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COMPARISON BETWEEN W_2 DISTANCE AND $\dot{\mathrm{H}}^{-1}$ NORM, AND LOCALIZATION OF WASSERSTEIN DISTANCE*

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Abstract. It is well known that the quadratic Wasserstein distance $\mathrm{W}_2(\boldsymbol{\cdot},\boldsymbol{\cdot})$ is formally equivalent, for infinitesimally small perturbations, to some weighted H^{-1} homogeneous Sobolev norm. In this article I show that this equivalence can be integrated to get non-asymptotic comparison results between these distances. Then I give an application of these results to prove that the W_2 distance exhibits some localization phenomenon: if μ and ν are measures on $\mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \to \mathbb{R}_+$ is some bump function with compact support, then under mild hypotheses, you can bound above the Wasserstein distance between $\varphi\cdot\mu$ and $\varphi\cdot\nu$ by an explicit multiple of $\mathrm{W}_2(\mu,\nu)$.

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1. Foreword

This article is divided into two sections, each of which having its own introduction. Section 2 deals with general results of comparison between Wasserstein distance and homogeneous Sobolev norm, while Section 3 handles an application to localization of W_2 distance.

2. Non-asymptotic equivalence between W_2 distance and $\dot{\mathrm{H}}^{-1}$ norm

2.1. Introduction

In all this section, M denotes a connected Riemannian manifold endowed with its distance $dist(\boldsymbol{\cdot},\boldsymbol{\cdot})$ and its standard measure λ provided by the volume form (so, in the case $M=\mathbb{R}^n$, λ is the Lebesgue measure). Let us give a few standard definitions which will be at the core of our work:

- For μ,ν two positive measures on M, denoting by $\Pi(\mu,\nu)$ the set of (positive) measures on $M\times M$ whose respective marginals are μ and ν, for $\pi\in\Pi(\mu,\nu)$ one defines

$$I(\pi) := \int_{M\times M} dist(x,y)^2 \pi(\mathrm{d}x,\mathrm{d}y) \tag{2.1}$$

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and then

$$W_2(\mu,\nu) := \inf\{I(\pi) | \pi \in \Pi(\mu,\nu)\}^{1/2}. \tag{2.2}$$

W_2 is a (possibly infinite) distance, called the *quadratic Wasserstein distance* ([13], Sect. 7.1). Note that this distance is finite only between measures having the same total mass.

- On the other hand, for μ a (positive) measure on M, if f is a C^1 real function on M, one denotes

$$\|f\|_{\dot{H}^1(\mu)} := \left(\int_M |\nabla f(x)|^2 \mu(\mathrm{d}x)\right)^{1/2}, \tag{2.3}$$

which defines a semi-norm; for ν a signed measure on M, one then denotes

$$\|\nu\|_{\dot{H}^{-1}(\mu)} := \sup\{|\langle f,\nu\rangle| \mid \|f\|_{\dot{H}^1(\mu)} \leqslant 1\}, \tag{2.4}$$

where the duality product $\langle f,\nu\rangle$ denotes the integral of the function f against the measure ν.[2] We observe that $\|\cdot\|_{\dot{H}^{-1}(\mu)}$ defines a (possibly infinite) norm, which we will call the $\dot{H}^{-1}(\mu)$ *weighted homogeneous Sobolev norm*. Note that this norm is finite only for measures having zero total mass. In the case μ is the standard measure, we will merely write "$\dot{H}^{-1}$" for "$\dot{H}^{-1}(\lambda)$".

The W_2 Wasserstein distance is an important object in analysis; but it is non-linear, which makes it harder to study. For infinitesimal perturbations however, the linearized behaviour of W_2 is well known: if μ is a positive measure on M and $\mathrm{d}\mu$ is an infinitesimally small perturbation of this measure,[3] one has formally (see [13], Sect 7.6, or [9], Sect. 7)

$$W_2(\mu,\mu+\mathrm{d}\mu) = \|\mathrm{d}\mu\|_{\dot{H}^{-1}(\mu)} + o(\mathrm{d}\mu).^4 \tag{2.5}$$

More precisely, one has the following equality, known as the *Benamou–Brenier formula* ([2], Prop. 1.1) (see [10] when M is a general Riemannian manifold): for two positive measures μ,ν on M,

$$W_2(\mu,\nu) = \inf\left\{\int_0^1 \|\mathrm{d}\mu\|_{\dot{H}^{-1}(\mu t)} \,\middle|\, \mu_0=\mu,\ \mu_1=\nu\right\}. \tag{2.6}$$

Then, a natural question is the following: are there *non-asymptotic* comparisons between the W_2 distance and the $\dot{H}^{-1}$ norm? Concretely, we are looking for inequalities like

$$C_{\mathrm{a}}\,\|\mu-\nu\|_{\dot{H}^{-1}(\mu)} \leqslant W_2(\mu,\nu) \leqslant C_{\mathrm{b}}\,\|\mu-\nu\|_{\dot{H}^{-1}(\mu)} \tag{2.7}$$

for constants $0 < C_{\mathrm{a}} \leqslant C_{\mathrm{b}} < \infty$, under mild assumptions on μ and ν.

2.2. Controlling W_2 by $\dot{H}^{-1}$

Theorem 2.1. *For any positive measures μ,ν on M,*

$$W_2(\mu,\nu) \leqslant 2\,\|\mu-\nu\|_{\dot{H}^{-1}(\mu)}. \tag{2.8}$$

Proof. We suppose that $\|\mu-\nu\|_{\dot{H}^{-1}(\mu)} < \infty$, otherwise there is nothing to prove. For $t \in [0,1]$, let

$$\mu_t := (1-t)\mu + t\nu, \tag{2.9}$$

[2] The rationale behind the use of duality notation in this article is that we cannot use the notation "$\mathrm{d}\mu$" to refer to the measure of a small volume: see indeed Footnote 3 below.

[3] Beware that here $\mathrm{d}\mu$ denotes a small measure on M, not the value of μ on a small volume.

[4] This formula has to be understood in the sense that, for every measure ν, one has $W_2(\mu,\mu+\varepsilon\nu) \overset{\varepsilon\to 0}{=} |\varepsilon|\,\|\nu\|_{\dot{H}^{-1}(\mu)} + o(\varepsilon)$. As explained in the references cited, some regularity assumptions on ν shall be required for that property to hold rigorously: in particular, one must have $\nu \ll \mu$ with a bounded and smooth enough density.

so that $\mu_0 = \mu$, $\mu_1 = \nu$ and $\mathrm{d}\mu_t = (\mu - \nu)\mathrm{d}t$. Then, by the Benamou–Brenier formula (2.6):

$$\mathrm{W}_2(\mu,\nu) \leqslant \int_0^1 \|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}(\mu_t)}\,\mathrm{d}t. \tag{2.10}$$

Now, we use the following key lemma, whose proof is postponed:

Lemma 2.2. *If μ, μ' are two measures such that $\mu' \geqslant \rho\mu$ for some $\rho > 0$, then $\|\cdot\|_{\dot{\mathrm{H}}^{-1}(\mu')} \leqslant \rho^{-1/2}\|\cdot\|_{\dot{\mathrm{H}}^{-1}(\mu)}$.*[5]

Here obviously $\mu_t \geqslant (1-t)\mu$, so

$$\mathrm{W}_2(\mu,\nu) \leqslant \int_0^1 (1-t)^{-1/2}\|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}(\mu)}\,\mathrm{d}t = 2\|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}(\mu)}. \tag{2.11}$$

□

Corollary 2.3. *If $\mu \geqslant \rho\lambda$ for some $\rho > 0$, then*

$$\mathrm{W}_2(\mu,\nu) \leqslant 2\rho^{-1/2}\|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}}. \tag{2.12}$$

