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GEOMETRIC CONSTRAINTS ON THE DOMAIN FOR A CLASS OF MINIMUM PROBLEMS *

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Abstract. We consider minimization problems of the form

$$\min_{u\in\varphi+W_0^{1,1}(\Omega)}\int_\Omega [f(Du(x))-u(x)]\,\mathrm{d}x$$

where $\Omega\subseteq\mathbb{R}^N$ is a bounded convex open set, and the Borel function $f\colon\mathbb{R}^N\to[0,+\infty]$ is assumed to be neither convex nor coercive. Under suitable assumptions involving the geometry of Ω and the zero level set of f, we prove that the viscosity solution of a related Hamilton–Jacobi equation provides a minimizer for the integral functional.

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Introduction

Let Ω be a bounded convex open subset of $\mathbb{R}^N$, $N\geq 1$, and let J be the integral functional

$$J(u)\doteq\int_\Omega [f(Du(x))-u(x)]\,\mathrm{d}x\,,$$

acting on the functions $u\colon\Omega\to\mathbb{R}$ belonging to the class $\varphi+W_0^{1,1}(\Omega)$, $\varphi\in C^1(\overline{\Omega})$.

If the function $f\colon\mathbb{R}^N\to[0,+\infty]$ is assumed to be convex and superlinear, then, by the direct method of Calculus of Variations, it can be shown that there exists at least one minimizer for J. On the other hand, in several problems of optimal shape design the Lagrangians do not obey these requirements (see, for example [3,15] and [16]). For this reason, a branch of the recent developments in the theory of Calculus of Variations is devoted to the study of such "nonstandard problems". Among others, we mention [8, 14, 18] and the references therein (see also [9–13] for radially symmetric problems).

Keywords and phrases: Calculus of Variations, existence, non-convex problems, non-coercive problems, viscosity solutions.

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The result presented in this paper fits into the framework introduced by Cellina in [8], and developed in [6, 7, 20, 21]. More precisely, we consider the problem

$$\min_{u\in\varphi+W_0^{1,1}(\Omega)} J(u)\,, \tag{1}$$

where f is a nonnegative Borel function, and φ is a concave function belonging to $C^1(\overline{\Omega})$. We emphasize that neither convexity nor superlinearity are required on f. Setting

$$Z_f \doteq \left\{\xi \in \mathbb{R}^N \mid f(\xi) = 0\right\}, \tag{2}$$

we shall assume that its convex hull K is a compact subset of $\mathbb{R}^N$ with nonempty interior, and that $D\varphi(x)$ belongs to the interior of K for every $x \in \overline{\Omega}$.

In the papers mentioned above it is always assumed that the boundary of K is entirely contained in Z_f, and it is proved that, if the inradius of Ω is sufficiently small (see condition (H6) below), then there exists a solution to problem (1). In particular, the result proved in [7] (which subsumes those obtained in [8, 20, 21]) states that, if ρ^0 is the Minkowski functional of the polar set of K, then the function

$$u_0(x) \doteq \inf_{y\in\partial\Omega} \left\{\varphi(y) + \rho^0(x-y)\right\}$$

is a solution to (1).

It can be shown that, if $Z_f \cap \partial K$ is a closed set strictly contained in ∂K, then the minimum problem (1) may have no solution (see Ex. 2.7 at the end of the paper). Anyhow, in [4] it is proved that, if $F\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function whose zero level set coincides with $Z_f \cap \partial K$, then u_0 is a $W^{1,\infty}(\Omega)$ viscosity solution (in the sense defined in [1, 2] and [17]) of the Hamilton–Jacobi equation

$$\begin{cases} F(Du) = 0 & \text{in } \Omega\,, \\ u = \varphi & \text{on } \partial\Omega\,, \end{cases} \tag{3}$$

provided that Ω satisfies suitable geometric constraints depending on Z_f and φ. We stress the fact that, if $\partial K \subseteq Z_f$, then no restrictions, other than convexity, are imposed on the geometry of Ω.

The key observation here is that, under the same geometric constraints, u_0 provides a solution to (1), even if Z_f does not contains ∂K. This result generalizes the one given in [7] when the datum φ is smooth and $D\varphi(x)$ belongs to the interior of the set K for every $x \in \overline{\Omega}$.

