem 4.3. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, and $f \in \mathrm{C}^0(\mathbb{R}^{n\times m})$ defines a sequentially weakly-$*$ lower semicontinuous functional on the Sobolev space $W^{1,\infty}(\Omega, \mathbb{R}^m)$: $\tilde{I}_f(w) = \int_\Omega f(\nabla w(x))\mathrm{d}x$, where $w : \Omega \to \mathbb{R}^m$. Let us take $A \in \mathbb{R}^{n\times m}$, two independent vectors: $\xi_1, \xi_2 \in \mathbb{R}^n$, and $a_1, a_2, a_3, a_4 \in \mathbb{R}^m$ and define $\lambda_i = \xi_i \otimes a_i$ for $i \in \{1, 2\}$ and $\lambda_3 = (\xi_1 + \xi_2) \otimes a_3$, $\lambda_4 = (\xi_1 - \xi_2) \otimes a_4$, where $\xi \otimes a = (\xi_i a_j)_{i=1,\dots,n, j=1,\dots,m} \in \mathbb{R}^{n\times m}$. Assume that all λ_i's are linearly independent. Then the mapping*

$$\mathbb{R}^4 \ni x = (x_1, \dots, x_4) \mapsto \tilde{f}(x) = f\left(A + \sum_{i=1}^{4} x_i \lambda_i\right)$$

is a $sub(\mathcal{A}, P, R_1, G)$ function.

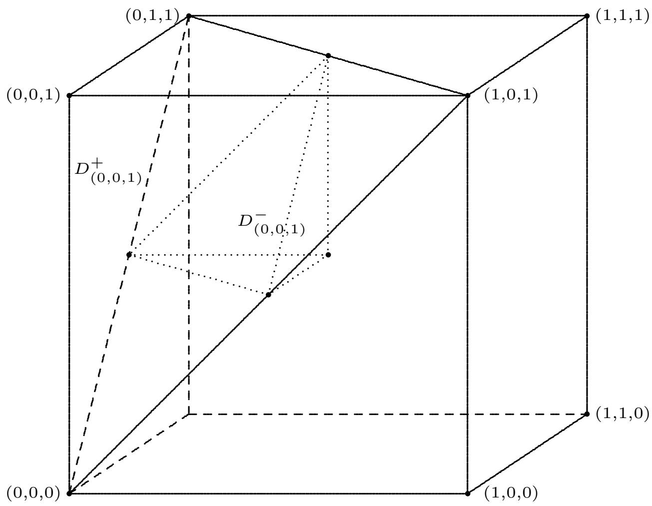

FIGURE 2. $D^+_{(0,0,1)}$ and $D^-_{(0,0,1)}$.

Proof. In this case the manifold V consists of pairs $(\xi, \xi \otimes a)$ where $\xi \in \mathbb{R}^n \setminus \{0\}$ and $a \in \mathbb{R}^m$. Hence the result follows directly from Theorem 4.2. □

Remark 4.6. Continuity assumptions can be violated in all our statements. It suffices to assume that f is Borel measurable and locally bounded. The argument is the following. Assume at first that $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, locally bounded and defines the lower semicontinuous functional $I_f(u) = \int_{[0,1]} f(u(x))\mathrm{d}x$ defined on the class of measurable and bounded functions $u : \mathbb{R} \to \mathbb{R}$. Let $u(x)$ be the periodic extension of $\chi_{[0,t]}A + \chi_{[t,1]}B$ where $t \in (0,1)$ and $A, B \in \mathbb{R}$ are taken arbitrary, and let $u^\nu(x) = u(\nu x)$ where $\nu \in \mathbb{N}$. If we apply lower semicontinuity assumption to this sequence we see (the argument follows directly from Riemann–Lebesgue's Theorem, see *e.g.* Lem. 1.2 in [16]) that f must be convex, so also continuous. This argument can be used in the proof of Theorem 1.1, where one restricts the general functional I_f to the subspace in ker P given by $\{A + v(< x, \xi >)\lambda, v \in \mathrm{L}^\infty\}$ where $(\xi, \lambda) \in V$. In particular, every Borel measurable and bounded integrand which defines a lower semicontinuous functional given by (1.4) must be Λ-convex, so also continuous in all Λ-directions. In particular if Λ spans all of $\mathbb{R}^m$ then f must be continuous. In our arguments we study the behaviour of f along subspaces spanned by Λ only (starting from an arbitrary point in $\mathbb{R}^m$), so continuity assumption is satisfied there. This remark was presented to us by Pietro Celada.

5. Final conclusions and remarks

There are a series of questions and remarks naturally arising after following these results. We state them below with the hope that the proposed research program will contribute to further development of this research field.

Three dimensional conditions

Remark 5.1. Let D be an arbitrary tetrahedron in $\mathbb{R}^3$ with three edges parallel to the axes. It is obtained by translation and dilation I of the cutted corner of the standard cube $Q = [0,1]^3$ (see Fig. 2).

Let us denote $D = I(D^+_{(0,0,1)})$, where $D^+_{(0,0,1)} = \mathrm{conv}\{(0,0,1), (0,1,1), (1,0,1), (0,0,0)\}$. Let $P^+_D f$ be the only polynomial from 7 dimensional space of polynomials $\mathrm{span}\{1, x_i, x_i x_j : i,j \in \{1,2,3\}, i \neq j\}$, which equals with f in 7 corners of the cube $\tilde{Q} = I(Q)$, all except $I(1,1,0)$.

Let $f \in \mathrm{C}^0(\mathbb{R}^3)$ and consider the following functional

$$I_f(u) = \int_{[0,1]^2} f(u_1(x_1), u_2(x_2), u_3(x_1+x_2))\mathrm{d}x_1\mathrm{d}x_2, \tag{5.1}$$

where $u_i \in \mathrm{L}^\infty(\mathbb{R})$.

It was proved in [34] that if the functional I_f is lower semicontinuous then the following inequality

$$f(x) \le P_D^+ f(x) \tag{5.2}$$

holds for every $x \in D$.

Obviously, such an inequality implies analogous inequalities for integrands defining lower semicontinuous functionals having the general form (1.4), satisfied on three dimensional symplexes, see Theorem 3.2 in [34]. We do not know if it is possible to obtain three dimensional conditions analogous to (5.2) as the direct consequence of our new four-dimensional conditions obtained in Lemma 4.2 and Theorem 4.2.

Remark 5.2. Three dimensional conditions (5.2) obtained in Theorem 3.1 in [34] were satisfied on all cutted corners of the standard cube $[0,1]^3$ (as for example $D^+_{(0,0,1)}$ under notation of Rem. 5.1). Our new four dimensional conditions obtained in Lemma 4.2 are satisfied on R_1 and his $\tilde{G}$ similar cousins. None of this symplexes is the cutted corner of the standard cube in $\mathbb{R}^4$.

Remark 5.3. Assume that $f \in \mathrm{C}^0(\mathbb{R}^3)$ defines a lower semicontinuous functional of the general form (1.3). The calculations in Section 3.4 suggest that there is a whole family of new conditions of the form $f(x) \le Pf(x)$, where P is a certain projection operator onto the space of sufficiently good polynomials, which are satisfied on three dimensional symplexes. We can easily deduce such one's with similar but different statements to (5.2) on symplexes obtained by transformations of T_2 in Lemma 6.6 (see the Appendix). This shows that even three dimensional conditions still require further investigations.

Perspectives for further geometric conditions

Remark 5.4. We think that if one considers functionals of the form

$$I_f(u) = \int_\Omega f(u_1(A_1x), \ldots, u_m(A_mx))\mathrm{d}x \tag{5.3}$$

where $u_i : \mathbb{R} \to \mathbb{R}$ and $A_i : \mathbb{R}^n \to \mathbb{R}$ are some linear operators then additional geometric conditions will appear. These conditions will be satisfied on symplexes with the specially prescribed geometry and probably will have a form of inequality between f on the symplex and some projection operator onto the space of weakly continuous functionals given by (5.3) like in Lemma 4.2. The relation between the structure of the A_i's, the geometry of these symplexes and the projection operators in this relation requires systematic study. Such issue should lead to the generalization of Theorems 4.2, 4.3 and Theorem 3.2 in [34].