Proof. Just use that $\|\cdot\|_{\dot{\mathrm{H}}^{-1}(\mu)} \leqslant \rho^{-1/2}\|\cdot\|_{\dot{\mathrm{H}}^{-1}}$ by Lemma 2.2. □

Proof of Lemma 2.2. Take $\mu' \geqslant \rho\mu$ and let ν be a signed measure on M such that $\mu+\nu$ is positive; then $\mu'+\rho\nu$ is also positive. For m a measure on M, we denote by $\mathrm{diag}(m)$ the measure on $M\times M$ supported by the diagonal whose marginals (which are equal) are m, *i.e.*:

$$\big(\mathrm{diag}(m)\big)(A\times B) := m(A\cap B); \tag{2.13}$$

with that notation,

$$\pi \in \Pi(\mu,\mu+\nu) \Rightarrow \rho\pi + \mathrm{diag}(\mu'-\rho\mu) \in \Pi(\mu',\mu'+\rho\nu), \tag{2.14}$$

and

$$I\big(\rho\pi + \mathrm{diag}(\mu'-\rho\mu)\big) = \rho I(\pi). \tag{2.15}$$

Therefore, taking infima,

$$\begin{aligned}\mathrm{W}_2(\mu',\mu'+\rho\nu)^2 &= \inf\{I(\pi')\,|\,\pi'\in\Gamma(\mu',\mu'+\rho\nu)\}\\ &\leqslant \inf\big\{I\big(\rho\pi+\mathrm{diag}(\mu'-\rho\mu)\big)\,\big|\,\pi\in\Gamma(\mu,\mu+\nu)\big\}\\ &= \rho\inf\{I(\pi)\,|\,\pi\in\Gamma(\mu,\mu+\nu)\} = \rho\mathrm{W}_2(\mu,\mu+\nu)^2.\end{aligned} \tag{2.16}$$

For infinitesimally small ν,[6] it follows by equation (2.5) that $\|\rho\nu\|^2_{\dot{\mathrm{H}}^{-1}(\mu')} \leqslant \rho\|\nu\|^2_{\dot{\mathrm{H}}^{-1}(\mu)}$, hence $\|\nu\|_{\dot{\mathrm{H}}^{-1}(\mu')} \leqslant \rho^{-1/2}\|\nu\|_{\dot{\mathrm{H}}^{-1}(\mu)}$. This relation remains true even for non-infinitesimal ν by linearity, which ends the proof. □

Remark 2.4. Lemma 2.2 could also be proved very quickly by using the definition (2.3)–(2.4) of the $\dot{\mathrm{H}}^{-1}(\mu)$ norm. The proof above, however, has the advantage that it does not need the precise expression of $\|\cdot\|_{\dot{\mathrm{H}}^{-1}(\mu)}$, but only the fact that it is the linearized W_2 distance.

[5]Beware that here '$\cdot$' stands for a *measure*, not for a function: otherwise the formula would be false.— When f is a function, $\|f\|_{\dot{\mathrm{H}}^{-1}(\mu)}$ stands for the $\dot{\mathrm{H}}^{-1}(\mu)$ norm of the measure having density f w.r.t. μ.

[6]To make rigorous the formal argument of taking an infinitestimally small ν, according to Footnote 4 above, one would have to replace ν by $\varepsilon\nu_1$, where ν_1 is a regular enough measure, and to let ε tend to 0; then the regularity assumption on ν_1 would be relaxed by a classical approximation argument. Anyway, Lemma 2.2 can also be proved easily and rigorously without referring to optimal transportation at all, *cf.* Remark 2.4 below.

2.3. Controlling $\dot{\mathrm{H}}^{-1}$ by W_2

Theorem 2.5. *Assume M has nonnegative Ricci curvature. Then for any positive measures μ,ν on M such that $\mu \leqslant \rho_0\lambda$ and $\nu \leqslant \rho_1\lambda$,*

$$\|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}} \leqslant \frac{2(\rho_0^{1/2}-\rho_1^{1/2})}{\ln(\rho_0\,/\,\rho_1)}\mathrm{W}_2(\mu,\nu). \tag{2.17}$$

(*For $\rho_1=\rho_0$, the right-hand side of* (2.17) *is to be taken as $\rho_0^{1/2}\mathrm{W}_2(\mu,\nu)$ by continuity*).

Remark 2.6. For $M=\mathbb{R}^n$ a similar result was already stated in ([7], Prop. 2.8), with a different proof.

Proof. Assume that $\mathrm{W}_2(\mu,\nu)<\infty$, otherwise there is nothing to prove. Let $(\mu_t)_{0\leqslant t\leqslant 1}$ be the displacement interpolation between μ and ν (*cf.* [14], Chapt. 7), which is such that $\mu_0=\mu,\ \mu_1=\nu$ and the infimum in (2.6) is attained with $\|\mathrm{d}\mu_t\|_{\dot{\mathrm{H}}^{-1}(\mu_t)}=\mathrm{W}_2(\mu,\nu)\mathrm{d}t\ \forall t$. Since Ricci curvature is nonnegative, the Lott−Sturm−Villani theory tells us that, denoting by $\|\mu\|_\infty$ the essential supremum of the density of μ w.r.t. λ, one has $\|\mu_t\|_\infty \leqslant \|\mu_0\|_\infty^{1-t}\,\|\mu_1\|_\infty^{t} \leqslant \rho_0^{1-t}\rho_1^t$ (see [14], Cor. 17.19 or [5], Lem. 6.1); so that $\|\cdot\|_{\dot{\mathrm{H}}^{-1}} \leqslant \rho_0^{(1-t)/2}\rho_1^{t/2}\,\|\cdot\|_{\dot{\mathrm{H}}^{-1}(\mu_t)}$ by Lemma 2.2.

Then, by the integral triangle inequality for normed vector spaces,

$$\begin{aligned}\|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}} = \left\|\int_0^1 \mathrm{d}\mu_t\right\|_{\dot{\mathrm{H}}^{-1}} &\leqslant \int_0^1 \|\mathrm{d}\mu_t\|_{\dot{\mathrm{H}}^{-1}} \leqslant \int_0^1 \rho_0^{(1-t)/2}\rho_1^{t/2}\,\|\mathrm{d}\mu_t\|_{\dot{\mathrm{H}}^{-1}(\mu_t)} \\ &= \left(\int_0^1 \rho_0^{(1-t)/2}\rho_1^{t/2}\mathrm{d}t\right)\mathrm{W}_2(\mu,\nu) = \frac{2(\rho_0^{1/2}-\rho_1^{1/2})}{\ln(\rho_0\,/\,\rho_1)}\mathrm{W}_2(\mu,\nu).\end{aligned} \tag{2.18}$$

□

Remark 2.7. Taking into account the dimension n of the manifold M, the bound on $\|\mu_t\|_\infty$ could be refined into

$$\|\mu_t\|_\infty \leqslant \big((1-t)\,\|\mu_0\|_\infty^{-1/n}+t\,\|\mu_1\|_\infty^{-1/n}\big)^{-n} \tag{2.19}$$

(*cf.* [8], Thm. 2.3), which would yield a slightly sharper bound in equation (2.17), namely:

$$\|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}} \leqslant \left(\int_0^1 \big((1-t)\rho_0^{-1/n}+t\rho_1^{-1/n}\big)^{-n/2}\mathrm{d}t\right)\mathrm{W}_2(\mu,\nu) = \begin{cases} \dfrac{\rho_0^{1/2-1/n}-\rho_1^{1/2-1/n}}{(n/2-1)(\rho_1^{-1/n}-\rho_0^{-1/n})}\mathrm{W}_2(\mu,\nu) & n\geqslant 2; \\ \dfrac{\log(\rho_1\,/\,\rho_0)}{2(\rho_0^{-1/2}-\rho_1^{-1/2})}\mathrm{W}_2(\mu,\nu) & n=2. \end{cases} \tag{2.20}$$

For $n=1$ it turns out that one can let tend $\rho_1\to\infty$ in (2.20) without making the integral diverge; which leads to a much more powerful result:

Theorem 2.8. *When M is an interval of $\mathbb{R}$, then under the sole assumption that $\mu\leqslant\rho_0\lambda$, one has for all positive measures ν on M*:

$$\|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}} \leqslant 2\rho_0^{1/2}\mathrm{W}_2(\mu,\nu). \tag{2.21}$$

Remark 2.9. For $n\geqslant 2$ there is no hope to get a bound valid for all ν, because then it can occur that $\mathrm{W}_2(\mu,\nu)<\infty$ but $\|\mu-\nu\|_{\dot{\mathrm{H}}^{-1}}=\infty$: for instance, take μ to be the uniform measure on the 2-dimensional sphere and ν a Dirac mass.