1. Preliminaries

In what follows $\langle\cdot,\cdot\rangle$ and $|\cdot|$ will denote respectively the standard scalar product and the Euclidean norm in $\mathbb{R}^N$, $N \geq 1$.

We shall denote by $\overline{A}$, $\operatorname{int} A$ and $\operatorname{co} A$ respectively the closure, the interior and the convex hull of a set A. The distance between a point $\xi \in \mathbb{R}^N$ and a set $A \subseteq \mathbb{R}^N$ will be denoted by $d(\xi, A)$. Finally, $\operatorname{ext} C$ will be the set of the extremal points of the convex set C.

Let $K \subset \mathbb{R}^N$ be a compact convex set with $0 \in \operatorname{int} K$. The Minkowski functional (or gauge) of K is defined by

$$\rho(\xi) \doteq \inf\left\{\lambda > 0 \mid \xi \in \lambda K\right\}\cdot$$

Notice that, if K is the unit ball centered at the origin, then $\rho(\xi) = |\xi|$. In general, when $0 \in \operatorname{int} K$, ρ is a continuous positively 1-homogeneous convex function such that $\rho(\xi) \leq 1$ if and only if $\xi \in K$, and $\rho(\xi) = 1$ if and only if $\xi \in \partial K$. By K^0 we denote the polar set of K, that is

$$K^0 \doteq \left\{\xi^* \in \mathbb{R}^N \mid \langle \xi, \xi^*\rangle \leq 1 \ \forall \xi \in K\right\}\cdot$$

We shall consider the minimization problem

$$\min_{u\in\varphi+W_0^{1,1}(\Omega)} J(u) \doteq \min_{u\in\varphi+W_0^{1,1}(\Omega)} \int_\Omega [f(Du(x)) - u(x)]\,\mathrm{d}x\,, \tag{4}$$

where Ω is a bounded convex open subset of $\mathbb{R}^N$.

Let us define the set

$$\mathcal{N} \doteq \{y \in \partial\Omega \mid \exists\ \nu(y) \text{ inward normal}\}\,\cdot \tag{5}$$

Since Ω is a convex set, then $\mathcal{N}$ differs from $\partial\Omega$ for a set of $(N-1)$-dimensional Hausdorff measure zero. Let Z_f be the zero level set of f defined in (2).

We start by listing the assumptions on the functions f and φ.

(H1) $f\colon \mathbb{R}^N \to [0,+\infty]$ is a Borel function;
(H2) $K \doteq \operatorname{co} Z_f$ is a compact subset of $\mathbb{R}^N$ and $Z_f \cap \partial K$ is closed;
(H3) $0 \in \operatorname{int} K$;
(H4) $\varphi \in C^1(\overline{\Omega})$ is a concave function, and $D\varphi(x) \in \operatorname{int} K$ for every $x \in \overline{\Omega}$;
(H5) for every $y \in \mathcal{N}$ there exists a unique $\lambda_0(y) > 0$ such that

$$D\varphi(y) + \lambda_0(y)\nu(y) \in Z_f \cap \partial K\,.$$

Hypothesis (H5) is the compatibility condition between the geometry of Ω and the zero level set of f introduced in [4], which imposes the geometrical constraints on Ω (see Ex. 1.7).

Let ρ and ρ^0 be respectively the Minkowski functionals of the set K defined in (H2) and of its polar set K^0. Fixed φ satisfying (H4) and (H5), let us consider the function u_0 defined by

$$u_0(x) \doteq \inf_{y\in\partial\Omega} \left\{\varphi(y) + \rho^0(x-y)\right\}\cdot \tag{6}$$

Notice that for every $x \in \overline{\Omega}$ the infimum in the definition of $u_0(x)$ is achieved, and $u_0 \in W^{1,\infty}(\Omega)$.

The last requirement needed in our existence result is a link between the oscillation of u_0 and the slope of the integrand f, defined by

$$\Lambda_K(f) \doteq \sup\left\{\lambda \geq 0 \mid f(\xi) \geq \lambda(\rho(\xi) - 1)\ \forall \xi \in \mathbb{R}^N\right\}\cdot \tag{7}$$

More precisely, we require that

(H6) $\max_{\overline{\Omega}} u_0 - \min_{\overline{\Omega}} u_0 \leq \Lambda_K(f)$.

