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# WEAK LINKING THEOREMS AND SCHRÖDINGER EQUATIONS WITH CRITICAL SOBOLEV EXPONENT

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**Abstract.** In this paper we establish a variant and generalized weak linking theorem, which contains more delicate result and insures the existence of bounded Palais–Smale sequences of a strongly indefinite functional. The abstract result will be used to study the semilinear Schrödinger equation $-\Delta u + V(x)u = K(x)|u|^{2^*-2}u + g(x,u), u \in W^{1,2}(\mathbf{R}^N)$, where $N \geq 4; V, K, g$ are periodic in $x_j$ for $1 \leq j \leq N$ and 0 is in a gap of the spectrum of $-\Delta + V$; $K > 0$. If $0 < g(x,u)u \leq c|u|^{2^*}$ for an appropriate constant $c$, we show that this equation has a nontrivial solution.

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## 1. Introduction

In this article, the aim is to study the following semilinear Schrödinger equation with critical Sobolev exponent and periodic potential:

$$-\Delta u + V(x)u = K(x)|u|^{2^*-2}u + g(x,u), \qquad u \in W^{1,2}(\mathbf{R}^N), \tag{S}$$

where $N \geq 4; 2^* := 2N/(N-2)$ is the critical Sobolev exponent and $g$ is of subcritical growth.

First of all, we recall that the equation

$$-\Delta u + \lambda u = |u|^{2^*-2}u, \qquad \lambda \neq 0, \tag{1.1}$$

**has only the trivial solution** $u = 0$ in $W^{1,2}(\mathbf{R}^N)$ (*cf.* [4]). Therefore, the existence of nontrivial solution of (S) is an interesting problem.

Before we state the main result, we introduce the following conditions:

($\mathbf{S}_1$) $V, K \in \mathcal{C}(\mathbf{R}^N, \mathbf{R}), g \in \mathcal{C}(\mathbf{R}^N \times \mathbf{R}, \mathbf{R}), k_0 := \inf_{x\in\mathbf{R}^N} K(x) > 0; V, K, g$ are 1-periodic in $x_j$ for $j = 1, ..., N$;

($\mathbf{S}_2$) $0 \notin \sigma(-\Delta + V)$ and $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset$, where $\sigma$ denotes the spectrum in $L^2(\mathbf{R}^N)$;

($\mathbf{S}_3$) $K(x_0) := \max_{x\in\mathbf{R}^N} K(x)$ and $K(x) - K(x_0) = o(|x - x_0|^2)$ as $x \to x_0$ and $V(x_0) < 0$;

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($\mathbf{S}_4$) $|g(x,u)| \le c_0(1+|u|^{p-1})$ for all $(x,u) \in \mathbf{R}^N \times \mathbf{R}$, where $c_0 > 0$ and $p \in (2, 2^*)$. Moreover, $g(x,u)/|u|^{2^*-1} \to 0$ as $u \to 0$ uniformly for $x \in \mathbf{R}^N$;

($\mathbf{S}_5$) $g(x,u)u > 0$ for all $x \in \mathbf{R}^N$ and $u \neq 0$.

The main result is the following:

**Theorem 1.1.** *Assume that* ($S_1$–$S_5$) *hold. If*

$$\frac{k_0}{m_g} \ge \frac{N-2}{2}, \qquad where \qquad m_g := \max_{x\in\mathbf{R}^N,\, u\in\mathbf{R}\setminus\{0\}} \frac{g(x,u)u}{|u|^{2^*}}, \tag{1.2}$$

*then equation* (S) *has a solution $u \neq 0$. Particularly, if $K(x) \equiv k_0 > 0$,* ($S_3$) *can be deleted and the same result holds.*

An equivalent form of Theorem 1.1 is the following:

**Corrolary 1.1.** *Assume that* ($S_1$–$S_5$) *hold. Then the following Schrödinger equation*

$$-\Delta u + V(x)u = K(x)|u|^{2^*-2}u + \beta g(x,u), \quad u \in W^{1,2}(\mathbf{R}^N),$$

*has a nontrivial solution for all $\beta \in (0, 2k_0/(m_g(N-2))]$. If $K(x) \equiv k_0 > 0$, condition* ($S_3$) *can be omitted and the same result holds.*

**Remark 1.1.** It is an open problem whether or not the results of the present paper remain true for the case of $N = 3$. This problem is also raised by Y.Y. Li in private communications.

Now we make some comments on this problem and the main results. Under the hypotheses on $V$ the spectrum of $-\Delta + V$ in $L^2(\mathbf{R}^N)$ is purely continuous and bounded below and is the union of disjoint closed intervals (*cf.* Th. XIII. 100 of [17] and Th. 4.5.9 of [13]), which makes the problem difficult to be dealt with.

Recently, equation (S) was studied in [6], which also generalized the early results obtained in [7]. In [6], the assumption

$$0 \le \gamma G(x,u) \le ug(x,u) \text{ on } \mathbf{R}^N \times \mathbf{R}, \tag{1.3}$$

where $\gamma = 2; G(x,u) := \int_0^u g(x,s)ds$, was imposed in order to prove the boundedness of the Palais–Smale sequence. Obviously, this condition contains the case of $g \equiv 0$. Condition (1.3) has three disadvantages: the first is that one has to compute the primitive function $G$ of $g$; the second is that one has to check the second inequality of (1.3); the third is that (1.3) does not contain the sublinear (at infinity) case and some asymptotically linear (at infinity) case. But sometimes, it is either impossible to compute $G$ so that (1.3) can be checked or the second inequality of (1.3) does not hold.

These cases happen on the following three examples:

$$\text{(i)}\ \ g(x,u) := \begin{cases} c|u|^{2^*}u\mathrm{e}^{-\sin^2 u} & |u| \le 1 \\ c|u|^{-2/3}u\mathrm{e}^{-\sin^2 u}(1+\ln|u|) & |u| \ge 1, \end{cases}$$

$$\text{(ii)}\ \ g(x,u) := \begin{cases} c|u|^{2^*}u & |u| \le 1 \\ c|u|^{-2/3}u & |u| \ge 1, \qquad \text{(sublinear at infinity)} \end{cases}$$

$$\text{(iii)}\ \ g(x,u) := \begin{cases} c|u|^{2^*}u & |u| \le 1 \\ \frac{c}{2}(u+|u|^{-2/3}u) & |u| \ge 1, \qquad \text{(asymptotically linear at infinity).} \end{cases}$$

However, we emphasize that the above examples satisfy the hypotheses of Theorem 1.1 of the present paper for appropriate $c > 0$. Moreover, conditions ($S_4$) and ($S_5$) permit the nonlinearity $g$ to be superlinear, asymptotically linear or sublinear.

Evidently, if we set

$$\bar{m}_g(r) = \max_{x\in\mathbf{R}^N,\, |u|\ge r \text{ or } |u|\le 1/r} \frac{g(x,u)u}{|u|^{2^*}},$$

then $k_0/\bar{m}_g(r) \to \infty$ as $r \to \infty$ . It is an **open** problem whether or not assumption (1.2) can be concealed or equivalently, Corollary 1.1 holds for all $\beta > 0$. On the other hand, it should be mentioned that (1.2) is the price to pay for relaxing (1.3).

Equation (S) with $K(x) \equiv 0$, *i.e.*, the nonlinear term is of subcritical growth, has been studied by several authors (for example, *cf.* [1–3, 5, 8, 10–12, 24, 26] and the references cited therein). In those papers, the Ambrosetti–Rabinowitz condition (1.3) with $\gamma > 2$ was needed. In [23], the authors considered the asymptotically linear case. In [25] (see also [3]), zero is an end point of $\sigma(-\Delta + V)$. In [14], the author studied a special case $-\Delta u = Ku^5$ in $\mathbf{R}^3$ (see also [15] for higher dimension case on $S^N$). Very little is known for (S) with critical Sobolev exponent and periodic potential.

Without (1.3) with $\gamma \geq 2$, the problem becomes more complicated. The main obstacle is how to get a bounded Palais–Smale sequence. To get over this road block, we establish a variant and generalized weak linking theorem. Roughly speaking, let $E$ be a Hilbert space, let $N \subset E$ be a separable subspace, and let $Q \subset N$ be a bounded open convex set, with $p_0 \in Q$. Let $F$ be a "*weak*" continuous map of $E$ onto $N$ such that $F|_Q = id$ and that $F(u - v) - (F(u) - F(v))$ is contained in a fixed finite-dimensional subspace of $E$ for all $u, v \in E$. Then under suitable hypotheses, $\partial Q$ links $F^{-1}(p_0)$ with respect to a suitable restricted class of deformations of $\bar{Q}$. We will define a family of $\mathcal{C}^1$-functional $\{H_\lambda\}_{\lambda \in [1,2]}$ which is related to problem (S). Since the spectrum of $-\Delta + V$ in $L^2(\mathbf{R}^N)$ is purely continuous, both positive and negative subspaces of the functional $H_\lambda$ are infinite-dimensional. Moreover, $H_\lambda$ is unbounded from both below and above, the so-called strongly indefinite functional. Furthermore, because of the weaker assumptions (particularly, without (1.3)), the usual minimax techniques (*e.g.* [1–7, 10, 11, 14, 15]), can not be used here. However, by using the new weak linking theorem, we show that $\{H_\lambda\}$ has a bounded Palais–Smale sequence for almost every $\lambda \in [1, 2]$. The main idea is the Monotonicity Trick due to [10, 11] (see also [21] for an earlier application). Other applications of this trick can be found in [23, 27, 28, 30–32]. We also give the estimates of the energy, *i.e.*, the energy lies in $[\inf_{F^{-1}(p_0)} H_\lambda, \sup_{\bar{Q}} H_1]$. By this way, there is no need to impose some strong conditions for proving the boundedness of Palais–Smale sequences. In other words, we permit much more freedom for the nonlinearity.

