

## ANISOTROPIC FUNCTIONS: A GENERICITY RESULT WITH CRYSTALLOGRAPHIC IMPLICATIONS

VICTOR J. MIZEL<sup>1</sup> AND ALEXANDER J. ZASLAVSKI<sup>2</sup>

**Abstract.** In the 1950's and 1960's surface physicists/metallurgists such as Herring and Mullins applied ingenious thermodynamic arguments to explain a number of experimentally observed surface phenomena in crystals. These insights permitted the successful engineering of a large number of alloys, where the major mathematical novelty was that the surface response to external stress was **anisotropic**. By examining step/terrace (**vicinal**) surface defects it was discovered through lengthy and tedious experiments that the stored energy density (**surface tension**) along a step edge was a smooth symmetric function  $\beta$  of the azimuthal angle  $\theta$  to the step, and that the positive function  $\beta$  attains its minimum value at  $\theta = \pi/2$  and its maximum value at  $\theta = 0$ . The function  $\beta$  provided the crucial thermodynamic parameters needed for the engineering of these materials. Moreover the minimal energy configuration of the step is determined by the values of the **stiffness function**  $\beta'' + \beta$  which ultimately leads to the magnitude and direction of surface mass flow for these materials. In the 1990's there was a dramatic improvement in electron microscopy which permitted real time observation of the meanderings of a step edge under Brownian heat oscillations. These observations provided much more rapid determination of the relevant thermodynamic parameters for the step edge, even for crystals at temperatures below their **roughening** temperature. Use of these tools led J. Hannon and his coexperimenters to discover that some crystals behave in a highly anti-intuitive manner as their temperature is varied. The present article is devoted to a model described by a class of variational problems. The main result of the paper describes the solutions of the corresponding problem for a generic integrand.

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### 1. INTRODUCTION

It is well-known in surface physics that when a crystalline substance is maintained at a temperature  $T$  above its *roughening temperature*  $T_R$  then the surface stored energy integrand, usually referred to as *surface tension*, is a smooth function  $\beta$  of the azimuthal angle of orientation  $\theta$ . Furthermore,  $\beta$  obeys the following:

$$\beta(-\theta) = \beta(\pi - \theta) = \beta(\theta), \quad 0 < \beta(\pi/2) \leq \beta(\theta) \leq \beta(0)$$

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<sup>1</sup> Department of Mathematical Sciences, Carnegie Mellon University, USA; e-mail: [Fage1cmu.edu](mailto:Fage1cmu.edu)

<sup>2</sup> Department of Mathematics, Technion-Israel Institute of Technology, 32000, Haifa, Israel; e-mail: [ajzasltx.technion.ac.il](mailto:ajzasltx.technion.ac.il)

(cf. for example [1, 2, 5]). We note that given any point  $P$  on an arc in the  $x, y$  plane, the convention here is that  $\theta(P) \in [-\pi, \pi]$  denotes the angle (measured counterclockwise) that the tangent line at  $P$  makes with the positive  $x$ -axis. The issue we proceed to discuss is the relative availability of smooth surface tension functions  $\beta$  for which the quantity  $\beta'' + \beta$ , usually referred to as the *stiffness*, is negative at the angle  $\theta = 0$  where  $\beta$  attains its maximum, i.e.,  $\beta''(0) + \beta(0) < 0$ . That condition is present in a type of Silicon crystal which exhibited very unorthodox behavior during experimental observations involving mass transport along the surface of the crystal (cf. [3, 4]). This phenomenon led us to undertake the present investigation.

For each function  $f : X \rightarrow R$  set  $\inf(f) = \inf\{f(x) : x \in X\}$ .

