ESAIM: COCV 24 (2018) 1675–1704
https://doi.org/10.1051/cocv/2017043

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ON THE APPROXIMATE BOUNDARY SYNCHRONIZATION FOR A COUPLED SYSTEM OF WAVE EQUATIONS: DIRECT AND INDIRECT CONTROLS *

Ta-Tsien Li[1,**] and Bopeng Rao[2]

Abstract. The approximate boundary synchronization by p-groups ($p \geq 1$) in the pinning sense has been introduced in [T.-T. Li and B. Rao, *Asymp. Anal.* **86** (2014) 199–224], in this paper the authors give a new and more natural definition on the approximate boundary synchronization by p-groups in the consensus sense for a coupled system of N wave equations with Dirichlet boundary controls. We show that the approximate boundary synchronization by p-groups in the consensus sense is equivalent to that in the pinning sense. Moreover, by means of a corresponding Kalman's criterion, the concept of the number of total (direct and indirect) controls is introduced. It turns out that in the case that the minimal number of total controls is equal to $(N-p)$, the existence of the approximately synchronizable state by p-groups as well as the necessity of the strong C_p-compatibility condition are the consequence of the approximate boundary synchronization by p-groups, while, in the opposite case, the approximate boundary synchronization by p-groups could imply some non-expected additional properties, called the induced approximate boundary synchronization.

Mathematics Subject Classification. 93B05, 93B07, 93C20, 35L53.

Received March 2, 2016. Accepted June 7, 2017.

1. Introduction and main results

The objective of this paper (also see [19]) is to investigate, *via* Kalman's criterion, the approximate boundary synchronization for the following coupled system of wave equations with Dirichlet boundary controls:

$$\begin{cases} U'' - \varDelta U + AU = 0 & \text{in} \quad (0,+\infty) \times \varOmega, \\ U = 0 & \text{on} \quad (0,+\infty) \times \varGamma_0, \\ U = DH & \text{on} \quad (0,+\infty) \times \varGamma_1 \end{cases} \tag{1.1}$$

Keywords and phrases. Direct and indirect controls, system of wave equations, Dirichlet boundary controls, approximate boundary synchronization by groups, strong C_p-compatibility condition, Kalman's criterion, induced approximate synchronization.

* *Projet supported by the National Basic Research Program of China (No. 2013CB834100), and the National Natural Science Foundation of China (No. 11121101).*

[1] School of Mathematical Sciences, Fudan University, Shanghai 200433, China; Shanghai Key Laboratory for Contemporary Applied Mathematic; Nonlinear Mathematical Modeling and Methods Laboratory.

[2] Département de Mathématique et d'Informatique, Institut de Recherche Mathématique Avancée, Université de Strasbourg, 67084 Strasbourg, France; School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
**Corresponding author: dqli@fudan.edu.cn

with the initial condition

$$t = 0: \qquad U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in} \quad \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ such that $\Gamma_1 \cap \Gamma_0 = \emptyset$, $U = (u^{(1)}, \dots, u^{(N)})^T$, $H = (h^{(1)}, \dots, h^{(M)})^T$ with $M \leq N$ are the state variables and the boundary controls acting on Γ_1, respectively, the coupling matrix A of order $N \times N$ and the full column-rank control matrix D of order $N \times M$ are both with constant entries.

The well-posedness and the approximate boundary null controllability of problem (1.1)−(1.2) have been considered in [14] that we recall as follows.

Definition 1.1. System (1.1) is approximately null controllable at the time $T > 0$, if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a sequence $\{H_n\}$ of boundary controls in $(L^2(0, +\infty; L^2(\Gamma_1))^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1.1)−(1.2) satisfies

$$U_n \to 0 \quad \text{in} \quad \big(C^0_{\text{loc}}([T, +\infty); L^2(\Omega))\big)^N \cap \big(C^1_{\text{loc}}([T, +\infty); H^{-1}(\Omega))\big)^N \tag{1.3}$$

as $n \to +\infty$.

Accordingly, let $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$. Consider the adjoint problem

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \Phi = 0 & \text{on} \quad (0, +\infty) \times \Gamma, \\ t = 0: \quad \Phi = \Phi_0, \qquad \Phi' = \Phi_1 & \text{in} \quad \Omega, \end{cases} \tag{1.4}$$

where A^T is the transpose of A.

Definition 1.2. The adjoint problem (1.4) is D-observable on the time interval $[0, T]$, if the observation

$$D^T \partial_\nu \Phi \equiv 0 \quad \text{on} \quad [0, T] \times \Gamma_1 \tag{1.5}$$

implies $\Phi \equiv 0$, where ∂_ν denotes the outward normal derivative on Γ_1.

Lemma 1.3 (see [12, 14])**.** *System (1.1) is approximately null controllable at the time $T > 0$ if and only if the adjoint problem (1.4) is D-observable on the time interval $[0, T]$.*

Lemma 1.4 (see [16, 18])**.** *If system (1.1) is approximately null controllable at the time $T > 0$, then we necessarily have the following Kalman's criterion*:

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N. \tag{1.6}$$

Here, we point out that Kalman's criterion (1.6) is also sufficient to the approximate boundary null controllability for some special systems (1.1) such as cascade systems, 2×2 systems and one-dimensional systems (see [18]).

For the exact boundary null controllability, the number $M = \text{rank}(D)$ should be equal to N, the number of state variables (see [13, 17]). However, we know (see [18, 20]) that the approximate boundary null controllability of system (1.1) could be realized if the number $M = \text{rank}(D)$ is very small, even if $M = \text{rank}(D) = 1$. However, Kalman's criterion (1.6) shows that if system (1.1) is approximately null controllable, then the enlarged matrix $(D, AD, \dots, A^{N-1}D)$, composed of the coupling matrix A and the boundary control matrix D, should be of full row-rank. That is to say, even if the rank of D might be small, but because of the existence and influence of the coupling matrix A, in order to realize the approximate boundary null controllability, the rank of the enlarged matrix $(D, AD, \dots, A^{N-1}D)$ should be still equal to N, the number of state variables. From this point of view, we may say that the rank M of D is the number of "direct" boundary controls acting on Γ_1, and $\text{rank}(D, AD, \ldots, A^{N-1}D)$ denotes the "total" number of direct and indirect controls, while the number of "indirect" controls is given by the difference: $\text{rank}(D, AD, \ldots, A^{N-1}D) - \text{rank}(D)$, which is equal to $(N-M)$ in the case of approximate boundary null controllability. It is different from the exact boundary null controllability, in which only the number $\text{rank}(D)$ of direct boundary controls is concerned and $M = \text{rank}(D)$ should be equal to N, that for the approximate boundary null controllability, we should consider not only the number of direct boundary controls, but also the number of indirect controls, namely, the total (direct and indirect) controls.

The above consideration on the approximate boundary null controllability have been used in [16, 18] to investigate the approximate boundary synchronization and the approximate boundary synchronization by p-groups. Since the approximate boundary synchronization can be considered as a special case ($p = 1$) of the approximate boundary synchronization by p-groups, in what follows, our attention will be focused on the approximate boundary synchronization by p-groups.

Let $p \geq 1$ be an integer and

$$0 = n_0 < n_1 < n_2 < \ldots < n_p = N. \tag{1.7}$$

Accordingly, we arrange the components of the state variable U into p groups:

$$(u^{(1)}, \ldots, u^{(n_1)}), \quad (u^{(n_1+1)}, \ldots, u^{(n_2)}), \ldots, (u^{(n_{p-1}+1)}, \ldots, u^{(n_p)}). \tag{1.8}$$

Definition 1.5. System (1.1) is approximately synchronizable by p-groups at the times $T > 0$ in the pinning sense, if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a sequence $\{H_n\}$ of boundary controls in $(L^2(0, +\infty; L^2(\Gamma_1)))^M$ with compact support in $[0, T]$ and some scalar functions $u_1, \ldots, u_p$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1.1)–(1.2) satisfies

$$u_n^{(k)} \to u_r \quad \text{in} \quad C^0_{\text{loc}}([T, +\infty); L^2(\Omega)) \cap C^1_{\text{loc}}([T, +\infty); H^{-1}(\Omega)) \tag{1.9}$$

as $n \to +\infty$ for all $n_{r-1} + 1 \leq k \leq n_r$ and $1 \leq r \leq p$, where $(u_1, \ldots, u_p)$, being unknown *a priori*, will be called the approximately synchronizable state by p-groups.

In Definition 1.5, the existence of the approximately synchronizable state by p-groups $(u_1, \ldots, u_p)$ is *a priori* assumed. Clearly, theses functions $u_1, \ldots, u_p$ depend on the initial data and on the applied boundary controls. In what follows, when we claim that $u_1, \ldots, u_p$ are linearly independent, it means that there exists at least one initial data such that the corresponding $u_1, \ldots, u_p$ are linearly independent whatever the sequence of applied boundary controls would be chosen, while, when we claim that $u_1, \ldots, u_p$ are linearly dependent, it means that they are linearly dependent for any given initial data and any given sequence of applied boundary controls.

Now let S_r be the full row-rank matrix of order $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$:

$$S_r = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}, \qquad 1 \leq r \leq p. \tag{1.10}$$

Then define the full row-rank matrix of order $(N-p) \times N$:

$$C_p = \begin{pmatrix} S_1 & 0 & \ldots & 0 \\ 0 & S_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & S_p \end{pmatrix}. \tag{1.11}$$

C_p will be called the matrix of synchronization by p-groups corresponding to the repartition (1.7). Let $\{\epsilon_1, \ldots, \epsilon_N\}$ be the canonical basis of $\mathbb{R}^N$. We define

$$e_r = \sum_{l=n_{r-1}+1}^{n_r} \epsilon_l, \qquad 1 \leq r \leq p. \tag{1.12}$$

Clearly, we have

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \dots, e_p\}. \tag{1.13}$$

We will say that A satisfies the C_p-compatibility condition, if there exists a unique matrix $\overline{A}_p$ of order $(N-p)$, such that

$$C_p A = \overline{A}_p C_p, \tag{1.14}$$

or equivalently, by Lemma 2.1 below, $\mathrm{Ker}(C_p)$ is an invariant subspace of A:

$$A\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p). \tag{1.15}$$

Then, setting $W_p = C_p U$ in system (1.1), we get the following self-closed reduced system:

$$\begin{cases} W_p'' - \Delta W_p + \overline{A}_p W_p = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ W_p = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ W_p = C_p D H & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ t=0: \quad W_p = C_p\widehat{U}_0, \quad W_p' = C_p\widehat{U}_1 & \text{in} \quad \Omega. \end{cases} \tag{1.16}$$

Clearly, the approximate boundary synchronization by p-groups of system (1.1) in the pinning sense implies the approximate boundary null controllability of the reduced system (1.16). Since we have *a priori* assumed the existence of the approximately synchronizable state by p-groups $(u_1, \dots, u_p)$, we can not directly conclude the equivalence between the approximate boundary synchronization of system (1.1) in the pinning sense and the approximate boundary null controllability of the reduced system (1.16). However, we have established the following

Lemma 1.6 (see [18, 20])**.** *Assume that the C_p-compatibility condition (1.14) holds. Then system (1.1) is approximately synchronizable by p-groups in the pining sense if and only if the reduced system (1.16) is approximately null controllable, or equivalently, if and only if the adjoint problem of the reduced problem (1.16):*

$$\begin{cases} \Phi_p'' - \Delta\Phi_p + \overline{A}_p^T\Phi_p = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ \Phi_p = 0 & \text{on} \quad (0,+\infty)\times\Gamma, \\ t=0: \quad \Phi_p = \Phi_{p0}, \quad \Phi_p' = \Phi_{p1} & \text{in} \quad \Omega \end{cases} \tag{1.17}$$

is C_pD-observable on the time tinterval $[0,T]$, namely, the observation

$$(C_pD)^T\partial_\nu\Psi_p \equiv 0 \qquad \text{on} \qquad [0,T]\times\Gamma_1 \tag{1.18}$$

implies $\Psi_p \equiv 0$.

By a similar way as in the proof of Lemma 1.6, possibly with a suitable invertible linear transformation of state variables, we can get a more general result as follows.