The paper is organized as follows: in Section 2, we establish a variant weak linking theorem. In Section 3, equation (S) will be studied. In Section 4, an Appendix will be given.

## 2. A variant weak linking theorem

Let $E$ be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and have an orthogonal decomposition $E = N \oplus N^\perp$, where $N \subset E$ is a closed and separable subspace. Since $N$ is separable, we can define a new norm $|v|_w$ satisfying $|v|_w \leq \|v\|, \forall v \in N$ and such that the topology induced by this norm is equivalent to the weak topology of $N$ on bounded subset of $N$ (see Appendix of Sect. 4). For $u = v + w \in E = N \oplus N^\perp$ with $v \in N, w \in N^\perp$, we define $|u|_w^2 = |v|_w^2 + \|w\|^2$, then $|u|_w \leq \|u\|, \forall u \in E$.

Particularly, if $(u_n = v_n + w_n)$ is $|\cdot|_w$-bounded and $u_n \overset{|\cdot|_w}{\to} u$, then $v_n \rightharpoonup v$ weakly in $N$, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in $E$ (*cf.* [9]).

Let $Q \subset N$ be a $\|\cdot\|$-bounded open convex subset, $p_0 \in Q$ be a fixed point. Let $F$ be a $|\cdot|_w$-continuous map from $E$ onto $N$ satisfying

- $F|_Q = id$; $F$ maps bounded sets to bounded sets;
- there exists a fixed finite-dimensional subspace $E_0$ of $E$ such that $F(u - v) - (F(u) - F(v)) \subset E_0, \forall v, u \in E$;
- $F$ maps finite-dimensional subspaces of $E$ to finite-dimensional subspaces of $E$.

We use the letter $c$ to denote various positive constants.

$$A := \partial Q, \quad B := F^{-1}(p_0),$$

where $\partial Q$ denotes the $\|\cdot\|$-boundary of $Q$.

**Remark 2.1.** There are many examples:

(i) let $N = E^-$, $N^\perp = E^+$, then $E = E^- \oplus E^+$ and let $Q := \{u \in E^- : \|u\| < R\}$, $p_0 = 0 \in Q$. For any $u = u^- \oplus u^+ \in E$, define $F : E \mapsto N$ by $Fu := u^-$, then $A := \partial Q$, $B := F^{-1}(p_0) = E^+$ satisfy the above conditions;

(ii) let $E = E^- \oplus E^+$, $z_0 \in E^+$ with $\|z_0\| = 1$. For any $u \in E$, we write $u = u^- \oplus sz_0 \oplus w^+$ with $u^- \in E^-, s \in \mathbf{R}$, $w^+ \in (E^- \oplus \mathbf{R}z_0)^\perp := E_1^+$. Let $N := E^- \oplus \mathbf{R}z_0$. For $R > 0$, let $Q := \{u := u^- + sz_0 : s \in \mathbf{R}^+, u^- \in E^-, \|u\| < R\}, p_0 = s_0 z_0 \in Q$, $s_0 > 0$. Let $F : E \mapsto N$ be defined by $Fu := u^- + \|sz_0 + w^+\|z_0$, then $F, Q, p_0$ satisfy the above conditions with

$$B = F^{-1}(s_0 z_0) = \{u := sz_0 + w^+ : s \geq 0, w^+ \in E_1^+, \|sz_0 + w^+\| = s_0\}\cdot$$

In fact, according to the definition, $F|_Q = id$ and $F$ maps bounded sets to bounded sets. On the other hand, for any $u, v \in E$, we write $u = u^- + sz_0 + w^+, v = v^- + tz_0 + w_1^+$, then

$$F(u) = u^- + \|sz_0 + w^+\|z_0, \quad F(v) = v^- + \|tz_0 + w_1^+\|z_0,$$

$$F(u - v) = u^- - v^- + \|(s-t)z_0 + w^+ - w_1^+\|z_0,$$

therefore,

$$\begin{aligned} F(u-v) - (F(u) - F(v)) &= \Big(\|(s-t)z_0 + w^+ - w_1^+\| - \|sz_0 + w^+\| + \|tz_0 + w_1^+\|\Big) z_0 \\ &\subset \mathbf{R}z_0 := E_0 \quad \text{(an 1-dimensional subspace).} \end{aligned}$$

For $H \in \mathcal{C}^1(E, \mathbf{R})$, we define

$$\begin{aligned} \Gamma := \Big\{ h : [0,1] \times \bar{Q} \mapsto E, h \text{ is } |\cdot|_w\text{-continuous. For any } (s_0, u_0) \in [0,1] \times \bar{Q}, \\ \text{there is a } |\cdot|_w - \text{ neighborhood } U_{(s_0,u_0)} \text{ such that} \\ \{u - h(t,u) : (t,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \bar{Q})\} \subset E_{\text{fin}}, \\ h(0,u) = u, H(h(s,u)) \leq H(u), \forall u \in \bar{Q} \Big\}, \end{aligned}$$

then $\Gamma \neq \emptyset$ since $id \in \Gamma$. Here and then, we use $E_{\text{fin}}$ to denote various finite-dimensional subspaces of $E$ whose exact dimensions are irrelevant and depend on $(s_0, u_0)$.

The variant weak linking theorem is:

**Theorem 2.1.** *The family of $\mathcal{C}^1$-functional $(H_\lambda)$ has the form*

$$H_\lambda(u) := I(u) - \lambda J(u), \qquad \forall \lambda \in [1,2].$$

*Assume*

(a) $J(u) \geq 0, \forall u \in E; H_1 := H$;

(b) $I(u) \to \infty$ *or* $J(u) \to \infty$ *as* $\|u\| \to \infty$;

(c) $H_\lambda$ *is* $|\cdot|_w$*-upper semicontinuous;* $H'_\lambda$ *is weakly sequentially continuous on* $E$*. Moreover,* $H_\lambda$ *maps bounded sets to bounded sets;*

(d) $\sup_A H_\lambda < \inf_B H_\lambda, \forall \lambda \in [1,2]$.

*Then for almost all $\lambda \in [1,2]$, there exists a sequence $(u_n)$ such that*

$$\sup_n \|u_n\| < \infty, \quad H'_\lambda(u_n) \to 0, \quad H_\lambda(u_n) \to C_\lambda;$$

*where*

$$C_\lambda := \inf_{h\in\Gamma} \sup_{u\in\bar{Q}} H_\lambda(h(1,u)) \in [\inf_B H_\lambda, \sup_{\bar{Q}} H].$$

Before proving this theorem, let us make some remarks.

**Remark 2.2.** Similar weak linking was developed in [18–20, 29]. In [18–20], conditions "$F|_N \equiv id$" and "$F(v-w) = v - Fw$ for all $v \in N, w \in E$" were stated but not needed. All that was used was $F|_Q \equiv id$ and $F(v-w) = v - Fw$ for all $v \in Q, w \in E$. This was noted in [29]. Particularly, we emphasize that because the monotonicity trick was not used in [18–20,29], the boundedness of Palais–Smale sequence was not a consequence of the Theorems. Therefore, some compactness conditions were introduced and played an important role. The results of [18–20, 29] can not be used to deal with equation (S).

**Remark 2.3.** In [12] (see also [26]), some theorems were given which contained only a particular linking and the boundedness of Palais–Smale sequence is also remained unknown. Therefore, in applications, Ambrosetti–Rabinowitz type condition (1.3) with $\gamma > 2$ is needed. In [12, 26], a $\tau$-topology is specially constructed to accommodate the splitting of $E$ into subspace and by this, a new degree of Leray–Schauder type is established. The new degree is also applied in [23, 25, 27, 28].