3. Application to localization of Wasserstein distance

3.1. Introduction

In all this section, we work in the Euclidian space $\mathbb{R}^n$, whose norm is denoted by $|\cdot|$. $dist(x, A) := \inf\{|x-y| \mid y \in A\}$ denotes the distance between a point x and a set A; A^{c} denotes the complement of A; λ denotes the Lebesgue measure. We will use the following notation to handle measures:

- For μ a measure on $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ a measurable map, $f_* \mu$ denotes the pushforward of μ by f, that is, $\big(f_* \mu\big)(A) := \mu(f^{-1}(A))$.
- For μ a measure on $\mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \to \mathbb{R}_+$ a nonnegative measurable function, $\varphi \cdot \mu$ denotes the measure such that $\big(\varphi \cdot \mu\big)(\mathrm{d}x) := \varphi(x)\mu(\mathrm{d}x)$.

We will also use the following norms on measures:

- $\|\mu\|_{\dot{H}^{-1}(\nu)}$ has the same definition as in Section 2;
- $\|\mu\|_1 := \int_{\mathbb{R}^n} |\mu(\mathrm{d}x)|$ is the total variation norm of μ;[7]
- For ν a positive measure with $\mu \ll \nu$, we define

$$\|\mu\|_{L^2(\nu)} := \left(\int_{\operatorname{supp}\nu} \left(\frac{\mu(\mathrm{d}x)}{\nu(\mathrm{d}x)}\right)^2 \nu(\mathrm{d}x) \right)^{1/2}. \tag{3.1}$$[8]

For $A \subset \mathbb{R}^n$, we also denote $\|\cdot\|_{L^2(A)}$ for $\|\cdot\|_{L^2(\mathbf{1}_A \cdot \lambda)}$.

The goal of this section is to give an application of Theorem 2.1 to the problem of *localization* of the quadratic Wasserstein distance. Morally, the question is the following: take two measures μ, ν on $\mathbb{R}^n$ being close to each other in the sense of W_2 distance; is it true that μ and ν remain close when you consider their restrictions to a subset of $\mathbb{R}^n$? Concretely, if φ is a non-negative real function on $\mathbb{R}^n$ with compact support (plus some technical assumptions to be specified later), we want to bound above $W_2(a\varphi \cdot \mu, \varphi \cdot \nu)$ by some multiple of $W_2(\mu, \nu)$ — where, in the former expression, a is a constant factor ensuring that $a\varphi \cdot \mu$ and $\varphi \cdot \nu$ have the same mass (for otherwise the distance between $\varphi \cdot \mu$ and $\varphi \cdot \nu$ is generically infinite).

This question, which was my initial motivation for the results of Section 2, was asked to me by Xavier Tolsa, who needed such a result for his paper [12] on characterizing uniform rectifiability in terms of mass transport. Actually Xavier managed to devise a proof of his own ([12], Thm. 1.1), but it was quite long (about thirty pages) and involved arguments of multi-scale analysis. With Theorem 2.1 at hand, however, the reasoning becomes far more direct; moreover we will be able to relax some of the assumptions of Xavier's theorem.

3.2. Statement of the theorem

Theorem 3.1. *Let μ, ν be (positive) measures on $\mathbb{R}^n$ having the same total mass; let B be a ball of $\mathbb{R}^n$ (whose radius will be denoted by R when needed). Assume that on B, the density of μ w.r.t. the Lebesgue measure is bounded above and below:*

$$\exists\ 0 < m_1 \leqslant m_2 < \infty \qquad \forall x \in B \qquad m_1\lambda(\mathrm{d}x) \leqslant \mu(\mathrm{d}x) \leqslant m_2\lambda(\mathrm{d}x). \tag{3.2}$$

Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}_+$ be a function such that:

(i) *φ is zero outside B;*
(ii) *There exist $0 < c_1 \leqslant c_2 < \infty$ such that for all $x \in B$, $c_1\, dist(x, B^{\mathrm{c}})^2 \leqslant \varphi(x) \leqslant c_2\, dist(x, B^{\mathrm{c}})^2$.*
(iii) *φ is k-Lipschitz for some $k < \infty$.*

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[7] Note that in the case μ is a positive measure on $\mathbb{R}^n$, then $\|\mu\|_1$ is nothing but $\mu(\mathbb{R}^n)$.

[8] What we denote here by $\mu(\mathrm{d}x)/\nu(\mathrm{d}x)$ here is what is commonly called $\big(\mathrm{d}\mu/\mathrm{d}\nu\big)(x)$: indeed, as we already told, in this article we reserve the use of "$\mathrm{d}\mu$" to denote a mass distribution of infinitesimally small magnitude, rather than for the mass of an infinitely small volume.

Then, denoting $a := \|\varphi\cdot\nu\|_1 \,/\, \|\varphi\cdot\mu\|_1$,

$$\mathrm{W}_2(a\varphi\cdot\mu,\varphi\cdot\nu) \leqslant C(n)\frac{c_2^{3/2}m_2^{3/2}}{c_1^{3/2}m_1^{3/2}}kc_1^{-1/2}\mathrm{W}_2(\mu,\nu), \tag{3.3}$$

for $C(n)<\infty$ *some absolute constant only depending on* n. *Moreover, one can bound explicitly* $C(n)$ *in such a way that* $C(n)=O(n^{1/2})$ *when* $n\to\infty$.[9]

Remark 3.2. Theorem 3.1 relaxes the assumptions of Theorem 1.1 of [12] on the following points: first, Tolsa's theorem required that $|\nabla\varphi|$ was bounded by a multiple of $dist(\cdot,B^{\mathsf{c}})$, while ours does not impose any specific control on $|\nabla\varphi|$ near the boundary of B; second, Tolsa's theorem worked only for radially symmetric φ. Also, contrary to [12], our conclusions state explicitly how the bound on $\mathrm{W}_2(a\varphi\cdot\mu,\varphi\cdot\nu)$ depends on the constants k, c_1, c_2, m_1, m_2 and on the dimension n.