Lemma 1.7. *Let $\bar{C}_{\bar{p}}$ be a full row-rank matrix of order $(N-\bar{p})\times N$ with Ker$(\bar{C}_{\bar{p}}) = \mathrm{Span}\{\bar{e}_1, \dots, \bar{e}_{\bar{p}}\}$. Assume that the coupling matrix A satisfies the $\bar{C}_{\bar{p}}$-compatibility condition, namely, there exists a unique matrix $\bar{A}_{\bar{p}}$ of order $(N-\bar{p})$, such that $\bar{C}_{\bar{p}}A = \bar{A}_{\bar{p}}\bar{C}_{\bar{p}}$. Then the corresponding reduced system*

$$\begin{cases} W_{\bar{p}}'' - \Delta W_{\bar{p}} + \bar{A}_{\bar{p}}W_{\bar{p}} = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ W_{\bar{p}} = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ W_{\bar{p}} = \bar{C}_{\bar{p}}DH & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ t=0: \quad W_{\bar{p}} = \bar{C}_{\bar{p}}\widehat{U}_0, \quad W_{\bar{p}} = \bar{C}_{\bar{p}}\widehat{U}_1 & \text{in} \quad \Omega, \end{cases}$$

in which $W_{\bar{p}} = \bar{C}_{\bar{p}} U$, *is approximately null controllable if and only if there exist some scalar functions* $\bar{u}_1, \ldots, \bar{u}_{\bar{p}}$, *such that*

$$U_n \to \sum_{r=1}^{\bar{p}} \bar{u}_r \bar{e}_r \quad in \quad (C^0_{\mathrm{loc}}([T, +\infty); L^2(\Omega)))^N \cap (C^1_{\mathrm{loc}}([T, +\infty); H^{-1}(\Omega)))^N$$

as $n \to +\infty$.

As pointed out before, Definition 1.5 is greatly based on the existence of the approximately synchronizable state by p-groups $(u_1, \ldots, u_p)$, which is a purely *a priori* hypothesis. However, under the C_p-compatibility condition (1.14), the equivalence between the approximate boundary synchronization by p-groups in the pinning sense of system (1.1) and the approximate boundary null controllability of the reduced system (1.16) was established in Lemma 1.6. But the necessity of the C_p-compatibility condition is a delicate problem, which intrinsically depends not only on the number of the employed total controls, but also on the algebraic structure of the coupling matrix A with respect to the synchronization matrix C_p.

In this paper, we will give a more natural definition of approximate boundary synchronization by p-groups *in the consensus sense* by

Definition 1.8. System (1.1) is approximately synchronizable by p-groups at the times $T > 0$ in the consensus sense, if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a sequence $\{H_n\}$ of boundary controls in $(L^2(0, +\infty; L^2(\Gamma_1)))^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1.1)−(1.2) satisfies

$$u^{(k)}_n - u^{(l)}_n \to 0 \quad \text{in} \quad C^0_{\mathrm{loc}}([T, +\infty); L^2(\Omega)) \cap C^1_{\mathrm{loc}}([T, +\infty); H^{-1}(\Omega)) \tag{1.19}$$

as $n \to +\infty$ for all $n_{r-1} + 1 \le k, l \le n_r$ and $1 \le r \le p$, or equivalently

$$C_p U_n \to 0 \quad \text{in} \quad (C^0_{\mathrm{loc}}([T, +\infty); L^2(\Omega)))^{N-p} \cap (C^1_{\mathrm{loc}}([T, +\infty); (H^{-1}(\Omega)))^{N-p} \tag{1.20}$$

as $n \to +\infty$.

Clearly, the approximate boundary synchronization by p-groups in the pinning sense implies that in the consensus sense, then the later seems to be weaker than the former. However, we will establish the equivalence between these two kinds of synchronizations.

First, it's easy to get

Lemma 1.9. *Under the* C_p*-compatibility condition* (1.14)*, system* (1.1) *is approximately synchronizable by* p*-groups in the consensus sense if and only if the reduced system* (1.16) *is approximately null controllable.*

When the coupling matrix A satisfies the C_p-compatibility condition (1.14), by Lemmas 1.6 and 1.9, the approximate boundary synchronization by p-groups in the consensus sense is equivalent to that in the pinning sense.

On the other hand, when the coupling matrix A does not satisfy the C_p-compatibility condition (1.14), we introduce a full column-rank matrix $\tilde{C}_{\tilde{p}}$ of order $(N - \tilde{p}) \times N (0 \le \tilde{p} < p)$ by

$$\mathrm{Im}(\tilde{C}^T_{\tilde{p}}) = \mathrm{Span}(C^T_p, A^T C^T_p, \ldots, (A^T)^{N-1} C^T_p). \tag{1.21}$$

The matrix $\tilde{C}_{\tilde{p}}$ will be called the extension matrix of C_p related to the matrix A. By Cayley−Hamilton's theorem, we have $A^T \mathrm{Im}(\tilde{C}^T_{\tilde{p}}) \subseteq \mathrm{Im}(\tilde{C}^T_{\tilde{p}})$, or equivalently, $A\mathrm{Ker}(\tilde{C}_{\tilde{p}}) \subseteq \mathrm{Ker}(\tilde{C}_{\tilde{p}})$ (see Rem. 2.2 below). Therefore, A satisfies the corresponding $\tilde{C}_{\tilde{p}}$-compatibility condition, namely, there exists a matrix $\tilde{A}_{\tilde{p}}$ of order $(N - \tilde{p})$, such that $\tilde{C}_{\tilde{p}}A = \tilde{A}_{\tilde{p}}\tilde{C}_{\tilde{p}}$. Thus setting $W_{\tilde{p}} = \tilde{C}_{\tilde{p}}U$, we get the reduced system

$$\begin{cases} W_{\tilde{p}}'' - \Delta W_{\tilde{p}} + \tilde{A}_{\tilde{p}} W_{\tilde{p}} = 0 & \text{in} \quad (0,+\infty)\times\Omega, \\ W_{\tilde{p}} = 0 & \text{on} \quad (0,+\infty)\times\Gamma_0, \\ W_{\tilde{p}} = \tilde{C}_{\tilde{p}} DH & \text{on} \quad (0,+\infty)\times\Gamma_1, \\ t=0: \qquad W_{\tilde{p}} = \tilde{C}_{\tilde{p}}\widehat{U}_0, \quad W_{\tilde{p}} = \tilde{C}_{\tilde{p}}\widehat{U}_1 & \text{in} \quad \Omega. \end{cases} \tag{1.22}$$

Since $\tilde{C}_{\tilde{p}}$ is an extension of C_p, we have that

$$\tilde{C}_{\tilde{p}} U_n \to 0 \quad \text{in} \quad (C^0_{\text{loc}}([T,+\infty); L^2(\Omega)))^{N-\tilde{p}} \cap (C^1_{\text{loc}}([T,+\infty); (H^{-1}(\Omega)))^{N-\tilde{p}} \tag{1.23}$$

as $n\to+\infty$ implies (1.20). Moreover, we will prove (see Thm. 4.1) that the inverse also holds true. Thus, we get

Lemma 1.10. *The approximate boundary synchronization by p-groups in the consensus sense for system* (1.1) *is equivalent to the approximate boundary null controllability of the reduced system* (1.22).

Noting that A satisfies the $\tilde{C}_{\tilde{p}}$-compatibility condition and (1.23) holds, by Lemma 1.7, we can get

Lemma 1.11. *System* (1.1) *is approximately synchronizable by p-groups in the pining sense if and only if the reduced system* (1.22) *is approximately null controllable.*

Thus the combination of Lemmas 1.10 and 1.11 concludes

Theorem 1.12. *The approximate boundary synchronization by p-groups in the consensus sense is equivalent to the approximate boundary synchronization by p-groups in the pinning sense* (see also Thm. *4.4).*

Thus, the approximate boundary synchronization by p-groups both in the pinning sense and in the consensus sense can be unifiedly called the approximate boundary synchronization by p-groups. However, we should further determine that the components $u_1,\dots,u_p$ of the approximately synchronizable state by p-groups are actually linearly dependent or linearly independent for system (1.1).

The following result indicates the lower bound on the minimal number of total controls necessary to the approximate boundary synchronization by p-groups for system (1.1), no matter whether the C_p-compatibility condition (1.14) is satisfied or not.

Theorem 1.13. *If system* (1.1) *is approximately synchronizable by p-groups, then we have*

$$\text{rank}(D, AD, \dots, A^{N-1}D) \geq N-p. \tag{1.24}$$

According to Theorem 1.13, it is natural to consider the approximate boundary synchronization by p-groups for system (1.1) with the minimal number $(N-p)$ of total controls. In this case, the coupling matrix A should possess some fundamental properties related to the synchronization matrix C_p.

Theorem 1.14. *Assume that system* (1.1) *is approximately synchronizable by p-groups under the minimal rank condition*

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N-p. \tag{1.25}$$

Then we have the following assertions:

(i) *The coupling matrix A satisfies the C_p-compatibility condition* (1.14).
(ii) *There exist some linearly independent functions $u_1, u_2, \dots, u_p$, independent of the applied boundary controls, such that* (1.9) *holds.*

(iii) *We have the rank condition*

$$\operatorname{rank}(C_pD, C_pAD, \dots, C_pA^{N-1}D) = N - p. \tag{1.26}$$

(iv) A^T *admits an invariant subspace, which is biorthonormal to* $Ker(C_p)$.

Definition 1.15. The coupling matrix A satisfies the strong C_p-compatibility condition, if A satisfies the C_p-compatibility condition (1.14), namely, $\operatorname{Ker}(C_p)$ is an invariant subspace of A; moreover, A^T has an invariant subspace W which is bi-orthogonal to $\operatorname{Ker}(C_p)$.

The assertions (i) and (iv) of Theorem 1.14 means that in this case the coupling matrix A satisfies the strong C_p-compatibility condition. Let $\mathbb{D}_p$ be the set of the matrices under which system (1.1) is approximately synchronizable by p-groups. We set

$$N_p = \inf_{D\in\mathbb{D}_p} \operatorname{rank}(D, AD, \dots, A^{N-1}D). \tag{1.27}$$

We will show (see Cor. 3.8) that $N_p = N - p$ if and only if A satisfies the strong C_p-compatibility condition. In the case that A doesn't satisfy the strong C_p-compatibility condition, we introduce a full row-rank matrix C_q^* of order $(N-q)\times N$, such that A satisfies the corresponding strong C_q^*-compatibility condition.

Definition 1.16. An $(N-q)\times N(0 \le q < p)$ full row-rank matrix C_q^* is called the enlarged matrix of C_p, related to the matrix A, if

(i) $\operatorname{Im}(C_q^{*T})$ contains $\operatorname{Im}(C_p^T)$.
(ii) $\operatorname{Im}(C_q^{*T})$ is an invariant subspace of A^T.
(iii) A admits an invariant subspace V, which is bi-orthogonal to $\operatorname{Im}(C_q^{*T})$.
(iv) $\operatorname{Im}(C_q^{*T})$ is the least one satisfying the previous three conditions.

The following result improves (1.24) and deeply reveals that the minimal number of total controls necessary to the approximate boundary synchronization by p-groups depends not only on the number p of groups, but also on the algebraic structure of the coupling matrix A with respect to the synchronization matrix C_p.

Theorem 1.17. *Assume that system* (1.1) *is approximately synchronizable by p-groups. Then, we necessarily have*

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) \ge N - q. \tag{1.28}$$

Thus, in order to realize the approximate boundary synchronization by p-groups, we *are required* to use at least $(N-q)$ total controls. This suggests us to consider the following generalized approximate boundary synchronization with respect to the enlarged matrix C_q^*.

Definition 1.18. System (1.1) is induced approximately synchronizable at the time $T' > T$, if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, there exists a sequence $\{H_n\}$ of boundary controls in $(L^2(0,+\infty; L^2(\Gamma_1)))^M$ with compact support in $[0, T']$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1.1)–(1.2) satisfies

$$C_q^* U_n \to 0 \quad \text{in} \quad (C^0_{\text{loc}}([T', +\infty); L^2(\Omega)))^{N-q} \cap (C^1_{\text{loc}}([T', +\infty); H^{-1}(\Omega)))^{N-q} \tag{1.29}$$

as $n \to +\infty$.

We observe that because of the lack of boundary controls and the finite speed of wave propagation, the action of the boundary controls must be on a time interval $[0, T']$ with $T' > T$ (see Thm. 4.4 in [20] for the estimation on T').

Since $\text{Im}(C_p) \subseteq \text{Im}(C_q^*)$, the convergence (1.29) implies obviously the convergence (1.20). If we can get the convergence (1.29) from (1.20), then (1.29) could provide some information on linear dependence of the components $u_1, \dots, u_p$ of the approximately shchronizable state by p-groups, so that we can actually get one of the following situations: the approximate boundary synchronization by $\tilde{p}$-groups with $\tilde{p} < p$, the approximate boundary synchronization and approximate boundary null controllability by groups or even the approximate boundary null controllability *etc.*

Since A satisfies the strong C_q^*-compatibility condition, there exists a unique matrix A_q^* of order $(N-q)$, such that $C_q^* A = A_q^* C_q^*$. Then setting $W_q = C_q^* U$, we get the following reduced problem:

$$\begin{cases} W_q'' - \varDelta W_q + A_q^* W_q = 0 & \text{in} \quad (0, +\infty) \times \varOmega, \\ W_q = 0 & \text{on} \quad (0, +\infty) \times \varGamma_0, \\ W_q = C_q^* D H & \text{on} \quad (0, +\infty) \times \varGamma_1, \\ t = 0: \quad W_q = C_q^* \widehat{U}_0, \quad W_q' = C_q^* \widehat{U}_1 \quad \text{in} \quad \varOmega. \end{cases} \tag{1.30}$$

Clearly, the induced approximate boundary synchronization of system (1.1) is equivalent to the approximate boundary null controllability of the reduced system (1.30). Then, by Lemma 1.4, we necessarily have the following rank condition:

$$\text{rank}(C_q^* D, C_q^* A D, \dots, C_q^* A^{N-1} D) = N - q. \tag{1.31}$$

If the rank condition (1.31) is sufficient to the approximate boundary null controllability of the reduced system (1.30), for example, as the reduced system (1.30) is a cascade system, a 2×2 system, or an one-dimensional system (see [8, 10]), then system (1.1) is induced approximately synchronizable. In this case, because of the minimality of the rank in (1.31), we can show (see Thm. 5.5) that the approximately synchronizable state $(u_1, u_2, \dots, u_p)$ is still independent of the employed boundary controls, but linearly dependent because of $p > q$ (see Examples 5.10 and 5.11).