*Proof of Theorem 2.1.*

**Step 1.** We prove that $C_\lambda \in [\inf_B H_\lambda, \sup_Q H]$. Evidently, by the definition of $C_\lambda$,

$$C_\lambda \leq \sup_{u\in\bar{Q}} H_\lambda(u) \leq \sup_{u\in\bar{Q}} H_1(u) \equiv \sup_{u\in\bar{Q}} H(u) < \infty.$$

To show $C_\lambda \geq \inf_B H_\lambda$ for all $\lambda \in [1,2]$, we have to prove that $h(1,\bar{Q}) \cap B \neq \emptyset$ for all $h \in \Gamma$. By hypothesis, the map $Fh : [0,1] \times \bar{Q} \to N$ is $|\cdot|_w$-continuous. Let $K := [0,1] \times \bar{Q}$. Then $K$ is $|\cdot|_w$-compact. In fact, since $K$ is bounded with respect to both norms $|\cdot|_w$ and $\|\cdot\|$, for any $(t_n, v_n) \in K$, we may assume that $v_n \rightharpoonup v_0$ weakly in $E$ and that $t_n \to t_0 \in [0,1]$. Then $v_0 \in \bar{Q}$ since $\bar{Q}$ is convex. Since on the bounded set $Q \subset N$, the $|\cdot|_w$-topology is equivalent to the weak topology, then $u_n \xrightarrow{|\cdot|_w} v_0$. So, $K$ is $|\cdot|_w$-compact. By the definition of $\Gamma$, for any $(s_0, u_0) \in K$, there is a $|\cdot|_w$-neighborhood $U_{(s_0,u_0)}$ such that

$$\{u - h(t,u) : (t,u) \in U_{(s_0,u_0)} \cap K\} \subset E_{\text{fin}},$$

here and then, we use $E_{\text{fin}}$ to denote various finite-dimensional subspaces of $E$ whose exact dimensions are irrelevant. Now, $K \subset \cup_{(s,u)\in K} U_{(s,u)}$. Since $K$ is $|\cdot|_w$-compact, $K \subset \cup_{i=1}^{j_0} U_{(s_i,u_i)}, (s_i, u_i) \in K$. Consequently,

$$\{u - h(t,u) : (t,u) \in K\} \subset E_{\text{fin}}.$$

Hence, by the basic assumptions on $F$,

$$F\{u - h(t,u) : (t,u) \in K\} \subset E_{\text{fin}}$$

and

$$\{u - Fh(t,u) : (t,u) \in K\} \subset E_{\text{fin}}.$$

Then we can choose a finite-dimensional subspace $E_{\text{fin}}$ such that $p_0 \in E_{\text{fin}}$ and that

$$Fh : [0,1] \times (\bar{Q} \cap E_{\text{fin}}) \to E_{\text{fin}}.$$

We *claim* that $Fh(t,u) \neq p_0$ for all $u \in \partial(\bar{Q} \cap E_{\text{fin}}) = \partial\bar{Q} \cap E_{\text{fin}}$ and $t \in [0,1]$. By way of negation, if there exist $t_0 \in [0,1]$ and $u_0 \in \partial\bar{Q} \cap E_{\text{fin}}$ such that $Fh(t_0,u_0) = p_0$, *i.e.*, $h(t_0,u_0) \in B$. It follows that

$$H_1(u_0) \geq H_1(h(t_0,u_0)) \geq \inf_B H_1 > \sup_{\partial\bar{Q}} H_1,$$

which contradicts the assumption (d). Thus, our *claim* is true. By the homotopy invariance of Brouwer degree, we get that

$$\begin{aligned}\deg(Fh(1,\cdot), Q\cap E_{\text{fin}}, p_0) &= \deg(Fh(0,\cdot), Q\cap E_{\text{fin}}, p_0)\\ &= \deg(id, Q\cap E_{\text{fin}}, p_0)\\ &= 1.\end{aligned}$$

Therefore, there exists $u_0 \in Q\cap E_{\text{fin}}$ such that $Fh(1,u_0)=p_0$.

**Step 2.** Evidently, $\lambda \mapsto C_\lambda$ is nonincreasing, hence $C'_\lambda = \dfrac{dC_\lambda}{d\lambda}$ exists for almost every $\lambda\in[1,2]$. We consider those $\lambda\in[1,2]$ where $C'_\lambda$ exists and use the monotonicity trick (see *e.g.* [21]).

Let $\lambda_n\in[1,2]$ be a strictly increasing sequence such that $\lambda_n\to\lambda$. Then there exists $n(\lambda)$ large enough such that

$$-C'_\lambda - 1 \le \frac{C_{\lambda_n}-C_\lambda}{\lambda-\lambda_n} \le -C'_\lambda + 1 \quad \text{for} \quad n\ge n(\lambda). \tag{2.1}$$

**Step 3.** There exists a sequence $h_n\in\Gamma$, $k:=k(\lambda)>0$ such that $\|h_n(1,u)\|\le k$ if $H_\lambda(h_n(1,u))\ge C_\lambda-(\lambda-\lambda_n)$. In fact, by the definition of $C_{\lambda_n}$, let $h_n\in\Gamma$ be such that

$$\sup_{u\in\bar Q} H_{\lambda_n}(h_n(1,u)) \le C_{\lambda_n} + (\lambda-\lambda_n). \tag{2.2}$$

Therefore, if $H_\lambda(h_n(1,u))\ge C_\lambda-(\lambda-\lambda_n)$ for some $u\in\bar Q$, then for $n\ge n(\lambda)$(large enough), by (2.1) and (2.2),

$$\begin{aligned}J(h_n(1,u)) &= \frac{H_{\lambda_n}(h_n(1,u)) - H_\lambda(h_n(1,u))}{\lambda-\lambda_n}\\ &\le \frac{C_{\lambda_n}-C_\lambda}{\lambda-\lambda_n}+2\\ &\le -C'_\lambda+3\end{aligned}$$

and

$$\begin{aligned}I(h_n(1,u)) &= H_{\lambda_n}(h_n(1,u)) + \lambda_n J(h_n(1,u))\\ &\le C_{\lambda_n} + (\lambda-\lambda_n) + \lambda_n(-C'_\lambda+3)\\ &\le C_\lambda - \lambda C'_\lambda + 3\lambda.\end{aligned}$$

By assumption (b), $\|h_n(1,u)\|\le k:=k(\lambda)$.

**Step 4.** By step 2 and (2.2)

$$\sup_{u\in\bar Q} H_\lambda(h_n(1,u)) \le \sup_{u\in\bar Q} H_{\lambda_n}(h_n(1,u)) \le C_\lambda + (2-C'_\lambda)(\lambda-\lambda_n).$$

**Step 5.** For $\varepsilon>0$, define

$$F_\varepsilon(\lambda) := \{u\in E : \|u\|\le k+4, |H_\lambda(u)-C_\lambda|\le\varepsilon\}\cdot \tag{2.3}$$

Then we claim, for $\varepsilon$ small enough, that $\inf\{\|H'_\lambda(u)\| : u\in F_\varepsilon(\lambda)\}=0$. Otherwise, there exists $\varepsilon_0>0$ such that $\|H'_\lambda(u)\|\ge\varepsilon_0$ for all $u\in F_{\varepsilon_0}(\lambda)$. Let $h_n\in\Gamma$ be as in Steps 3, 4 and $n$ be large enough such that $\lambda-\lambda_n\le\varepsilon_0$ and $(2-C'_\lambda)(\lambda-\lambda_n)\le\varepsilon_0$. Define

$$F^*_{\varepsilon_0}(\lambda) := \{u\in E : \|u\|\le k+4, C_\lambda-(\lambda-\lambda_n)\le H_\lambda(u)\le C_\lambda+\varepsilon_0\}\cdot \tag{2.4}$$

Clearly, $F^*_{\varepsilon_0}(\lambda) \subset F_{\varepsilon_0}(\lambda)$. Now, we consider

$$F^*(\lambda) := \{u \in E : H_\lambda(u) < C_\lambda - (\lambda - \lambda_n)\} \tag{2.5}$$

and $F^*_{\varepsilon_0}(\lambda) \cup F^*(\lambda)$. Since $\|H'_\lambda(u)\| \geq \varepsilon_0$ for $u \in F^*_{\varepsilon_0}(\lambda)$, we let

$$h_\lambda(u) := \frac{2H'_\lambda(u)}{\|H'_\lambda(u)\|^2} \qquad \text{for } u \in F^*_{\varepsilon_0}(\lambda).$$

Then $\langle H'_\lambda(u), h_\lambda(u)\rangle = 2$ for $u \in F^*_{\varepsilon_0}(\lambda)$. Since $H'_\lambda$ is weakly sequentially continuous, if $\{u_n\}$ is $\|\cdot\|$-bounded and $u_n \overset{|\cdot|_w}{\to} \bar{u}$, then $u_n \rightharpoonup \bar{u}$ in $E$, hence

$$\langle H'_\lambda(u_n), h_\lambda(u)\rangle \to \langle H'_\lambda(\bar{u}), h_\lambda(u)\rangle$$

as $n \to \infty$. It follows that $\langle H'_\lambda(\cdot), h_\lambda(u)\rangle$ is $|\cdot|_w$-continuous on sets bounded in $E$. Therefore, there is an open $|\cdot|_w$-neighborhood $\mathcal{N}_u$ of $u$ such that

$$\langle H'_\lambda(v), h_\lambda(u)\rangle > 1 \quad \text{for } v \in \mathcal{N}_u, u \in F^*_{\varepsilon_0}(\lambda).$$

On the other hand, since $H_\lambda$ is $|\cdot|_w$-upper semi-continuous, $F^*(\lambda)$ is $|\cdot|_w$-open. Consequently,

$$\mathcal{N}_\lambda := \{\mathcal{N}_u : u \in F^*_{\varepsilon_0}(\lambda)\} \cup F^*(\lambda)$$

is an open cover of $F^*_{\varepsilon_0}(\lambda) \cup F^*(\lambda)$. Now we may find a $|\cdot|_w$-locally finite and $|\cdot|_w$ open refinement $(\mathcal{U}_j)_{j\in J}$ with a corresponding $|\cdot|_w$-Lipschitz continuous partition of unity $(\beta_j)_{j\in J}$. For each $j$, we can either find $u_j \in F^*_{\varepsilon_0}(\lambda)$ such that $\mathcal{U}_j \subset \mathcal{N}_{u_j}$, or if such $u$ does not exist, then we have $\mathcal{U}_j \subset F^*(\lambda)$. In the first case we set $w_j(u) = h_\lambda(u_j)$; in the second case, $w_j(u) = 0$. Let $U^* = \cup_{j\in J}\mathcal{U}_j$, then $U^*$ is $|\cdot|_w$ -open and $F^*_{\varepsilon_0}(\lambda) \cup F^*(\lambda) \subset U^*$. Define

$$Y_\lambda(u) := \sum_{j\in J} \beta_j(u) w_j(u), \tag{2.6}$$

then $Y_\lambda : U^* \mapsto E$ is a vector field which has the following properties:

(1) $Y_\lambda$ is locally Lipschitz continuous in both $\|\cdot\|$ and $|\cdot|_w$ topology;
(2) $\langle H'_\lambda(u), Y_\lambda(u)\rangle \geq 0, \forall u \in U^*$;
(3) $\langle H'_\lambda(u), Y_\lambda(u)\rangle \geq 1, \forall u \in F^*_{\varepsilon_0}(\lambda)$;
(4) $|Y_\lambda(u)|_w \leq \|Y_\lambda(u)\| \leq 2/\varepsilon_0$ for $u \in U^*$ and all $\lambda \in [1, 2]$.