We stress that (H6) is a growth condition on f in an external neighborhood of K. This assertion will be clarified in Example 1.5 below.

Remark 1.1. Notice that, under our assumptions, the set $Z_f \cap \partial K$ is not empty. Namely $\operatorname{ext} K \neq \emptyset$ because of the compactness of K, and $\operatorname{ext} K \subseteq Z_f \cap \partial K$ (see [19], Cor. 18.3.1).

Remark 1.2. If f is a lower semicontinuous function, then Z_f is a closed set, so that, in this case, in (H2) the only requirement is the compactness of K.

Remark 1.3. The hypothesis (H3) can be replaced by

$$\operatorname{int} K \neq \emptyset\,, \tag{8}$$

which is more natural in view of the requirement (H4) on the boundary datum. Namely, if $0 \notin \operatorname{int} K \neq \emptyset$, then, fixing $\xi_0 \in \operatorname{int} K$, we can consider the function $\tilde{f}(\xi) \doteq f(\xi + \xi_0)$, so that $Z_{\tilde{f}} = Z_f - \xi_0$ and $\tilde{K} = K - \xi_0$,

$0 \in \operatorname{int} \tilde{K}$. For every $u \in \varphi + W_0^{1,1}(\Omega)$ we consider the function

$$v(x) \doteq u(x) - \langle \xi_0, x \rangle \in \varphi - \langle \xi_0, \cdot \rangle + W_0^{1,1}(\Omega)\,.$$

Then we have

$$J(u) = \int_\Omega [\tilde{f}(Dv(x)) - v(x)]\,\mathrm{d}x + \left\langle \xi_0,\ \int_\Omega x\,\mathrm{d}x \right\rangle = \tilde{J}(v) + c\,.$$

Hence problem (4) is equivalent to the problem

$$\min_{v \in \varphi - \langle \xi_0, \cdot \rangle + W_0^{1,1}(\Omega)} \tilde{J}(v)\,,$$

where $\tilde{f}$ and the boundary datum $\varphi - \langle \xi_0, \cdot \rangle$ satisfy (H1–H5). Even if (8) is more general than (H3), we prefer to deal with (H3) for sake of simplicity.

Remark 1.4. The hypothesis that $Z_f \cap \partial K$ is a closed set, together with (H5), can be replaced by the following assumption: for every $y \in \mathcal{N}$ there exists a unique $\lambda_0(y) > 0$ such that

$$D\varphi(y) + \lambda_0(y)\nu(y) \in Z\,,$$

where Z is a closed set satisfying $\operatorname{ext} K \subseteq Z \subseteq Z_f \cap \partial K$ (see the proof of Th. 2.1 for details).

Example 1.5. Let us consider the radial case $f(\xi) = g(|\xi|)$, where $g\colon [0, +\infty[\to [0, +\infty]$ is a Borel function satisfying $g(R) = 0$ for some $R > 0$, and $g(s) \geq \mu(s - R)$ for some $\mu > 0$ and every $s \geq 0$. It is clear that $\partial B_R(0) \subset Z_f \subset \overline{B}_R(0)$, hence $K = \overline{B}_R(0)$, so that (H1, H2) and (H3) are fulfilled. Moreover, $Z_f \cap \partial K = \partial B_R(0)$, which implies that (H5) is satisfied for every boundary datum φ. The Minkowski functionals of K and its polar set $K^0 = \overline{B}_{1/R}(0)$ are respectively $\rho(\xi) = |\xi|/R$ and $\rho^0(\xi^*) = R|\xi^*|$. The constant $\Lambda_K(f)$ is given by $\sup\{\lambda > 0 \mid g(s) \geq \lambda(s/R - 1),\ \forall s \geq 0\}$. In terms of the bipolar function g^{**} of $t \mapsto g(|t|)$ we have that $\Lambda_K(f)$ is the right derivative $(g^{**})'_+(R)$ (see Fig. 1).