Denote by  $\mathcal{M}$  the set of all functions  $\beta \in C^2(R)$  which satisfy the following assumption:

(A)

$$\beta(t) \geq 0 \text{ for all } t \in R, \tag{1.1}$$

$$\beta(\pi/2) \leq \beta(t) \leq \beta(0) \text{ for all } t \in R, \tag{1.2}$$

$$\beta(t) = \beta(-t) \text{ for all } t \in R, \tag{1.3}$$

$$\beta(t + \pi) = \beta(t) \text{ for all } t \in R, \tag{1.4}$$

$$\beta(0) + \beta''(0) \leq 0. \tag{1.5}$$

For each  $\beta_1, \beta_2 \in \mathcal{M}$  set

$$\rho(\beta_1, \beta_2) = \sup \left\{ |\beta_1^{(i)}(t) - \beta_2^{(i)}(t)| : t \in R, i = 0, 1, 2 \right\}. \tag{1.6}$$

It is not difficult to see that the metric space  $(\mathcal{M}, \rho)$  is complete.

Denote by  $\mathcal{M}_r$  the set of all  $\beta \in \mathcal{M}$  such that

$$\beta(t) > 0 \text{ for all } t \in R, \tag{1.7}$$

$$\beta(0) + \beta''(0) < 0. \tag{1.8}$$

It is obvious that  $\mathcal{M}_r$  is nonempty.

**Proposition 1.1.**  $\mathcal{M}_r$  is an open everywhere dense subset of  $(\mathcal{M}, \rho)$ .

*Proof.* Evidently  $\mathcal{M}_r$  is an open subset of  $(\mathcal{M}, \rho)$ . Let us show that  $\mathcal{M}_r$  is an everywhere dense subset of  $(\mathcal{M}, \rho)$ .

Let  $\beta \in \mathcal{M}$  and  $\tilde{\beta} \in \mathcal{M}_r$ . Then for each natural number  $n$  the function  $\beta + (n)^{-1}\tilde{\beta} \in \mathcal{M}_r$ ,

$$(\beta + (n)^{-1}\tilde{\beta})(t) \geq (n)^{-1}\tilde{\beta}(t) > 0$$

for all  $t \in R$  and

$$(\beta + (n)^{-1}\tilde{\beta})(0) + (\beta + (n)^{-1}\tilde{\beta})''(0) = \beta(0) + \beta''(0) + [\tilde{\beta}(0) + \tilde{\beta}''(0)]/n < 0.$$

Thus  $\beta + n^{-1}\tilde{\beta} \in \mathcal{M}_r$  for all natural numbers  $n$ . It is easy to see that  $\beta + n^{-1}\tilde{\beta} \rightarrow \beta$  as  $n \rightarrow \infty$  in  $(\mathcal{M}, \rho)$ . Therefore  $\mathcal{M}_r$  is an everywhere dense subset of  $(\mathcal{M}, \rho)$ . Proposition 1.1 is proved.  $\square$

Let  $\beta \in \mathcal{M}_r$ . Define

$$G_\beta(z) = \beta(\arctan(z))(1 + z^2)^{1/2}, z \in R. \tag{1.9}$$

Clearly  $G_\beta$  is a continuous function and

$$G_\beta(z) \rightarrow \infty \text{ as } z \rightarrow \pm\infty. \tag{1.10}$$

**Remark 1.1.** Note that  $\inf(G_\beta) < \beta(0)$  (see [4]).

The classical model for the free energy of certain crystals is given by

$$J(y) = \int_0^S \beta(\theta) ds$$

where  $s$  is arclength and  $y$  is a function defined on a fixed interval  $[0, L]$  whose graph is the locus under consideration:

$$y \in W^{1,1}(0, L), \theta = \arctan(y') \in [-\pi/2, \pi/2].$$

We can rewrite  $J$  in the form

$$J(y) = \int_0^L G_\beta(y') dx.$$

It was shown in [4] that  $y \in W^{1,1}(0, L)$  is a minimizer of  $J$  if and only if

$$|y'| \in \{z \in R : G_\beta(z) = \inf(G_\beta)\} \text{ a.e.}$$

In this paper we show that for a generic function  $\beta$  the set

$$\{z \in R : G_\beta(z) = \inf(G_\beta)\} = \{z_\beta, -z_\beta\}$$

where  $z_\beta$  is a unique positive number depending only on  $\beta$ .