Summarizing the above discussion, we can proceed the reduction by two ways. The first one is done by the synchronization matrix C_p. Then we get the reduced problem (1.16), which gives the approximate boundary synchronization by p-groups without any additional information. While, as we use the enlarged matrix C_q^* to proceed the reduction, then we get another reduced problem (1.30). If the rank condition (1.31) is sufficient for the approximate boundary null controllability of the reduced system (1.30), then we can not only obtain the approximate boundary synchronization by p-groups, but also recover some additional relations, which are lost in the first procedure of reduction by C_p.

The phenomenon of synchronization was first observed by Huygens [9] in 1665, however, the previous studies focused only on systems described by ODEs. The synchronization in the PDEs case was first studied for a coupled system of wave equations with Dirichlet boundary controls by Li and Rao in [13, 15, 17] for the exact boundary synchronization, later in [14] for the approximate boundary synchronization.

One of the motivation of studying the synchronization consists of establishing the controllability in the case of fewer boundary controls. When the number of boundary controls is fewer than the number of state variables, the non-exact boundary controllability of a coupled system of wave equations with Dirichlet/Neumann boundary controls in the usual energy space was established in [13, 20]. However, if the components of initial data are allowed to have different levels of energy, then the exact boundary controllability for a system of two wave equations by means of only one boundary control was established in Alabau−Boussouira [1], Liu and Rao [23], Rosier and de Teresa [25], or for a cascade system of N wave equations by means of only one boundary control in Alabau−Boussouira [2]. In a recent work [5], Dehman *et al.* established the controllability of two coupled wave equations on a compact manifold with only one local distributed control. The optimal time of controllability and the controllable spaces are given in the cases that the waves propagate with the same speed or with different speeds.

However, the approximate boundary null controllability is more flexible with respect to the number of boundary controls. It was shown in Li and Rao [13, 20] that for a coupled system of wave equations with Dirichlet/Neuman boundary controls, some basic properties could be characterized by means of Kalman's criterion. Although the criterion is only necessary for the approximate boundary null controllability in general, it opens an important way for the research on the unique continuation of partial differential equations.

In contrast with hyperbolic systems, Kalman's criterion is sufficient to the exact boundary null controllability for systems of parabolic equations (see [3, 7]).

In [27] (see also [24]), Zuazua proposed the average controllability as another way to deal with the controllability with fewer controls. The observability inequality is particularly interesting for a trial on the decay rate of approximate controllability. However the decay rate of the convergence (1.3) of a system of wave equations with fewer controls is still an open problem and seems to be a very interesting topic.

The rest of the paper is organized as follows. We regroup some frequently used preliminaries in Section 2. In Section 3, we investigate the approximate boundary synchronization by p-groups under the minimal rank condition and establish the necessity of the C_p-compatibility condition, the existence of approximately synchronizable state $(u_1, u_2, \dots, u_p)$ which is linearly independent, and the algebraic property on the coupling matrix A. In Section 4, we consider the approximate boundary synchronization by p-groups without the C_p-compatibility condition and establish the equivalence of the two kinds of synchronizations. In Section 5, by introducing the induced approximate boundary synchronization, we determine the minimal number of total controls necessary to the approximate boundary synchronization by p-groups in the general case, and recover the information lost in the procedure of reduction by the matrix C_p. Some technic details are given in Appendix.

2. Preliminaries

In this section, we give some results which will be frequently used in what follows.

Lemma 2.1 (see [23])**.** *Let A be a matrix of order N and C be a full row-rank matrix of order $(N-p)\times N$ with $1 \le p < N$. Then the following assertions are equivalent*:

(i) *The kernel of C is an invariant subspace of A*:

$$A Ker(C) \subseteq Ker(C).$$

(ii) *There exists a unique matrix $\overline{A}$ of order $(N-p)$*:

$$\overline{A} = CAC^T(CC^T)^{-1},$$

called the reduced matrix of A by C, such that

$$CA = \overline{A}C.$$

Remark 2.2. Since $\mathrm{Ker}(C) = \{\mathrm{Im}(C^T)\}^{\perp}$, the assertions mentioned in Lemma 2.1 are also equivalent to the fact that $\mathrm{Im}(C^T)$ is an invariant subspace of A^T.

Lemma 2.3. *Let C be a matrix of order $(N-p)\times N$ with $1 \le p < N$ and $\mathcal{K}$ be a matrix of order $N \times K$. Then, the equality*

$$\mathrm{rank}(C\mathcal{K}) = \mathrm{rank}(\mathcal{K})$$

holds if and only if

$$Ker(C) \cap Im(\mathcal{K}) = \{0\}.$$

Proof. Define the linear map $\mathcal{C}$ by $\mathcal{C}x = Cx$ for all $x \in \mathrm{Im}(\mathcal{K})$. Then we have

$$\mathrm{Im}(\mathcal{C}) = \mathrm{Im}(C\mathcal{K}), \qquad \mathrm{Ker}(\mathcal{C}) = \mathrm{Ker}(C) \cap \mathrm{Im}(\mathcal{K}).$$

From the rank-nullity theorem:

$$\dim \mathrm{Im}(\mathcal{C}) + \dim \mathrm{Ker}(\mathcal{C}) = \mathrm{rank}(\mathcal{K}),$$

we easily get the conclusion of Lemma 2.3.

Lemma 2.4. *Let A be a matrix of order N and C be a matrix of order $(N-p) \times N$ with $1 \leq p < N$. Assume that there exists a matrix $\overline{A}$ of order $(N-p)$, such that $CA = \overline{A}C$. Then for any matrix D of order $N \times M$, we have*

$$\mathrm{rank}(CD, \overline{A}CD, \ldots, \overline{A}^{N-p-1}CD) = \mathrm{rank}(CD, CAD, \ldots, CA^{N-1}D).$$

Proof. By Cayley−Hamilton's Theorem, we have

$$\mathrm{rank}(CD, \overline{A}CD, \ldots, \overline{A}^{N-p-1}CD) = \mathrm{rank}(CD, \overline{A}CD, \ldots, \overline{A}^{N-1}CD).$$

Then, noting $CA^l = \overline{A}^l C$ for any given integer $l \geq 0$, we get

$$(CD, \overline{A}CD, \ldots, \overline{A}^{N-1}CD) = (CD, CAD, \ldots, CA^{N-1}D).$$

The proof is complete.

Lemma 2.5. *Let A be a matrix of order N and D be a matrix of order $N \times M$. Let $d \geq 0$ be an integer.*

(i) The rank condition

$$\mathrm{rank}(D, AD, \ldots, A^{N-1}D) \geq N - d$$

holds if and only if the dimension of any given invariant subspace of A^T, contained in $\mathrm{Ker}(D^T)$ does not exceed d.

(ii) The rank condition

$$\mathrm{rank}(D, AD, \ldots, A^{N-1}D) = N - d$$

holds if and only if d is the largest dimension of invariant subspaces of A^T, contained in $\mathrm{Ker}(D^T)$.

Proof. The proof of (i) can be found in [18,20]. In order to get (ii), it suffices to write the rank condition in (ii) as

$$\mathrm{rank}(D, AD, \ldots, A^{N-1}D) \geq N - d \quad \text{and} \quad \mathrm{rank}(D, AD, \ldots, A^{N-1}D) \leq N - d,$$

then apply the assertion (i) and its counterpart.

Remark 2.6. The Hautus criterion (see [8]) corresponds to the particular case $d = 0$ of the assertion (ii) of Lemma 2.4.

Definition 2.7. Two subspaces V, W of $\mathbb{R}^N$ are biorthonormal if there exist a basis $(\epsilon_1, \ldots, \epsilon_d)$ of V and a basis $(\eta_1, \ldots, \eta_d)$ of W, such that

$$(\epsilon_k, \eta_l) = \delta_{kl}, \qquad 1 \leq k, l \leq d,$$

where δ_{kl} is the Kronecker symbol.

Lemma 2.8 (see [19]). *Two subspaces V, W of $\mathbb{R}^N$ are biorthonormal if and only if*

$$V^\perp \cap W = V \cap W^\perp = \{0\}.$$

Lemma 2.9 (see [19]). *Let A be a matrix of order N and V be an invariant subspace of A. Then V admits a supplement which is also invariant for A if and only if A^T admits an invariant subspace W which is biorthonormal to V. In particular, the matrix A is diagonalizable by blocks according to the decomposition $V \oplus W^\perp$.*

By duality, Lemma 2.9 can be formulated as

Lemma 2.10. *Let A^T be a matrix of order N and W be an invariant subspace of A^T. Then W admits a supplement which is also invariant for A^T if and only if A admits an invariant subspace V which is biorthonormal to W. In particular, the matrix A^T is diagonalizable by blocks according to the decomposition $W \oplus V^\perp$.*

Lemma 2.11. *Let W_0 be an invariant subspace of A^T. Then there exists a least subspace $\overline{W}$ with $W_0 \subseteq \overline{W}$, which admits a supplement $\overline{W}'$, such that both $\overline{W}$ and $\overline{W}'$ are invariant for A^T.*

Proof. Let $\mathcal{W}$ denote the set of all the subspaces W, which contains W_0 and admits a supplement W' such that both W and W' are invariant for A^T.

Let $W_1, W_2 \in \mathcal{W}$. By Lemma 2.10, there exist two subspaces V_1, V_2 which are invariant for A and biorthonormal to W_1, W_2, respectively. Using the relation $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$, we get

$$(V_1 \cap V_2)^\perp \cap (W_1 \cap W_2) = (V_1^\perp + V_2^\perp) \cap (W_1 \cap W_2) \subseteq V_1^\perp \cap W_1 + V_2^\perp \cap W_2.$$

Since V_1, V_2 are biorthonormal to W_1, W_2, respectively, by Lemma 2.8 we get

$$V_1 \cap W_1^\perp = \{0\}, \qquad V_2 \cap W_2^\perp = \{0\}.$$

It follows that

$$(V_1 \cap V_2)^\perp \cap (W_1 \cap W_2) = \{0\}.$$

Similarly, we can show that

$$(V_1 \cap V_2) \cap (W_1 \cap W_2)^\perp = \{0\}.$$

Thus by Lemma 2.8, $V_1 \cap V_2$ is biorthonormal to $W_1 \cap W_2$.

On the other hand, noting that $W_1 \cap W_2$ is invariant for A^T and $V_1 \cap V_2$ is invariant for A, by Lemma 2.10, $W_1 \cap W_2$ admits a supplement which is also invariant for A^T. Moreover, since $W_0 \subseteq W_1 \cap W_2$, we have $W_1 \cap W_2 \in \mathcal{W}$. Thus, we can define $\overline{W}$ as the intersection of all the subspaces in $\mathcal{W}$:

$$\overline{W} = \bigcap_{W \in \mathcal{W}} W.$$

Clearly, $\overline{W}$ satisfies the requirements of Lemma 2.10. The proof is complete.

3. Approximate boundary synchronization by groups with the minimal rank condition

We begin the consideration with the minimal rank condition (1.25), and we will show the necessity of the C_p-compatibility condition (1.14), the existence of approximately synchronizable state by p-groups as well as the algebraic structure of the matrix A related to the matrix C_p. The general situation will be discussed in Section 5.

We first establish a lower bound on the minimal rank necessary to the approximate boundary synchronization by p-groups of system (1.1) in the consensus sense, no matter whether the C_p-compatibility condition (1.14) is satisfied or not.

Theorem 3.1. *Assume that system (1.1) is approximately synchronizable by p-groups in the consensus sense under the action of a boundary control matrix D. Then the rank condition (1.24) holds true.*

Proof. Let $\tilde{C}_{\tilde{p}}$ be defined by (1.21). Consider the reduced system (1.22), which is approximately null controllable because of Theorem 4.1 below. Then by Lemma 1.4, we have

$$\mathrm{rank}(\tilde{C}_{\tilde{p}}D, \tilde{A}_{\tilde{p}}\tilde{C}_{\tilde{p}}D, \dots, \tilde{A}_{\tilde{p}}^{N-\tilde{p}-1}\tilde{C}_{\tilde{p}}D) = N - \tilde{p},$$

which, together with Lemma 2.4, implies that

$$\mathrm{rank}(D, AD, \dots, A^{N-1}D) \geq \mathrm{rank}(\tilde{C}_{\tilde{p}}D, \tilde{C}_{\tilde{p}}AD, \dots, \tilde{C}_{\tilde{p}}A^{N-1}D) = N - \tilde{p}. \tag{3.1}$$

We get then (1.24) because of $\tilde{p} \leq p$. The proof is complete.