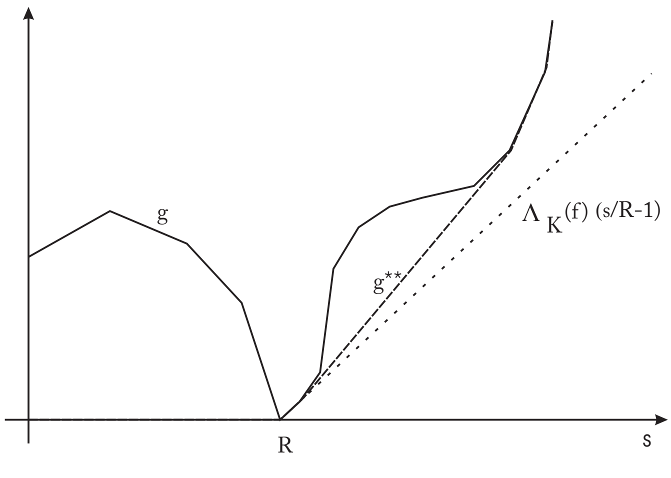

Figure 1.

In the homogeneous case $\varphi = 0$, the assumption (H6) now becomes $R \max\{d(x, \partial\Omega);\ x \in \overline{\Omega}\} \leq \Lambda_K(f)$, which is a condition for the existence of a solution introduced in [8]. In [6] it is proved that, if this condition is violated then, in general, problem (4) has not a solution.

Remark 1.6. It is clear that (H6) prevents f from being smooth even in the convex case. As a consequence, the Euler–Lagrange conditions associated to (4) can be only written in terms of differential inclusions: a solution u of the minimum problem is an integral solution of the system

$$p(x) \in \partial f(Du(x)), \qquad \operatorname{div} p(x) = -1. \tag{9}$$

For example, in the settings of Example 1.5 with the piecewise affine function $g(s) = \max\{0, \Lambda(s-1)\}$, the first inclusion in (9) can be rewritten as $p(x) = \alpha(x)Du(x)/|Du(x)|$, with $\alpha(x) = 0$ if $|Du(x)| < 1$, $\alpha(x) \in [0, \Lambda]$ if $|Du(x)| = 1$, and $\alpha(x) = \Lambda$ if $|Du(x)| > 1$. These information do not seem sufficient in order to obtain the explicit solution u_0 even in this simple case.

Example 1.7. Let us clarify the meaning of the compatibility condition (H5) with the following example in two dimensions. Let Z be the set composed by the four points $(-1,-1)$, $(-1,1)$, $(1,-1)$, $(1,1)$, and let us consider the function $f(\xi) = d(\xi, Z)$. It is clear that $Z_f = Z$ and $K = [-1,1]^2$ (see Fig. 2).

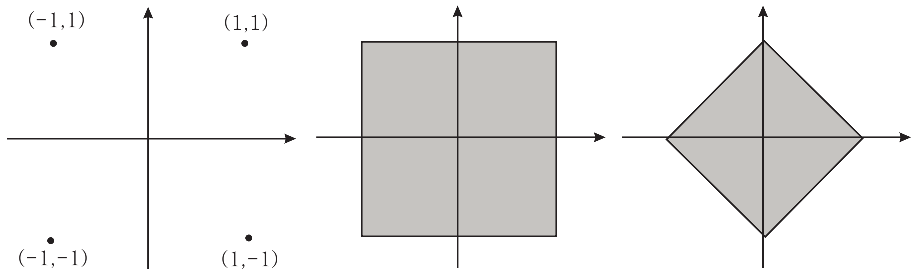

FIGURE 2. The sets Z_f, K, K^0.

For $\varphi = 0$, the convex domains satisfying (H5) are only the rectangles with sides orthogonal to the directions of Z_f. For example, the first domain in Figure 3 satisfies (H5), whereas the second one has the horizontal side whose normal is not parallel to the directions of Z_f.