Remark 3.3. Actually the constraint that the support of φ is a ball is of little importance: we could assume as well that it would be a cube, a simplex, or many other shapes, as the corollary below shows:

Corollary 3.4. *Make the same assumptions as in Theorem 3.1, except that* B *need not be a ball: instead, we only assume that, denoting by* $B_\circ$ *the (true) ball having the same volume as* B, *there exists a bijection* $\Phi\colon B\leftrightarrow B_\circ$ *mapping the uniform measure on* B *onto the uniform measure on* $B_\circ$ *(i.e. such that* $\Phi_*(\mathbf{1}_B\cdot\lambda)=\mathbf{1}_{B_\circ}\cdot\lambda$*) such that* Φ *is bi-Lipschitz (i.e. such that both* Φ *and* Φ^{-1} *are Lipschitz). Denote by* $\|\Phi\|_{\mathrm{Lip}}$ *and* $\|\Phi^{-1}\|_{\mathrm{Lip}}$ *the optimal Lipschitz constants for resp.* Φ *and* Φ^{-1}. *Then, the conclusion of Theorem 3.1 remains true, except that now you have to replace the factor* $C(n)$ *by*

$$(\|\Phi\|_{\mathrm{Lip}}\,\|\Phi^{-1}\|_{\mathrm{Lip}})^5 C(n). \tag{3.4}$$

Proof. Consider the measures $\mu_\circ := \Phi_*\,\mu$ and $\nu_\circ := \Phi_*\,\nu$, and the bump function $\varphi_\circ := \varphi\circ\Phi^{-1}$; then, $\mu_\circ$, $\nu_\circ$ and $\varphi_\circ$ satisfy the original assumptions of Theorem 3.1, the roles of 'm_1' and 'm_2' (in the ball situation) being held by m_1 and m_2 (in the general situation) themselves, the role of 'k' being held by $\|\Phi^{-1}\|_{\mathrm{Lip}}\,k$, and the roles of '$c_1$' and '$c_2$' being held by $c_1 \,/\, \|\Phi\|_{\mathrm{Lip}}^2$ and $c_2\,\|\Phi^{-1}\|_{\mathrm{Lip}}^2$. Therefore, applying (3.3):

$$\mathrm{W}_2(a\varphi_\circ\cdot\mu_\circ,\varphi_\circ\cdot\nu_\circ) \leqslant C(n)\,\|\Phi\|_{\mathrm{Lip}}^4\,\|\Phi^{-1}\|_{\mathrm{Lip}}^4\,\frac{c_2^{3/2}m_2^{3/2}}{c_1^{3/2}m_1^{3/2}}\mathrm{W}_2(\mu_\circ,\nu_\circ). \tag{3.5}$$

But the optimal transportation plan from μ to ν, with cost $\mathrm{W}_2(\mu,\nu)^2$, can be pushed forward by Φ into a (not optimal in general) transportation plan from $\mu_\circ$ to $\nu_\circ$, whose cost will then be $\leqslant \|\Phi\|_{\mathrm{Lip}}^2\,\mathrm{W}_2(\mu,\nu)^2$; so $\mathrm{W}_2(\mu_\circ,\nu_\circ)\leqslant\|\Phi\|_{\mathrm{Lip}}\,\mathrm{W}_2(\mu,\nu)$. Similarly $\mathrm{W}_2(a\varphi\cdot\mu,\varphi\cdot\nu)\leqslant\|\Phi^{-1}\|_{\mathrm{Lip}}\,\mathrm{W}_2(a\varphi_\circ\cdot\mu_\circ,\varphi_\circ\cdot\nu_\circ)$. The announced result follows. □

3.3. Proof of the main theorem

In the sequel we will shorthand $\mathrm{W}_2(\mu,\nu) =: w$, and also $\varphi\cdot\mu =: \hat\mu$, resp. $\varphi\cdot\nu =: \hat\nu$. Let $g =: \mathrm{id}+S$ be a map achieving optimal transportation from ν to μ, *i.e.* such that $\mu = g_*\,\nu$ with $\int_{\mathbb{R}^n}|S(y)|^2\nu(\mathrm{d}y)=w^2$.[10]

Our strategy will consist in transforming $\hat\nu$ into $a\hat\mu$ according to the following procedure:

① We apply the transportation plan g to $\hat\nu$; this transforms $\hat\nu$ into some measure $\hat\mu^*$. The measure $\hat\mu^*$ is not supported by B *a priori*, so we split it into $\hat\mu^*_B + \hat\mu^*_{\mathsf{c}} := \mathbf{1}_B\cdot\hat\mu^* + \mathbf{1}_{B^{\mathsf{c}}}\cdot\hat\mu^*$.

[9] For instance, with the estimates of this article, one finds that $C(n) := 47n^{1/2}$ fits—though this may be strongly suboptimal.

[10] Actually such an g does not always exist, as it can occur that the optimal transportation plan from ν to μ "splits points" if ν is not regular enough. However it would suffice to use the general formalism of transportation plans to handle that case: we do not do it here to keep notation light, but this is straightforward. Also note that it is not obvious that the infimum in (2.2) is attained: again, that is not a real problem as our proof still works by considering a sequence of transportation plans approaching optimality.

② Denoting $a_{\mathsf{c}} := \|\hat{\mu}^*_{\mathsf{c}}\|_1 \,/\, \|\hat{\mu}\|_1$, we then transform $\hat{\mu}^*_{\mathsf{c}}$ into $a_{\mathsf{c}}\hat{\mu}$ according to an arbitrary transference plan.

③ Finally, denoting $a_B := \|\hat{\mu}^*_B\|_1 \,/\, \|\hat{\mu}\|_1$,[11] we transform $\hat{\mu}^*_B$ into $a_B\hat{\mu}$ according to the optimal transference plan: the cost of this operation is $\mathrm{W}_2(\hat{\mu}^*_B, a_B\hat{\mu})$, which we bound above by $2\,\|\hat{\mu}^*_B - a_B\hat{\mu}\|_{\dot{\mathrm{H}}^{-1}(a_B\hat{\mu})}$ thanks to Theorem 2.1.

Then, denoting by $\mathrm{W}_2(①), \mathrm{W}_2(②), \mathrm{W}_2(③)$ the respective Wasserstein distances of these steps, we shall have $\mathrm{W}_2(\hat{\nu}, a\hat{\mu}) \leqslant \mathrm{W}_2(①) + (\mathrm{W}_2(②)^2 + \mathrm{W}_2(③)^2)^{1/2}$.

Let us begin with bounding the cost of Step ①. The squared cost of this step is

$$\mathrm{W}_2(①)^2 = \int |S(y)|^2 \hat{\nu}(\mathrm{d}y) = \int |S(y)|^2 \varphi(y)\nu(\mathrm{d}y) \leqslant \sup\varphi \times \int |S(y)|^2 \nu(\mathrm{d}y) = \sup\varphi \times w^2 \leqslant c_2 R^2 w^2, \tag{3.6}$$

whence $\mathrm{W}_2(①) \leqslant c_2^{1/2} R w$.

Now consider Step ②. As $a_{\mathsf{c}}\hat{\mu}$ is supported by B, one has obviously

$$\mathrm{W}_2(②)^2 \leqslant \int \big(\mathit{dist}(x,B) + 2R\big)^2 \hat{\mu}^*_{\mathsf{c}}(\mathrm{d}x) = \int_{B^{\mathsf{c}}} \big(\mathit{dist}(x,B) + 2R\big)^2 \hat{\mu}^*(\mathrm{d}x). \tag{3.7}$$

From that we deduce that $\mathrm{W}_2(②) \leqslant 2c_2^{1/2} R w$ by the following computation:

$$\begin{aligned}
\int_{B^{\mathsf{c}}} \big(\mathit{dist}(x,B) + 2R\big)^2 \hat{\mu}^*(\mathrm{d}x) &= \int_{g(y)\notin B} \big(\mathit{dist}(g(y),B) + 2R\big)^2 \varphi(y)\nu(\mathrm{d}y) \\
&\leqslant c_2 \int_{\substack{y\in B\\ g(y)\notin B}} \big(\mathit{dist}(g(y),B) + 2R\big)^2 \,\mathit{dist}(y,B^{\mathsf{c}})^2 \nu(\mathrm{d}y) \\
&\leqslant c_2 \int_{\substack{y\in B\\ g(y)\notin B}} \big(R\,\mathit{dist}(g(y),B) + 2R\,\mathit{dist}(y,B^{\mathsf{c}})\big)^2 \nu(\mathrm{d}y) \\
&\leqslant 4c_2R^2 \int_{\substack{y\in B\\ g(y)\notin B}} \big(\mathit{dist}(g(y),B) + \mathit{dist}(y,B^{\mathsf{c}})\big)^2 \nu(\mathrm{d}y) \\
&\leqslant 4c_2R^2 \int |y - g(y)|^2 \nu(\mathrm{d}y) = 4c_2R^2w^2.
\end{aligned} \tag{3.8}$$