New conditions and CW-structures

Remark 5.5. Note that the new geometric conditions have CW-structural nature: they are given inside symplexes, and on their lower order skeleton's. For example Λ-convexity condition is the condition given on the subset of the 1-skeleton of the symplex.

Some other estimates

Remark 5.6. In [34] also other conditions were obtained. It was shown in Theorem 3.2 there that under notations of Remark 5.1 we have for every $x \in D^- = I(D^-_{(0,0,1)})$

$$f(x) \le (P_D^- f)(x), \tag{5.4}$$

where $D^-_{(0,0,1)} = \mathrm{conv}\{(1/2,1/2,1/2),\ (1/2,0,1/2),\ (0,1/2,1/2),\ (1/2,1/2,1)\}$, and $P^-_D f(x)$ is the disturbation of $P^+_D f(x)$: $P^-_D f(x) = P^+_D f(x) + Rf(x)$, where $Rf(x) = \mathrm{dist}(x,D)^2 \cdot \Delta^3 f$, $\mathrm{dist}(x,D)$ is the distance function, and $\Delta^3 f$ is certain combination of values of f in corners of $\tilde{Q}$ (the exact definition is given in [34]). In particular $P^-_D f(x)$ does not define weakly continuous special functional of the form (5.1), as it was the case of $P^+_D f(x)$. It would be interesting to know if one may obtain (5.4) as the consequence of inequalities like (5.2).

One can also obtain variants of (5.4) on four dimensional symplexes (they cannot be relatives to the nobel R_1), with regular term disturbed by square roots of distances from walls of those symplexes. We have seen this in our calculations when writing this paper, but because of technical difficulties we did not take care about their exact form.

Geometric conditions and elliptic systems

Remark 5.7. Let us look at the inequality in Definition 4.1: $f(x) \le (g^*Pf)(x)$ more closely. Its right hand side, $h = g^*Pf$, is the solution to the elliptic system: $\frac{\partial^2}{\partial x_i^2} h = 0$ for $i \in \{1,2,3,4\}$, $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} h = 0$ for different i,j,k, and h agrees with f on certain given set (related to $g(R_1)$, it consists of corners of $g(Q)$ where Q is the standard cube). Inequality holds in the certain set of dimension 4. We think we have to do with a kind of subsolutions to elliptic systems. Recall that if one deals with a single elliptic equation then the subsolution has the property that if it coincides with the solution to this equation on the boundary of the sufficiently regular set Ω then inside Ω it is less or equal than this solution (see *e.g.* [26,28]). Here we deal with similar property. We refer *e.g.* to [2,3,12,13,27,44,47,49], and their references for various maximum principles for solutions of linear and nonlinear elliptic systems.

Numerical treatment and Morrey's conjecture

Remark 5.8. In the literature there are several of examples of functions which are known to be rank-one convex but it is not confirmed whether they are quasiconvex (see *e.g.* [1,10,18,19,29,75]). Now one could check at least numerically if those functions satisfy the new three and four dimensional geometric conditions obtained in [34] and in Theorem 4.2. The numerical verification of three dimensional conditions was done in [74].

Šverák's example and locality

Remark 5.9. It was shown in [34] that the rank-one convex function which is not quasiconvex constructed by Šverák in his famous paper [77] does not satisfy the new geometric conditions on three dimensional symplexis (see Rem. 4.1 in [34]). It is possible to verify that also four dimensional conditions of our Theorem 4.3 cannot be satisfied by this function either. The sketch of the proof is the following. If we relax the dependence of f on x_4 in Lemma 4.2 we arrive at the same inequality as (5.2) but on a subsymplex of D. Now we can use the same arguments as in Remark 4.1 in [34] and show that Šverák's function (after the slight modification to make it strongly rank-one convex) does not satisfy this three dimensional condition. Thus it cannot satisfy the four dimensional conditions either. This shows also that our four dimensional conditions cannot be local: there exists strongly rank-one convex function which is not quasiconvex and does not satisfy this condition. But as was shown by Kristensen (see [39] for details) every strongly rank-one convex function agrees with quasiconvex functions on balls covering $\mathbb{R}^{n\times m}$. As each of them satisfies four dimensional conditions, it follows that our conditions restricted to functions defined on $n \times m$ matrices where $n \ge 2$ and $m \ge 3$ cannot be local (see also Rem. 4.5 in [34]).

New conditions and null-Lagrangians

Remark 5.10. It is known that if $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ defines weakly continuous variational functional given by (1.1) then f is a null-lagrangian, that is f belongs to the linear space spanned by $\{1, \lambda_{ij}, m_{I,J}\}$ where $m_{I,J}$ are minors of matrices in $\mathbb{R}^{n\times m}$ (see *e.g.* [5,6,11,16,17,23,80,81]). In particular if $n = m = 2$ the space of null-lagrangians is 6 dimensional. As the determinant function is affine along every rank-one direction and is the polynomial of degree 2, we see that under the notation of Theorem 4.3 the function $\tilde{f}$ belongs to $\mathcal{A}$ for arbitrary four rank-one matrices $\lambda_1, \dots, \lambda_4$ like in Theorem 4.3. Since $\mathcal{A}$ is 11 dimensional space, we do not think we can expect that for an arbitrary quasiconvex function $\tilde{f}$ the mapping $P\tilde{f}$ will be a null-largangian. It would be interesting to understand the relations between projections $P\tilde{f}$ of quasiconvex integrands and null-lagrangians. We refer *e.g.* to [30–32, 45, 55, 71, 85] and their references for selected results on null-lagrangians.

Are our conditions sufficient for lower semicontinuity?

Remark 5.11. Let us consider the variational case. If we deal with gradients of scalar functions of two variables, that is $m = 1$ under the notation of (1.1) then quasiconvexity condition is equivalent to the usual convexity. It would be interesting to know what is the answer on two related questions:

1. Is it true that if we deal with second gradients of scalar functions of two variables, that is $\nabla u \in \mathbb{R}^{2\times 2}$ and ∇u is symmetric, then rank-one tetrahedral convexity condition (the condition on three dimensional symplexes) of Theorem 4.1 in [34] is equivalent to quasiconvexity?
2. Is it true that in the case $n = m = 2$ (gradients are 2×2 matrices) our new geometric necessary conditions on four dimensional symplexes are also sufficient for quasiconvexity?

For answers on both questions it would be helpful to know if our new convexity conditions on three and four dimensional symplexes from Theorem 3.1 in [34] and Lemma 4.2 are also sufficient for lower semicontinuity of functionals $I_f(u) = \int_{[0,1]^2} f(u_1(\tau_1), u_2(\tau_2), u_3(\tau_1 + \tau_2))\mathrm{d}\tau$ and $I_f(u) = \int_{[0,1]^2} f(u_1(\tau_1),\ u_2(\tau_2), u_3(\tau_1 + \tau_2), u_4(\tau_1 - \tau_2))\mathrm{d}\tau$ respectively.

Hulls, envelops and other remarks

Remark 5.12. Let $f \geq 0$ and f^{qc} and f^{rc} be the quasiconvex and rank-one convex envelop of f, that is the largest quasiconvex and rank-one convex minorant of f respectively. By the Fundamental Relaxation Theorem we know that the minimum value of I_f given by (4.3) is the same as the minimum of $I_{f^{qc}}$ in the class $\{u \in W^{1,\infty}(\Omega, \mathbb{R}^m) : u \equiv Fx \text{ on } \partial\Omega\}$ (see *e.g.* [16, 72]). It is not known how to compute in general the quasiconvex envelop of f (see *e.g.* [36, 37, 43, 70] for some of the very few results), but as quasiconvexity implies rank-one convexity we have $f^{qc} \leq f^{rc} \leq f$. Hence the minimum of I_f is the same as the minimum of $I_{f^{rc}}$ and it becomes important to be able to compute the rank-one convex envelop of f when one looks for minimum of I_f. Now we can introduce envelopes of f which will be related to the new geometric conditions. Let us denote them by f^x. As these conditions imply rank-one convexity condition we will have $f^{qc} \leq f^x \leq f^{rc} \leq f$, so f^x will be closer to the quasiconvex envelope of f than f^{rc}. Perhaps the new envelopes will be more helpful for numerical computations, for example it will be easier to find the minimizing sequence for I_{f^x} than for $I_{f^{rc}}$. We refer *e.g.* to [9, 20, 21, 36, 37, 46, 67] and their references for the approach related to rank-one convex and quasiconvex envelopes and their computation.