Denote by  $\mathcal{F}$  the set of all  $\beta \in \mathcal{M}_r$  which satisfy the following condition:

There is  $z_\beta \in R$  such that

$$G_\beta(z) > G_\beta(z_\beta) \text{ for all } z \in R \setminus \{z_\beta, -z_\beta\}. \tag{1.11}$$

We will establish the following result.

**Theorem 1.1.**  $\mathcal{F}$  is a countable intersection of open everywhere dense subsets of  $(\mathcal{M}, \rho)$ .

## 2. PRELIMINARY RESULTS

**Proposition 2.1.** Let  $\beta \in \mathcal{M}_r$ . Then there exist  $M_0 > 0$  and a neighborhood  $\mathcal{U}$  of  $\beta$  in  $\mathcal{M}$  such that  $\mathcal{U} \subset \mathcal{M}_r$  and the following assertion holds:

if  $\phi \in \mathcal{U}$ ,  $z \in R$  and  $G_\phi(z) \leq \inf(G_\phi) + 1$ , then  $|z| \leq M_0$ .

*Proof.* There is  $c_0 > 0$  such that

$$\beta(t) \geq c_0 \text{ for all } t \in R. \tag{2.1}$$

There is a neighborhood  $\mathcal{U}$  of  $\beta$  in  $(\mathcal{M}, \rho)$  such that

$$\mathcal{U} \subset \mathcal{M}_r \tag{2.2}$$

and for each  $\phi \in \mathcal{U}$

$$\phi(t) \geq c_0/2 \text{ for all } t \in R, \tag{2.3}$$

$$\phi(0) \leq 2\beta(0). \tag{2.4}$$

Assume that  $\phi \in \mathcal{U}$ ,  $z \in R$ ,

$$G_\phi(z) \leq \inf(G_\phi) + 1. \tag{2.5}$$

Then (2.3), (2.4) hold. By (1.9), (2.3), (2.5), (2.4)

$$(1 + z^2)^{1/2} c_0/2 \leq \phi(\arctan(z))(1 + z^2)^{1/2} = G_\phi(z) \leq G_\phi(0) + 1 = \phi(0) + 1 \leq 2\beta(0) + 1$$

and

$$|z| \leq 2(2\beta(0) + 1)c_0^{-1}.$$

Thus the assertion of Proposition 2.1 holds with  $M_0 = 2(2\beta(0) + 1)c_0^{-1}$ .  $\square$

It is not difficult to see that the next proposition is true.

**Proposition 2.2.** *Let  $\beta \in \mathcal{M}_r$ ,  $\epsilon, M > 0$ . Then there exists a neighborhood  $\mathcal{U}$  of  $\beta$  in  $\mathcal{M}$  such that  $\mathcal{U} \subset \mathcal{M}_r$  and the following assertion holds:*

*if  $\phi \in \mathcal{U}$ ,  $z \in R$ ,  $|z| \leq M$ , then*

$$|G_\phi(z) - G_\beta(z)| \leq \epsilon.$$

**Proposition 2.3.** *Let  $\beta \in \mathcal{M}_r$ ,  $\epsilon > 0$ . Then there exists a neighborhood  $\mathcal{U}$  of  $\beta$  in  $\mathcal{M}$  such that  $\mathcal{U} \subset \mathcal{M}_r$  and the following assertion holds:*

*if  $\phi_1, \phi_2 \in \mathcal{U}$ ,  $z \in R$ ,*

$$G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1, \quad (2.6)$$

*then*

$$|G_{\phi_1}(z) - G_{\phi_2}(z)| \leq \epsilon.$$

*Proof.* By Proposition 2.1 there exist  $M > 0$  and a neighborhood  $\mathcal{U}_1$  of  $\beta$  in  $(\mathcal{M}, \rho)$  such that

$$\mathcal{U}_1 \subset \mathcal{M}_r \quad (2.7)$$

and the following property holds:

(P1) if  $\phi \in \mathcal{U}_1$ ,  $z \in R$ ,  $G_\phi(z) \leq \inf(G_\phi) + 1$ , then  $|z| \leq M$ .