Step ③ is the difficult one. We begin with observing that it is easy to bound the $\mathrm{L}^2(B)$ distance between $\hat{\mu}^*_B$ and $\hat{\mu}$: indeed, denoting by $f =: \mathrm{id} + T$ the inverse map of g,[12]

$$\begin{aligned}
\|\hat{\mu}^*_B - \hat{\mu}\|^2_{\mathrm{L}^2(\mathbf{1}_B\cdot\mu)} = \int_B \left(\frac{\hat{\mu}^*(\mathrm{d}x) - \varphi(x)\mu(\mathrm{d}x)}{\mu(\mathrm{d}x)}\right)^2 \mu(\mathrm{d}x) &= \int_B \big(\varphi(f(x)) - \varphi(x)\big)^2 \mu(\mathrm{d}x) \\
&\leqslant k^2 \int_{\mathbb{R}^n} |x - f(x)|^2 \mu(\mathrm{d}x) = k^2 \int |T(x)|^2 \mu(\mathrm{d}x) = k^2 w^2,
\end{aligned} \tag{3.9}$$

(where we used that $\hat{\mu}^*(\mathrm{d}x) = \hat{\nu}\big(\mathrm{d}(f(x))\big) = \varphi(f(x))\nu\big(\mathrm{d}(f(x))\big) = \varphi(f(x))\mu(\mathrm{d}x)$), so that

$$\|\hat{\mu}^*_B - \hat{\mu}\|^2_{\mathrm{L}^2(B)} \leqslant k^2 m_2 w^2. \tag{3.10}$$ [13]

[11] Observe that $a_B + a_{\mathsf{c}} = a$.

[12] For f to exist, g should be bijective, which is not always true *stricto sensu*; but we can safely carry out the reasoning with pretending so, by the same argument as in Footnote 10 on page 1494.

[13] Remember that when ν stands for a *measure*, $\|\nu\|_{\mathrm{L}^2(\mu)}$ means what is more commonly denoted by $\|\mathrm{d}\nu/\mathrm{d}\mu\|_{\mathrm{L}^2(\mu)}$, so that the relation $\mu \leqslant m\lambda$ implies that $\|\nu\|^2_{\mathrm{L}^2(\lambda)} \leqslant m\,\|\nu\|^2_{\mathrm{L}^2(\mu)}$—while on the other hand, when f stands for a *function*, one has $\|f\|^2_{\mathrm{L}^2(\mu)} \leqslant m\,\|f\|^2_{\mathrm{L}^2(\lambda)}$.

Now we have to link $\|\cdot\|_{\mathrm{L}^2(B)}$ with $\|\cdot\|_{\dot{\mathrm{H}}^{-1}(\mu)}$. This is achieved by the following lemma, whose proof is postponed:

Lemma 3.5. *Define $\hat{\lambda}$ to be the measure on B such that $\hat{\lambda}(\mathrm{d}x) := dist(x, B^{\mathrm{c}})^2\lambda(\mathrm{d}x)$. Then, for any signed measure m on B having total mass zero*:

$$\|m\|_{\dot{\mathrm{H}}^{-1}(\hat{\lambda})} \leqslant C_1(n)^{1/2}\,\|m\|_{\mathrm{L}^2(B)}\,, \tag{3.11}$$

where $C_1(n)$ is some absolute constant only depending on n. Moreover, taking $C_1(n) := \big((2e+1)n-1\big) \vee 8e$ fits.

Thanks to Theorem 2.1 and Lemma 3.5, we have that

$$\mathrm{W}_2(③) \leqslant 2\,\|a_B\hat{\mu} - \hat{\mu}^*_B\|_{\dot{\mathrm{H}}^{-1}(a_B\hat{\mu})} \leqslant 2(a_Bc_1m_1)^{-1/2}\,\|a_B\hat{\mu} - \hat{\mu}^*_B\|_{\dot{\mathrm{H}}^{-1}(\hat{\lambda})} \leqslant 2C_1(n)^{1/2}(a_Bc_1m_1)^{-1/2}\|a_B\hat{\mu} - \hat{\mu}^*_B\|_{\mathrm{L}^2(B)}. \tag{3.12}$$

Next, we compute

$$\begin{aligned}\|a_B\hat{\mu} - \hat{\mu}^*_B\|_{\mathrm{L}^2(B)} &= \left\|\frac{\|\hat{\mu}^*_B\|_1}{\|\hat{\mu}\|_1}\hat{\mu} - \hat{\mu}^*_B\right\|_{\mathrm{L}^2(B)} \leqslant \frac{|\,\|\hat{\mu}^*_B\|_1 - \|\hat{\mu}\|_1\,|}{\|\hat{\mu}\|_1}\,\|\hat{\mu}\|_{\mathrm{L}^2(B)} + \|\hat{\mu}^*_B - \hat{\mu}\|_{\mathrm{L}^2(B)}\\ &\leqslant \frac{\|\hat{\mu}\|_{\mathrm{L}^2(B)}}{\|\hat{\mu}\|_1}\,\|\hat{\mu}^*_B - \hat{\mu}\|_1 + \|\hat{\mu}^*_B - \hat{\mu}\|_{\mathrm{L}^2(B)} \leqslant \left(\frac{\|\hat{\mu}\|_{\mathrm{L}^2(B)}}{\|\hat{\mu}\|_1}\lambda(B)^{1/2} + 1\right)\|\hat{\mu}^*_B - \hat{\mu}\|_{\mathrm{L}^2(B)}\\ &\leqslant \left(\frac{c_2m_2}{c_1m_1}\frac{\lambda(B)^{1/2}\left\|\hat{\lambda}\right\|_{\mathrm{L}^2(B)}}{\left\|\hat{\lambda}\right\|_1} + 1\right)\|\hat{\mu}^*_B - \hat{\mu}\|_{\mathrm{L}^2(B)} \underset{14}{\leqslant} \left(\sqrt{6}\frac{c_2m_2}{c_1m_1} + 1\right)\|\hat{\mu}^*_B - \hat{\mu}\|_{\mathrm{L}^2(B)}\\ &\underset{(3.10)}{\leqslant} (\sqrt{6}+1)\frac{c_2m_2}{c_1m_1}km_2^{1/2}w,\end{aligned} \tag{3.13}$$

so that, combining (3.12) and (3.13), we have got:

$$\mathrm{W}_2(③) \leqslant (2\sqrt{6}+2)C_1(n)^{1/2}a_B^{-1/2}\frac{c_2m_2^{3/2}}{c_1m_1^{3/2}}\frac{k}{c_1^{1/2}}w. \tag{3.14}$$

Equation (3.14) is the kind of bound we were looking for, *provided* $a_B \lesssim 1$. Though this will be the case in practice (since we are mainly interested in cases where ν is close to μ and thus $\hat{\mu}^*$ is close to $\hat{\mu}$), this is not quite satisfactory yet. So, what can we do when $a_B \ll 1$, that is, when $\|\hat{\mu}^*_B\|_1 \ll \|\hat{\mu}\|_1$? In fact that case is easier, because transportation between small measures has low cost, while w has to be large to make $\hat{\mu}^*_B$ very different from $\hat{\mu}$.