Remark 5.13. Let $K \subseteq \mathbb{R}^{n\times m}$ be the closed subset. In variational problems of martenstic phase transitions and material microstructures, in the general theory of Partial Differential Inclusions solved by the method of Convex Integration due to Gromow and its applications to construct wild solutions of nonlinear elliptic and parabolic systems (see *e.g.* [7, 8, 15, 35, 59–63, 79]) one introduces various semiconvex hulls of sets. Those hulls are defined as quasiconvex, rank-one convex and lamination convex hulls respectively. The first two hulls are defined as cosets of quasiconvex and rank-one convex functions respectively (see *e.g.* [79, 86] for details). The lamination convex hull is defined as the smallest set $\hat{K}$ with the following property: $K \subseteq \hat{K}$ and for all $A, B \in \hat{K}$ which satisfy $\operatorname{rank}(A-B) = 1$ where the interval $[A, B]$ is also contained in $\hat{K}$. Now one can define hulls described with the help of new geometric conditions. For example instead of adding the rank-one intervals in the construction of rank-one convex hulls one can add symplexes instead and obtain richer sets. Perhaps this new hulls will be helpful in the computation of quasiconvex hulls of sets. We refer *e.g.* to [22, 38, 41, 86] and their references for the related works.

Remark 5.14. Some other questions related to three dimensional conditions for integrands defying lower semicontinuous functionals were stated in [34]. One can forward them and use four dimensional conditions instead.

6. Appendix

This section is devoted to find a possibly shorter way to compute special distributions of measures stated in Lemma 3.2. The result will be achieved in several steps presented successively in the proceeding subsections.

6.1. **Properties of the $\mathcal{S}^k$ spaces**

Although our point of interest is the case $k \in \{3,4\}$ only, the calculations for this special cases are not essentially shorter.

We will deal with compositions of operators Π_i^ϵ and Π_i.

At first let us describe the superpositions of operators s_ϵ^i. For $1 \leq i_1 \leq \cdots \leq i_t \leq l$ and $\epsilon_1, \ldots, \epsilon_t \in \{0,1\}$ we define the transformation of indices by putting $\epsilon_1, \ldots, \epsilon_t$ on place $i_1, \ldots, i_t$. Namely, the operator $s_{i_1,\ldots,i_t}^{\epsilon_1,\ldots,\epsilon_t} : \{0,1\}^{l-t} \to \{0,1\}^l$ is given by

$$s_{i_1,\ldots,i_t}^{\epsilon_1,\ldots,\epsilon_t}(j) = s_{i_1} \circ \cdots \circ s_{i_t}(j).$$

This operation induces the mapping $\Pi_{i_1,\ldots,i_t}^{\epsilon_1,\ldots,\epsilon_t} : \mathcal{S}^l \to \mathcal{S}^{l-t}$, given by

$$(\Pi_{i_1,\ldots,i_t}^{\epsilon_1,\ldots,\epsilon_t}\{h\})_j := \{h\}_{s_{i_1,\ldots,i_t}^{\epsilon_1,\ldots,\epsilon_t}(j)} = (\Pi_{i_1}^{\epsilon_1} \circ \cdots \circ \Pi_{i_t}^{\epsilon_t}\{h\})_j.$$

The above definition makes sense for arbitrary $l \geq i_t$ and l is not included in the notation. We also set

$$\Pi_{i_1,\ldots,i_t}^1 := \Pi_{i_1,\ldots,i_t}^{1,\ldots,1} \quad \text{and} \quad \Pi_{i_1,\ldots,i_t}^0 := \Pi_{i_1,\ldots,i_t}^{0,\ldots,0}.$$

For example if $\{h\} \in \mathcal{S}^3$ then $\Pi_{1,2}^1\{h\} = \{h_{(1,1,j)}\}_{j\in\{0,1\}} \in \mathcal{S}^1$. Note that in most situations $\Pi_{1,2}^\epsilon = \Pi_1^\epsilon \circ \Pi_2^\epsilon \neq \Pi_2^\epsilon \circ \Pi_1^\epsilon$.

We introduce operators of discrete integration in $i_1, \ldots, i_t$ direction:

$$\Pi_{i_1,\ldots,i_t} := \Pi_{i_1} \circ \cdots \circ \Pi_{i_t} : \mathcal{S}^l \to \mathcal{S}^{l-t},$$

where $1 \leq i_1 \leq \cdots \leq i_t \leq l$.

We will also deal with compositions of operators $\Pi_{r_1,\ldots,r_t}^0$, $\Pi_{l_1,\ldots,l_s}^1$ and $\Pi_{t_1,\ldots,t_m}$.

Namely, let $A = \{r_1, \ldots, r_t\}$, $B = \{l_1, \ldots, l_s\}$ and $C = \{t_1, \ldots, t_m\}$ be disjoint subsets of $\{1, \ldots, k\}$, $r_1 \leq \cdots \leq r_t$, $l_1 \leq \cdots \leq l_s$, $t_1 \leq \cdots \leq t_m$ (in particular $t + s + m = N \leq k$). For $i \in A \cup B \cup B$ we define operators $\Pi_i^{A,B,C}$ by

$$\Pi_i^{A,B,C} = \begin{cases} \Pi_i \text{ if } i \in A \\ \Pi_i^1 \text{ if } i \in B \\ \Pi_i^0 \text{ if } i \in C. \end{cases}$$

As their domain one can consider $\mathcal{S}^l$ with an arbitrary l such that for the given i we have: $i \leq l$.

Let us write the numbers: $r_1, \ldots, r_t, l_1, \ldots, l_s, t_1, \ldots, t_m$ in an increasing order: $a_1 < a_2 < \cdots < a_N$. Then we define

$$\bigcirc \Pi_{r_1,\ldots,r_t} \Pi_{l_1,\ldots,l_s}^1 \Pi_{t_1,\ldots,t_m}^0 = \Pi_{a_1}^{A,B,C} \circ \cdots \circ \Pi_{a_N}^{A,B,C} : \mathcal{S}^k \to \mathcal{S}^{k-N}. \tag{6.1}$$

The described above operators will be sometimes also denoted by $\bigcirc \Pi_{i\in A} \Pi_{i\in B}^1 \Pi_{i\in C}^0$.

For example if $\{h\} \in \mathcal{S}^7$, we have $(\bigcirc \Pi_{2,3} \Pi_{1,5}^1 \Pi_{4,6}^0 \{h\})_j = \sum_{\epsilon_1,\epsilon_2\in\{0,1\}} h_{(1,\epsilon_1,\epsilon_2,0,1,0,j)}$, where $j \in \{0,1\}$. In other words, the operator $\bigcirc \Pi_{i\in A} \Pi_{i\in B}^1 \Pi_{i\in C}^0 \{h\}$ restricts $\{h\}$ (treated as mapping defined on $\{0,1\}^k$) to the subset of those $j \in \{0,1\}^k$ which have 1 in all places from B and 0 on each places from C, then integrates such restricted $\{h\}$ in all directions from the set A, with respect to the counting measure.

6.2. The $\mathcal{T}^k$ spaces

Let $r, k \in \mathbb{N}$ and $r \leq k$. By $s(r,k)$ we denote the set of all subsets of $\{1, \ldots, k\}$ consisting of r elements; we will identify this subsets with ordered sequences: $l_1 \ldots l_r$ where $1 \leq l_1 < \cdots < l_r \leq k$. This set consists of $\binom{k}{r}$ elements.