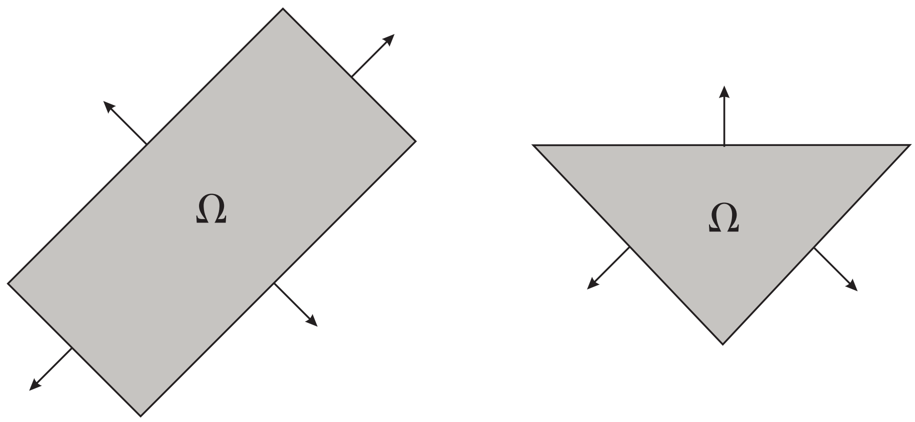

FIGURE 3.

We shall show in Example 2.7 that, if (H5) is not satisfied, then the functional J may have no minimizers.

2. The result

In this section we shall prove the following existence result:

Theorem 2.1. *Under the assumptions (*H1–H6*), the function u_0 defined in* (6) *provides a solution to problem* (4).

The proof of Theorem 2.1 relies on the following result proved in [4].

Theorem 2.2. *Let $F\colon \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that the set*

$$Z_F \doteq \{\xi \in \mathbb{R}^N \mid F(\xi) = 0\}$$

is bounded and contained in $\partial(\operatorname{co} Z_F)$. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded convex open set, let $\mathcal{N} \subseteq \partial\Omega$ be the set defined in (5)*, and let $\varphi \in C^1(\overline{\Omega})$ satisfy $D\varphi(x) \in \operatorname{int}(\operatorname{co} Z_F)$ for every $x \in \overline{\Omega}$. Assume that, for every $y \in \mathcal{N}$, there exists a unique $\lambda_0(y) > 0$ such that*

$$D\varphi(y) + \lambda_0(y)\nu(y) \in Z_F\,.$$

Then the function u_0 defined in (6) *is a $W^{1,\infty}(\Omega)$ viscosity solution to the Hamilton–Jacoby equation* (3).

The definition of viscosity solution can be found in [1, 2] and [17]. To our aim, it is enough to recall that a $W^{1,\infty}(\Omega)$ viscosity solution of (3) satisfies

$$Du_0(x) \in Z_F \qquad \text{a.e. } x \in \Omega\,. \tag{10}$$

The key point in the proof of Theorem 2.1 is to relate this result about viscosity solutions with the following existence result for minima of integral functionals proved in [7].

Theorem 2.3. *Assume that f satisfies (*H1–H3*), and, in addition, that*

$$\partial K \subseteq Z_f\,. \tag{11}$$

Let $\varphi\colon \overline{\Omega} \to \mathbb{R}$ be a Lipschitz continuous concave function such that $D\varphi(x) \in K$ for a.e. $x \in \Omega$. Then, if (H6) holds, the function u_0 defined in (6) *provides a solution to the minimum problem* (4).

Notice that Theorem 2.1 generalizes Theorem 2.3 in the following sense. In order to apply Theorem 2.2, we need stronger regularity assumptions on the boundary datum φ, but, on the other hand, we relax the condition (11). Indeed, if (H2–H4) and (11) are fulfilled, then (H5) holds for every convex domain Ω. Namely, for every $y \in \mathcal{N}$, since $D\varphi(y) \in \operatorname{int} K$, there exists a unique $\lambda_0(y) > 0$ such that $D\varphi(y) + \lambda_0(y)\nu(y) \in \partial K$.