The computations are the following. First, it is obvious that

$$\mathrm{W}_2(③) = \mathrm{W}_2(\hat{\mu}^*_B, a_B\hat{\mu}) \leqslant 2R\,\|\hat{\mu}^*_B\|_1^{1/2}. \tag{3.15}$$

[14] This step comes from the computation $\lambda(B)^{1/2}\left\|\hat{\lambda}\right\|_{\mathrm{L}^2(B)} / \left\|\hat{\lambda}\right\|_1 = (\int_0^1 r^{n-1}\mathrm{d}r)^{1/2}\big(\int_0^1(1-r)^4r^{n-1}\mathrm{d}r\big)^{1/2} / \big(\int_0^1(1-r)^2r^{n-1}\mathrm{d}r\big) = (6(1+n)(2+n)\,/\,(3+n)(4+n))^{1/2} \leqslant \sqrt{6}\ \forall n$.

Next, observing that $\varphi(f(x)) \geqslant \frac{c_1}{c_2}\varphi(x) - 2c_1\, dist(x, B^c)|T(x)|$,[15] we compute that

$$\begin{aligned}\|\hat{\mu}^*_B\|_1 &= \int_B \varphi(f(x))\mu(\mathrm{d}x) \geqslant \int_B \left(\frac{c_1}{c_2}\varphi(x) - 2c_1\, dist(x,B^c)|T(x)|\right)\mu(\mathrm{d}x)\\ &\geqslant \frac{c_1}{c_2}\|\hat{\mu}\|_1 - 2c_1\left(\int_B dist(x,B^c)^2\mu(\mathrm{d}x)\right)^{1/2}\left(\int_B |T(x)|^2\mu(\mathrm{d}x)\right)^{1/2}\\ &= \frac{c_1}{c_2}\|\hat{\mu}\|_1 - 2c_1\left\|dist(\cdot,B^c)^2\cdot\mu\right\|_1^{1/2} w \geqslant \frac{c_1}{c_2}\|\hat{\mu}\|_1 - 2c_1 m_2^{1/2}\left\|\hat{\lambda}\right\|_1^{1/2} w,\end{aligned} \tag{3.17}$$

whence

$$w \geqslant \frac{\left(\frac{c_1}{c_2}\|\hat{\mu}\|_1 - \|\hat{\mu}^*_B\|_1\right)^+}{2c_1 m_2^{1/2}\left\|\hat{\lambda}\right\|_1^{1/2}} = \frac{\left(\frac{c_1}{c_2} - a_B\right)^+\|\hat{\mu}\|_1}{2c_1 m_2^{1/2}\left\|\hat{\lambda}\right\|_1^{1/2}} \geqslant \frac{m_1^{1/2}}{2c_1^{1/2}m_2^{1/2}}\left(\frac{c_1}{c_2} - a_B\right)^+\|\hat{\mu}\|_1^{1/2}. \tag{3.18}$$

So,

$$\mathrm{W}_2(③) \leqslant 2R\|\hat{\mu}^*_B\|_1^{1/2} = 2Ra_B^{1/2}\|\hat{\mu}\|_1^{1/2} \leqslant 4Rc_1^{1/2}\frac{m_2^{1/2}}{m_1^{1/2}}\frac{a_B^{1/2}}{(\frac{c_1}{c_2} - a_B)^+}w. \tag{3.19}$$

In the end, choosing either (3.14) if $a_B \geqslant c_1 / 2c_2$ or (3.19) if $c_1 / 2c_2$, and observing that $c_1 \leqslant kR^{-1}$, one has always:

$$\mathrm{W}_2(③) \leqslant \left((4\sqrt{3}+2\sqrt{2})C_1(n)^{1/2} \vee 4\sqrt{2}\right)\frac{c_2^{3/2}m_2^{3/2}}{c_1^{3/2}m_1^{3/2}}\frac{k}{c_1^{1/2}}w. \tag{3.20}$$

Remark 3.6. To bound $\mathrm{W}_2(③)$ in the situation where $a_B \ll 1$, we could also have started from "$\varphi(f(x)) \geqslant \varphi(x) - k|T(x)|$" (instead of "$\varphi(f(x)) \geqslant \frac{c_1}{c_2}\varphi(x) - 2c_1\, dist(x,B^c)|T(x)|$") to get another bound analogous to (3.17). Following such an approach, the factor $(c_2 / c_1)^{3/2}$ in (3.19) would be improved into (c_2 / c_1) in the analogous formula; however the dimensional factor would behave in $O(n)$ rather than in $O(n^{1/2})$.

3.4. Proof of Lemma 3.5

It still remains to prove Lemma 3.5, whose statement we recall to be:

Lemma 3.7. *Denoting $\hat{\lambda} := dist(\cdot, B^c)^2 \cdot \lambda$, one has, for any signed measure m on B having total mass zero*:

$$\|m\|_{\dot{\mathrm{H}}^{-1}(\hat{\lambda})} \leqslant \left(\left((2e+1)n - 1\right) \vee 8e\right)^{1/2}\|m\|_{\mathrm{L}^2(B)}. \tag{3.21}$$

—In the sequel, "$\left((2e+1)n-1\right) \vee 8e$" will be shorthanded into "$C_1(n)$".

Remark 3.8. The bound (3.21) is within a constant factor of being optimal, uniformly in n, as one sees by taking a linear function f in (3.24).

Proof of the lemma. We begin with translating the lemma into a functional analysis statement by a duality argument. Recall the duality definition of $\|m\|_{\dot{\mathrm{H}}^{-1}(\hat{\lambda})}$ from Section 2:

$$\|m\|_{\dot{\mathrm{H}}^{-1}(\hat{\lambda})} := \sup\{|\langle f, m\rangle| \mid \|f\|_{\dot{\mathrm{H}}^1(\hat{\lambda})} \leqslant 1\}. \tag{3.22}$$

[15] This follows from the computation:

$$\varphi(f(x)) \geqslant c_1\, dist(f(x),B^c)^2 \geqslant c_1\left(\left(dist(x,B^c) - |T(x)|\right)^+\right)^2 \geqslant c_1\, dist(x,B^c)^2 - 2c_1\, dist(x,B^c)|T(x)| \geqslant \frac{c_1}{c_2}\varphi(x) - 2c_1\, dist(x,B^c)|T(x)|. \tag{3.16}$$

There is a similar duality formula for $\|m\|_{\mathrm{L}^2(B)}$:

$$\|m\|_{\mathrm{L}^2(B)} = \sup\{|\langle f, m\rangle| \mid \|f\|_{\mathrm{L}^2(B)} \leqslant 1\}, \tag{3.23}$$

where, for f a *function*, $\|f\|_{\mathrm{L}^2(B)}$ has its usual meaning, namely $\|f\|_{\mathrm{L}^2(B)} := \big(\int_B f(x)^2 \lambda(\mathrm{d}x)\big)^{1/2}$. Since m is assumed to have total mass zero, $|\langle f, m\rangle|$ does not change when one adds a constant to f. On the other hand, when f describes the set $\{\|f_0 + a\| \mid a \in \mathbb{R}\}$, $\|f\|_{\mathrm{L}^2(B)}$ is minimal when a is such that f has zero mean on B, while the value of $\|f\|_{\dot{\mathrm{H}}^1(\hat{\lambda})}$ remains constant.[16] As a consequence, we can restrict the supremum in (3.22) and (3.23) to those f *having zero mean on* B. Thus, the lemma will be implied[17] by proving that

$$\langle f, \mathbf{1}_B \cdot \lambda\rangle = 0 \quad \Rightarrow \quad \|f\|_{\mathrm{L}^2(B)} \leqslant C_1(n)^{1/2} \, \|f\|_{\dot{\mathrm{H}}^1(\hat{\lambda})} \,. \tag{3.24}$$

Going back to the definitions of $\|\cdot\|_{\dot{\mathrm{H}}^{-1}(\hat{\lambda})}$ and $\|\cdot\|_{\mathrm{L}^2(B)}$, relaxing the condition on f to be centred by projecting it orthogonally in $\mathrm{L}^2(B)$ onto the subspace of centred functions, and denoting by P the uniform probability measure on B, Equation (3.24) turns into:

$$\forall f \qquad \mathrm{Var}_P(f) \leqslant C_1(n) \int \mathit{dist}(x, B^{\mathrm{c}})^2 |\nabla f(x)|^2 P(\mathrm{d}x), \tag{3.25}$$

which we recognize to be a so-called "improved Poincaré inequality" [3, 6]. In general, Poincaré inequalities, bounding the variance of f by a quadratic integral of its first derivative, are linked with the exponential convergence of a certain diffusion Markov process towards equilibrium (*cf.* [1], Chap. 2): that probabilistic vision initially guided me to tackle Equation (3.25), although this will not be apparent in the sequel.