By $\mathcal{T}^{r,k}$ we denote all functions from $s(r,k)$ to $\mathbb{R}$, that is all finite sequences $\{t_s\}_{s \in s(r,k)}$. Note that the space $\mathcal{T}^{r,k}$ is isomorphic to $\mathbb{R}^N$ with $N = \binom{k}{r}$, in particular $\mathcal{T}^{k,k}$ can be identified with $\mathbb{R}$. For our convenience we also denote $\mathcal{T}^{0,k} = \mathbb{R}$.

Let $j = (j_1, \ldots, j_k) \in \{0,1\}^k$. If $|j| = r$ and j has 1 on place $t_1, \ldots, t_r$ where $t_1 < t_2 < \cdots < t_r$, we denote $d(j) = \{t_1, \ldots, t_r\} \in s(r,k)$; in particular every $j \in \{0,1\}^k$ determines uniquely an element of $s(|j|, k)$.

Let us introduce the special subsets of $s(r,k)$:

$$s(r,k,j) = \{s \in s(r,k) : d(j) \subset s\}$$

(note that $s(r,k,j) = \emptyset$ if $|j| > r$), and define the following operations on the space $\mathcal{T}^{r,k}$: if $A = \{A_s\}_{s \in s(r,k)} \in \mathcal{T}^{r,k}$, we set

$$\sum A = \sum_{s \in s(r,k)} A_s, \qquad \sum^{+j} A = \sum_{s \in s(r,k,j)} A_s \tag{6.2}$$

(if $|j| > r$ the second sum is zero).

We set $\mathcal{T}^k = \mathcal{T}^{0,k} \times \mathcal{T}^{1,k} \times \cdots \times \mathcal{T}^{k,k}$. As this space is isomorphic to $\mathbb{R}^N$ with $N = 2^k$, it follows that it is also isomorphic to $\mathcal{S}^k$. Elements $\{A\} \in \mathcal{T}^k$ will be identified with long vectors $\{A\} = \{A^r\}_{r=0}^k$ where $A^r = \{A^r_s\}_{s \in s(r,k)} \in \mathcal{T}^{r,k}$.

6.3. Special isomorphism

Our techniques will be based on the lemma about isomorphism presented below.

Lemma 6.1 (Lemma about isomorphism). *Let $k \in \mathbb{N}$, $\{A\} = \{A^r\}_{r=0,\ldots,k} \in \mathcal{T}^k$, $\bigcirc\Pi_{i \notin s}\Pi^1_{i \in s}\{h\}$ be as in (6.1) and $\sum^{+j}$ be as in (6.2), and consider the system of 2^k linear equations with unknown $\{h\} \in \mathcal{S}^k$:*

$$\begin{cases} \Pi_{1,\ldots,k}\{h\} = A^0, \\ \bigcirc\Pi_{i \notin s}\Pi^1_{i \in s}\{h\} = A^r_s, \ r \in \{1, \ldots, k\}, \ s \in s(r,k). \end{cases} \tag{6.3}$$

The solutions to this system are uniquely determined by the formulae

$$h_j = (-1)^{|j|} \sum_{r=|j|}^{k} (-1)^r \sum^{+j} A^r. \tag{6.4}$$

Proof. Let $B : \mathcal{S}^k \to \mathcal{T}^k$ be the linear mapping associated to the system (6.3), so that (6.3) reads as $B\{h\} = \{A\}$. Let $j_0 \in \{0,1\}^k$, and $\{\delta_{j_0}\} \in \mathcal{S}^k$ be such that $(\delta_{j_0})_j = 0$ if $j \neq j_0$ and $(\delta_{j_0})_j = 1$ if $j = j_0$. Obviously, vectors $\{\delta_j\}_{j \in \{0,1\}^k}$ form the basis in $\mathcal{S}^k$. Moreover, an easy calculation shows that $B\{\delta_{j_0}\} = \{A(j_0)\}$ where

$$(A(j_0))^r_s = \begin{cases} 1 \text{ if } & r = 0 \\ 1 \text{ if } & r \geq 1 \text{ and } s \subseteq d(j_0) \\ 0 \text{ if } & r \geq 1 \text{ and } s \not\subseteq d(j_0), \end{cases}$$

for $r \in \{0, \ldots k\}$ and $s \in s(r,k)$, in particular vectors $\{A(j_0)\}_{j_0 \in \{0,1\}^k}$ are linearly independent. This implies that B is the isomorphism. To justify that the formulae (6.4) holds true at first we write it in the form: $C\{A\} = \{h\}$ where $\{A\} \in \mathcal{T}^k$ and $\{h\} \in \mathcal{S}^k$; then we show that $C = B^{-1}$. This will be done if we prove that

$$C\{A(j_0)\} = \{\delta_{j_0}\} \text{ for every } j_0 \in \{0,1\}^k, \tag{6.5}$$

as this implies that C agrees with B^{-1} on the basis $\{A(j_0)\}_{j_0\in\{0,1\}^k}$ of $\mathcal{T}^k$. The justification of (6.5) follows by easy calculations which are left to the reader. $\square$

Remark 6.1. For the reader's convenience we list the solutions to the system (6.3) for $k = 3$ and $k = 4$ in tables below, putting for simplicity $A^3_{123} = A^3$ and $A^4_{1234} = A^4$.

j	h_j
$(0,0,0)$	$A^0 - \sum A^1 + \sum A^2 - A^3$
$d(j) = \{s\}$	$A^1_s - \sum_{\{j_1j_2:s\in\{j_1,j_2\}\}} A^2_{j_1j_2} + A^3$
$d(j) = \{s_1, s_2\}$	$A^2_{s_1s_2} - A^3$
$j = (1,1,1)$	A^3

Solutions for $k = 3$.

j	h_j
$(0,0,0,0)$	$A^0 - \sum A^1 + \sum A^2 - \sum A^3 + A^4$
$d(j) = \{s\}$	$A^1_s - \sum_{\{j_1j_2:s\in\{j_1,j_2\}\}} A^2_{j_1j_2} + \sum_{\{j_1j_2j_3:s\in\{j_1,j_2,j_3\}\}} A^3_{j_1j_2j_3} - A^4$
$d(j) = \{s_1, s_2\}$	$A^2_{s_1s_2} - \sum_{\{j_1j_2j_3:s_1,s_2\in\{j_1,j_2,j_3\}\}} A^3_{j_1j_2j_3} + A^4$
$d(j) = \{s_1, s_2, s_3\}$	$A^3_{s_1s_2s_3} - A^4$
$j = (1,1,1,1)$	A^4

Solutions for $k = 4$.

Our further calculations will be based on the following observation.

Corollary 6.1. *Suppose that $\{h\} \in \mathcal{S}^4(Q)$ is unknown, but that we know all expressions: $\sum_j h_j(x) = A^0(x)$, $\sum_{j:j_k=1} h_j(x) = A^1_k(x)$ where $k \in \{1,2,3,4\}$, $\sum_{j:j_{s_1}=1,j_{s_2}=1} h_j(x) = A^2_{s_1s_2}(x)$ where $s_1 < s_2$ and $s_i \in \{1,2,3,4\}$, $\sum_{j:j_{s_1}=1,j_{s_2}=1,j_{s_3}=1} h_j(x) = A^3_{s_1s_2s_3}(x)$ where $s_1 < s_2 < s_3$ and $s_i \in \{1,2,3,4\}$, and $h_{1111}(x) = A^4(x)$. Then all the values of h_j's can be reconstructed from the formulae (6.4) (and the table given in Rem. 6.1) from the given quantities.*

6.4. Some preliminary calculations

In this and the remaining subsections we will calculate expressions $A^i(x)$ in Corollary 6.1 for $i \in \{1,2,3,4\}$ in the selected subregions of Q. We will be interested in those regions only where all those quantities agree with special polynomials from the space $\mathcal{A}$. Let us start with the following lemma.