By Proposition 2.2 there is a neighborhood  $\mathcal{U}$  of  $\beta$  in  $(\mathcal{M}, \rho)$  such that

$$\mathcal{U} \subset \mathcal{U}_1$$

and the following property holds:

(P2) If  $\phi_1, \phi_2 \in \mathcal{U}$ ,  $z \in R$ ,  $|z| \leq M$ , then

$$|G_{\phi_1}(z) - G_{\phi_2}(z)| \leq \epsilon. \quad (2.8)$$

Now assume that  $\phi_1, \phi_2 \in \mathcal{U}$ ,  $z \in R$  and (2.6) holds. By (2.6) and the property (P1),

$$|z| \leq M. \quad (2.9)$$

By (2.9) and property (P2), the inequality (2.8) is true. Proposition 2.3 is proved.  $\square$

**Proposition 2.4.** *Let  $\beta \in \mathcal{M}_r$ ,  $\epsilon > 0$ . Then there exists a neighborhood  $\mathcal{U}$  of  $\beta$  in  $\mathcal{M}$  such that  $\mathcal{U} \subset \mathcal{M}_r$  and for each  $\phi \in \mathcal{U}$ ,*

$$|\inf(G_\phi) - \inf(G_\beta)| \leq \epsilon.$$

*Proof.* Let a neighborhood  $\mathcal{U}$  of  $\beta$  in  $\mathcal{M}$  be as guaranteed by Proposition 2.3. Let  $\phi_1, \phi_2 \in \mathcal{U}$ . It is enough to show that

$$\inf(G_{\phi_2}) \leq \inf(G_{\phi_1}) + \epsilon.$$

By the choice of  $\mathcal{U}$  and Proposition 2.3 the inequality (2.8) holds for any  $z \in R$  satisfying (2.6). This implies that

$$\begin{aligned} \inf(G_{\phi_2}) &\leq \inf\{G_{\phi_2}(z) : z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\} \\ &\leq \inf\{G_{\phi_1}(z) + \epsilon : z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\} \\ &= \epsilon + \inf\{G_{\phi_1}(z) : z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\} \\ &= \epsilon + \inf(G_{\phi_1}) \end{aligned}$$

and

$$\inf(G_{\phi_2}) \leq \inf(G_{\phi_1}) + \epsilon.$$

This completes the proof of Proposition 2.4. □

**Proposition 2.5.** *Let  $\beta \in \mathcal{M}_r$ ,  $\bar{z} \in R$ ,*

$$G_\beta(z) > G_\beta(\bar{z}) \text{ for all } z \in R \setminus \{\bar{z}, -\bar{z}\}. \tag{2.10}$$

*Let  $\epsilon > 0$ . Then there exist a neighborhood  $\mathcal{U}$  of  $\beta$  in  $\mathcal{M}$  and  $\delta > 0$  such that  $\mathcal{U} \subset \mathcal{M}_r$  and that for each  $\phi \in \mathcal{U}$  and each  $z \in R$  satisfying*

$$G_\phi(z) \leq \inf(G_\phi) + \delta \tag{2.11}$$

*the inequality*

$$\min\{|z - \bar{z}|, |z + \bar{z}|\} \leq \epsilon$$

*is true.*

*Proof.* Let us assume the converse. Then for each natural number  $n$  there exist  $\phi_n \in \mathcal{M}_r$  and  $z_n \in R$  such that

$$\rho(\beta, \phi_n) \leq 1/n, \tag{2.12}$$

$$G_{\phi_n}(z_n) \leq \inf(G_{\phi_n}) + 1/n \tag{2.13}$$

and

$$\min\{|z_n - \bar{z}|, |z_n + \bar{z}|\} > \epsilon. \tag{2.14}$$

By (2.12), (2.13) and Proposition 2.1 the sequence  $\{z_n\}_{n=1}^\infty$  is bounded. Extracting a subsequence and reindexing, if necessary, we may assume without loss of generality that there exists