As a corollary of Theorem 2.2, we obtain the following result:

Proposition 2.4. *Let $Z \subseteq \mathbb{R}^N$ be a compact set such that $Z \subseteq \partial(\operatorname{co} Z)$ and $\operatorname{int}(\operatorname{co} Z) \neq \emptyset$. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded convex open set, and let $\varphi \in C^1(\overline{\Omega})$ satisfy*

(i) *$D\varphi(x) \in \operatorname{int}(\operatorname{co} Z)$, for every $x \in \overline{\Omega}$;*
(ii) *for every $y \in \mathcal{N}$ there exists a unique $\lambda_0(y) > 0$ such that*

$$D\varphi(y) + \lambda_0(y)\nu(y) \in Z\,.$$

Then the function u_0 defined in (6) *satisfies*

$$Du_0(x) \in Z\,, \qquad \text{a.e. } x \in \Omega\,. \tag{12}$$

Proof. Let $F(\xi) \doteq d(\xi, Z)$, $\xi \in \mathbb{R}^N$. Since $Z_F = \overline{Z} = Z$, by Theorem 2.2, u_0 provides a $W^{1,\infty}(\Omega)$ viscosity solution to (3), so that (12) holds. □

Proof of Theorem 2.1. Let $g\colon \mathbb{R}^N \to [0, +\infty]$ be the function defined by

$$g(\xi) \doteq \begin{cases} \Lambda_K(f)(\rho(\xi) - 1) & \xi \notin K\,, \\ 0 & \xi \in K\,, \end{cases}$$

where $\Lambda_K(f) \in]0, +\infty]$ is the constant defined in (7) (notice that, by (H6), $\Lambda_K(f) > 0$). We have that $Z_g = K$, g satisfies (H1–H3), and $\Lambda_K(g) = \Lambda_K(f)$, so that (H6) holds. Then we can apply Theorem 2.3, obtaining that u_0 is a minimizer of the functional

$$G(u) \doteq \int_\Omega [g(Du(x)) - u(x)]\, \mathrm{d}x\,,$$

in the class $\varphi + W_0^{1,1}(\Omega)$.

By the very definition of $\Lambda_K(f)$, and since f is nonnegative, we deduce that $f \geq g$ in $\mathbb{R}^N$. Moreover, $\mathrm{co}(Z_f \cap \partial K) = K$. Namely, by the compactness of K, we have that $K = \mathrm{co}(\mathrm{ext}\, K)$ (see [19], Cor. 18.5.1), and, since $\mathrm{ext}\, K \subseteq Z_f \cap \partial K$, one gets $K \subseteq \mathrm{co}(Z_f \cap \partial K)$. The other inclusion is trivial.

Then we can apply Proposition 2.4 with $Z = Z_f \cap \partial K$, obtaining

$$Du_0(x) \in Z_f \cap \partial K\,, \qquad \text{a.e. } x \in \Omega\,.$$

Hence $f(Du_0(x)) = g(Du_0(x)) = 0$ for almost every $x \in \Omega$, so that, for every $u \in \varphi + W_0^{1,1}(\Omega)$, one has

$$J(u_0) = G(u_0) \leq G(u) \leq J(u)\,,$$

which concludes the proof. □

Remark 2.5. The existence result stated in Theorem 2.1 holds for more general minimum problems of the form

$$\min_{u \in \varphi + W_0^{1,1}(\Omega)} \int_\Omega [f(Du(x)) + h(x, u(x))]\, \mathrm{d}x\,, \tag{13}$$

where f and φ satisfy (H1–H5), while the function $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to x for every fixed $u \in \mathbb{R}$, non increasing with respect to u for a.e. fixed $x \in \Omega$, $h(\cdot, 0) \in L^1(\Omega)$, and there exists a constant $L > 0$ such that

$$|h(x, u) - h(x, v)| \leq L|u - v|\,, \qquad a.e.\ x \in \Omega\,,\ \forall u, v \in \mathbb{R}\,.$$

Finally, the condition (H6) must be replaced by

$$L \left(\max_{\overline{\Omega}} u_0 - \min_{\overline{\Omega}} u_0 \right) \leq \Lambda_K(f).$$

Under these assumptions, in [5] it is proved that the function u_0 provides a solution to (13), and the proof of Theorem 2.1 can be carried out in the very same way.