Lemma 6.2. *Functions h_j defined by Lemma 2.5 satisfy the following relations:*

i) $\displaystyle\sum_{j\in\{0,1\}^4} h_j(x) = 1,$ **ii)** $\forall l \in \{1,2,3,4\}$ $\displaystyle\sum_{j\in\{0,1\}^4,\, j_l=1} h_j(x) = x_l,$

iii) $\forall k, l \in \{1,2,3,4\}$ $k \neq l$ $\displaystyle\sum_{j\in\{0,1\}^4,\, j_k=1,\, j_l=1} h_j(x) = x_kx_l.$

Proof. In the same manner as in Lemma 3.2 in [34] part i) follows from substitution $f(\lambda) = 1$, part ii) from substitution $f(\lambda) = \lambda_l$ and part iii) from substitution $f(\lambda) = \lambda_k\lambda_l$. (The case $(k,l) = (3,4)$ is geometrically explained in Fig. 3.) $\square$

As a corollary we obtain the following fact.

Corollary 6.2. *Let $D \subseteq Q$ be such subregion that all expressions $h^i_{111}(x)$ (where $i \in \{1,2,3,4\}$) and $h_{1111}(x)$ agree with polynomials from the space $\mathcal{A}$ when $x \in D$. Then for every $j \in \{0,1\}^4$ the function $D \ni h_j(x)$ agrees with a function from the space $\mathcal{A}$.*

Proof. Combine Lemmas 6.2 and 6.1. $\square$

Remark 6.2. However for every $j \in \{0,1\}^4$ the function $h_j(x)$ is a piecewise polynomial function of degree not bigger than two, in general it does not belong to the space $\mathcal{A}$ in the region where it behaves polynomially.

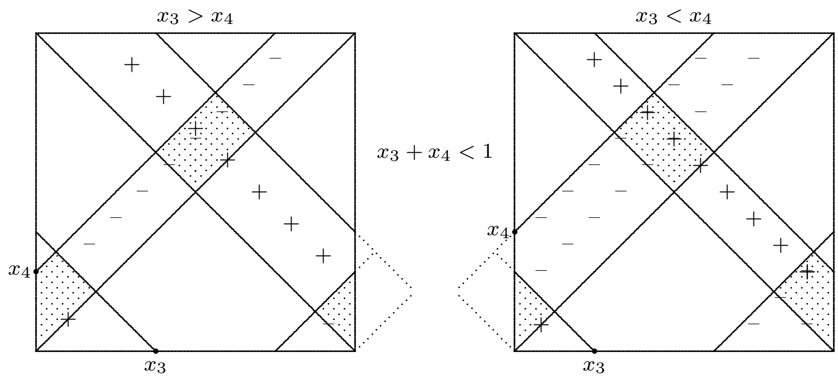

FIGURE 3. $2 \cdot \frac{x_3}{\sqrt{2}} \frac{x_4}{\sqrt{2}} = x_3 x_4$.

Now we will compute the expressions $h^3_{111}(x)$ and $h^4_{111}(x)$, leaving the more complicated calculations for $h^1_{111}(x)$, $h^2_{111}(x)$ and that of $h_{1111}(x)$ to remaining subsections.

Lemma 6.3. *The mapping* $x = (x_1, x_2, x_3) \mapsto h^4_j(x)$ *belongs to* $\mathcal{A}$ *for every* $j \in \{0,1\}^3$ *if and only if* $x \in D_4 = \{(x_1, x_2, x_3) \in [0,1]^3 : x_1 + x_2 < 1, x_3 > x_1 + x_2\}$ *or* $x \in S(D_4)$. *The respective values of* $\{h^4_j(x)\}$ *in the sets* D_4 *and* $S(D_4)$ *are given in the two tables below.*

j	$(0,0,0)$	$(0,0,1)$	$(1,0,0)$	$(1,0,1)$
$h^4_j(x)$	$(1-x_3)(1-(x_1+x_2))$	$x_3(1-x_1)$ $+x_2(x_1-x_3)$	$(1-x_3)x_1$	$(x_3-x_2)x_1$
j	$(0,1,0)$	$(0,1,1)$	$(1,1,0)$	$(1,1,1)$
$h^4_j(x)$	$(1-x_3)x_2$	$x_2(x_3-x_1)$	0	x_1x_2

h^4_j *in the set* D_4

j	$(0,0,0)$	$(0,0,1)$	$(1,0,0)$	$(1,0,1)$
$h^4_j(x)$	$(1-x_1)(1-x_2)$	0	$(1-x_2)(x_1-x_3)$	$x_3(1-x_2)$
j	$(0,1,0)$	$(0,1,1)$	$(1,1,0)$	$(1,1,1)$
$h^4_j(x)$	$(1-x_1)(x_2-x_3)$	$x_3(1-x_1)$	$(1-x_2)(x_3-x_1)$ $+(1-x_3)x_1$	$x_3(x_1+x_2-1)$

h^4_j *in the set* $S(D_4)$

The proof of this fact is given in [34], but for the reader's convenience we include the sketch.

Proof. According to Lemmas 6.1 and 6.2 it suffices to calculate $h^4_{111}(x)$ and use table given in Remark 6.1 for $k = 3$. Then we use the result about invariances (Lem. 3.1), which shows that calculations can be reduced to the region $\{x_1 + x_2 < 1, x_1 > x_2\}$. $\square$

Our next lemma immediately follows from Lemmas 3.1 and 6.3.

Lemma 6.4. *We have* $x = (x_1, x_2, x_4) \mapsto h^3_j(x) \in \mathcal{A}$ *for every* $j \in \{0,1\}^3$ *if and only if* $x \in D_3$ *or* $x \in S(D_3)$, *where* $D_3 = (x_1, x_2, x_4) \in [0,1]^3 : x_1 - x_2 < 0, x_4 > x_1 + 1 - x_2\}$ *and there we have* $h^3_{111}(x) = h^4_{101}(x_1, 1-x_2, x_4) = (x_4 + x_2 - 1)x_1$, *while for* $x \in S(D_3)$ *the value of* $h^3_{111}(x)$ *equals* $x_2 x_4$.

6.5. The computation of h^1_{111} and h^2_{111}

We are now computing the functions h^1_{111} and h^2_{111} in the selected regions of $[0,1]^3$. The values of h^1_{111} are obtained with the help of the following lemma.

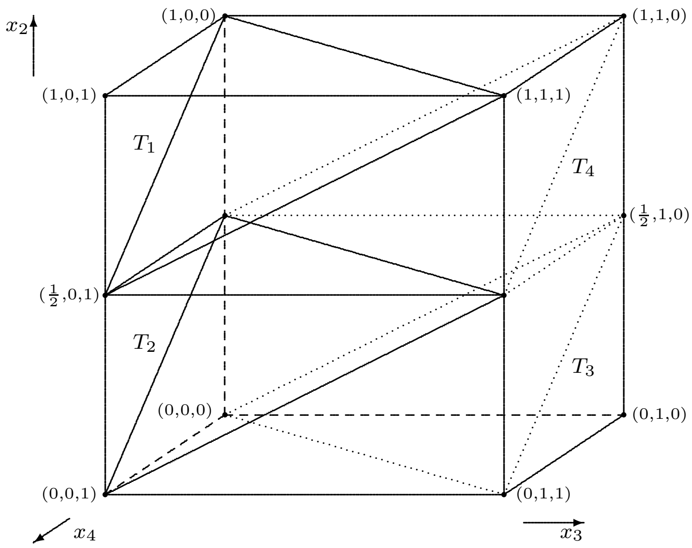

FIGURE 4. The tetrahedrons T_1, T_2, T_3, T_4.

Lemma 6.5. *The mapping* $x = (x_2, x_3, x_4) \mapsto h^1_{111}(x)$ *belongs to the space* $\mathcal{A}$ *if and only if* $x \in \bigcup_{k=1}^4 T_k$, *where* T_k *for* $k = 1, 2, 3, 4$ *are disjoint tetrahedrons*

$$T_1 = \left\{(x_2, x_3, x_4) \in [0,1]^3 : x_2 > 1 + \frac{x_3 - x_4}{2}\right\},$$
$$T_2 = T_1 - \left(\frac{1}{2}, 0, 0\right) = \left\{\frac{1}{2} > x_2 > \frac{1 + x_3 - x_4}{2}\right\},\ T_3 = S(T_1),\ T_4 = S(T_2).$$

Under the above notation we have

$$h^1_{111}(x_2, x_3, x_4) = \begin{cases} x_3(x_4 + x_2 - 1) & \text{if } (x_2, x_3, x_4) \in T_1 \\ x_3\left(x_2 - \dfrac{1 - x_4}{2}\right) & \text{if } (x_2, x_3, x_4) \in T_2 \\ x_2 x_4 & \text{if } (x_2, x_3, x_4) \in T_3 \\ x_4\left(x_2 - \dfrac{1 - x_3}{2}\right) & \text{if } (x_2, x_3, x_4) \in T_4. \end{cases}$$

Remark 6.3. The figure presented below shows the position of sets T_i in the cube $[0, 1]^3$.