According to the lower bound (1.24), we consider the approximate boundary synchronization by p-groups of system (1.1) in the consensus sense under the minimal rank $(N - p)$.

Theorem 3.2. *Assume that system* (1.1) *is approximately synchronizable by p-groups in the consensus sense under the minimal rank condition* (1.25). *Then we have the following assertions*:

(i) *There exist linearly independent functions $u_1, u_2, \dots, u_p$, which are independent of the applied boundary controls, such that the convergence* (1.9) *holds true.*
(ii) *The coupling matrix A satisfies the strong C_p-compatibility condition.*
(iii) *The rank condition* (1.26) *holds true.*

Proof.

(i) By the assertion (ii) of Lemma 2.5 with $d = p$, the rank condition (1.25) guarantees the existence of an invariant subspace V of A^T, with dimension p and contained in $\mathrm{Ker}(D^T)$. Let $\{E_1, \dots, E_p\}$ be a basis of V, such that

$$A^T E_r = \sum_{s=1}^{p} \alpha_{rs} E_s, \qquad D^T E_r = 0, \qquad 1 \leq r \leq p.$$

Applying E_r to problem (1.1)–(1.2) with $U = U_n$ and $H = H_n$, and setting $\phi_r = (E_r, U_n)$ for $r = 1, \dots, p$, we get

$$\begin{cases} \phi_r'' - \mathit{\Delta}\phi_r + \sum_{s=1}^{p} \alpha_{rs}\phi_s = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \phi_r = 0 & \text{on} \quad (0, +\infty) \times \Gamma, \\ t = 0: \qquad \phi_r = (E_r, \widehat{U}_0), \quad \phi_r' = (E_r, \widehat{U}_1) & \text{in} \quad \Omega. \end{cases} \tag{3.2}$$

We claim that $\mathrm{Im}(C_p^T) \cap V = \{0\}$. Otherwise, without loss of generality, we can assume that $E_1 \in \mathrm{Im}(C_p^T) \cap V$. Then there exists a vector $R \in \mathbb{R}^{N-p}$, such that

$$E_1 = C_p R.$$

It follows that

$$\phi_1 = (E_1, U_n) = (R, C_p U_n) \to 0 \qquad \text{as} \quad n \to +\infty.$$

Since ϕ_1 is independent of n, then

$$\phi_1 \equiv 0 \qquad t \geq T. \tag{3.3}$$

On the other hand, for any given $t \geq 0$, problem (3.2) defines an isomorphism in the space $(H_0^1(\Omega))^p \times (L^2(\Omega))^p$. Thus, as the initial data of (3.2) varies in $(H_0^1(\Omega))^p \times (L^2(\Omega))^p$, the state variable $(\phi_1(t), \dots, \phi_p(t))$ will run through the space $(H_0^1(\Omega))^p$. This contradicts (3.3). Then we can write

$$U_n = \begin{pmatrix} C_p \\ E_1^T \\ \vdots \\ E_p^T \end{pmatrix}^{-1} \begin{pmatrix} C_p U_n \\ (E_1, U_n) \\ \vdots \\ (E_p, U_n) \end{pmatrix} \to \begin{pmatrix} C_p \\ E_1^T \\ \vdots \\ E_p^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} = U$$

in $(C^0_{\rm loc}([T,+\infty);(L^2(\Omega))^N \cap (C^1_{\rm loc}([T,+\infty);(H^{-1}(\Omega)))^N$ as $n \to +\infty$. Clearly, U is independent of the applied boundary controls.

Moreover, noting $C_pU = 0$ for $t \geq T$ and the structure of $\mathrm{Ker}(C_p)$ in (1.13), there exist some scalar functions $u_1, \dots, u_p$ such that

$$U = \sum_{r=1}^{p} u_r e_r \qquad t \geq T.$$

Thus,

$$U_n \to \sum_{r=1}^{p} u_r e_r \tag{3.4}$$

in $(C^0_{\rm loc}([T,+\infty);(L^2(\Omega))^N \cap (C^1_{\rm loc}([T,+\infty);(H^{-1}(\Omega)))^N$ as $n \to +\infty$. Finally, noting (1.12), the convergence (3.4) means exactly (1.9).

(ii) If A does not satisfy the C_p-compatibility condition, then $\tilde{p} < p$. This contradicts the rank condition (1.25). On the other hand, from (3.4) we get

$$t \geq T: \qquad \phi_s = \sum_{r=1}^{p} (E_s, e_r) u_r, \qquad s = 1, \dots, p.$$

Since the state variable $(\phi_1(t), \dots, \phi_p(t))$ runs through the space $(H^1_0(\Omega))^p$ as the initial data of (3.2) varies in $(H^1_0(\Omega))^p \times (L^2(\Omega))^p$, so is the synchronizable state $(u_1, \dots, u_p)$. In particular, the matrix $((E_s, e_r))_{1 \leq s,r \leq p}$ is invertible, the inverse of which is denoted by $(\alpha_{sr})_{1 \leq s,r \leq p}$. Then setting the new basis of $\mathrm{Ker}(C_p)$ by

$$\epsilon_r = \sum_{q=1}^{p} \alpha_{qr} e_q, \qquad 1 \leq r \leq p,$$

we have

$$(E_s, \epsilon_r) = \sum_{q=1}^{p} \alpha_{qr} (E_s, e_q) = \delta_{sr}, \quad 1 \leq s, r \leq p.$$

The subspaces $\mathrm{Ker}(C_p) = \mathrm{Span}\{\epsilon_1, \dots, \epsilon_p\}$ and $V = \mathrm{Span}\{E_1, \dots, E_p\}$ are biorthonormal.

(iii) Clearly, under the C_p-compatibility condition (1.14), the approximate boundary synchronization by p-groups for system (1.1) in the consensus sense is equivalent to the approximate boundary null controllability of the reduced system (1.16). Then, it follows from Lemma 1.4 that

$$\mathrm{rank}(C_pD, \overline{A}_p C_p D, \dots, \overline{A}_p^{N-p-1} C_p D) = N - p,$$

which, together with Lemma 2.4, implies (1.26). The proof is then complete.

Remark 3.3. The rank condition (1.25) indicates that the number of total controls is equal to $(N-p)$, but the state variable U of system (1.1) has N independent components, so if system (1.1) is approximately synchronizable by p-groups, there should exist p directions $E_1, \dots, E_p$, on which the projections $(E_1, U_n), \dots, (E_p, U_n)$ of the solution U_n to problem (1.1)–(1.2) are independent of the $(N-p)$ boundary controls, therefore, converge. This is why a weaker requirement (1.19) could actually imply the existence of linearly independent functions $u_1, \dots, u_p$ in (1.9).

Remark 3.4. Noting that the rank condition (1.26) indicates that the reduced problem (1.16) of $(N-p)$ equations is still submitted to $(N-p)$ total controls, which is necessary for the corresponding approximate boundary null controllability. The procedure of reduction from (1.1) to (1.16) reduces only the number of equations, but not the number of total controls, so that we get a reduced system of $(N-p)$ equations submitted to $(N-p)$ total controls, that is just what we want.

Remark 3.5. Since the invariant subspace $\text{Span}\{E_1, E_2, \dots, E_p\}$ of A^T is biorthonormal to the invariant subspace $\text{Ker}(C_p) = \text{Span}\{e_1, e_2, \dots, e_p\}$ of A, by Lemma 2.8, the invariant subspace $\text{Span}\{E_1, E_2, \dots, E_p\}^\perp$ of A is a supplement of $\text{Span}\{e_1, e_2, \dots, e_p\}$. Therefore, A is diagonalizable by blocs according to the decomposition $\text{Span}\{e_1, e_2, \dots, e_p\} \oplus \text{Span}\{E_1, E_2, \dots, E_p\}^\perp$.

Theorem 3.6. *Assume that A satisfies the strong C_p-compatibility condition. Then there exists a matrix D which satisfies the minimal rank condition* (1.25)*, and realizes the approximate boundary synchronization by p-groups for system* (1.1)*.*

Proof. By Lemmas 1.6 and 1.9, under the C_p-compatibility condition (1.14), the approximate boundary synchronization by p-groups of system (1.1) in the consensus sense is equivalent to the C_pD-observability of the adjoint problem (1.17) of the reduced problem (1.16).

Let D be chosen by

$$\text{Ker}(D^T) = W, \tag{3.5}$$

where W is a subspace which is invariant for A^T and bi-orthogonal to $\text{Ker}(C_p)$. Clearly, W is the only invariant subspace of A^T, with the maximal dimension p and contained in $\text{Ker}(D^T)$. Then by the assertion (ii) of Lemma 2.4 with $d = p$, the rank condition (1.25) holds for this choice of D. On the other hand, the biorthonomality of W with $\text{Ker}(C_p)$ implies that $\text{Ker}(C_p) \cap \text{Im}(D) = \{0\}$. Therefore, by Lemma 2.3 we have

$$\text{rank}(D) = \text{rank}(C_pD) = N - p.$$

Then the C_pD-observation (1.18) becomes the complete observation:

$$\partial_\nu \Psi = 0 \qquad \text{on} \quad [0, T] \times \Gamma_1,$$

which implies well $\Psi \equiv 0$ because of Holmgren's uniqueness Theorem (see [24]). The proof is thus complete.

Remark 3.7. The matrix D defined by (3.5) is of rank $(N - p)$. So, we can realize the approximate boundary synchronization by p-groups for system (1.1) by means of $(N - p)$ direct boundary controls. But we are more interested in using fewer direct boundary controls to realize the approximate boundary synchronization by p-groups for system (1.1). We will give later in Appendix a matrix D with the minimal rank, such that the rank conditions (1.25) and (1.26) are simultaneously satisfied. We point out that Kalman's criterion (1.26) is indeed sufficient for the approximate boundary null controllability of the reduced problem (1.16), for some special reduced systems such as cascade systems, 2×2 systems and one-dimensional systems (see [18, 20]).

Let $\mathbb{D}_p$ be the set of all matrices D which realize the approximate boundary synchronization by p-groups for system (1.1) in the consensus sense. Define the minimal number of total controls for the approximate boundary synchronization by p-groups for system (1.1) by

$$N_p = \inf_{D \in \mathbb{D}_p} \text{rank}(D, AD, \dots, A^{N-1}D). \tag{3.6}$$

Then, summarizing the results obtained in Theorems 3.1, 3.2 and 3.6, we have

Corollary 3.8. *The equality*

$$N_p = N - p \tag{3.7}$$

holds, if and only if A satisfies the strong C_p-compatibility condition.

In the opposite case, the determination of the number N_p is more complicated, and we will discuss it in the next sections.

Proposition 3.9. *The rank conditions* (1.25) *and* (1.26) *simultaneously hold for some control matrix D, if and only if A satisfies the strong C_p-compatibility condition.*

Proof. By the assertion (ii) of Lemma 2.5 with $d = p$, the rank conditions (1.25) implies the existence of an invariant subspace W of A^T, with the dimension p and contained in $\mathrm{Ker}(D^T)$. It is easy to see that

$$W \subseteq \mathrm{Ker}(D, AD, \dots, A^{N-1}D)^T$$

and

$$\dim(W) = \dim\, \mathrm{Ker}(D, AD, \dots, A^{N-1}D)^T = p,$$

then we have

$$W = \mathrm{Ker}(D, AD, \dots, A^{N-1}D)^T. \tag{3.8}$$

By Lemma 2.3, the rank conditions (1.25) and (1.26) imply that

$$\mathrm{Ker}(C_p) \cap W^{\perp} = \mathrm{Ker}(C_p) \cap \mathrm{Im}(D, AD, \dots, A^{N-1}D) = \{0\},$$

or equivalently, we have

$$\{\mathrm{Ker}(C_p)\}^{\perp} \cup W = \mathbb{R}^N.$$

Since $\dim(W) = p$ and $\dim\{\mathrm{Ker}(C_p)\}^{\perp} = N - p$, we get

$$\{\mathrm{Ker}(C_p)\}^{\perp} \cap W = \{0\}.$$

Therefore, by Lemma 2.8, W is biorthonormal to $\mathrm{Ker}(C_p)$.

Conversely, assume that W is an invariant subspace of A^T, and biorthonormal to $\mathrm{Ker}(C_p)$, therefore with dimension p. Define a full column-rank matrix D of order $N \times (N - p)$ by

$$\mathrm{Ker}(D^T) = W.$$

Clearly, W is an invariant subspace of A^T, with dimension p and contained in $\mathrm{Ker}(D^T)$. Moreover, the dimension of $\mathrm{Ker}(D^T)$ is equal to p. Then by the assertion (ii) of Lemma 2.5 with $d = p$, the rank condition (1.25) holds. Keeping in mind that (3.8) remains true in the present situation, it follows that

$$\mathrm{Ker}(C_p) \cap \mathrm{Im}(D, AD, \dots, A^{N-1}D) = \mathrm{Ker}(C_p) \cap W^{\perp} = \{0\},$$

which, by Lemma 2.3, implies the rank condition (1.26). The proof is then complete.