To prove (3.25), the first key idea (inspired by [4]) is to separate radial and spherical coordinates. This is, considering the bijection

$$\begin{aligned} \varphi\colon (0, R) \times \mathbb{S}^{n-1} &\to B \setminus \{0\} \\ (r, \theta) &\mapsto r\theta \end{aligned} \tag{3.26}$$

(the origin of space being set at the center of B), we introduce the measure $\tilde{P} := {\varphi^{-1}}_* P$, which is obviously the product measure $\tilde{P}_r \otimes \tilde{P}_\theta$, where $\tilde{P}_r$ is the probability measure on $(0, R)$ such that $\tilde{P}_r(\mathrm{d}r) := nR^{-n} r^{n-1} \mathrm{d}r$, resp. $\tilde{P}_\theta$ is the uniform measure on the sphere $\mathbb{S}^{n-1}$. With this notation, we perform can a change of variables to see that (3.25) is equivalent to proving that, for all $g \in \mathrm{L}^2(\tilde{P})$:

$$C_1(n)^{-1} \, \mathrm{Var}_{\tilde{P}}(g) \leqslant \int_0^R \int_{\mathbb{S}^{n-1}} (R - r)^2 \big(|\nabla_r g(r, \theta)|^2 + r^{-2} |\nabla_\theta g(r, \theta)|^2\big) \tilde{P}_r(\mathrm{d}r) \tilde{P}_\theta(\mathrm{d}\theta), \tag{3.27}$$

where ∇_r and ∇_θ denote the gradient along resp. the r coordinate and the θ coordinate[18]. We will denote the right-hand side of (3.27) by $\mathcal{E}(g, g)$.

Because $\tilde{P} = \tilde{P}_r \otimes \tilde{P}_\theta$, we know that $\mathrm{L}^2(\tilde{P})$ can be seen as (the closure of) the tensor product of $\mathrm{L}^2(\tilde{P}_r)$ and $\mathrm{L}^2(\tilde{P}_\theta)$:

$$\mathrm{L}^2(\tilde{P}) = \mathrm{cl}(\mathrm{L}^2(\tilde{P}_r) \overset{\perp}{\otimes} \mathrm{L}^2(\tilde{P}_\theta)), \tag{3.28}$$

where the symbol '$\overset{\perp}{\otimes}$' means that the Hilbertian structure of $\mathrm{L}^2(\tilde{P})$ is compatible with the Hilbertian structures of $\mathrm{L}^2(\tilde{P}_r)$ and $\mathrm{L}^2(\tilde{P}_\theta)$—*i.e.*, that $\langle h_{\mathrm{a}} \otimes u_{\mathrm{a}}, h_{\mathrm{b}} \otimes u_{\mathrm{b}}\rangle_{\mathrm{L}^2(\tilde{P})} = \langle h_{\mathrm{a}}, h_{\mathrm{b}}\rangle_{\mathrm{L}^2(\tilde{P}_r)} \times \langle u_{\mathrm{a}}, u_{\mathrm{b}}\rangle_{\mathrm{L}^2(\tilde{P}_\theta)}$. Now consider the spherical harmonics $Y_0, Y_1, \ldots$, which by definition are an orthonormal basis, in $\mathrm{L}^2(\tilde{P}_\theta)$, of eigenfunctions of the

[16] Here we implicitly assume that $\int_B |f(x)| \lambda(\mathrm{d}x) < \infty$, which is legitimate since an approximation argument allows to restrict the suprema in (3.22) and (3.23) to those f having a C^∞ continuation on $\mathrm{cl}(B)$.

[17] Actually there is even equivalence.

[18] In the latter case, we have to use the Riemannian definition of the gradient on $\mathbb{S}^{n-1}$.

Laplace−Beltrami operator Δ on $\mathbb{S}^{n-1}$; and call $\ell_0, \ell_1, \dots$ the associated eigenvalues, which are known to be such that (up to permuting indices) $Y_0 \equiv 1$ with $\ell_0 = 0$, and $\ell_i \leqslant -(n-1)\ \forall i \neq 0$ (see for instance [11]). By construction, $L^2(\tilde{P}_\theta) = \mathrm{cl}\big(\overset{\perp}{\bigoplus}_{i\in\mathbb{N}}(\mathbb{R}\cdot Y_i)\big)$; therefore, one has that

$$L^2(\tilde{P}) = \mathrm{cl}\left(\overset{\perp}{\bigoplus_{i\in\mathbb{N}}} L^2(\tilde{P}_r)\cdot Y_i\right): \tag{3.29}$$

in other words, the functions of $L^2(\tilde{P})$ are those of the form

$$g(r,\theta) = \sum_{i\in\mathbb{N}} h_i(r) Y_i(\theta), \tag{3.30}$$

with $\sum_i \|h_i\|^2_{L^2(\tilde{P}_r)} < \infty$, and the correspondence is bijective. An interesting point is that, then, one has:

$$\mathrm{Var}_{\tilde{P}}(g) = \mathrm{Var}_{\tilde{P}_r}(h_0) + \sum_{i\neq 0} \|h_i\|^2_{L^2(\tilde{P}_r)}\,. \tag{3.31}$$

On the other hand, one has

$$\mathcal{E}(g,g) = -\langle Lg, g\rangle_{L^2(\tilde{P})}, \tag{3.32}$$

where

$$\big(Lg\big)(r,\theta) := (R-r)^2\Delta_r g + \left((n-1)\frac{(R-r)^2}{r} - 2(R-r)\right)\boldsymbol{e}_r\cdot\nabla_r g + \frac{(R-r)^2}{r^2}\Delta_\theta g. \tag{3.33}$$

From (3.33) we see that, since the Y_i are eigenfunctions of Δ_θ, all the $L^2(\tilde{P}_r)\cdot Y_i$ are invariant by L, and that one has:

$$\mathcal{E}(g,g) = \sum_{i\in\mathbb{N}}\int_0^R \left((R-r)^2|\nabla h_i(r)|^2 - \ell_i\frac{(R-r)^2}{r^2}h_i(r)^2\right)\tilde{P}_r(\mathrm{d}r). \tag{3.34}$$

So, proving (3.27) becomes equivalent to proving that both following formulas hold for all $h \in L^2(\tilde{P}_r)$:

$$\mathrm{Var}_{\tilde{P}_r}(h) \leqslant C_1(n)\int_0^R (R-r)^2|\nabla h(r)|^2\tilde{P}_r(\mathrm{d}r); \tag{3.35}$$

$$\|h\|^2_{L^2(\tilde{P}_r)} \leqslant C_1(n)\int_0^R \left((R-r)^2|\nabla h(r)|^2 + (n-1)\frac{(R-r)^2}{r^2}h(r)^2\right)\tilde{P}_r(\mathrm{d}r). \tag{3.36}$$