Consider the following initial value problem

$$\frac{\mathrm{d}\eta(t,u)}{\mathrm{d}t} = -Y_\lambda(\eta), \quad \eta(0,u) = u,$$

for all $u \in F^*(\lambda) \cup F(\lambda, \varepsilon_0)$, where $F^*(\lambda)$ is given by (2.5) and

$$F(\lambda, \varepsilon_0) := \{u \in E : \|u\| \leq k, C_\lambda - (\lambda - \lambda_n) \leq H_\lambda(u) \leq C_\lambda + \varepsilon_0\} \subset F^*_{\varepsilon_0}(\lambda). \tag{2.7}$$

Then by classical theory of ordinary differential equations and the properties of $Y_\lambda$, for each $u$ as above, there exists a unique solution $\eta(t,u)$ as long as it does not approach the boundary of $U^*$. Furthermore, $t \mapsto H_\lambda(\eta(t,u))$ is nonincreasing.

**Step 6.** We prove that $\eta(t,u)$ is $|\cdot|_w$-continuous for $t\in[0,2\varepsilon_0], u\in F(\lambda,\varepsilon_0)\cup F^*(\lambda)$. For fixed $t_0\in[0,2\varepsilon_0], u_0\in F(\lambda,\varepsilon_0)\cup F^*(\lambda)$, we see that

$$\eta(t,u)-\eta(t,u_0)=u-u_0+\int_0^t\Big(Y_\lambda(\eta(s,u_0))-Y_\lambda(\eta(s,u))\Big)\mathrm{d}s. \tag{2.8}$$

Since the set $\Lambda:=\eta([0,2\varepsilon_0]\times\{u_0\})$ is compact and $|\cdot|_w$-compact and $Y_\lambda$ is $|\cdot|_w$-locally $|\cdot|_w$-Lipschitz, there exist $r_1>0, r_2>0$ such that $\{u\in E:\inf_{e\in\Lambda}|u-e|_w<r_1\}\subset U^*$ and $|Y_\lambda(u)-Y_\lambda(v)|_w\le r_2|u-v|_w$ for any $u,v\in\Lambda$. Suppose that $\eta(s,u)\in U^*$ for $0\le s\le t$. Then by (2.8),

$$\begin{aligned}|\eta(t,u)-\eta(t,u_0)|_w &\le |u-u_0|_w+\int_0^t|Y_\lambda(\eta(s,u_0))-Y_\lambda(\eta(s,u))|_w\mathrm{d}s\\ &\le |u-u_0|_w+r_2\int_0^t|\eta(s,u_0)-\eta(s,u)|_w\mathrm{d}s.\end{aligned}$$

By the Gronwall inequality (see *e.g.*, Lem. 6.9 of [26]),

$$|\eta(t,u)-\eta(t,u_0)|_w\le|u-u_0|_w\mathrm{e}^{r_2t}\le|u-u_0|_w\mathrm{e}^{r_2}.$$

If $|u-u_0|_w<\delta$, where $0<\delta<r_1e^{-r_2}$, then $|\eta(t,u)-\eta(t,u_0)|_w<r_1$. Therefore, if $|t-t_0|<\delta$,

$$\begin{aligned}|\eta(t,u)-\eta(t_0,u_0)|_w &\le |\eta(t,u)-\eta(t,u_0)|_w+|\eta(t,u_0)-\eta(t_0,u_0)|_w\\ &\le |\eta(t,u)-\eta(t,u_0)|_w+\left|\int_{t_0}^t Y_\lambda(\eta(s,u_0))\mathrm{d}s\right|_w\\ &\le \delta\mathrm{e}^{r_2}+\delta c\\ &\to 0\quad \text{as } \delta\to 0.\end{aligned}$$

**Step 7.** Consider

$$\eta^*(t,u)=\begin{cases}h_n(2t,u) & 0\le t\le 1/2\\ \eta(4\varepsilon_0t-2\varepsilon_0,h_n(1,u)) & 1/2\le t\le 1.\end{cases}$$

We prove that $\eta^*\in\Gamma$.

Evidently, for $u\in\bar{Q}$, we have either $h_n(1,u)\in F^*(\lambda)$ or $C_\lambda-(\lambda-\lambda_n)\le H_\lambda(h_n(1,u))$. For the later case, we observe that $\|h_n(1,u)\|\le k$ by Step 3 and $H_\lambda(h_n(1,u))\le C_\lambda+\varepsilon_0$ by Step 4, hence, $h_n(1,u)\in F(\lambda,\varepsilon_0)$. In view of Step 6, $\eta^*$ is $|\cdot|_w$-continuous satisfying $\eta^*(0,u)=u$ and $H(\eta^*(t,u))\le H(u)$. Now for any $(s_0,u_0)\in[0,1]\times\bar{Q}$, since $h_n\in\Gamma$, we first find a $|\cdot|_w$-neighborhood $U^1_{(s_0,u_0)}$ such that

$$\{u-h_n(s,u):(s,u)\in U^1_{(s_0,u_0)}\cap([0,1]\times\bar{Q})\}\subset E_{\text{fin}}. \tag{2.9}$$

Furthermore, it is easy to see that there exists a $|\cdot|_w$-neighborhood $U^2_{(s_0,u_0)}$ of $(s_0,u_0)$ such that

$$\{h_n(s,u)-h_n(2s,u):(s,u)\in U^2_{(s_0,u_0)}\cap([0,1]\times\bar{Q})\}\subset E_{\text{fin}}. \tag{2.10}$$

Next, we have to estimate $h_n(t,u)-\eta(4\varepsilon_0t-2\varepsilon_0,h_n(1,u))$ for $t\in[1/2,1]$. If $H_\lambda(h_n(1,u_0))<C_\lambda-(\lambda-\lambda_n)$, then

$$H_\lambda(\eta(t,h_n(1,u_0)))\le H_\lambda(h_n(1,u_0))<C_\lambda-(\lambda-\lambda_n),\qquad \text{for } t\in[0,2\varepsilon_0]. \tag{2.11}$$

Particularly, $\eta(t,h_n(1,u_0))\in F^*(\lambda)$ (see (2.5)).

If $H_\lambda(h_n(1,u_0)) \geq C_\lambda - (\lambda - \lambda_n)$, then by Step 3, $\|h_n(1,u_0)\| \leq k$ and by Step 4,

$$h_n(1,u_0) \in F(\lambda, \varepsilon_0) \subset F^*_{\varepsilon_0}(\lambda). \tag{2.12}$$

Since

$$\begin{aligned}\|\eta(t,h_n(1,u_0)) - h_n(1,u_0)\| &= \|\int_0^t \mathrm{d}\eta(s,h_n(1,u_0))\| \\ &\leq \int_0^t \|Y_\lambda(\eta(s,h_n(1,u_0)))\|\mathrm{d}s \\ &\leq \frac{2t}{\varepsilon_0},\end{aligned}$$

hence

$$\|\eta(t,h_n(1,u_0))\| \leq \|h_n(1,u_0)\| + \frac{2t}{\varepsilon_0} \leq k+4, \qquad \text{for } t \in [0,2\varepsilon_0]. \tag{2.13}$$

Further, by Step 4, $H_\lambda(\eta(t,h_n(1,u_0)) \leq H_\lambda(h_n(1,u_0)) \leq C_\lambda + \varepsilon_0$. Therefore, for this case,

$$\eta(t,h_n(1,u_0)) \in F^*_{\varepsilon_0}(\lambda) \cup F^*(\lambda), \quad t \in [0,2\varepsilon_0]. \tag{2.14}$$

Consider $\Lambda_1 := \{\eta([0,2\varepsilon_0],h_n(1,u_0))\}$, which is $|\cdot|_w$-compact and contained in $U^*$ of Step 5 because of (2.11) and (2.14). Moreover, there are $r_3>0, r_4>0$ such that

- $\Lambda_2 := \{u \in E : |u - \Lambda_1|_w < r_3\} \subset U^*$;
- $|Y_\lambda(u) - Y_\lambda(v)|_w \leq r_4|u-v|_w, \quad \forall u,v \in \Lambda_2$;
- $Y_\lambda(\Lambda_2) \subset E_{\mathrm{fin}}$.