Example 3.10. Consider the approximate boundary synchronization of the following system

$$\begin{cases} u'' - \Delta u + v = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ v'' - \Delta v - u + 2v = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ u = v = 0 & \text{on} \quad (0, +\infty) \times \Gamma_0, \\ u = \alpha h, \quad v = \beta h & \text{on} \quad (0, +\infty) \times \Gamma_1. \end{cases} \tag{3.9}$$

We have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and

$$C_1 = (1, -1), \qquad \mathrm{Ker}(C_1) = \mathrm{Span}\{e_1\} \quad \text{with} \quad e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Clearly, A satisfies the C_1-compatibility condition with $\overline{A}_1 = 1$. The reduced system

$$\begin{cases} w'' - \varDelta w + w = 0 & \text{in} \quad (0,+\infty)\times\varOmega, \\ w = 0 & \text{on} \quad (0,+\infty)\times\varGamma_0, \\ w = (\alpha-\beta)h & \text{on} \quad (0,+\infty)\times\varGamma_1 \end{cases} \tag{3.10}$$

is approximately null controllable, so, by Lemma 1.6, system (3.9) is approximately synchronizable, provided that $\alpha \neq \beta$ and $T > 0$ is large enough.

On the other hand, from the expressions

$$(D, AD) = \begin{pmatrix} \alpha & \beta \\ \beta & 2\beta-\alpha \end{pmatrix}, \qquad \det(D, AD) = -(\alpha-\beta)^2,$$

we observe that $\alpha \neq \beta$ if and only if $\mathrm{rank}(D, AD) = 2$. This means that we have to use two total controls to realize the approximate boundary synchronization of system (3.9). Noting that Kalman's criterion is sufficient for the approximate boundary null controllability of 2×2 systems (see Thm. 4.4 in [20]), system (3.9) is not only approximately synchronizable, but also approximately null controllable under the action of the same control matrix D on a time interval $[0, T']$ with $T' > T$.

4. Approximate boundary synchronization by groups without C_p-compatibility condition

In this section we will establish the equivalence between the approximate boundary synchronization by p-groups in the consensus sense and that in the pinning sense.

When A does not satisfy the C_p-compatibility condition (1.14), we have introduced in (1.21) an extension matrix $\tilde{C}_{\tilde{p}}$, such that the coupling matrix A satisfies the $\tilde{C}_{\tilde{p}}$-compatibility condition: there exists a matrix of order $(N-\tilde{p})$, such that $\tilde{C}_{\tilde{p}}A = \tilde{A}_{\tilde{p}}\tilde{C}_{\tilde{p}}$. Then setting $W_{\tilde{p}} = \tilde{C}_{\tilde{p}}U$ we get the reduced system (1.22). Moreover, the following theorem shows that the reduced system (1.22) is approximately null controllable.

Theorem 4.1. *Assume that system (1.1) is approximately synchronizable by p-groups in the consensus sense. Then, for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\varOmega))^N \times (H^{-1}(\varOmega))^N$, *there exists a sequence* $\{H_n\}$ *of boundary controls in* $(L^2(0,+\infty; L^2(\varGamma_1)))^M$ *with compact support in* $[0, T]$, *such that the converge (1.23) holds true for the corresponding sequence* $\{U_n\}$ *of solutions to problem (1.1)−(1.2).*

Remark 4.2. Since $\mathrm{Im}(C_p^T) \subseteq \mathrm{Im}(\tilde{C}_{\tilde{p}}^T)$, (1.23) contains more information than (1.20). On the other hand, we will perceive that (1.23) follows from (1.20) and the non C_p-compatibility.

The proof of Theorem 4.1 will be given later on. We first use it to get the following

Corollary 4.3. *Assume that A does not satisfy the C_1-compatibility condition. If system (1.1) is approximately synchronizable, then it is approximately null controllable.*

Proof. Since $p = 1$, the extension matrix $\tilde{C}_{\tilde{p}} = I_N$. Then (1.23) gives the approximate boundary null controllability.

Theorem 4.4. *Assume that system (1.1) is approximately synchronizable by p-groups in the consensus sense. Then for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\varOmega))^N \times (H^{-1}(\varOmega))^N$, *there exists a sequence* $\{H_n\}$ *of boundary controls in* $(L^2(0,+\infty; L^2(\varGamma_1)))^M$ *with compact support in* $[0, T]$ *and some scalar functions* $u_1, \ldots, u_p$, *such that the convergence (1.9) holds true for the corresponding sequence* $\{U_n\}$ *of solutions to problem (1.1)−(1.2).*

Proof. By Theorem 4.1, the reduced system (1.22) is approximately null controllable. Since A satisfies the $\tilde{C}_{\tilde{p}}$-compatibility condition, by Lemma 1.7, there exist some scalar functions $\tilde{u}_1, \dots, \tilde{u}_{\tilde{p}}$ such that

$$U_n \to \sum_{r=1}^{\tilde{p}} \tilde{u}_r \tilde{e}_r \quad \text{in} \quad (C^0_{\text{loc}}([T, +\infty); L^2(\Omega)))^N \cap (C^1_{\text{loc}}([T, +\infty); H^{-1}(\Omega)))^N \tag{4.1}$$

as $n \to +\infty$, where $\text{Ker}(\tilde{C}_{\tilde{p}}) = \{\tilde{e}_1, \dots, \tilde{e}_{\tilde{p}}\}$. Moreover, noting $\text{Ker}(\tilde{C}_{\tilde{p}}) \subseteq Ker(C_p)$, we can write

$$\tilde{e}_r = \sum_{s=1}^{p} \alpha_{rs} e_s \qquad r = 1, \dots, \tilde{p}. \tag{4.2}$$

Then, setting

$$u_s = \sum_{r=1}^{\tilde{p}} \alpha_{rs} \tilde{u}_r \qquad s = 1, \dots, p \tag{4.3}$$

in (4.1), we get

$$U_n \to \sum_{s=1}^{p} u_s e_s \tag{4.4}$$

in $(C^0_{\text{loc}}([T, +\infty); L^2(\Omega)))^N \cap (C^1_{\text{loc}}([T, +\infty); H^{-1}(\Omega)))^N$ as $n \to +\infty$, which, noting (1.12), means exactly the pinning convergence (1.9). The proof is then complete.

Remark 4.5. From now on, we no longer distinguish the two (pinning and consensus) kinds of approximate boundary synchronizations.

Remark 4.6. When A doesn't satisfy the C_p-compatibility condition, we have $\tilde{p} < p$. So, it is different from the assertion (ii) in Theorem 3.2 that the functions $u_1, \dots, u_p$ in (4.4) are certainly linearly dependent.

Now let us go back to the proof of Theorem 4.1. We first generalize Definition 1.1 for weaker initial data.

Definition 4.7. Let $m \geq 0$ be an integer. System (1.1) is approximately null controllable in the space $(H^{-2m}(\Omega))^N \times (H^{-(2m+1)}(\Omega))^N$ at the time $T > 0$, if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H^{-2m}(\Omega))^N \times (H^{-(2m+1)}(\Omega))^N$, there exists a sequence $\{H_n\}$ of boundary controls in $(L^2(0, +\infty; L^2(\Gamma_1))^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1.1)–(1.2) satisfies

$$U_n \to 0 \quad \text{in} \quad C^0_{\text{loc}}([T, +\infty); H^{-2m}(\Omega))^N \cap C^1_{\text{loc}}([T, +\infty); H^{-(2m+1)}(\Omega))^N \tag{4.5}$$

as $n \to +\infty$.

Correspondingly, we give

Definition 4.8. Let $m \geq 0$ be an integer. The adjoint problem (1.4) is D-observable in the space $(H^{2m+1}_0(\Omega))^N \times (H^{2m}_0(\Omega))^N$ on the time interval $[0, T]$, if for $(\Phi_0, \Phi_1) \in (H^{2m+1}_0(\Omega))^N \times (H^{2m}_0(\Omega))^N$, the observation (1.5) implies $\Phi \equiv 0$.

Similarly to Lemma 1.3 for the case $m = 0$, we can establish the following

Proposition 4.9. *Let $m \geq 0$ be an integer. System (1.1) is approximately null controllable in the space $(H^{-2m}(\Omega))^N \times (H^{-(2m+1)}(\Omega))^N$ at the time $T > 0$ if and only if the adjoint problem (1.4) is D-observable in the space $(H^{2m+1}_0(\Omega))^N \times (H^{2m}_0(\Omega))^N$ on the time interval $[0, T]$.*

Proposition 4.10. *Let $m \geq 0$ be an integer. Then system (1.1) is approximately null controllable in the space $(H^{-2m}(\Omega))^N \times (H^{-(2m+1)}(\Omega))^N$ if and only if it is approximately null controllable in the space $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$.*

Proof. By Proposition 4.9, it is sufficient to show that the adjoint system (1.4) is D-observable in the space $(H_0^{(2m+1)}(\Omega))^N \times (H_0^{2m}(\Omega))^N$ if and only if it is D-observable in the space $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$.

Assume that the adjoint system (1.4) is D-observable in the space $(H_0^{(2m+1)}(\Omega))^N \times (H_0^{2m}(\Omega))^N$, then the following expression

$$\|(\Phi_0, \Phi_1)\|_F^2 = \int_0^T \int_{\Gamma_1} |D^T \partial_\nu \Phi|^2 \mathrm{d}\Gamma \mathrm{d}t$$

defines a Hilbert norm in the space $(H_0^{(2m+1)}(\Omega))^N \times (H_0^{2m}(\Omega))^N$. Let $\mathcal{F}$ be the closure of $(H_0^{(2m+1)}(\Omega))^N \times (H_0^{2m}(\Omega))^N$ with respect to the F-norm.

By the hidden regularity obtained in [15] for the adjoint system (1.4), we have

$$\int_0^T \int_{\Gamma_1} |\partial_\nu \Phi|^2 \mathrm{d}\Gamma \mathrm{d}t \leq c\|(\Phi_0, \Phi_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)},$$

then

$$(H_0^1(\Omega))^N \times (L^2(\Omega))^N \subseteq \mathcal{F}.$$

Since system (1.4) is D-observable in $\mathcal{F}$, it is still D-observable in its subspace $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$. The converse is trivial. The proof is complete.

Remark 4.11. Similar results on the exact boundary controllability can be found in [6].

Proposition 4.12. *Assume that system (1.1) is approximately synchronizable by p-groups in the consensus sense. Then, for any given integer $l \geq 0$ and any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, we have*

$$C_p A^l U_n \to 0 \tag{4.6}$$

in $(C^0_{\mathrm{loc}}([T, +\infty); H^{-2l}(\Omega)))^{N-p} \cap (C^1_{\mathrm{loc}}([T, +\infty); H^{-(2l+1)}(\Omega)))^{N-p}$ as $n \to +\infty$.

Proof. Noting that U_n satisfies the homogeneous system

$$\begin{cases} U_n'' - \Delta U_n + AU_n = 0 & \text{in} \quad (T, +\infty) \times \Omega, \\ U_n = 0 & \text{on} \quad (T, +\infty) \times \Gamma, \end{cases} \tag{4.7}$$

we have

$$\|BU_n''\|_{C^0_{\mathrm{loc}}([T,+\infty);H^{-2}(\Omega))} \sim \|BU_n\|_{C^0_{\mathrm{loc}}([T,+\infty);L^2(\Omega))} \tag{4.8}$$

for any given matrix B with constant entries.

Aplying $C_p A^{l-1}$ to system (4.7) and using (4.8), it follows that

$$\begin{aligned} &\|C_p A^l U_n\|_{(C^0_{\mathrm{loc}}([T,+\infty);H^{-2l}(\Omega)))^{N-p}} \\ \leq &\|C_p A^{l-1} U_n''\|_{(C^0_{\mathrm{loc}}([T,+\infty);H^{-2l}(\Omega))^{N-p}} \\ &+ \|\Delta C_p A^{l-1} U_n\|_{(C^0_{\mathrm{loc}}([T,+\infty);H^{-2l}(\Omega))^{N-p}} \\ \leq &c\|C_p A^{l-1} U_n\|_{(C^0_{\mathrm{loc}}([T,+\infty);H^{-2(l-1)}(\Omega))^{N-p}} \\ \leq &c^l \|C_p U_n\|_{(C^0_{\mathrm{loc}}([T,+\infty);L^2(\Omega))^{N-p}}, \end{aligned} \tag{4.9}$$

where $c > 0$ is a positive constant. Similar result can be shown for $C_p A^l U_n'$. The proof is complete.

Proof of Theorem 4.1.