$$z_* = \lim_{n \rightarrow \infty} z_n. \tag{2.15}$$

By (2.12) and Proposition 2.4

$$\lim_{n \rightarrow \infty} \inf(G_{\phi_n}) = \inf(G_\beta). \tag{2.16}$$

Since the sequence  $\{z_n\}_{n=1}^\infty$  is bounded it follows from (2.12) and Proposition 2.2 that

$$\lim_{n \rightarrow \infty} [G_{\phi_n}(z_n) - G_\beta(z_n)] = 0. \tag{2.17}$$

It follows from (2.15), (2.17), (2.13), (2.16) that

$$G_\beta(z_*) = \lim_{n \rightarrow \infty} G_\beta(z_n) = \lim_{n \rightarrow \infty} G_{\phi_n}(z_n) = \lim_{n \rightarrow \infty} \inf(G_{\phi_n}) = \inf(G_\beta).$$

Thus

$$G_\beta(z_*) = \inf(G_\beta).$$

By (2.10) either  $z_* = \bar{z}$  or  $z_* = -\bar{z}$  and

$$\text{either } z_n \rightarrow \bar{z} \text{ or } z_n \rightarrow -\bar{z} \text{ as } n \rightarrow \infty.$$

This contradicts (2.14). The contradiction we have reached proves Proposition 2.5. □

3. A BASIC LEMMA

For  $\psi \in C^2(R)$  set

$$\|\psi\|_{C^2(R)} = \sup \left\{ |\psi^{(i)}(t)| : t \in R, i = 0, 1, 2 \right\}.$$

**Lemma 3.1.** *Let  $\beta \in \mathcal{M}_r, \epsilon > 0,$*

$$\beta(t) > \beta(\pi/2) \text{ for all } t \in [0, \pi/2]. \tag{3.1}$$

*Then there exist  $\phi \in \mathcal{M}_r, \bar{z} \in R$  such that*

$$\begin{aligned} \rho(\phi, \beta) &\leq \epsilon, \\ G_\phi(z) &> G_\phi(\bar{z}) \text{ for all } z \in R \setminus \{\bar{z}, -\bar{z}\}. \end{aligned}$$

*Proof.* There is  $\bar{z} \in R$  such that

$$G_\beta(\bar{z}) = \inf(G_\beta). \tag{3.2}$$

By Remark 1.1  $\bar{z} \neq 0$ . We may assume that

$$\bar{z} > 0. \tag{3.3}$$

Set

$$\bar{\theta} = \arctan(\bar{z}) \in (0, \pi/2). \tag{3.4}$$

There exists  $\psi \in C^\infty(R)$  such that

$$0 \leq \psi(t) \leq 1 \text{ for all } t \in R, \psi(t) = 0 \text{ if } |t| \geq 1, \psi(t) = 1, \text{ if } |t| \leq 1/2. \tag{3.5}$$

Set

$$\psi_1(t) = \psi(t)(1 - t^2), t \in R. \tag{3.6}$$

Clearly  $\psi_1 \in C^\infty(R),$

$$0 \leq \psi_1(t) \leq 1 \text{ for all } t \in R, \psi_1(t) = 0 \text{ if } |t| \geq 1, \psi_1(t) = 1 - t^2, \text{ if } |t| \leq 1/2, \tag{3.7}$$

$$1 = \psi_1(0) > \psi_1(t) \text{ for all } t \in R \setminus \{0\}. \tag{3.8}$$

By (3.4), (3.1), and the relation  $\beta \in \mathcal{M}_r,$  we can choose positive constants  $c_0, c_1$  such that  $c_0 > 1, c_1 < 1,$

$$[\bar{\theta} - c_0^{-1}, \bar{\theta} + c_0^{-1}] \subset (0, \pi/2), \tag{3.9}$$

$$\inf \{ \beta(t) : t \in [\bar{\theta} - c_0^{-1}, \bar{\theta} + c_0^{-1}] \} - \beta(\pi/2) > 4c_1, c_1 < -[\beta(0) + \beta''(0)], \tag{3.10}$$

$$\|\psi_1\|_{C^2(R)} c_1 c_0^2 < \epsilon. \tag{3.11}$$

Consider the function

$$\psi_2(t) = c_1 - c_1 \psi_1(c_0(t - \bar{\theta})), t \in R. \tag{3.12}$$

Clearly  $\psi_2 \in C^\infty(R).$  By (3.12), (3.7)

$$0 \leq \psi_2(t) \leq c_1, t \in R, \tag{3.13}$$

$$\psi_2(t) = c_1 \text{ for each } t \in R \text{ satisfying } |t - \bar{\theta}| \geq c_0^{-1}. \tag{3.14}$$