Proof. The proof follows by steps: 1) We assume that $x_3 \le x_4, x_3 + x_4 < 1$. 2) We complete the proof of the lemma.

Proof of step 1. Let us denote for simplicity $I = h^1_{111}(x_2, x_3, x_4)$, and consider the following five sets

$$\begin{aligned} A_0 &= \{(\tau_1, \tau_2) : \tau_1 + \tau_2 < x_3\} \\ A_1 &= \{(\tau_1, \tau_2) : 0 < \tau_1 + \tau_2 - 1 < x_3\} \\ B_0 &= \{(\tau_1, \tau_2) : 0 < \tau_1 - \tau_2 < x_4\} \\ B_1 &= \{(\tau_1, \tau_2) : 0 < 1 + \tau_1 - \tau_2 < x_4\} \\ H_2 &= \{(\tau_1, \tau_2) : \tau_2 \in [0, x_2]\}. \end{aligned}$$

Figure 5 represents geometrically the position of the sets A_0, A_1, B_0 and B_1 in the cube $[0, 1]^2$. □

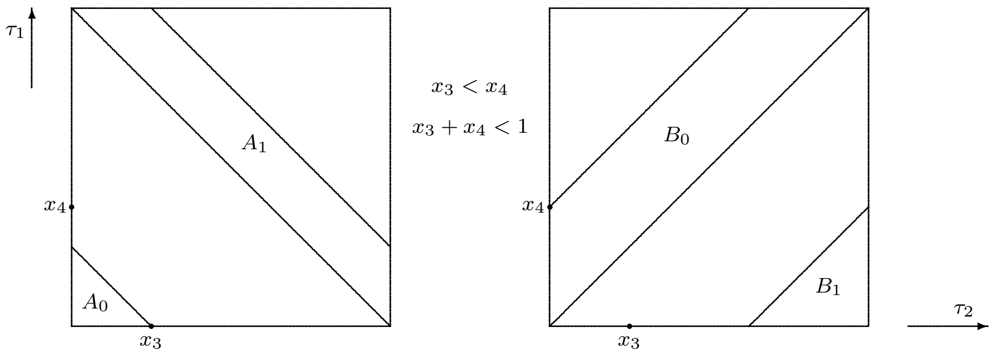

FIGURE 5. The sets A_0, A_1, B_0, B_1.

Then $I = I_1+I_2+I_3+I_4$ where $I_1 = |A_0 \cap B_0 \cap H_2|$, $I_2 = |A_0 \cap B_1 \cap H_2|$, $I_3 = |A_1 \cap B_0 \cap H_2|$, $I_4 = |A_1 \cap B_1 \cap H_2|$. As for every $a, b, c, d, \alpha, \beta \in \mathbb{R}$, we have

$$\mathcal{L}(a, b, c, d, \alpha, \beta) = \int_a^b \left(\int_{c+\alpha\tau_2}^{d+\beta\tau_2} \mathrm{d}\tau_1 \right) \mathrm{d}\tau_2 = (b-a)\left[(d-c) + \frac{1}{2}(\beta - \alpha)(b+a) \right]. \tag{6.6}$$

After easy computation we find that

$$I_1 = \int_0^{\min\{x_2, \frac{x_3}{2}\}} \left(\int_{\tau_2}^{x_3-\tau_2} \mathrm{d}\tau_1 \right) \mathrm{d}\tau_2 = \begin{cases} x_2 x_3 - x_2^2 & \text{if } x_2 \le \dfrac{x_3}{2} \\ \frac{x_3^2}{4} & \text{if } x_2 > \dfrac{x_3}{2}. \end{cases}$$

To calculate I_2 we observe that $A_0 \cap B_1 = \emptyset$, so immediately $I_2 = 0$.

Let us calculate I_3. Subtracting the inequalities defining sets A_1 and B_0 we get $1 - x_4 < 2\tau_2$, so for $2x_2 < 1 - x_4$ we obtain $I_3 = 0$. Now, for $x_2 > (1-x_4)/2$ we conclude that I_3 has the following form

$$I_3 = \int_{\frac{1-x_4}{2}}^{\min\{x_2, \frac{1+x_3}{2}\}} \left(\int_{\max\{\tau_2, 1-\tau_2\}}^{\min\{x_3+1-\tau_2, x_4+\tau_2\}} \mathrm{d}\tau_1 \right) \mathrm{d}\tau_2.$$

Let us decompose $[0,1]$ into 5 subintervals $C_i = [a_{i-1}, a_i]$ where $i \in \{1, \dots, 5\}$, with the help of 6 ordered numbers: $a_0 = 0 < a_1 = (1-x_4)/2 < a_2 = (1+x_3-x_4)/2 < a_3 = 1/2 < a_4 = (1+x_3)/2 < a_5 = 1$. Then

$$\max\{\tau_2, 1-\tau_2\} = \begin{cases} 1-\tau_2 & \text{on } C_1 \cup C_2 \cup C_3 \\ \tau_2 & \text{on } C_4 \cup C_5, \end{cases} \quad \text{while}$$

$$\min\{x_3 + 1 - \tau_2, x_4 + \tau_2\} = \begin{cases} x_4 + \tau_2 & \text{on } C_1 \cup C_2 \\ x_3 + 1 - \tau_2 & \text{on } C_3 \cup C_4 \cup C_5. \end{cases}$$

Now using (6.6) we easily compute the values of I_3 on every set C_i and present them in table below.

C_j	$[0, \frac{1-x_4}{2}]$	$[\frac{1-x_4}{2}, \frac{1+x_3-x_4}{2}]$	$[\frac{1+x_3-x_4}{2}, \frac{1}{2}]$	$[\frac{1}{2}, \frac{1+x_3}{2}]$	$[\frac{1+x_3}{2}, 1]$
I_3	0	$(x_2 + \frac{x_4-1}{2})^2$	$-\frac{x_3^2}{4} + x_3(x_2 - \frac{1-x_4}{2})$	$\frac{x_3 x_4}{2} - (\frac{1+x_3}{2} - x_2)^2$	$\frac{x_3 x_4}{2}$

In a similar manner we calculate I_4. First we observe that for all $(\tau_1, \tau_2) \in A_1 \cap B_1$ we have $2 - x_4 < 2\tau_2$, so for $2x_2 < 2 - x_4$ we have $I_4 = 0$, while for $x_2 > (2 - x_4)/2$

$$I_4 = \int_{1-\frac{x_4}{2}}^{x_2} \left(\int_{1-\tau_2}^{\min\{x_4-1+\tau_2, 1+x_3-\tau_2\}} \mathrm{d}\tau_1 \right) \mathrm{d}\tau_2.$$

After the simple calculation based on (6.6) we find the values of I_4 on subintervals D_i of $[0,1]$, and present them in table below

D_j	$[0, 1-\frac{x_4}{2}]$	$[1-\frac{x_4}{2}, 1-\frac{x_4}{2}+\frac{x_3}{2}]$	$[1-\frac{x_4}{2}+\frac{x_3}{2}, 1]$
I_4	0	$(x_2+\frac{x_4}{2}-1)^2$	$-\frac{x_3^2}{4}+x_3(x_2-1+\frac{x_4}{2})$