Evidently, by the $|\cdot|_w$ continuity of $Y_\lambda, \eta$, and $h_n$, there exists a $|\cdot|_w$-neighborhood $U^3_{(s_0,u_0)}$ such that

$$\eta(t,h_n(1,u)) \subset \Lambda_2 \tag{2.15}$$

for $t \in [0,2\varepsilon_0]$ and $u \in U^3_{(s_0,u_0)}$. For $t \in [1/2,1]$, note that

$$\begin{aligned}&h_n(t,u) - \eta(4\varepsilon_0 t - 2\varepsilon_0, h_n(1,u)) \\ &= h_n(t,u) - h_n(1,u) + \int_0^{4\varepsilon_0 t - 2\varepsilon_0} Y_\lambda(\eta(s,h_n(1,u)))\mathrm{d}s,\end{aligned}$$

we conclude by (2.15) that

$$\{h_n(t,u) - \eta(4\varepsilon_0 t - 2\varepsilon_0, h_n(1,u)) : (t,u) \in U^3_{(s_0,u_0)} \cap ([1/2,1] \times \bar{Q})\} \subset E_{\mathrm{fin}}. \tag{2.16}$$

According to the definition of $\eta^*$,

$$u - \eta^*(t,u) = u - h_n(t,u) + h_n(t,u) - h_n(2t,u), \quad t \in [0,1/2];$$

$$u - \eta^*(t,u) = u - h_n(t,u) + h_n(t,u) - \eta(4\varepsilon_0 t - 2\varepsilon_0, h_n(1,u)), \quad t \in [1/2,1].$$

Therefore, by combining (2.9, 2.10) and (2.16), we obtain that

$$\{u - \eta^*(t,u) : (t,u) \in \tilde{U}^*_{(s_0,u_0)} \cap ([0,1] \times \bar{Q})\} \subset E_{\mathrm{fin}},$$

which implies that $\eta^* \in \Gamma$, where $\tilde{U}^*_{(s_0,u_0)} = U^1_{(s_0,u_0)} \cap U^2_{(s_0,u_0)}$ or $\tilde{U}^*_{(s_0,u_0)} = U^1_{(s_0,u_0)} \cap U^3_{(s_0,u_0)}$

**Step 8.** We will get a contradiction in this step.

**Case 1:** if $H_\lambda(h_n(1,u)) < C_\lambda - (\lambda - \lambda_n)$ for some $u \in \bar{Q}$, then $h_n(1,u) \in F^*(\lambda)$ (see (2.5)) and

$$H_\lambda(\eta^*(1,u)) = H_\lambda(\eta(2\varepsilon_0, h_n(1,u)) \le H_\lambda(h_n(1,u))) < C_\lambda - (\lambda - \lambda_n). \tag{2.17}$$

**Case 2:** if $H_\lambda(h_n(1,u)) \ge C_\lambda - (\lambda - \lambda_n)$ for some $u \in \bar{Q}$, then by Step 3 and Step 4, $\|h_n(1,u)\| \le k$ and $\sup_{u\in\bar{Q}} H_\lambda(h_n(1,u)) \le C_\lambda + \varepsilon_0$. Then, $h_n(1,u) \in F^*_{\varepsilon_0}(\lambda)$. Assume that $H_\lambda(\eta^*(1,u)) \ge C_\lambda - (\lambda - \lambda_n)$, then for $0 \le t \le 2\varepsilon_0$, we have,

$$\begin{aligned} C_\lambda - (\lambda - \lambda_n) &\le H_\lambda(\eta^*(1,u)) \\ &= H_\lambda(\eta(2\varepsilon_0, h_n(1,u))) \\ &\le H_\lambda(\eta(t, h_n(1,u))) \\ &\le H_\lambda(\eta(0, h_n(1,u))) \\ &= H_\lambda(h_n(1,u)) \\ &\le C_\lambda + \varepsilon_0. \end{aligned} \tag{2.18}$$

Furthermore, for any $t \in [0, 2\varepsilon_0]$, by Property (4) of $Y_\lambda$ (see (2.6)),

$$\begin{aligned} \|\eta(t, h_n(1,u)) - h_n(1,u)\| &= \left\| \int_0^t \frac{\mathrm{d}\eta(s, h_n(1,u))}{\mathrm{d}s} \mathrm{d}s \right\| \\ &\le \int_0^t \|Y_\lambda(\eta(s, h_n(1,u))\| \mathrm{d}s \\ &\le 2t/\varepsilon_0, \end{aligned}$$

it follows that

$$\|\eta(t, h_n(1,u))\| \le 2t/\varepsilon_0 + \|h_n(1,u)\| \le k + 4 \quad \text{for } t \in [0, 2\varepsilon_0]. \tag{2.19}$$

Hence, equations (2.18) and (2.19) imply that $\eta(t, h_n(1,u)) \in F^*_{\varepsilon_0}(\lambda)$ for $t \in [0, 2\varepsilon_0]$. Since on $F^*_{\varepsilon_0}(\lambda)$, $\langle H'_\lambda(u), Y_\lambda(u)\rangle > 1$, then

$$\begin{aligned} H_\lambda(\eta(2\varepsilon_0, h_n(1,u))) - H_\lambda(h_n(1,u))) &= \int_0^{2\varepsilon_0} \frac{\mathrm{d}}{\mathrm{d}t} H_\lambda(\eta(t, h_n(1,u)))\mathrm{d}t \\ &= -\int_0^{2\varepsilon_0} \langle H'_\lambda(\eta(t, h_n(1,u))), Y_\lambda(\eta(t, h_n(1,u)))\rangle \mathrm{d}t \\ &\le -2\varepsilon_0. \end{aligned}$$

Therefore, by Step 4,

$$\begin{aligned} H_\lambda(\eta(2\varepsilon_0, h_n(1,u))) &\le H_\lambda(h_n(1,u)) - 2\varepsilon_0 \\ &\le C_\lambda - \varepsilon_0 \\ &\le C_\lambda - (\lambda - \lambda_n). \end{aligned} \tag{2.20}$$

Combining (2.17) and (2.20), we find

$$H_\lambda(\eta^*(1,u)) = H_\lambda(\eta(2\varepsilon_0, h_n(1,u))) \le C_\lambda - (\lambda - \lambda_n)$$

for any $(t,u) \in [0,1] \times \bar{Q}$, which contradicts the definition of $C_\lambda$. □

## 3. Schrödinger equation

Let $E := W^{1,2}(\mathbf{R}^N)$. It is well known that there is a one-to-one correspondence between solutions of (S) and critical points of the $\mathcal{C}^1(E, \mathbf{R})$-functional

$$H(u) := \frac{1}{2}\int_{\mathbf{R}^N}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \frac{1}{2^*}\int_{\mathbf{R}^N} K(x)|u|^{2^*}\mathrm{d}x - \int_{\mathbf{R}^N} G(x,u)\mathrm{d}x. \tag{3.1}$$

Let $(E(\lambda))_{\lambda\in\mathbf{R}}$ be the spectral family of $-\Delta+V$ in $L^2(\mathbf{R}^N)$. Let $E^- := E(0)L^2\cap E$ and $E^+ := (id-E(0))L^2\cap E$, then the quadratic form $\int_{\mathbf{R}^N}(|\nabla u|^2 + Vu^2)\mathrm{d}x$ is positive definite on $E^+$ and negative definite on $E^-$ (*cf.* [22]). By introducing a new inner product $\langle\cdot,\cdot\rangle$ in $E$, the corresponding norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{1,2}$, the usual norm of $W^{1,2}(\mathbf{R}^N)$. Moreover, $\int_{\mathbf{R}^N}(|\nabla u|^2 + Vu^2)\mathrm{d}x = \|u^+\|^2 - \|u^-\|^2$, where $u^\pm \in E^\pm$. Then functional (3.1) can be rewritten as

$$H(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2^*}\int_{\mathbf{R}^N} K(x)|u|^{2^*}\mathrm{d}x - \int_{\mathbf{R}^N} G(x,u)\mathrm{d}x. \tag{3.2}$$

In order to use Theorem 2.1, we consider the family of functional defined by

$$H_\lambda(u) = \frac{1}{2}\|u^+\|^2 - \lambda\left(\frac{1}{2}\|u^-\|^2 + \frac{1}{2^*}\int_{\mathbf{R}^N} K(x)|u|^{2^*}\mathrm{d}x + \int_{\mathbf{R}^N} G(x,u)\mathrm{d}x\right) \tag{3.3}$$

for $\lambda \in [1,2]$.