By (3.12), (3.8)

$$\psi_2(\bar{\theta}) = 0, \tag{3.15}$$

$$\psi_2(t) > 0 \text{ for each } t \in R \setminus \{\bar{\theta}\}. \tag{3.16}$$

It is not difficult to see that there exists a function  $\psi_3 : R \rightarrow R$  such that

$$\psi_3(t) = \psi_2(t), t \in [0, \pi/2], \tag{3.17}$$

$$\psi_3(-t) = \psi_3(t), t \in R,$$

$$\psi_3(t + \pi) = \psi_3(t), t \in R.$$

It is not difficult to see that  $\psi_3 \in C^\infty(\mathbb{R})$ ,

$$0 \leq \psi_3(t) \leq c_1, \quad t \in \mathbb{R}, \quad (3.18)$$

$$\begin{aligned} \psi_3(\bar{\theta}) &= 0, \\ \psi_3(t) &> 0 \text{ for all } t \in [0, \pi/2] \setminus \{\bar{\theta}\}, \\ \psi_3(t) &> 0 \text{ for all } t \in [-\pi/2, 0] \setminus \{-\bar{\theta}\}. \end{aligned} \quad (3.19)$$

Define

$$\phi(t) = \beta(t) + \psi_3(t), \quad t \in \mathbb{R}. \quad (3.20)$$

Clearly  $\phi \in C^2(\mathbb{R})$ . By (3.20), (1.1), (3.18), (1.3), (3.17), (1.4) we have

$$\phi(t) \geq 0 \text{ for all } t \in \mathbb{R} \text{ and } \phi(t) = \phi(-t) = \phi(t + \pi) \text{ for all } t \in \mathbb{R}. \quad (3.21)$$

By (3.20), the relation  $\beta \in \mathcal{M}_r$ , (3.18)

$$\phi(t) > 0 \text{ for all } t \in \mathbb{R}. \quad (3.22)$$

We show that

$$\phi(0) + \phi''(0) < 0. \quad (3.23)$$

By (3.20), (3.17)

$$\begin{aligned} \phi(0) + \phi''(0) &= \beta(0) + \beta''(0) + \psi_3(0) + \psi_3''(0) \\ &= \beta(0) + \beta''(0) + \psi_2(0) + \psi_2''(0). \end{aligned} \quad (3.24)$$

By (3.14), (3.9),

$$\psi_2(0) = c_1, \quad \psi_2''(0) = 0. \quad (3.25)$$

Combined with (3.24), (3.10) this implies that

$$\phi(0) + \phi''(0) = \beta(0) + \beta''(0) + c_1 < 0.$$

Thus (3.23) is true. We show that

$$\phi(\pi/2) \leq \phi(t) \leq \phi(0) \quad (3.26)$$

for all  $t \in \mathbb{R}$ . Clearly it is enough to show that this inequality holds for all  $t \in [0, \pi/2]$ .

Let  $t \in [0, \pi/2]$ . Then by (3.20), (3.17)

$$\phi(t) = \beta(t) + \psi_2(t). \quad (3.27)$$

By (3.27), (3.13), (3.25), (1.2), (3.17), (3.20)

$$\begin{aligned} \phi(t) &\leq \beta(t) + c_1 = \beta(t) + \psi_2(0) \leq \beta(0) + \psi_2(0) \\ &= \beta(0) + \psi_3(0) = \phi(0). \end{aligned}$$

Thus

$$\phi(t) \leq \phi(0). \quad (3.28)$$

By (3.20), (3.17), (3.14), (3.9)

$$\phi(\pi/2) = \beta(\pi/2) + \psi_2(\pi/2) = \beta(\pi/2) + c_1. \quad (3.29)$$

There are two cases:

- (a)  $|t - \bar{\theta}| \geq c_0^{-1}$ ; (b)  $|t - \bar{\theta}| < c_0^{-1}$ .