Adding all pieces together we obtain the following result

$$I = I_1 + I_2 + I_3 + I_4 \in \mathcal{A} \Leftrightarrow \quad x_2 > 1 + \frac{x_3 - x_4}{2} \quad \text{and} \quad I = x_3(x_4 + x_2 - 1)$$
$$\text{or } \frac{1 + x_3 - x_4}{2} < x_2 \le \frac{1}{2} \text{ and } I = x_3\left(\frac{x_4}{2} + x_2 - \tfrac{1}{2}\right).$$

Proof of step 2. Using Lemmas 6.2 and 6.1 for $k = 3$ we calculate all the h_j^1's in the regions $T_1 \cap \{x_3 \le x_4\} \cap \{x_3 + x_4 \le 1\}$ and $T_2 \cap \{x_3 \le x_4\} \cap \{x_3 + x_4 \le 1\}$. The obtained values are presented in the next two tables below. □

j	$(0,0,0)$	$(0,0,1)$
$h_j^1(x)$	$(1-x_2)(1-x_4)$	$(1-x_2)(x_4-x_3)$
j	$(1,0,0)$	$(1,0,1)$
$h_j^1(x)$	$(1-x_4)(x_2-x_3)$	$x_2(x_4-x_3)+x_3(1-x_4)$
j	$(0,1,0)$	$(0,1,1)$
$h_j^1(x)$	0	$x_3(1-x_2)$
j	$(1,1,0)$	$(1,1,1)$
$h_j^1(x)$	$x_3(1-x_4)$	$x_3(x_4+x_2-1)$

$h_j^1(x_2, x_3, x_4)$ in the set T_1.

j	$(0,0,0)$	$(0,0,1)$
$h_j^1(x)$	$(1-x_4)(1-x_2-\frac{x_3}{2})$	$x_4(1-\frac{x_3}{2}-x_2)+x_3(x_2-\frac{1}{2})$
j	$(1,0,0)$	$(1,0,1)$
$h_j^1(x)$	$(1-x_4)(x_2-\frac{x_3}{2})$	$x_2(x_4-x_3)+\frac{x_3}{2}(1-x_4)$
j	$(0,1,0)$	$(0,1,1)$
$h_j^1(x)$	$\frac{x_3}{2}(1-x_4)$	$\frac{1}{2}x_3x_4+x_3(\frac{1}{2}-x_2)$
j	$(1,1,0)$	$(1,1,1)$
$h_j^1(x)$	$\frac{x_3}{2}(1-x_4)$	$\frac{1}{2}x_3x_4-x_3(\frac{1}{2}-x_2)$

$h_j^1(x_2, x_3, x_4)$ in the set T_2.

Now we use Lemma 3.1. Since the isometry $B_1(x_2, x_3, x_4) = (x_2, 1-x_4, 1-x_3)$ transforms the region $\{x_3 \le x_4\}$ into itself and the region $\{x_3 + x_4 < 1\}$ into $\{x_3 + x_4 > 1\}$, we find the whole region where $h_{111}^1 \in \mathcal{A}$ in the set $\{x_3 \le x_4\}$. It is easy to see (Rem. 6.3) that this is the set $T_1 \cup T_2$. Moreover, using the identity $h_j(x) = h_{B_1 j}(B_1 x)$ we obtain the formulas for all h_j^1 in $T_1 \cup T_2$. Those formulas are the same as presented in the two tables above. Now having described the whole region where $h_{111}^1 \in \mathcal{A}$ in the set $\{x_3 \le x_4\}$ we use the symmetry with respect to the point $(1/2, 1/2, 1/2)$: $S = B_1 \circ B_2$, and according to the formulae $h_j^1(x) = h_{Sj}^1(Sx)$

we find all functions h^1_j in sets T_3 and T_4. Obviously, T_1, T_2, T_3 and T_4 are the only regions where we have $h_j \in \mathcal{A}$ for all j's.

j	$(0,0,0)$	$(0,0,1)$
$h^1_j(x)$	$(1-x_3)(1-x_2-x_4)$	$(1-x_3)x_4$
j	$(1,0,0)$	$(1,0,1)$
$h^1_j(x)$	$(1-x_3)x_2$	0
j	$(0,1,0)$	$(0,1,1)$
$h^1_j(x)$	$(1-x_2)(x_3-x_4)+x_4(1-x_3)$	$x_4(x_3-x_2)$
j	$(1,1,0)$	$(1,1,1)$
$h^1_j(x)$	$x_2(x_3-x_4)$	x_2x_4

$h^1_j(x_2, x_3, x_4)$ in the set T_3.

j	$(0,0,0)$	$(0,0,1)$
$h^1_j(x)$	$(1-x_3)(1-x_2-\frac{x_4}{2})$	$\frac{1}{2}x_4(1-x_3)$
j	$(1,0,0)$	$(1,0,1)$
$h^1_j(x)$	$(1-x_3)(x_2-\frac{x_4}{2})$	$\frac{1}{2}x_4(1-x_3)$
j	$(0,1,0)$	$(0,1,1)$
$h^1_j(x)$	$(1-x_2)(x_3-x_4)+\frac{1}{2}x_4(1-x_3)$	$x_4(\frac{1}{2}-x_2+\frac{x_3}{2})$
j	$(1,1,0)$	$(1,1,1)$
$h^1_j(x)$	$(x_2-\frac{1}{2})(x_3-x_4)+\frac{1}{2}x_3(1-x_4)$	$x_4(x_2-\frac{1}{2}+\frac{x_3}{2})$

$h^1_j(x_2, x_3, x_4)$ in the set T_4.

□

Now we will calculate $h^2_j(x_1, x_3, x_4)$. We have

Lemma 6.6. *The mapping $x \mapsto h^2_j(x)$ belongs to the space $\mathcal{A}$ for every j if and only if $x \in \cup_{i=1}^4 \hat{S}(T_i)$ where T_i are the tetrahedrons from Lemma 6.5 and $\hat{S}$ denotes the symmetry with respect to the plane $\{x_4 = \frac{1}{2}\}$. Moreover, we have*

$$h^2_{111}(x) = \begin{cases} x_3x_4 & \text{if } x \in \hat{S}(T_1) = \{x_1 > \dfrac{x_3+x_4+1}{2}\} \\ \dfrac{1}{2}x_3x_4 & \text{if } x \in \hat{S}(T_2) = \{\dfrac{1}{2} > x_1 > \dfrac{x_3+x_4}{2}\} \\ x_1(x_3+x_4-1) & \text{if } x \in \hat{S}(T_3) = \{x_1 < \dfrac{x_3+x_4-1}{2}\} \\ (x_1-\dfrac{1}{2})(x_3+x_4-1)+\dfrac{1}{2}x_3x_4 & \text{if } x \in \hat{S}(T_4) = \{\dfrac{1}{2} < x_1 < \dfrac{x_3+x_4}{2}\} \end{cases}$$

where $x = (x_1, x_3, x_4)$.

Proof. This follows immediately from Lemmas 3.1 and 6.5. □

6.6. The computation of h_{1111} and proof of Lemma 3.2

Now we are ready to complete the proof of Lemma 3.2.

Proof of Lemma 3.2. The proof follows by steps:

1) We show that the maximal $\mathcal{A}$-regular subset in

$$\Omega_1 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 < 1,\ x_2 \le x_1,\ x_3 + x_4 < 1\}$$

is contained in

$$R_1 = \{x \in Q : x_1 + x_2 < 1, x_3 + x_4 < 1, x_3 > x_1 + x_2, x_1 \leq 1/2, 2x_1 > x_3 + x_4\}.$$

2) R_1 is the maximal $\mathcal{A}$-regular subset of Ω_1, and for $x \in R_1$ we have

$$h_{1111}(x) = x_2 x_4.$$

3) We compute h_j's for $x \in R_1$.
4) We complete the proof of the lemma.