**Lemma 3.1.** *$H_\lambda$ is $|\cdot|_w$-upper semicontinuous. $H'_\lambda$ is weakly sequentially continuous.*

*Proof.* Noting that $u_n := u_n^- + u_n^+ \overset{|\cdot|_w}{\to} u$ implies that $u_n \rightharpoonup u$ weakly in $E$ and $u_n^+ \to u^+$ strongly in $E$, then the proof is the same as that in [23] (see also [6,12]). The second conclusion is due to [6]. ☐

Let

$$\varphi_\varepsilon(x) := \frac{c_N\psi(x)\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}},$$

where $c_N = (N(N-2))^{(N-2)/4}, \varepsilon > 0$ and $\psi \in \mathcal{C}_0^\infty(\mathbf{R}^N, [0,1])$ with $\psi(x) = 1$ if $|x| \le r/2$; $\psi(x) = 0$ if $|x| \ge r, r$ small enough (*cf. e.g.* pp. 35 and 52 of [26]). Write $\varphi_\varepsilon = \varphi_\varepsilon^+ + \varphi_\varepsilon^-$ with $\varphi_\varepsilon^+ \in E^+, \varphi_\varepsilon^- \in E^-$. Then

$$\|\varphi_\varepsilon^-\| \to 0, \|\varphi_\varepsilon^+\|_{2^*}^{2^*} \to S^{N/2} \quad \text{as } \varepsilon \to 0 \ (\textit{cf. } \text{Prop. 4.2 of [6]}),$$

where

$$S := \inf_{u\in E\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}\cdot$$

The following lemma can be found in Proposition 4.2 of [6].

**Lemma 3.2.** *Set*

$$I_1(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2^*}\int_{\mathbf{R}^N} K(x)|u|^{2^*}\mathrm{d}x, \qquad u \in E, \tag{3.4}$$

*then*

$$\sup_{Z_\varepsilon} I_1 < c^* := \frac{S^{N/2}}{N\|K\|_\infty^{(N-2)/2}},$$

*for $\varepsilon$ small enough, where $Z_\varepsilon := E^- \oplus \mathbf{R}\varphi_\varepsilon^+$.*

To carry forward, we prepare an auxiliary results.

**Lemma 3.3.** *Assume that $g(x,u)/u \to 0$ as $|u| \to 0$ uniformly for $x \in \mathbf{R}^N$ and that $g$ is of subcritical Sobolev exponent growth. If a bounded sequence $(w_n) \subset E$ and $\lambda_n \in [1,2]$ satisfy*

$$\lambda_n \to \lambda, \quad H'_{\lambda_n}(w_n) \to 0, \quad H_{\lambda_n}(w_n) \to c(\lambda),$$

*where $0 < c(\lambda) < c^*_\lambda := \dfrac{S^{N/2}}{N\|\lambda K\|_\infty^{(N-2)/2}}$, then $(w_n)$ is nonvanishing, i.e., there exist $r, \eta > 0$ and a sequence $(y_n) \subset \mathbf{R}^N$, a sequence of open ball $(B(y_n, r))$ centered at $y_n$ with radius $r$, such that*

$$\limsup_{n\to\infty} \int_{B(y_n,r)} w_n^2 \mathrm{d}x \geq \eta.$$

*Proof.* The idea is essentially due to Proposition 4.1 of [6]. We give the sketch for the reader's convenience.

If $(w_n)$ is not nonvanishing, then $w_n \to 0$ in $L^r(\mathbf{R}^N)$ for $2 < r < 2^*$ by Lions' lemma ([16], Lem 1.21). By standard arguments,

$$\int_{\mathbf{R}^N} g(x, w_n) v_n \mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbf{R}^N} G(x, w_n)\mathrm{d}x \to 0 \tag{3.5}$$

whenever $(v_n) \subset E$ is bounded. Hence

$$H_{\lambda_n}(w_n) - \frac{1}{2}\langle H'_{\lambda_n}(w_n), w_n\rangle = \frac{\lambda_n}{N}\int_{\mathbf{R}^N} K(x)|w_n|^{2^*}\mathrm{d}x + o(1) \to c(\lambda). \tag{3.6}$$

For any $\delta > 0$, we choose $\mu > \|V\|_\infty(1+\delta)/\delta$. Write $w_n = w_n^+ + w_n^- \in E^+ \oplus E^-$, and let $w_n^+ = \tilde{w}_n + \tilde{z}_n$, with $\tilde{w}_n \in E(\mu)L^2, \tilde{z}_n \in (id - E(\mu))L^2$, where $(E(\lambda))_{\lambda\in\mathbf{R}}$ is the spectral family of $-\Delta + V$ in $L^2$. By Proposition 2.4 of [6], $\tilde{w}_n \in E$ and

$$\|w_n^-\|_q \leq c\|w_n^-\|_2 \leq c\|w_n\| \quad \text{and} \quad \|\tilde{w}_n\|_q \leq c\|\tilde{w}_n\|_2 \leq c\|w_n\|, \tag{3.7}$$

where $q = 2N/(N-4)$ if $N > 4$ and $q$ may be chosen arbitrarily large if $N = 4$. Therefore,

$$\begin{aligned}\lambda_n\|w_n^-\|^2 &= -\langle H'_{\lambda_n}(w_n), w_n^-\rangle - \lambda_n\int_{\mathbf{R}^N} K(x)|w_n|^{2^*-2}w_n w_n^-\mathrm{d}x - \lambda_n\int_{\mathbf{R}^N} g(x,w_n)w_n^-\mathrm{d}x\\ &\leq 2\|K\|_\infty\|w_n\|_r^{2^*-1}\|w_n^-\|_q + o(1)\\ &\to 0,\end{aligned}$$

where $r$ satisfies $(2^*-1)/r + 1/q = 1$, hence $2 < r < 2^*$. By the same reasoning,

$$\|\tilde{w}_n\| \to 0, \quad \text{hence,} \quad w_n - \tilde{z}_n \to 0. \tag{3.8}$$

It follows that

$$\begin{aligned}\|\tilde{z}_n\|^2 &= \int_{\mathbf{R}^N}(|\nabla\tilde{z}_n|^2 + V\tilde{z}_n^2)\mathrm{d}x\\ &= \lambda_n\int_{\mathbf{R}^N} K(x)|w_n|^{2^*-2}w_n\tilde{z}_n\mathrm{d}x + o(1)\\ &= \lambda_n\int_{\mathbf{R}^N} K(x)|w_n|^{2^*}\mathrm{d}x\end{aligned} \tag{3.9}$$

On the other hand, by (4.6) of [6], for any $\delta > 0$ and $\mu > \|V\|_\infty(1+\delta)/\delta$, we have that

$$(1-\delta)\int_{\mathbf{R}^N}|\nabla\tilde{z}_n|^2\mathrm{d}x \leq \int_{\mathbf{R}^N}(|\nabla\tilde{z}_n|^2 + V\tilde{z}_n^2)\mathrm{d}x. \tag{3.10}$$

By (3.9, 3.8) and (3.10), we have that

$$\begin{aligned}\left(\lambda\int_{\mathbf{R}^N} K(x)|w_n|^{2^*}\mathrm{d}x\right)^{2/2^*} &\le (\lambda\|K\|_\infty)^{2/2^*}\|w_n\|_{2^*}^2\\ &= (\lambda\|K\|_\infty)^{2/2^*}\|\tilde z_n\|_{2^*}^2+o(1)\\ &\le (\lambda\|K\|_\infty)^{2/2^*}\|\nabla\tilde z_n\|_2^2/S+o(1)\\ &\le \frac{(\lambda\|K\|_\infty)^{2/2^*}}{S(1-\delta)}\lambda\int_{\mathbf{R}^N}K(x)|w_n|^{2^*}\mathrm{d}x+o(1).\end{aligned}$$

If we let $n\to\infty$ and use (3.6), it follows that

$$(Nc(\lambda))^{2/2^*}\le\frac{(\lambda\|K\|_\infty)^{2/2^*}}{S(1-\delta)}Nc(\lambda),$$

which implies that either $c(\lambda)=0$ or $c(\lambda)\ge(1-\delta)^{N/2}c_\lambda^*$. Either way, we get a contradiction since $\delta$ is chosen arbitrarily. □

Choose $z_0:=\varphi_\varepsilon^+/\|\varphi_\varepsilon^+\|\in E^+$. For $R>0$, set $Q:=\{u=u^-+sz_0:\|u\|<R,u^-\in E^-,s\in\mathbf{R}^+\}$. Let $p_0=s_0z_0\in Q, s_0>0$. For any $u\in E$, we write $u=u^-+sz_0+w$ with $u^-\in E^-,w\in(E^-\oplus\mathbf{R}z_0)^\perp,s\in\mathbf{R}$. Consider a map $F:E\to E^-\oplus\mathbf{R}z_0$ defined by

$$F(u^-+sz_0+w)=u^-+\|sz_0+w\|z_0.$$

Let $B:=F^{-1}(p_0)$, then

$$B=\{u=sz_0+w:w\in(E^-\oplus\mathbf{R}z_0)^\perp,\|u\|=s_0\}\cdot$$

It is easy to check that $F,p_0,B$ satisfy the basic assumptions in Section 2. By hypotheses ($S_4$) and ($S_5$), the proof of the next lemma is trivial.