(i) By definition of extension matrix $\tilde{C}_{\tilde{p}}$ given by (1.21) and the convergence (4.6) with $0 \leq l \leq N-1$, we get

$$\tilde{C}_{\tilde{p}} U_n \to 0 \tag{4.10}$$

in the space

$$(C^0_{\text{loc}}([T,+\infty); H^{-2(N-1)}(\Omega)))^{N-\tilde{p}} \cap (C^1_{\text{loc}}([T,+\infty); H^{-(2N-1)}(\Omega)))^{N-\tilde{p}} \tag{4.11}$$

as $n \to +\infty$ for any given initial data $(\widehat{U}_0, \widehat{U}_1)$ in the space

$$(L^2(\Omega))^N \times (H^{-1}(\Omega))^N. \tag{4.12}$$

(ii) Let $(\widehat{U}_0, \widehat{U}_1) \in (H^{-2(N-1)}(\Omega))^N \times (H^{-(2N-1)}(\Omega))^N$. By density, there exists a sequence $\{(U_0^m, U_1^m)\}_{m \geq 0}$ in $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, such that

$$(U_0^m, U_1^m) \to (\widehat{U}_0, \widehat{U}_1) \quad \text{in} \quad (H^{-2(N-1)}(\Omega))^N \times (H^{-(2N-1)}(\Omega))^N \tag{4.13}$$

as $m \to +\infty$. For each fixed m, there exists a sequence of boundary controls $\{H_n^m\}_{n \geq 0}$, such that the corresponding sequence $\{U_n^m\}_{n\geq 0}$ of solutions to problem (1.1)–(1.2) with the initial data (U_0^m, U_1^m) satisfies

$$\tilde{C}_{\tilde{p}} U_n^m \to 0 \tag{4.14}$$

in the space (4.11) as $n \to +\infty$.

(iii) Let $\mathcal{R}$ denote the resolution of problem (1.1)–(1.2):

$$\mathcal{R}: \quad (\widehat{U}_0, \widehat{U}_1; H_n) \to (U_n, U_n'), \tag{4.15}$$

which is continuous from $(H^{-2(N-1)}(\Omega))^N \times (H^{-(2N-1)}(\Omega))^N \times (L^2(0,T; L^2(\Gamma_1)))^M$ into $(C^0_{\text{loc}}([T,+\infty); H^{-2(N-1)}(\Omega)))^N \cap C^1_{\text{loc}}([T,+\infty); H^{-(2N-1)}(\Omega))\big)^N$.

Now, for any given $(\widehat{U}_0, \widehat{U}_1) \in (H^{-2(N-1)}(\Omega))^N \times (H^{-(2N-1)}(\Omega))^N$, we write

$$\mathcal{R}(\widehat{U}_0, \widehat{U}_1; H_n^m) = \mathcal{R}(U_0^m, U_1^m, H_n^m) + \mathcal{R}(\widehat{U}_0 - U_0^m, \widehat{U}_1 - U_1^m; 0). \tag{4.16}$$

By the well-posedness, we have

$$\|\mathcal{R}(\widehat{U}_0 - U_0^m, \widehat{U}_1 - U_1^m; 0)(t)\| \leq c_S \|(\widehat{U}_0 - U_0^m, \widehat{U}_1 - U_1^m)\|, \qquad 0 \leq t \leq T \leq S \tag{4.17}$$

with respect to the norm of $(H^{-2(N-1)}(\Omega))^N \times (H^{-(2N-1)}(\Omega))^N$, where c_S is a positive constant depending only on S. Then, noting (4.13) and (4.14), we can chose a diagonal subsequence $\{H_{n_k}^{m_k}\}_{k\geq 0}$ such that

$$\tilde{C}_{\tilde{p}} \mathcal{R}(\widehat{U}_0, \widehat{U}_1; H_{n_k}^{m_k}) \to 0 \tag{4.18}$$

in the space (4.11) as $k \to +\infty$. Hence, the reduced system (4.2) is approximately null controllable in the space $(H^{-2(N-1)}(\Omega))^{N-\tilde{p}} \times (H^{-(2N-1)}(\Omega))^{N-\tilde{p}}$, therefore, by Proposition 4.10, it is also approximately null controllable in the space $(L^2(\Omega))^{N-\tilde{p}} \times (H^{-1}(\Omega))^{N-\tilde{p}}$. The proof is complete.

5. Induced approximate boundary synchronization

In Section 3, we have established a quite complete theory on the approximate boundary synchronization by p-groups under the minimal rank condition (1.25). The objective of this section is to investigate the approximate boundary synchronization by p-groups in the case that

$$N_p = \inf_{D\in\mathbb{D}_p} \operatorname{rank}(D, AD, \dots, A^{N-1}D) > N - p, \tag{5.1}$$

or equivalently, by Corollary 3.8, in the case that either $\operatorname{Ker}(C_p)$ is not an invariant subspace of A, or (and) A^T does not admit any invariant subspace which is bi-orthonormal to $\operatorname{Ker}(C_p)$.

The key point of the study is to determine the minimal number N_p of total controls necessary to the approximate boundary synchronization by p-groups. The basic idea is to introduce the enlarged matrix C_q^* in Definition 1.16, such that A satisfies the strong C_q^*-compatibility condition. We can thus adapt back the steps in Section 3 by introducing the induced approximate boundary synchronization as in Definition 1.18.

Let $\tilde{C}_{\tilde{p}}$ be the extension matrix of C_p defined by (1.21) and $\tilde{C}_{\tilde{q}}^*$ be the enlarged matrix of $\tilde{C}_{\tilde{p}}$. Since $\operatorname{Im}(\tilde{C}_{\tilde{p}}^T)$ is the least invariant subspace of A^T, containing $\operatorname{Im}(C_p^T)$, it is easy to check that $C_q^* = \tilde{C}_{\tilde{q}}^*$. So, in what follows, without loss of generality, we always assume that A satisfies the C_p-compatibility condition.

Existence and uniqueness of the enlarged matrix C_q^*. Let $W_0 = \operatorname{Im}(C_p^T)$, which is an invariant subspace of A^T. Let W be the least subspace defined in Lemma 2.11. Then, setting the enlarged matrix C_q^* by

$$\operatorname{Im}(C_q^{*T}) = W, \tag{5.2}$$

we check easily that C_q^* verifies well all the requirements given in Definition 1.16.

Explicit construction of the enlarged matrix C_q^*. Because of the importance of C_q^* in the coming consideration, we give its explicit construction as follows.

Let $E_1^{(1)}, \dots, E_1^{(r)}$ be the eigenvectors of A^T contained in $\operatorname{Im}(C_p^T)$:

$$A^T E_1^{(j)} = \lambda_j E_1^{(j)}, \qquad j = 1, \dots, r.$$

For each $j = 1, \dots, r$, let $E_1^{(j)}, E_2^{(j)}, \dots, E_{m_j}^{(j)}$ be the Jordan chain associated with the jth eigenvector $E_1^{(j)}$:

$$E_0^{(j)} = 0, \qquad A^T E_i^{(j)} = \lambda_j E_i^{(j)} + E_{i-1}^{(j)}, \qquad i = 1, \dots, m_j.$$

Thus, we can define the matrix C_q^* of order $(N-q)\times N$ by

$$\operatorname{Im}(C_q^{*T}) = \operatorname{Span} \bigcup_{1\le j\le r} \left\{E_1^{(j)}, E_2^{(j)}, \dots, E_{m_j}^{(j)}\right\} \tag{5.3}$$

with

$$q = N - \sum_{j=1}^{r} m_j. \tag{5.4}$$

Now we give the following basic result.

Proposition 5.1. *Assume that A satisfies the C_p-compatibility condition. Let D satisfy the rank condition (1.26). Then, we necessarily have the rank condition (1.28).*

Proof. By the assertion (i) of Lemma 2.5 with $d = p$, it is sufficient to show that the dimension of any invariant subspace W of A^T, contained in $\operatorname{Ker}(D^T)$, does not exceed q.

Since A satisfies the C_p-compatibility condition, there exists a reduced matrix $\overline{A}_p$ of order $(N-p)$, such that $CA = \overline{A}_p C$. By Lemma 2.4, the rank condition (1.26) is equivalent to

$$\operatorname{rank}\left(C_p D, \overline{A}_p C_p D, \dots, \overline{A}_p^{N-p-1} C_p D\right) = N - p,$$

which, by the assertion (ii) of Lemma 2.5 with $d = p$, implies that $\operatorname{Ker}(C_p D)^T$ does not contain any non-trivial invariant space of $\overline{A}_p^T$.

Now let W be any given invariant subspace of A^T, contained in $\operatorname{Im}(C_p^T)$. The projected subspace

$$\overline{W} = (C_p C_p^T)^{-1} C_p W = \{\overline{x} : \quad C_p^T \overline{x} = x, \quad \forall x \in W\}$$

is an invariant subspace of $\overline{A}_p^T$. In particular, we have

$$\overline{W} \cap \operatorname{Ker}(C_p D)^T = 0.$$

For any given $x \in W$, there exists $\overline{x} \in \overline{W}$, such that $x = C^T \overline{x}$, then we have

$$D^T x = D^T C_p^T \overline{x} = (C_p D)^T \overline{x}.$$

Thus we have

$$W \cap \operatorname{Ker}(D^T) = \{0\} \tag{5.5}$$

for any given invariant subspace W of A^T, contained in $\operatorname{Im}(C_p^T)$.

Now let W^* be an invariant subspace of A^T, contained in $\operatorname{Im}(C_q^{*T}) \cap \operatorname{Ker}(D^T)$. Since $W^* \cap \operatorname{Im}(C_p^T)$ is also an invariant subspace of A^T, contained in $\operatorname{Im}(C_p^T) \cap \operatorname{Ker}(D^T)$, because of (5.5), we have $W^* \cap \operatorname{Im}(C_p^T) = \{0\}$. Then, it follows that

$$W^* \subseteq \operatorname{Im}(C_q^{*T}) \setminus \operatorname{Im}(C_p^T). \tag{5.6}$$

Since $\operatorname{Im}(C_q^{*T}) \setminus \operatorname{Im}(C_p^T)$ does not contain any eigenvector of A^T, it follows that

$$W^* = \{0\}. \tag{5.7}$$

Finally, let W be an invariant subspace of A^T, contained in $\operatorname{Ker}(D^T)$. Since $W \cap \operatorname{Im}(C_q^{*T})$ is an invariant subspace of A^T, contained in $\operatorname{Im}(C_q^{*T}) \cap \operatorname{Ker}(D^T)$, it follows from (5.6) that

$$W \cap \operatorname{Im}(C_q^{*T}) = \{0\}. \tag{5.8}$$

Then, we get

$$\dim \operatorname{Im}(C_q^{*T}) + \dim(W) = N - q + \dim\ (W) \leq N,$$

which implies that $\dim(W) \leq q$. This achieves the proof.

Theorem 5.2. *Assume that system (1.1) is approximately synchronizable by p-groups. Then we necessarily have the rank condition (1.28).*

Proof. Without loss of generality, we may assume that A satisfies the C_p-compatibility condition (1.14). Then the approximate boundary synchronization by p-groups of system (1.1) is equivalent to the approximate boundary null controllability of the reduced system (1.16). By Lemma 1.4, we have

$$\operatorname{rank}\left(C_p D, \overline{A}_p C_p D, \dots, \overline{A}_p^{N-p-1} C_p D\right) = N - p,$$

which, by means of Lemma 2.4, implies (1.26):

$$\operatorname{rank}\left(C_p D, C_p A D, \dots, C_p A^{N-1} D\right) = N - p.$$

Then, applying Proposition 5.1, we get (1.28). The proof is complete.

Proposition 5.3. *Assume that system* (1.1) *is approximately synchronizable by p-groups under the minimal rank condition*:

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) = N - q, \tag{5.9}$$

then, the rank condition (1.31) *holds true.*

Proof. The rank condition (5.9) implies the existence of an invariant subspace W of A^T, contained in $\operatorname{Ker}(D^T)$ and with dimension q. On the other hand, since the reduced system (1.16) is approximately null controllable, by Lemmas 1.4 and 2.4, the rank condition (1.26) holds. So, the relation (5.8) remains still true. Then, since

$$\dim \operatorname{Im}(C_q^{*T}) + \dim(W) = (N - q) + q = N,$$

W is a supplement of $\operatorname{Im}(C_q^{*T})$.

Furthermore, from the definition of W, we have

$$W \subseteq \operatorname{Ker}(D, AD, \dots, A^{N-1}D)^T,$$

or equivalently,

$$\operatorname{Im}(D, AD, \dots, A^{N-1}D) \subseteq W^\perp,$$

which together with the rank condition (5.9) imply

$$\operatorname{Im}(D, AD, \dots, A^{N-1}D) = W^\perp.$$

But $W^\perp$ is a supplement of $\operatorname{Ker}(C_q^*)$, then

$$\operatorname{Im}(D, AD, \dots, A^{N-1}D) \cap \operatorname{Ker}(C_q^*) = \{0\}.$$

By Lemma 2.3, we get

$$\operatorname{rank}(C_q^* D, C_q^* AD, \dots, C_q^* A^{N-1}D) = \operatorname{rank}(D, AD, \dots, A^{N-1}D) = N - q.$$

This achieves the proof.