Let us start with (3.35). In all the sequel of the proof, we introduce

$$b := 1 - n^{-1}. \tag{3.37}$$

By the Cauchy−Schwarz inequality, one has, for all $r \in (bR, R)$:

$$\begin{aligned}\big(h(r) - h(bR)\big)^2 = \left(\int_{bR}^r h'(s)\mathrm{d}s\right)^2 &\leqslant \left(\int_{bR}^r (R-s)^{-3/2}\mathrm{d}s\right)\times\int_{bR}^r (R-s)^{3/2}|\nabla h(s)|^2\mathrm{d}s\\ &\leqslant 2\Big((R-r)^{-1/2} - (R-bR)^{-1/2}\Big)\int_{bR}^r (R-s)^{3/2}|\nabla h(s)|^2\mathrm{d}s\\ &\leqslant 2(R-r)^{-1/2}\int_{bR}^r (R-s)^{3/2}|\nabla h(s)|^2\mathrm{d}s.\end{aligned} \tag{3.38}$$

Integrating and using Fubini's formula, it follows that

$$\begin{aligned}\int_{bR}^{R}\big(h(r)-h(bR)\big)^2\tilde{P}_r(\mathrm{d}r) &\leqslant 2\int_{s=bR}^{R}\left(\int_{r=s}^{R} nR^{-n}(R-r)^{-1/2}r^{n-1}\mathrm{d}r\right)(R-s)^{3/2}|\nabla h(s)|^2\mathrm{d}s\\ &\leqslant 2\int_{s=bR}^{R}\left(\int_{r=s}^{R} nR^{-n}(b^{-1}s)^{n-1}(R-r)^{-1/2}\mathrm{d}r\right)(R-s)^{3/2}|\nabla h(s)|^2\mathrm{d}s\\ &= 2b^{-(n-1)}\int_{s=bR}^{R}\left(\int_{r=s}^{R}(R-r)^{-1/2}\mathrm{d}r\right)(R-s)^{3/2}|\nabla h(s)|^2\tilde{P}_r(\mathrm{d}s)\\ &= 4b^{-(n-1)}\int_{s=bR}^{R}(R-s)^2|\nabla h(s)|^2\mathrm{d}s. \end{aligned} \tag{3.39}$$

One can apply the same line of reasoning for $r\in(0,bR)$: the (unweighted this time) Cauchy−Schwarz inequality then yields $\big(h(r)-h(bR)\big)^2\leqslant(bR-r)\int_r^{bR}|\nabla h(s)|^2\mathrm{d}s$, whence:

$$\begin{aligned}\int_0^{bR}\big(h(r)-h(bR)\big)^2\tilde{P}_r(\mathrm{d}r) &\leqslant \int_{s=0}^{bR}\left(\int_{r=0}^{s} nR^{-n}(bR-r)r^{n-1}\mathrm{d}r\right)|\nabla h(s)|^2\mathrm{d}s\\ &\leqslant bR^{-(n-1)}\int_{s=0}^{bR}\left(\int_{r=0}^{s} nr^{n-1}\mathrm{d}r\right)|\nabla h(s)|^2\mathrm{d}s = bR\int_0^{bR}|\nabla h(s)|^2 s^n\mathrm{d}s\\ &\leqslant bn^{-1}R^2\int_0^{bR}|\nabla h(s)|^2\tilde{P}_r(\mathrm{d}s)\leqslant b(1-b)^{-2}n^{-1}\int_0^{bR}(R-s)^2|\nabla h(s)|^2\tilde{P}_r(\mathrm{d}s). \end{aligned} \tag{3.40}$$

Summing (3.39) and (3.40), we get that

$$\int_0^{R}\big(h(r)-h(bR)\big)^2\tilde{P}_r(\mathrm{d}r)\leqslant\big(4b^{-(n-1)}\vee b(1-b)^{-2}n^{-1}\big)\int_0^{s}(R-s)^2|\nabla h(s)|^2\tilde{P}_r(\mathrm{d}s), \tag{3.41}$$

where $\big(4b^{-(n-1)}\vee b(1-b)^{-2}n^{-1}\big)$ can itself be bounded by $\big((n-1)\vee 4e\big)$. The left-hand-side of (3.41) being an upper bound for $\mathrm{Var}_{\tilde{P}_r}(h)$, this proves (3.35).

Now we turn to (3.36). For $r\in(bR,R)$ we have, similarly to (3.38), that

$$\big(h(r)-h(br)\big)^2\leqslant 2(R-r)^{-1/2}\int_{br}^{r}(R-s)^{3/2}|\nabla h(s)|^2\mathrm{d}s, \tag{3.42}$$

so that

$$h(r)^2\leqslant 2h(br)^2+4(R-r)^{-1/2}\int_{br}^{r}(R-s)^{3/2}|\nabla h(s)|^2\mathrm{d}s. \tag{3.43}$$

Then, integrating and applying Fubini's formula:

$$\int_{bR}^{R}h(r)^2\tilde{P}_r(\mathrm{d}r)\leqslant 2\int_{bR}^{R}h(br)^2\tilde{P}_r(\mathrm{d}r)+4\int_{s=b^2R}^{R}\left(\int_{r=s\vee bR}^{b^{-1}s\wedge R}nR^{-n}r^{n-1}(R-r)^{-1/2}\mathrm{d}r\right)(R-s)^{3/2}|\nabla h(s)|^2\mathrm{d}s. \tag{3.44}$$

By change of variables, the first term of the right-hand side of (3.44) is equal to $2b^{-n}\int_{b^2R}^{bR}h(s)^2\tilde{P}_r(\mathrm{d}s)$, which we can bound by

$$2b^{-(n-2)}\frac{(1-b)^{-2}}{n-1}\int_{b^2R}^{bR}(n-1)\frac{(R-r)^2}{r^2}h(s)^2\tilde{P}_r(\mathrm{d}s)\leqslant 2ne\int_0^{R}(n-1)\frac{(R-r)^2}{r^2}h(s)^2\tilde{P}_r(\mathrm{d}s). \tag{3.45}$$

The second term of the right-hand side of (3.44) is itself bounded by

$$4b^{-(n-1)} \int_{s=b^2R}^{R} \left(\int_{r=s}^{R} (R-r)^{-1/2} \mathrm{d}r \right) (R-s)^{3/2} |\nabla h(s)|^2 \tilde{P}_r(\mathrm{d}s) \leqslant 8e \int_0^R (R-s)^2 |\nabla h(s)|^2 \tilde{P}_r(\mathrm{d}s). \tag{3.46}$$

This way, we have bounded $\int_{bR}^{R} h(r)^2 \tilde{P}_r(\mathrm{d}r)$.

On the other hand, it is trivial that, for $r \leqslant bR$,

$$h(r)^2 \leqslant \frac{b^2}{(n-1)(1-b)^2} \times (n-1) \frac{(R-r)^2}{r^2} h(r)^2, \tag{3.47}$$

whence:

$$\int_0^{bR} h(r)^2 \tilde{P}_r(\mathrm{d}r) \leqslant (n-1) \int_0^R (n-1) \frac{(R-r)^2}{r^2} h(r)^2 \tilde{P}_r(\mathrm{d}r). \tag{3.48}$$

Combining (3.45), (3.46) and (3.48), we finally get the wanted bound (3.36). □

Remark 3.9. At the time I wrote that proof I was not aware of the already existing results on improved Poincaré inequalities, in particular ([6], Thm. 1.3), which equation (3.25) is actually a particular case of; nor were the people whom I had asked about such inequalities. Compared to the result of [6] however, my equation (3.25) states an explicit value for the constant in the inequality, which moreover is within a constant factor of being optimal, uniformly in the dimension n; also, it uses a quite different proof, which may be interesting ☺

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