Proof of step 1. Let us denote the maximal $\mathcal{A}$-regular set under consideration by P_1. According to Lemma 6.3 we have

$$P_1 \subseteq \Omega_2 = \Omega_1 \cap \{x \in \Omega : x_3 > x_1 + x_2\}.$$

Then by Lemma 6.4

$$P_1 \subseteq \Omega_3 = \Omega_2 \cap \{x \in Q : x_1 > x_2 + x_4\}.$$

An easy computation shows that if $x \in P_1$ we have $1 > x_3 > x_1 + x_2 > 2x_2 + x_4$, which implies $x_2 \leq 1/2$, and $x_4 \leq x_3$. Now we apply Lemma 6.5. As $x_2 \leq 1/2$, we have $P_1 \subseteq \Omega_3 \cap (T_2 \cup T_3)$ (we identify T_i's with its embeddings in $Q \subseteq \mathbb{R}^4$). But $\Omega_3 \cap T_2 = \emptyset$, because $T_2 \subseteq \{x_4 \geq x_3\}$ (see Fig. 3). Hence

$$P_1 \subseteq \Omega_4 = \Omega_3 \cap \{x \in Q : 2x_2 < x_3 - x_4\}.$$

Then we use Lemma 6.6, which implies that

$$P_1 \subseteq \Omega_4 \cap (\hat{S}(T_1) \cup \hat{S}(T_2) \cup \hat{S}(T_3) \cup \hat{S}(T_4)).$$

Now it suffices to verify that

$$\Omega_4 \cap \hat{S}(T_i) = \emptyset, \text{ for } i = 1, 3, 4. \tag{6.7}$$

The property (6.7) for $i = 1$ follows from the sequence of inequalities: $2x_1 > x_3 + x_4 + 1 > x_1 + x_2 + x_4 + 1$, which implies the impossible one: $x_1 > x_2 + x_4 + 1 > 1$; (6.7) for $i = 3$ and $i = 4$ contradicts the fact that $x_3 + x_4 < 1$ on P_1. Summing up all conditions describing P_1 we get

$$\begin{aligned} P_1 \subseteq R_1 = \{&x \in Q : x_1 + x_2 < 1, x_2 \leq x_1, x_3 + x_4 < 1, \\ &x_4 \leq x_3, x_3 > x_1 + x_2, x_1 > x_2 + x_4, x_4 \leq 1/2, \\ &x_2 \leq 1/2, 2x_2 < x_3 - x_4, x_1 \leq 1/2, 2x_1 > x_3 + x_4\}. \end{aligned}$$

Now it suffices to eliminate those conditions in the description of P_1, which are implied by the other. To do this let us denote inequalities defying P_1 by 1)–11) respectively. Then it suffices to note that 6) implies 2), 5) and 6) imply 4) and 9), 6) and 10) imply 7) and 8), 5) and 11) imply 6). Since R_1 is the nonempty intersection of 5 halfspaces, it is the symplex in $\mathbb{R}^4$.

Proof of step 2. According to Lemma 6.1, Corollary 6.1 and Lemma 6.2 we have $D \ni x \mapsto h_j(x) \in \mathcal{A}$ for every j if and only if $D \ni x \mapsto h^i_{111}(x) \in \mathcal{A}$ for $i = \{1, \ldots, 4\}$ and $D \ni x \mapsto h_{1111}(x) \in \mathcal{A}$. We have shown in the proof of step 1 that the mapping $R_1 \ni x \mapsto h^i_{111}(x)$ is represented by elements of $\mathcal{A}$. Hence it suffices to show that $h_{1111}(x) \in \mathcal{A}$ for $x \in R_1$. We will show it by direct computation of $h_{1111}(x)$. For simplicity we put $I = h_{(1,1,1,1)}(x)$. Then an easy computation shows that for $x \in R_1$ we have $I = \int_0^1 \int_0^1 g(\tau_1, \tau_2) \mathrm{d}\tau_1 \mathrm{d}\tau_2$, where $g(\tau) = f_0(\tau) f_1(\tau) f_2(\tau) f_3(\tau)$, and for $\tau = (\tau_1, \tau_2)$

$$\begin{aligned} f_0(\tau) &= \chi_{\tau_1 \in [0, x_1], \tau_2 \in [0, x_2]}, \\ f_1(\tau) &= \chi_{(\tau_1 + \tau_2) < x_3}, \\ f_2(\tau) &= \chi_{[\tau_1 - \tau_2]_1 \in [0, x_4]}, \\ f_3(\tau) &= \chi_{\tau_1 > \tau_2} \end{aligned}$$

(as for $\tau_1 \le \tau_2$ and $x_4 \le 1/2$ the function f_2 vanishes). Note that $f_0 f_3 \not\equiv 0$ only if $\tau_2 < x_1$, while $f_1 \not\equiv 0$ if $\tau_2 < x_3$ and $\tau_1 < x_3 - \tau_2$, so $f_1 f_3 \not\equiv 0$ only if $\tau_2 < x_3 - \tau_2$, which is equivalent to: $\tau_2 < x_3/2$. Hence

$$I = \int_0^{\min(x_1,x_2,x_3/2)} \int_{\tau_2}^{h(\tau_2)} \mathrm{d}\tau_1 \mathrm{d}\tau_2, \quad \text{where} \quad h(\tau) = \min(x_1, x_3 - \tau, x_4 + \tau).$$

But for $x \in R_1$ we have $\min\{x_1, x_2, x_3/2\} = x_2$, and $\min\{x_1, x_3 - \tau, x_4 + \tau\} = x_4 + \tau$ if $\tau < x_2$. Thus $I = x_2 x_4$.

Proof of step 3. Lemmas: 6.3, 6.4, 6.5, 6.6 give for $x \in R_1$: $h^1_{111}(x) = x_2 x_4$, $h^2_{111}(x) = 1/2 x_3 x_4$, $h^3_{111}(x) = x_2 x_4$, $h^4_{111}(x) = x_1 x_2$, $h_{1111}(x) = x_2 x_4$. Thus we can use Remark 6.1 and calculate all the remaining coefficients h_j on R_1. Note that according to our notation we have $A^0 = 1$, $A^1_j = x_j$, $A^2_{j_1 j_2} = x_{j_1} x_{j_2}$, $A^3_{123} = h^4_{111}$, $A^3_{124} = h^3_{111}$, $A^3_{134} = h^2_{111}$, $A^3_{234} = h^1_{111}$, $A^4_{1234} = h_{1111}$.

Proof of step 4. Consider the following sets

$$\begin{aligned} C_1 &= \{x_1 + x_2 < 1, x_2 \le x_1, x_3 + x_4 < 1\} \\ C_2 &= \{x_1 + x_2 < 1, x_2 \le x_1, x_3 + x_4 > 1\}. \end{aligned}$$

Since the isometry A_1 in Lemma 3.1 transforms C_1 onto C_2, it follows from Lemma 3.1 and previous steps that $R_1 \cup (R_2 = A_1(R_1))$ is the maximal $\mathcal{A}$-regular subset of

$$\{x_1 + x_2 < 1, x_2 \le x_1\}.$$

Now, if D is the maximal regular subset of $\{x_1 + x_2 < 1, x_2 \le x_1\}$, then $A_2(D)$ is the maximal $\mathcal{A}$-regular subset of $\{x_1 + x_2 < 1, x_2 \ge x_1\}$. In particular $R_1 \cup R_2 \cup (R_3 = A_2(R_1)) \cup (R_4 = A_2(R_2))$ is the maximal $\mathcal{A}$-regular subset of $\{x_1 + x_2 < 1\}$. Finally, it suffices to use the isometry $A = A_1 \circ A_2 \circ A_1 \circ A_2$, which transforms the region $\{x_1 + x_2 < 1\}$ onto $\{x_1 + x_2 > 1\}$. Then we obtain the maximal $\mathcal{A}$-regular subset of Q: $\cup_{i=1}^8 R_i$ where $R_{4+i} = A(R_i)$, where $i = 1, \ldots, 4$. The coefficients on every symplex R_i for $i = 2, \ldots, 8$ are calculated according to the role $h_j(x) = h_{Aj}(Ax)$ in Lemma 3.1 from those given on R_1, where we take $A = E_2, E_3, \ldots, E_8$ for calculations on R_2, R_3, $\ldots$, R_7 and R_8 respectively. □

Remark 6.4. It is easy to check that the set of coefficients $\{h_j(x)\}_j$ for all symplexis R_i where $i = 1, \ldots, 8$ is different for different i.

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