Consider the case (a). Then it follows from (3.17), (3.14) that

$$\psi_3(t) = \psi_2(t) = c_1.$$

Combined with (3.20), (1.2), (3.29) this implies that

$$\phi(t) = \beta(t) + c_1 \geq \beta(\pi/2) + c_1 = \phi(\pi/2)$$

and

$$\phi(t) \geq \phi(\pi/2). \tag{3.30}$$

Consider the case (b) with

$$|t - \bar{\theta}| < c_0^{-1}. \tag{3.31}$$

By (3.20), (3.17), (3.31), (3.13), (3.10), (3.29)

$$\phi(t) = \beta(t) + \psi_2(t) \geq \beta(t) > \beta(\pi/2) + 4c_1 > \phi(\pi/2)$$

and (3.30) is true. Thus (3.30) is true in both cases. (3.30), (3.28) imply (3.26).

We have shown that  $\phi \in \mathcal{M}_r$ . By (1.6), (3.20), (3.17), (3.13), (3.11), (3.12)

$$\begin{aligned} \rho(\beta, \phi) &= \sup\{\psi_3(t), |\psi_3'(t)|, |\psi_3''(t)| : t \in [-\pi/2, \pi/2]\} \\ &= \sup\{\psi_2(t), |\psi_2'(t)|, |\psi_2''(t)| : t \in [0, \pi/2]\} \\ &\leq \max\{c_1, \|\psi_1\|_{C^2(R)} c_1 c_0, \|\psi_1\|_{C^2(R)} c_1 c_0^2\} = c_1 c_0^2 \|\psi_1\|_{C^2} < \epsilon. \end{aligned}$$

Thus

$$\rho(\phi, \beta) < \epsilon. \tag{3.32}$$

We have

$$\phi(t) \geq \beta(t) \text{ for all } t \in R. \tag{3.33}$$

This implies that

$$G_\phi(t) \geq G_\beta(t) \text{ for all } t \in R. \tag{3.34}$$

By (1.9), (3.4), (3.20), (3.17), (3.15), (3.2)

$$\begin{aligned} G_\phi(\bar{z}) &= \phi(\arctan(\bar{z}))(1 + \bar{z}^2)^{1/2} = \phi(\bar{\theta})(1 + \bar{z}^2)^{1/2} \\ &= (\beta + \psi_3)(\bar{\theta})(1 + \bar{z}^2)^{1/2} = \beta(\bar{\theta})(1 + \bar{z}^2)^{1/2} = G_\beta(\bar{z}) = \inf(G_\beta). \end{aligned} \tag{3.35}$$

(3.35), (3.34) imply that

$$\inf(G_\phi) = \inf(G_\beta) = G_\phi(\bar{z}) = G_\beta(\bar{z}). \tag{3.36}$$

Let

$$z \in R \setminus \{\bar{z}, -\bar{z}\}. \tag{3.37}$$

It follows from (1.9), (3.20) that

$$\begin{aligned} G_\phi(z) &= \phi(\arctan(z))(1 + z^2)^{1/2} = \beta(\arctan(z))(1 + z^2)^{1/2} \\ &\quad + \psi_3(\arctan(z))(1 + z^2)^{1/2} = G_\beta(z) + \psi(\arctan(z))(1 + z^2)^{1/2}. \end{aligned} \tag{3.38}$$

By (3.37), (3.19)

$$\psi_3(\arctan(z)) > 0.$$



Combined with (3.38), (3.36) this inequality implies that

$$G_\phi(z) > G_\beta(z) \geq G_\beta(\bar{z}) = G_\phi(\bar{z}).$$

Lemma 3.1 is proved. □

**Lemma 3.2.** *Let  $\beta \in \mathcal{M}_r$ ,  $\epsilon > 0$ . Then there exists  $\tilde{\beta} \in \mathcal{M}_r$  such that  $\rho(\beta, \tilde{\beta}) < \epsilon$ ,  $\tilde{\beta}(t) > \tilde{\beta}(\pi/2)$  for all  $t \in [0, \pi/2)$ .*