**Lemma 3.4.** *There exist $R>0,s_0>0$, such that*

$$\inf_B H_\lambda>0,\quad \sup_{\partial\bar Q}H_\lambda\le 0,\qquad \text{for all } \lambda\in[1,2].$$

**Lemma 3.5.** *For almost all $\lambda\in[1,2]$, there exists $\{u_n\}\in E$ such that*

$$\sup_n\|u_n\|<\infty,\quad H_\lambda'(u_n)\to 0\quad\text{ and }\quad H_\lambda(u_n)\to C_\lambda,$$

*where $C_\lambda\in[\inf_B H_\lambda,\sup_{\bar Q}H]$. Furthermore, there exists $\delta_0>0$ small enough such that, for almost all $\lambda\in[1,1+\delta_0]$, there exists $u_\lambda\neq 0$ such that*

$$H_\lambda'(u_\lambda)=0,\qquad H_\lambda(u_\lambda)\le\sup_{\bar Q}H.$$

*Proof.* The first conclusion follows immediately from Lemmas 3.1, 3.4, 3.5 and Theorem 2.1. Now we prove the second conclusion. Since $g(x,u)u\ge 0$ and $\bar Q\subset Z_\varepsilon$, we get that

$$0<C_\lambda\le\sup_{\bar Q}H\le\sup_{Z_\varepsilon}I_1<c^*,\tag{3.11}$$

where $I_1$, $c^*$ and $Z_\varepsilon$ come from Lemma 3.2. Therefore, there exists $\delta_0>0$ such that $0<C_\lambda<c_\lambda^*$ for almost all $\lambda\in[1,1+\delta_0]$, where $c_\lambda^*$ comes from Lemma 3.3. For those $\lambda$, by Lemma 3.3, $\{u_n\}$ is nonvanishing, that is,

there exist $y_n \in \mathbf{R}^N, \alpha > 0, R_1 > 0$ such that

$$\limsup_{n\to\infty} \int_{B(y_n,R_1)} |u_n|^2 \mathrm{d}x \ge \alpha > 0.$$

We find $\bar{y}_n \in \mathbf{Z}^N$ such that

$$\limsup_{n\to\infty} \int_{B(0,2R_1)} |v_n|^2 \mathrm{d}x \ge \alpha > 0,$$

where $v_n(x) := u_n(x + \bar{y}_n)$. By the periodicity of $V, K$ and $g$, $\{v_n\}$ is still bounded and

$$\lim_{n\to\infty} H_\lambda(v_n) \in \left[\inf_B H_\lambda, \sup_{\bar{Q}} H\right], \quad \lim_{n\to\infty} H'_\lambda(v_n) = 0.$$

We may suppose that $v_n \rightharpoonup u_\lambda$. Since $E$ is embedded compactly in $L^t_{\mathrm{loc}}(\mathbf{R}^N)$ for $2 \le t < 2^*$, then

$$0 < \alpha \le \lim_{n\to\infty} \int_{B(0,2R_1)} |v_n|^2 \mathrm{d}x = \int_{B(0,2R_1)} |u_\lambda|^2 \mathrm{d}x \le |u_\lambda|_2^2,$$

therefore, $u_\lambda \ne 0$. Since $H'_\lambda$ is weakly sequentially continuous, $H'_\lambda(u_\lambda) = 0$. Finally, by Fatou's lemma,

$$\begin{aligned}
H_\lambda(u_\lambda) &= H_\lambda(u_\lambda) - \frac{1}{2}\langle H'_\lambda(u_\lambda), u_\lambda\rangle \\
&= \lambda \int_{\mathbf{R}^N} \left(\frac{1}{2}(K(x)|u_\lambda|^{2^*} + g(x,u_\lambda)u_\lambda) - \frac{1}{2^*}K(x)|u_\lambda|^{2^*} - G(x,u_\lambda)\right) \mathrm{d}x \\
&= \lambda \int_{\mathbf{R}^N} \lim_{n\to\infty} \left(\frac{1}{2}(K(x)|v_n|^{2^*} + g(x,v_n)v_n) - \frac{1}{2^*}K(x)|v_n|^{2^*} - G(x,v_n)\right) \mathrm{d}x \\
&\le \lim_{n\to\infty} \left(H_\lambda(v_n) - \frac{1}{2}\langle H'_\lambda(v_n), v_n\rangle\right) \\
&\le \lim_{n\to\infty} H_\lambda(v_n) \\
&\le \sup_{\bar{Q}} H.
\end{aligned}$$

□

**Lemma 3.6.** *There exist $\lambda_n \in [1, 1+\delta_0]$ with $\lambda_n \to 1$, and $z_n \in E\backslash\{0\}$ such that*

$$H'_{\lambda_n}(z_n) = 0, \quad H_{\lambda_n}(z_n) \le \sup_{\bar{Q}} H.$$

*Proof.* It is an immediately consequence of Lemma 3.5. □

**Lemma 3.7.** *$\{z_n\}$ is bounded.*

*Proof.* Let $g_1(x,u) := K(x)|u|^{2^*-2}u + g(x,u)$ and $G_1(x,u) := \int_0^u g_1(x,s)\mathrm{d}s$. Then by the assumption ($\mathrm{S}_4$), we see that

$$\lim_{u\to 0} \frac{g_1(x,u)u}{G_1(x,u)} = 2^* \quad \text{uniformly for } x \in \mathbf{R}^N.$$

Let $\varepsilon_1 > 0$ be such that $2^* - \varepsilon_1 > 2$. Hence, there exists $R_1 > 0$ such that

$$g_1(x,u)u \ge (2^* - \varepsilon_1)G_1(x,u), \quad \text{for } x \in \mathbf{R}^N, |u| \le R_1. \tag{3.12}$$

On the other hand, since $g(x,u)$ is of subcritical growth,

$$\lim_{u\to\infty} \frac{g_1(x,u)u - 2G_1(x,u)}{|u|^{2^*}} = (1-\frac{2}{2^*})K(x) \geq c > 0 \tag{3.13}$$

uniformly for $x \in \mathbf{R}^N$. Furthermore, condition (1.2) implies that

$$0 < g(x,u)u \leq \frac{2}{N-2} k_0 |u|^{2^*} \quad \text{ for all } x \in \mathbf{R}^N, u \neq 0,$$

hence

$$g_1(x,u)u - 2G_1(x,u) > 0 \quad \text{ for all } x \in \mathbf{R}^N, u \neq 0. \tag{3.14}$$

Therefore (3.13) and (3.14) imply that there exists $c$ small enough, such that

$$g_1(x,u)u - 2G_1(x,u) \geq c|u|^{2^*} \quad \text{ for all } x \in \mathbf{R}^N, |u| \geq R_1. \tag{3.15}$$

Recall that $H_{\lambda_n}(z_n) \leq \sup\limits_{\bar{Q}} H$ and $H'_{\lambda_n}(z_n) = 0$, then

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{2^*-\varepsilon_1}\right)\left(\|z_n^+\|^2 - \lambda_n\|z_n^-\|^2\right) + \lambda_n\left(\frac{1}{2^*-\varepsilon_1} - \frac{1}{2^*}\right)\int_{\mathbf{R}^N} K(x)|z_n|^{2^*}\mathrm{d}x \\ + \lambda_n \int_{\mathbf{R}^N}\left(\frac{1}{2^*-\varepsilon_1} g(x,z_n)z_n - G(x,z_n)\right)\mathrm{d}x \leq \sup_{\bar{Q}} H. \end{aligned}\tag{3.16}$$

By (3.12, 3.14) and (3.16),

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{2^*-\varepsilon_1}\right)\left(\|z_n^+\|^2 - \lambda_n\|z_n^-\|^2\right) &\leq c + c\left(\int_{|z_n|\leq R_1} + \int_{|z_n|\geq R_1}\right)\left(G_1(x,z_n) - \frac{1}{2^*-\varepsilon_1} g_1(x,z_n)z_n\right)\mathrm{d}x \\ &\leq c + c\int_{|z_n|\geq R_1}\left(G_1(x,z_n) - \frac{1}{2^*-\varepsilon_1} g_1(x,z_n)z_n\right)\mathrm{d}x \\ &\leq c + c\int_{|z_n|\geq R_1}\left(\frac{1}{2} g_1(x,z_n)z_n - \frac{1}{2^*-\varepsilon_1} g_1(x,z_n)z_n\right)\mathrm{d}x \\ &= c + c\int_{|z_n|\geq R_1} g_1(x,z_n)z_n \mathrm{d}x. \end{aligned}\tag{3.17}$$

Since, by ($\mathrm{S}_4$), $|g(x,z)z| \leq c|z|^{2^*}$ for all $(x,z) \in \mathbf{R}^N \times \mathbf{R}$, (3.17) implies that

$$\begin{aligned}\|z_n^+\|^2 - \lambda_n\|z_n^-\|^2 &\leq c + c\int_{|z_n|\geq R_1} g_1(x,z_n)z_n \mathrm{d}x \\ &\leq c + c\int_{|z_n|\geq R_1}\left(K(x)|z_n|^{2^*} + g(x,z_n)z_n\right)\mathrm{d}x \\ &\leq c + c\int_{|z_n|\geq R_1} |z_n|^{2^*}\mathrm{d}x. \end{aligned}\tag{3.18}$$

However (3.14) and (3.15) imply that

$$\begin{aligned}\sup_{\bar{Q}} H &\geq H_{\lambda_n}(z_n) - \frac{1}{2}\langle H'_{\lambda_n}(z_n), z_n\rangle \\ &= \int_{\mathbf{R}^N}\left(\frac{1}{2}g_1(x,z_n)z_n - G_1(x,z_n)\right)\mathrm{d}x \\ &\geq \int_{|z_n|\geq R_1}\left(\frac{1}{2}g_1(x,z_n)z_n - G_1(x,z_n)\right)\mathrm{d}x \\ &\geq c\int_{|z_n|\geq R_1}|z_n|^{2^*}\mathrm{d}x. \end{aligned} \tag{3.19}$$