Remark 5.4. The rank condition (1.31) is only necessary but not sufficient in general to the approximate boundary null controllability of the reduced system (1.30), therefore, to the induced approximate boundary synchronization of the original system (1.1). In order to guarantee the induced approximate boundary synchronization, we have to employ a control matrix D with stronger rank.

Theorem 5.5. *There exists a boundary control matrix D with the minimal rank condition* (5.9)*, such that the corresponding system* (1.1) *is induced approximately synchronizable by the enlarged matrix C_q^*.*

Proof. Since the induced approximate boundary synchronization of system (1.1) is equivalent to the approximate boundary null controllability of the reduced system (1.30), then by Lemma 1.3, equivalent to the C_q^*D-observability of its adjoint problem

$$\begin{cases} \Psi'' - \Delta\Psi + A_q^{*T}\Psi = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \Psi = 0 & \text{on} \quad (0, +\infty) \times \Gamma, \\ t = 0: \quad \Psi = \Psi_0, \quad \Psi' = \Psi_1 & \text{in} \quad \Omega. \end{cases} \tag{5.10}$$

Let W be a subspace which is invariant for A^T and biorthonormal to $\operatorname{Ker}(C_q^*)$. Clearly, W and $\operatorname{Ker}(C_q^*)$ have the same dimension q. Setting

$$\operatorname{Ker}(D^T) = W, \tag{5.11}$$

we see that W is the largest invariant subspace of A^T, contained in $\mathrm{Ker}(D^T)$. By the assertion (ii) of Lemma 2.5 with $d = q$, we have the rank condition (5.9).

On the other hand, since W is biorthonormal to $\mathrm{Ker}(C_q^*)$, by Lemma 2.8 we have

$$\mathrm{Im}(D) \cap \mathrm{Ker}(C_q^*) = W^\perp \cap \mathrm{Ker}(C_q^*) = \{0\}.$$

Then by Lemma 2.3 we have

$$\mathrm{rank}(C_q^* D) = \mathrm{rank}(D) = N - q.$$

Therefore the $C_q^* D$-observation becomes the complete observation

$$\partial_\nu \Psi \equiv 0 \quad \text{on} \quad [0, T] \times \Gamma_1, \tag{5.12}$$

which, because of Holmgren's uniqueness Theorem (see [24]), implies the $C_q^* D$-observability of the adjoint problem (5.10). The proof is then complete.

Remark 5.6. Theorem 5.5 means that system (1.1) can be induced approximately synchronizable by means of a matrix D of rank $(N-q)$. But, we prefer to use a control matrix D which has the minimal rank (see Appendix for the construction and discussions).

As a direct consequence of Theorems 5.2 and 5.5, we have

Corollary 5.7. *We have*

$$N_p = N - q. \tag{5.13}$$

Theorem 5.8. *Assume that system (1.1) is induced approximately synchronizable by the enlarged matrix C_q^* under the minimal rank condition (5.9). Then there exist some linearly independent functions $u_1^*, u_2^*, \ldots, u_q^*$, which are independent of the applied boundary controls, such that the corresponding solutions U_n to problem (1.1)–(1.2) satisfies*

$$U_n \to \sum_{r=1}^{q} u_r^* e_r^* \quad \textit{in} \quad (C^0_{\mathrm{loc}}([T', +\infty); L^2(\Omega)))^N \cap (C^1_{\mathrm{loc}}([T', +\infty); H^{-1}(\Omega)))^N \tag{5.14}$$

as $n \to +\infty$, where $T' > T$ and $\mathrm{Ker}(C_q^*) = \mathrm{Span}\{e_1^*, \ldots, e_q^*\}$.

Proof. Since the proof is quite similar to that of Theorem 3.2, we only give a brief sketch of it.

First, by the assertion (ii) of Lemma 2.5 with $d = q$, the rank condition (5.9) guarantees the existence of an invariant subspace W of A^T, with dimension q and contained in $\mathrm{Ker}(D^T)$. Let $\{E_1, \ldots, E_q\}$ be a basis of W, such that

$$A^T E_r = \sum_{s=1}^{q} \alpha_{rs} E_s, \qquad D^T E_r = 0, \qquad 1 \le r \le q.$$

Applying E_r to problem (1.1)–(1.2) with $U = U_n$ and $H = H_n$, and setting $\phi_r = (E_r, U_n)$ for $r = 1, \ldots, q$, we get

$$\begin{cases} \phi_r'' - \Delta \phi_r + \sum_{s=1}^{q} \alpha_{rs} \phi_s = 0 & \text{in} \quad (0, +\infty) \times \Omega, \\ \phi_r = 0 & \text{on} \quad (0, +\infty) \times \Gamma, \\ t = 0: \qquad \phi_r = (E_r, \widehat{U}_0), \quad \phi_r' = (E_r, \widehat{U}_1) & \text{in} \quad \Omega. \end{cases} \tag{5.15}$$

Clearly, $\phi_r (r = 1, \ldots, q)$ are independent of the applied boundary controls.

On the other hand, since system (1.1) is approximately synchronizable by p-groups, the reduced system (1.16) is approximately null controllable. Then, by Lemmas 1.4 and 2.4, the rank condition (1.26) holds true, so, (5.8) remains still true. Then we can write

$$U_n = \begin{pmatrix} C_q^* \\ E_1^T \\ \vdots \\ E_q^T \end{pmatrix}^{-1} \begin{pmatrix} C_q^* U_n \\ (E_1, U_n) \\ \vdots \\ (E_q, U_n) \end{pmatrix} \to \begin{pmatrix} C_q^* \\ E_1^T \\ \vdots \\ E_q^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \phi_1 \\ \vdots \\ \phi_q \end{pmatrix} =: U$$

in $(C^0_{\text{loc}}([T,+\infty);(L^2(\Omega)))^N \cap (C^1_{\text{loc}}([T,+\infty);(H^{-1}(\Omega)))^N$ as $n \to +\infty$. Moreover, noting $C_q^* U = 0$ for $t \geq T$, there exist some linearly independent functions $u_1^*, u_2^*, \dots, u_q^*$ which are independent of the applied boundary controls (because the homogeneous problem (5.15) is time invertible), such that

$$U = \sum_{r=1}^{q} u_r^* e_r^* \qquad t > T'.$$

The proof is then complete.

Remark 5.9. Since $\text{Ker}(C_q^*) \subseteq \text{Ker}(C_p)$, we can write

$$e_r^* = \sum_{s=1}^{p} \alpha_{rs} e_s \qquad r = 1, \dots, q.$$

Then, setting

$$u_s = \sum_{r=1}^{q} \alpha_{rs} u_r^* \qquad s = 1, \dots, p \tag{5.16}$$

in (5.14), we get

$$U_n \to \sum_{s=1}^{p} u_s e_s \tag{5.17}$$

in $(C^0_{\text{loc}}([T,+\infty);L^2(\Omega)))^N \cap (C^1_{\text{loc}}([T,+\infty);H^{-1}(\Omega)))^N$ as $n \to +\infty$. Noting that $p > q$ in (5.16), there exist some constants coefficients $\beta_{rs}(r = 1, \dots, p-q; s = 1, \dots, p)$ such that

$$\sum_{s=1}^{p} \beta_{rs} u_s = 0, \qquad r = 1, \dots, p-q. \tag{5.18}$$

This is just what we called the additional properties, which depend on the structure of $\text{Ker}(C_q^*)$.

Finally, we illustrate the additional properties hidden in the induced approximate boundary synchronization by the following two examples.

Example 5.10. Let $N = 4, M = 1, p = 2$,

$$A = \begin{pmatrix} 2 & -2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$C_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{with} \quad e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

First, it is easy to see that A satisfies the C_2-compatibility condition, the corresponding reduced matrix

$$\overline{A}_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

is similar to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of cascade type.

In order to determine the minimal number of total controls necessary to the approximate boundary synchronization by 2-groups, we exhibit the system of root vectors of the matrix A:

$$E_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \qquad E_1^{(2)} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad E_2^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad E_3^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Since $E_1^{(1)} \notin \mathrm{Im}(C_2^T)$ and $E_1^{(2)} \in \mathrm{Im}(C_2^T)$, the enlarged matrix C_1^* given by Definition 1.18 can be chosen as (see (5.3) for the construction):

$$C_1^* = C_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

This, by Theorem 5.2, justifies well that we have to use 3 (instead of 2 !) total controls to realize the approximate boundary synchronization by 2-groups.

All the one-column matrices D satisfying the following rank conditions:

$$\mathrm{rank}(C_2D, \overline{A}_2C_2D) = 2 \qquad \text{and} \quad \mathrm{rank}(D, AD, A^2D, A^3D) = 3$$

are given either by

$$D = \begin{pmatrix} \alpha+\beta \\ \alpha \\ \beta \\ 1 \end{pmatrix}, \qquad \forall \alpha, \beta \in \mathbb{R},$$

or by

$$D = \begin{pmatrix} \gamma \\ \alpha \\ \beta \\ 0 \end{pmatrix}, \qquad \forall \alpha, \beta, \gamma \in \mathbb{R} \text{ such that } \gamma \neq \alpha+\beta.$$

Since the above one-column matrix D provides only 3 total controls, by Lemma 1.4, system (1.1) is not approximately null controllable. However, the corresponding Kalman's criterion $\mathrm{rank}(C_2D, \overline{A}_2C_2D) = 2$ is sufficient (see Thm. 3.6 in [18]) for the approximate boundary null controllability of the corresponding reduced system (1.16) of cascade type. Noting that A satisfies the C_2-compatibility condition, by Lemma 1.6, the original system (1.1) is approximately synchronizable by 2-groups in the pinning sense. Thus, there exist scalar functions u and v such that

$$u_n^{(1)} \to u, \quad u_n^{(2)} \to u \quad \text{and} \quad u_n^{(3)} \to v, \quad u_n^{(4)} \to v$$

in $C^0_{\mathrm{loc}}([T, +\infty); L^2(\Omega)) \cap C^1_{\mathrm{loc}}([T, +\infty); H^{-1}(\Omega))$ as $n \to +\infty$.

On the other hand, the reduced matrix A_1^* by $C_1^*(= C_1)$ is given by

$$A_1^* = C_1^*AC_1^{*T}(C_1^*C_1^{*T})^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

which is similar to the matrix of cascasde type

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Moreover, by Proposition 5.3, we necessarily have Kalman's criterion

$$\operatorname{rank}(C_1^* D, C_1^* AD, C_1^* A^2 D, C_1^* A^3 D) = \operatorname{rank}(C_1^* D, A_1^* C_1^* D, {A_1^*}^2 C_1^* D) = 3,$$

which is sufficient (see Thm. 3.6 in [18]) for the approximate boundary null controllability of the corresponding reduced system (1.30) of cascade type, then, sufficient for the induced approximate boundary synchronization of the original system (1.1). Moreover, noting that

$$\operatorname{Ker}(C_1^*) = \operatorname{Span}\{e_1^*\} \quad \text{with} \quad e_1^* = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

by Theorem 5.8, there exists a scalar function $u^* \not\equiv 0$, independent of employed boundary controls, such that

$$u_n^{(1)} \to u^*, \quad u_n^{(2)} \to u^*, \quad u_n^{(3)} \to u^*, \quad u_n^{(4)} \to u^*$$

in $C^0_{\mathrm{loc}}([T', +\infty); L^2(\Omega)) \cap C^1_{\mathrm{loc}}([T', +\infty); H^{-1}(\Omega))$ as $n \to +\infty$, where $T' > T$.

Example 5.11. Let $N = 4, M = 1, p = 2$,

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

and

$$C_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{with} \quad e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

First, the rank of Kalman's matrix

$$(D, AD, A^2 D, A^3 D) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ -1 & 0 & 4 & 0 \end{pmatrix}$$

is equal to 3, then by Lemma 1.4, system (1.1) is not approximately null controllable under the action of the control matrix D.

Next, it is easy to see that the coupling matrix A satisfies the C_2-compatibility condition (1.14) with the reduced matrix

$$\overline{A}_2 = C_2 A C_2^T (C_2 C_2^T)^{-1} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

and

$$(C_2 D, \overline{A}_2 C_2 D) = \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}.$$

The reduced matrix $\overline{A}_2$ is of cascade type and then the corresponding Kalman's criterion $\text{rank}(C_2D, \overline{A}_2C_2D) = 2$ is sufficient (see Thm. 3.6 in [18]) for the approximate boundary null controllability of the corresponding reduced system (1.16). Since A satisfies the C_2-compatibility condition, by Lemma 1.6, the original system (1.1) is approximately synchronizable by 2-groups in the pinning sense. Thus, there exists some scalar functions u and v such that

$$u_n^{(1)} \to u, \quad u_n^{(2)} \to u \quad \text{and} \quad u_n^{(3)} \to v, \quad u_n^{(4)} \to v$$

in $C^0_{\text{loc}}([T, +\infty); L^2(\Omega)) \cap C^1_{\text{loc}}([T, +\infty); H^{-1}(\Omega))$ as $n \to +\infty$.