When $\varphi = 0$, the solution u_0 to (4) turns out to be the distance function from $\partial\Omega$ associated to the convex set K^0. More precisely, for every non empty subset A of $\mathbb{R}^N$, we introduce the distance function from A with respect to ρ^0

$$d_{K^0}(x, A) \doteq \inf_{y \in A} \rho^0(x - y)\,. \tag{14}$$

If $\varphi = 0$, the function u_0 defined in (6) coincides with the distance function $d_{K^0}(x, \partial\Omega)$, and Theorem 2.1 can be rewritten as follows:

Corollary 2.6. *Assume that (*H1, H2*), and (*H3*) hold. Suppose that*

$$\max_{x\in\overline{\Omega}} d_{K^0}(x, \partial\Omega) \le \Lambda_K(f) \tag{15}$$

and that

$$\frac{\nu(y)}{\rho(\nu(y))} \in Z_f \qquad \forall y \in \mathcal{N}\,. \tag{16}$$

Then the function $u_0(x) \doteq d_{K^0}(x, \partial\Omega)$ *is a solution of problem* (4).

The compatibility condition (H5) is a necessary condition for the existence of a minimizer of J, in the sense explained below.

Example 2.7. Let us assume that $\varphi = 0$, and let $Z \subseteq \mathbb{R}^N$ be a compact set such that $0 \in \operatorname{int}(\operatorname{co} Z)$ and

$$\partial(\operatorname{co} Z) \setminus Z \neq \emptyset\,. \tag{17}$$

We are going to show that there exist a convex set Ω and a function f, with $Z_f = Z$, satisfying all the assumptions of Corollary 2.6 but (16), and such that problem (4) has no solution. Let $\zeta \in \partial(\operatorname{co} Z) \setminus Z$, and let Ω be a cube with one face C having $\frac{\zeta}{|\zeta|}$ as inward normal, so that (16) is trivially not satisfied.

Define the function f by

$$f(\xi) \doteq \begin{cases} 0\,, & \xi \in Z\,, \\ +\infty\,, & \xi \notin Z\,. \end{cases}$$

As $\Lambda_{\operatorname{co} Z}(f) = +\infty$, the assumption (15) is satisfied.

We claim that, in this case,

$$\inf_{u\in W_0^{1,1}(\Omega)} J(u) = -\int_\Omega u_0(x)\,\mathrm{d}x\,,$$

where u_0 is the function considered in Corollary 2.6, but the infimum is not achieved. Let f^{**} be the bipolar function of f, given by

$$f^{**}(\xi) = \begin{cases} 0\,, & \xi \in \operatorname{co} Z\,, \\ +\infty\,, & \xi \notin \operatorname{co} Z\,, \end{cases}$$

and let us consider the relaxed functional

$$\overline{J}(u) \doteq \int_\Omega [f^{**}(Du(x)) - u(x)]\,\mathrm{d}x\,, \qquad u \in W_0^{1,1}(\Omega)\,.$$

Since $Z_{f^{**}} = \operatorname{co} Z$, all the assumptions of Corollary 2.6 are satisfied, hence u_0 is a minimizer of $\overline{J}$ in $W_0^{1,1}(\Omega)$.

Actually u_0 is the unique minimizer of $\overline{J}$. Namely, let $v \in W_0^{1,1}(\Omega)$ be another minimizer of $\overline{J}$, and define

$$w^- \doteq \min\{u_0, v\}\,, \qquad w^+ \doteq \max\{u_0, v\}\,\cdot$$

Clearly, $w^- \leq w^+$ and $w^- = w^+$ if and only if $u_0 = v$. From the fact that

$$\overline{J}(w^-) + \overline{J}(w^+) = \overline{J}(u_0) + \overline{J}(v),$$

we deduce that also w^- and w^+ are minimizers of $\overline{J}$. Henceforth $\overline{J}(w^-) = \overline{J}(w^+) < +\infty$, so that

$$\int_\Omega w^-(x)\,\mathrm{d}x = \int_\Omega w^+(x)\,\mathrm{d}x\,,$$

which implies that $u_0 = v$.

Since $\inf J = \min \overline{J}$, the claim is proved if we show that $J(u_0) > \overline{J}(u_0)$. By Lemma 2.9 in [4], for every $x \in \Omega$ with $d(x, C)$ small enough, we have that $Du_0(x) = \zeta$, hence $J(u_0) = +\infty$.

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