Then, combining (3.18) and (3.19), we obtain that

$$\|z_n^+\|^2 - \lambda_n\|z_n^-\|^2 \leq c. \tag{3.20}$$

Noting that $\langle H'_{\lambda_n}(z_n), z_n\rangle = 0$, we see that

$$\begin{aligned}\|z_n^+\|^2 - \lambda_n\|z_n^-\|^2 &= \lambda_n\int_{\mathbf{R}^N}\left(K(x)|z_n|^{2^*} + g(x,z_n)z_n\right)\mathrm{d}x \\ &\geq c\int_{\mathbf{R}^N}|z_n|^{2^*}\mathrm{d}x. \end{aligned} \tag{3.21}$$

So, by (3.20) and (3.21), $\int_{\mathbf{R}^N}|z_n|^{2^*}\mathrm{d}x \leq c$. Noting that $\langle H'_{\lambda_n}(z_n), z_n^+\rangle = 0$ and $(\mathrm{S}_4)$, we obtain, by Hölder's inequality and (3.21), that

$$\begin{aligned}\|z_n^+\|^2 &= \lambda_n\int_{\mathbf{R}^N}K(x)|z_n|^{2^*-2}z_nz_n^+\mathrm{d}x + \lambda_n\int_{\mathbf{R}^N}g(x,z_n)z_n^+\mathrm{d}x \\ &\leq c\int_{\mathbf{R}^N}|z_n|^{2^*-1}|z_n^+| \\ &\leq c\|z_n\|_{2^*}^{2^*-1}\|z_n^+\|_{2^*} \\ &\leq c\|z_n^+\|. \end{aligned}$$

Therefore $\|z_n^+\| \leq c$, and hence, $\|z_n^-\| \leq c$ by (3.21). □

**Lemma 3.8.** $\{z_n\}$ *is nonvanishing.*

*Proof.* Since $(z_n)$ is bounded, we may assume that

$$H_{\lambda_n}(z_n) \to c_1 \leq \sup_{\bar{Q}} H < c^* \ (cf.\ (3.11)). \tag{3.22}$$

If $\{z_n\}$ is not nonvanishing (*i.e.*, is vanishing), then it follows from Lions' lemma (*cf.* [16], Lem. 1.21) that $z_n \to 0$ in $L^r$ whenever $2 < r < 2^*$. The assumption $(\mathrm{S}_4)$ implies that

$$\int_{\mathbf{R}^N} g(x,z_n)z_n\mathrm{d}x \to 0, \quad \int_{\mathbf{R}^N} G(x,z_n)\mathrm{d}x \to 0, \tag{3.23}$$

and consequently

$$H_{\lambda_n}(z_n) - \frac{1}{2}\langle H'_{\lambda_n}(z_n), z_n\rangle = \frac{\lambda_n}{N}\int_{\mathbf{R}^N}K(x)|z_n|^{2^*}\mathrm{d}x + o(1) \to c_1. \tag{3.24}$$

Since $K(x) > 0$, $c_1 \geq 0$.

**Case 1:** If $c_1 > 0$, then by (3.22) and Lemma 3.3, $z_n$ is nonvanishing.

**Case 2:** If $c_1 = 0$, then (3.24) implies that

$$\int_{\mathbf{R}^N} |z_n|^{2^*} \mathrm{d}x \to 0. \tag{3.25}$$

Since $H'_{\lambda_n}(z_n) = 0$, for any $\varepsilon > 0$, by $(S_4)$, we have that

$$\begin{aligned} \|z_n^+\|^2 &= \lambda_n \int_{\mathbf{R}^N} \Big(K(x)|z_n|^{2^*-2} z_n z_n^+ + g(x, z_n) z_n^+\Big) \mathrm{d}x \\ &\leq c \int_{\mathbf{R}^N} |z_n|^{2^*-1} |z_n^+| \mathrm{d}x + \varepsilon \|z_n\| \|z_n^+\| + c\|z_n\|_p^{p-1} \|z_n^+\| \\ &\leq c\|z_n\|^{2^*-1} \|z_n^+\| + \varepsilon \|z_n\|^2 + \varepsilon \|z_n^+\|^2 + \|z_n^- + z_n^+\|^{p-1} \|z_n^+\|. \end{aligned}$$

Since $\|z_n^-\| \leq \|z_n^+\|$ (see (3.21)) and $\varepsilon$ is arbitrary,

$$c\|z_n^+\|^2 \leq c\|z_n^+\|^p + c\|z_n^+\|^{2^*},$$

which implies that $\|z_n^+\| \geq c > 0$. But, $H'_{\lambda_n}(z_n) = 0$, and $(S_4)$ implies that

$$\begin{aligned} \|z_n^+\|^2 &= \lambda_n \int_{\mathbf{R}^N} \Big(K(x)|z_n|^{2^*-2} z_n z_n^+ + g(x, z_n) z_n^+\Big) \mathrm{d}x \\ &\leq c\|z_n\|_{2^*}^{2^*-1} \|z_n^+\|_{2^*} + \varepsilon c \|z_n\| \|z_n^+\| + c\|z_n\|_p^{p-1} \|z_n^+\|. \end{aligned}$$

By the vanishing of $\{z_n\}$ and (3.25), $\|z_n^+\| \to 0$, a contradiction. Therefore, $\{z_n\}$ is nonvanishing. □

*Proof of Theorem 1.1.* Since $\{z_n\}$ is nonvanishing, there exist $r > 0, \alpha > 0$ and $y_n \in \mathbf{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B(y_n, r)} z_n^2 \mathrm{d}x \geq \alpha. \tag{3.26}$$

We may assume that $y_n \in \mathbf{Z}^N$ by taking a large $r$ if necessary. Now set $\tilde{z}_n(x) := z_n(x + y_n)$, since $H_\lambda$ is invariant with respect to the translation of $x$ by elements of $\mathbf{Z}^N$ (*i.e.*, $H_\lambda(u(\cdot)) = H_\lambda(u(\cdot + y))$ whenever $y \in \mathbf{Z}^N$), $\|z_n\| = \|\tilde{z}_n\|$, $H_{\lambda_n}(z_n) = H_{\lambda_n}(\tilde{z}_n)$. Without loss of generality, we may suppose, up to a subsequence, that $\tilde{z}_n \rightharpoonup z^*$, then (3.26) implies that $z^* \neq 0$ and $H'_1(z^*) = 0$, *i.e.*, $H'(z^*) = 0$. □

## 4. Appendix

In this Appendix, we give the proof of the existence of the new norm $|\cdot|_w$ satisfying $|v|_w \leq \|v\|, \forall v \in N$ and such that the topology induced by this norm is equivalent to the weak topology of $N$ on *bounded* subset of $N$, more details can be found in [9].

Let $\{e_k\}$ be an orthonormal basis for $N$. Define

$$|v|_w = \sum_{k=1}^{\infty} \frac{|(v, e_k)|}{2^k}, \quad v \in N.$$

Then $|v|_w$ is a norm on $N$ and satisfies $|v|_w \leq \|v\|, \quad v \in N$. If $v_j \to v$ weakly in $N$, then there is a $C > 0$ such that

$$\|v_j\|, \|v\| \leq C, \quad \forall j > 0.$$

For any $\varepsilon > 0$, there exist $K > 0, M > 0$, such that $1/2^K < \varepsilon/(4C)$ and $|(v_j - v, e_k)| < \varepsilon/2$ for $1 \leq k \leq K, j > M$. Therefore,

$$\begin{aligned}
|v_j - v|_w &= \sum_{k=1}^{\infty} \frac{|v_j - v, e_k)|}{2^k} \\
&\leq \sum_{k=1}^{K} \frac{\varepsilon/2}{2^k} + \sum_{k=K+1}^{\infty} \frac{2C}{2^k} \\
&\leq \frac{\varepsilon}{2} \sum_{k=1}^{\infty} \frac{1}{2^k} + \frac{2C}{2^K} \sum_{k=1}^{\infty} \frac{1}{2^k} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore, $v_j \to v$ weakly in $N$ implies $|v_j - v|_w \to 0$.

Conversely, let $\|v_j\|, \|v\| \leq C$ for all $j > 0$ and $|v_j - v|_w \to 0$. Let $\varepsilon > 0$ be given. If $h = \sum_{k=1}^{\infty} \alpha_k e_k \in N$, take $K$ so large that $\|h_K\| < \varepsilon/(4C)$, where $h_K = \sum_{k=K+1}^{\infty} \alpha_k e_k$. Take $M$ so large that $|v_j - v|_w < \varepsilon/(2 \max_{1 \leq k \leq K} 2^k |\alpha_k|)$ for all $j > M$. Then

$$\begin{aligned}
|(v_j - v, h - h_K)| = |\sum_{k=1}^{K} \alpha_k (v_j - v, e_k)| &\leq \max_{1 \leq k \leq K} 2^k |\alpha_k| \sum_{k=1}^{K} \frac{|(v_j - v, e_k)|}{2^k} \\
&< \varepsilon/2
\end{aligned}$$

for $j > M$. Also, $|(v_j - v, h_K)| \leq 2C\|h_K\| < \varepsilon/2$. Therefore,

$$|(v_j - v, h)| < \varepsilon, \quad \forall j > M,$$

that is, $v_j \to v$ weakly in $N$. $\square$

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