*Proof.* Consider the function

$$\beta_0(t) = \cos(2t) + 3/2, t \in R.$$

Clearly  $\beta_0 \in \mathcal{M}_r$ . For all  $t \in [0, \pi/2)$

$$\beta_0(t) = \cos(2t) + 3/2 > (\cos \pi) + 3/2 = \beta_0(\pi/2).$$

For each natural number  $n$  set

$$\beta_n(t) = \beta(t) + n^{-1}\beta_0(t), t \in R.$$

Clearly for all natural  $n$ ,  $\beta_n \in \mathcal{M}_r$ ,  $\beta_n(t) > \beta_n(\pi/2)$  for all  $t \in [0, \pi/2)$ ,

$$\beta_n \rightarrow \beta \text{ as } n \rightarrow \infty \text{ in } (\mathcal{M}, \rho).$$

Lemma 3.2 is proved. □

Lemmas 3.1 and 3.2 imply

**Lemma 3.3** (basic lemma). *Let  $\beta \in \mathcal{M}_r$ ,  $\epsilon > 0$ . Then there exists  $\phi \in \mathcal{M}_r$ ,  $\bar{z} \in R$  such that  $\rho(\phi, \beta) < \epsilon$ ,*

$$G_\phi(z) > G_\phi(\bar{z}) \text{ for all } z \in R \setminus \{\bar{z}, -\bar{z}\}.$$

#### 4. PROOF OF THEOREM 1.1

By Proposition 1.1 and Lemma 3.3,  $\mathcal{F}$  is an everywhere dense subset of  $(\mathcal{M}, \rho)$ . Let  $\beta \in \mathcal{F}$ ,  $z_\beta > 0$  satisfy (1.11),  $n$  be a natural number.

By Proposition 2.5 there are an open neighborhood  $\mathcal{U}(\beta, n)$  of  $\beta$  in  $(\mathcal{M}, \rho)$  and a number  $\delta(\beta, n) > 0$  such that

$$\mathcal{U}(\beta, n) \subset \mathcal{M}_r$$

and the following property holds:

(P3) if  $\phi \in \mathcal{U}(\beta, n)$ ,  $z \in R$ ,  $G_\phi(z) \leq \inf(G_\phi) + \delta(\beta, n)$ , then

$$\min\{|z - z_\beta|, |z + z_\beta|\} \leq 1/n.$$

Set

$$\mathcal{F}_0 = \bigcap_{n=1}^\infty \mathcal{U}(\beta, n) \cup \{\mathcal{U}(\beta, n) : \beta \in \mathcal{F}, n \text{ is a natural number}\}. \tag{4.1}$$

Clearly  $\mathcal{F} \subset \mathcal{F}_0$  and  $\mathcal{F}_0$  is a countable intersection of open everywhere dense subsets of  $(\mathcal{M}, \rho)$ . Let  $\phi \in \mathcal{F}_0$ ,  $z_1, z_2 \in [0, \infty)$ ,

$$G_\phi(z_1) = G_\phi(z_2) = \inf(G_\phi). \tag{4.2}$$

Let  $n \geq 1$  be an integer. By (4.1) there is  $\beta \in \mathcal{F}$  such that

$$\phi \in \mathcal{U}(\beta, n). \tag{4.3}$$

By (4.3), (P3), (4.2),

$$\begin{aligned} |z_i - z_\beta| &\leq 1/n, \quad i = 1, 2, \\ |z_1 - z_2| &\leq 2/n. \end{aligned}$$

Since  $n$  is any natural number we conclude that  $z_1 = z_2$ ,  $\phi \in \mathcal{F}$  and  $\mathcal{F}_0 = \mathcal{F}$ . This completes the proof of Theorem 1.1.

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