Now, we exhibit the system of root vectors of the matrix A^T:

$$E_1^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \qquad E_1^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \end{pmatrix}, \quad E_2^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad E_3^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $E_1^{(1)} \notin \text{Im}(C_2^T)$ and $E_2^{(1)} \in \text{Im}(C_2^T)$, then the extension matrix C_1^* given by Definition 1.18 can be chosen as (see (5.3) for the construction):

$$C_1^* = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The reduced matrix A_1^* by C_1^*, given by

$$A_1^* = C_1^* A C_1^{*T} (C_1^* C_1^{*T})^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is of lower-cascade type. By Proposition 5.3, we have

$$\text{rank}(C_1^*D, C_1^*AD, C_1^*A^2D, C_1^*A^3D) = \text{rank}(C_1^*D, A_1^*C_1^*D, {A_1^*}^2C_1^*D) = 3.$$

Since Kalman's criteria is sufficient for the approximate boundary null controllability for the corresponding reduced system (1.30), the original system (1.1) is induced approximately synchronizable. Moreover, noting that

$$\text{Ker}(C_1^*) = \text{Span}\{e_1^*\} \quad \text{with} \quad e_1^* = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

by Theorem 5.8, there exists a scalar function $u^* \not\equiv 0$, independent of employed boundary controls, such that

$$u_n^{(1)} \to u^*, \quad u_n^{(2)} \to u^* \quad \text{and} \quad u_n^{(3)} \to 0, \quad u_n^{(4)} \to 0$$

in $C^0_{\text{loc}}([T', +\infty); L^2(\Omega)) \cap C^1_{\text{loc}}([T', +\infty); H^{-1}(\Omega))$ as $n \to +\infty$, where $T' > T$.

Remark 5.12. The nature of the induced approximate boundary synchronization is determined by the structure of $\text{Ker}(C_q^*)$. In Example 5.10, because $C_1^* = C_1$, the induced approximate synchronization becomes the approximate boundary synchronization. This is purely a matter of chance. In fact, in Example 5.11, because $C_1^* \neq C_1$, the induced approximate synchronization implies the approximate boundary synchronization for the first group and the approximate boundary null controllability for the second one. Other examples could be constructed to illustrate more complicated situations.

6. Appendix

The following result gives the minimal number of direct boundary controls necessary to the approximate boundary null controllability.

Proposition 6.1. *Assume that the rank condition (1.6) holds. Then we have the following sharp lower bound estimate*

$$\text{rank}(D) \geq \mu, \tag{6.1}$$

where

$$\mu = \max_{\lambda \in Sp(A)} \textit{dim Ker}(A - \lambda I)$$

is called the largest geometrical multiplicity of the eigenvalues of A.

Proof. Let λ be an eigenvalue of A^T, with the largest geometrical multiplicity μ, and V_λ be the subspace composed of all the eigenvectors associated with the eigenvalue λ. By the assertion (ii) of Lemma 2.5 with $d = 0$, the rank condition (1.6) implies that there does not exist any non-trivial invariant subspace of A^T, contained in $\text{Ker}(D^T)$. Therefore, we have

$$\dim \text{Ker}(D^T) + \dim (V_\lambda) \leq N,$$

which yields the lower bound estimate (6.1).

We next show the sharpness of (6.1). Let $\lambda_1, \dots, \lambda_d$ be the distinct eigenvalues of A^T, associated with the corresponding eigenvectors:

$$A^T x_r^{(k)} = \lambda_r x_r^{(k)}, \qquad r = 1, \dots, d, \quad k = 1, \dots, \mu_r.$$

Without loss of generality, we may assume that

$$(x_r^{(k)}, x_r^{(l)}) = \delta_{kl}, \quad 1 \leq r \leq d, \quad 1 \leq k, l \leq \mu_r. \tag{6.2}$$

Let

$$D = \left(x_1^{(1)}, \dots, x_1^{(\mu_1)}, \dots, x_d^{(1)}, \dots, x_d^{(\mu_d)}\right) \overline{D}, \tag{6.3}$$

where $\overline{D}$ is a $(\mu_1 + \ldots + \mu_d) \times \mu$ matrix defined by

$$\overline{D} = \begin{pmatrix} I_{\mu_1} & 0 \\ I_{\mu_2} & 0 \\ \vdots & \vdots \\ I_{\mu_d} & 0 \end{pmatrix}, \tag{6.4}$$

in which, if $\mu_r = \mu$, then the rth zero submatrix will disappear.

Now let

$$x_r = \sum_{k=1}^{\mu_r} \alpha_k x_r^{(k)} \quad \text{with} \quad (\alpha_1, \dots, \alpha_{\mu_r}) \neq 0$$

be an eigenvector corresponding to the eigenvalue λ_r. By (6.2)–(6.4), we have

$$x_r^T D = (\dots, \alpha_1, \alpha_2, \dots, \alpha_{\mu_r}, \dots) \neq 0.$$

Thus, $\text{Ker}(D^T)$ does not contain any eigenvector of $\overline{A}^T$, therefore, any non-trivial invariant subspace of A^T either. The proof is complete.

Remark 6.2. A "good" coupling matrix A should have the distinct eigenvalues, or at least the geometrical multiplicity of eigenvalues should be as small as possible. For example, if all the eigenvalues of A are simple: $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ with the corresponding eigenvectors $x_1, x_2, \ldots, x_N$. The rank-one matrix D can be taken as (see (6.3)−(6.4)):

$$D = \sum_{i=1}^{N} x_i.$$

More generally, we can construct a matrix D with the minimal rank and satisfying the rank conditions (1.31) and (5.9) necessary to the the induced approximate boundary synchronization.

Noting that the coupling matrix A always satisfies the C_q^*-compatibility condition, by Lemma 2.4, the rank condition (1.31) is equivalent to

$$\operatorname{rank}(C_q^* D, A_q^* C_q^* D, \ldots, A_q^{*N-q-1} C_q^* D) = N - q. \tag{6.5}$$

Noting that A_q^* is of order $(N-q)$, by the sharpness of the lower bound (6.1) in Proposition 6.1, there exists a matrix D^* of order $(N-q) \times M$ with the minimal rank μ^*, μ^* being the largest geometrical multiplicity of the eigenvalues of A_q^*, such that

$$\operatorname{rank}(D^*, A_q^* D^*, \ldots, (A_q^*)^{N-q-1} D^*) = N - q, \tag{6.6}$$

On the other hand, A^T admits an invariant subspace W which is biorthonormal to $\operatorname{Ker}(C_q^*)$. Then $\{\operatorname{Ker}(C_q^*)\}^{\perp} = \operatorname{Im}(C_q^{*T})$ is an invariant subspace of A^T, and $W^{\perp} = \{e_{q+1}, \ldots, e_N\}$ is an invariant subspace of A and biorthonormal to $\operatorname{Im}(C_q^{*T})$, namely, we have

$$C_q^*(e_{q+1}, \ldots, e_N) = I_{N-q}.$$

Then, from (6.2) we get that

$$D = (e_{q+1}, \ldots, e_N) D^* \tag{6.7}$$

satisfies (6.5), therefore (1.31) holds.

Finally, since $\operatorname{Im}(D) \in \operatorname{Span}\{e_{q+1}, \ldots, e_N\}$, which is invariant for A, so

$$\operatorname{Im}(A^k D) \subseteq \operatorname{Span}\{e_{q+1}, \ldots, e_N\}, \qquad \forall k \geq 0.$$

Noting that $\operatorname{Span}\{e_{q+1}, \ldots, e_N\}$ is a supplement of $\operatorname{Ker}(C_q^*)$, it follows that

$$\operatorname{Ker}(C_q^*) \cap \operatorname{Im}(D, AD, \ldots, A^{N-1} D) = \{0\},$$

which, thanks to Lemma 2.3, implies the equality (5.9) and the sharpness of the rank of D:

$$\operatorname{rank}(C_q^* D) = \operatorname{rank}(D) = \mu^*.$$

Acknowledgements. The authors would like to thank the referees for their very valuable comments and remarks.

References

[1] F. Alabau−Boussouira, A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems. *SIAM J. Control Optimiz.* **42** (2003) 871–903.

[2] F. Alabau−Boussouira, A hierarchic multi-level energy method for the control of bidiagonal and mixed n-coupled cascade systems of PDE's by a reduced number of controls. *Adv. Diff. Equ.* **18** (2013) 1005–1072.

[3] F. Ammar Khodja, A. Benabdallah and C. Dupaix, Null-controllability of some reaction-diffusion systems with one control force. *J. Math. Anal. Appl.* **320** (2006) 928-944.

[4] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optimiz.* **30** (1992) 1024-1063.

[5] B. Dehman; J. Le Rousseau and M. Léautaud, Controllability of two coupled wave equations on a compact manifold. *Arch. Ration. Mech. Anal.* **211** (2014) 113–187.

[6] S. Ervedoza and E. Zuazua, *A systematic method for building smooth controls for smooth data. Discrete Contin. Dyn. Syst. Ser. B* **14** (2010) 1375–1401.

[7] E. Fernéndez−Cara, M. González−Burgos and L. de Teresa, boundary controllability of parabolic coupled equations. *J. Func Anal.* **259** (2010) 1720–1758

[8] M.L.J. Hautus, Controllability and observability conditions for linear autonomous systems *Indag. Math.*)*N.S.*) **31** (1969) 443-445.

[9] Ch. Huygens, Oeuvres Complètes, Vol.15, Swets & Zeitlinger B.V., Amsterdam (1964).

[10] R.E. Kalman, *Contributions to the theory of optimal control. Bol. Soc. Mat. Mexicana* **5** (1960) 102–119.

[11] T.-T. Li, X. Lu and B. Rao, Approximate controllability and asymptotic synchronization for a coupled system of wave equations with Neumann boundary controls. Contemporary Computational Mathematics – a celebration of the 80th Birthday of Ian Sloan (Edited by Josef Dick, Frances Y. Kuo, Henryk Wozniakowski), Springer-Verlag, 2018.

[12] T.-T. Li and B. Rao, Asymptotic controllability for linear hyperbolic systems. *Asymptotic Anal.* **72** (2011) 169–185.

[13] T.-T. Li and B. Rao, Exact synchronization for a coupled system of wave equations with Dirichlet boundary controls. *Chin. Ann. Math.* **34B** (2013) 139–160; Partial differential equations: Theory, Control and Approximation, 295–321, Springer, Dordrecht (2013).

[14] T.-T. Li and B. Rao, Asymptotic controllability and asymptotic synchronization for a coupled system of wave equations with Dirichlet boundary controls. *Asymp. Anal.* **86** (2014) 199–224.

[15] T.-T. Li and B. Rao, A note on the exact synchronization by groups for a coupled system of wave equations. *Math. Methods Appl. Sci.* **38** (2015) 241–243.

[16] T.-T. Li and B. Rao, Critères du type de Kalman pour la contrôlabilité approchée et la synchronisation approchée d'un système couplé d'équations des ondes. *C. R. Math. Acad. Sci.* Paris **353** (2015) 63–68.

[17] T.-T. Li and B. Rao, Exact synchronization by groups for a coupled system of wave equations with Dirichlet boundary controls. *J. Math. Pures Appl.* **105** (2016) 86–101.

[18] T.-T. Li and B. Rao, Kalman-type criteria for the approximate controllability and approximate synchronization of a coupled system of wave equations. *SIAM J. Control Optimiz.* **54** (2016) 49–72.

[19] T.-T. Li and B. Rao, Une nouvelle approche pour la synchronizaation approchée d'un système couplé d'équations des ondes: contrôles directs et indirects. *C.R. Math. Acad. Sci.* Paris **354** (2016) 1006–1012.

[20] T.-T. Li and B. Rao, *Exact boundary controllability for a coupled system of wave equations with Neumann boundary controls*, to appear in Chin. Ann. Math.

[21] T.-T. Li, B. Rao and Y. Wei, Generalized exact boundary synchronization for a coupled system of wave equations. *Discrete Contin. Dyn. Syst.* **34** (2014) 2893–2905.

[22] J.-L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués*, Vol. 1, Masson, Paris (1985).

[23] Z. Liu and B. Rao, A spectral approach to the indirect boundary control of a system of weakly coupled wave equations. *Discrete Contin. Dyn. Syst.* **23** (2009) 399–413.

[24] Q. Lu and E. Zuazua, Averaged controllability for random evolution partial differential equations. *J. Math. Pures Appl.* **105** (2016) 367–414.

[25] L. Rosier and L. de Teresa, Exact controllability of a cascade system of conservative equations. *C.R. Math. Acad. Sci.* Paris **349** (2011) 291–295.

[26] D.L. Russell, Controllability and stabilization theory for linear partial differential equations: Recent progress and open questions. *SIAM Rev.* **20** (1978) 639–739.

[27] E. Zuazua, Averaged control. *Automatica J. IFAC* **50** (2014) 3077–3087.