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OCCUPATIONAL MEASURES AND AVERAGED SHAPE OPTIMIZATION

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Abstract. We consider the minimization of averaged shape optimization problems over the class of sets of finite perimeter. We use occupational measures, which are probability measures defined in terms of the reduced boundary of sets of finite perimeter, that allow to transform the minimization into a linear problem on a set of measures. The averaged nature of the problem allows the optimal value to be approximated with sets with unbounded perimeter. In this case, we show that we can also approximate the optimal value with convex polytopes with $n+1$ faces shrinking to a point. We derive conditions under which we show the existence of minimizers and we also analyze the appropriate spaces in which to study the problem.

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1. Introduction

In this paper we study averaged shape optimization problems of the type

$$\inf_{E\subset\overline{\Omega}} \frac{1}{\mathcal{H}^{n-1}(\partial^* E)} \int_{\partial^* E} f\left(x, \boldsymbol{\nu}_E\left(x\right)\right) \mathrm{d}\mathcal{H}^{n-1}\left(x\right), \tag{1.1}$$

where the sets E are considered to be of finite perimeter with interior normal vector $\boldsymbol{\nu}_E$. This problem includes, for example, the minimization of the averaged flux of a physical quantity in the case when $f(x,\boldsymbol{\nu}_E) = \boldsymbol{F}(x)\cdot\boldsymbol{\nu}_E$. Throughout the paper, unless otherwise specified, we assume that $f \in C(\bar{\Omega}\times\mathbb{S}^{n-1})$ and Ω is an open bounded set with Lipschitz boundary. In the more general setting, when f depends on both x and $\boldsymbol{\nu}_E$, the optimal value for (1.1) need not be attained by a set $E\subset\bar{\Omega}$. Moreover, the averaged feature of the problem allows the situation where the optimal value could be approximated by a sequence of sets with perimeter increasing to infinity. We show (see Thm. 4.14) that in this case the value can also be approximated with a sequence of convex polytopes Δ_i with $n+1$ faces, shrinking to a point $x_0\in\overline{\Omega}$, in the sense that $\lim_{i\to\infty}\sup_{y\in\Delta_i}|y-x_0| = 0$. Therefore, the infimum value can always be approximated with a sequence of sets having uniformly bounded perimeter.

Our main approximation result is Theorem 4.14 for the general case when f depends on both x and $\boldsymbol{\nu}_E$. For the special case where f depends only on the normal $\boldsymbol{\nu}_E$, we show that the optimal value can always

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be approximated by convex polytopes Δ_i with $n+1$ faces shrinking to a point $x_0 \in \overline{\Omega}$ (see Cor. 4.15). For the case of space-dependent costs $f(x,v)=f(x)$, we show that if the infimum is not attained then it can be approximated by any sequence of sets E_i shrinking to a point $x_0 \in \bar{\Omega}$ (see Thm. 5.3).

Our results rely on the analysis of occupational measures, which are probability measures defined in terms of the reduced boundary of sets of finite perimeter. Occupational measures appear in the study of stochastic processes, and also in the context of optimization in the study of infinite horizon optimal control (see Finlay-Gaitsgory−Lebedev [20], Artstein−Bright [6], Gaitsgory−Quincampoix [21] and the references therein). The benefit of the use of these measures is in turning the optimization problem (1.1) in to a linear problem on the set of measures.

A key component in our results is an estimate of the integral of the normal over the boundary of a set of finite perimeter (see Bright-Torres [8]). An application of the Gauss−Green Theorem shows that the integral over the reduced boundary of any set of finite perimeter $E \subset \mathbb{R}^n$ of the normal vector field is the zero vector namely, $\int_{\partial^* E} \boldsymbol{\nu}_E(x) d\mathcal{H}^{n-1}(x) = \mathbf{0}$. With this observation, we obtained in [8] estimates of the integral of the normal over the boundary of a set of finite perimeter (see Thm. 2.14). The bound in Theorem 2.14 extends a previous bound by Bright−Lee [7] from the smooth to non-smooth settings. We used this bound in [8] to study the limit of sets with perimeter growing to infinity (see Thm. 2.15). With these results at hand, we study in this paper the averaged shape optimization problem (1.1).

The analysis for (1.1) also holds for the perturbed problem

$$\inf_{E\subset\overline{\Omega}} V(E), \quad V(E) = \frac{1}{\mathcal{H}^{n-1}(\partial^* E)} \left[\int_{\partial^* E} f\left(x, \boldsymbol{\nu}_E\left(x\right)\right) \mathrm{d}\mathcal{H}^{n-1}\left(x\right) + \int_E g(x)\mathrm{d}x \right], \tag{1.2}$$

where $g \in L^n(\Omega)$. The assumption that g belongs to $L^n(\Omega)$ guarantees that, if a sequence E_i of sets of finite perimeter satisfies $|E_i| \to 0$ then $\frac{\int_{E_i} g\mathrm{d}x}{\mathcal{H}^{n-1}(\partial^* E_i)} \to 0$ (see Lem. 4.16). This property allows to add a Cheeger type term to (1.1) and consider the perturbed problem (1.2). An application of (1.2) can be seen as follows. Let $\boldsymbol{F}$ be a bounded divergence-measure field, that is, $\boldsymbol{F} \in L^\infty$ and $\operatorname{div} \boldsymbol{F}$ is a measure. We can define (see [15,16,27]),

$$f(x,v) := \lim_{r\to 0} \frac{n}{\alpha_{n-1} r^n} \int_{B(x,v,r)} \boldsymbol{F}(y) \cdot \frac{y-x}{|y-x|} \mathrm{d}y$$

where $B(x,v,r) := B(x,r) \cap \{y \in \mathbb{R}^n : (y-x)\cdot v > 0\}$ and α_{n-1} is the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$. Then, $f(x,\boldsymbol{\nu}(x))$, $x \in \partial^* E$, defines the normal trace of $\boldsymbol{F}$ on $\partial^* E$, which we denote as $\mathscr{F} \cdot \boldsymbol{\nu}$. The function $\mathscr{F}\cdot\boldsymbol{\nu} \in L^\infty(\partial^* E)$ is actually the classical dot product $\boldsymbol{F}\cdot\boldsymbol{\nu}$ if $\boldsymbol{F}$ is a continuous vector field. Using the Gauss−Green formula for divergence-measure fields we can combine the averaged surface integral and the Cheeger term in a single term as $V(E) = \frac{\int_{E^1} \operatorname{div} \boldsymbol{F} + g}{\mathcal{H}^{n-1}(\partial^* E)}$, where E^1 is the measure theoretic interior of the set E.

The perturbed problem (1.2) includes Cheeger sets, which are solutions of the problem

$$\max_{E\subset\overline{\Omega}} \frac{\mathcal{L}^n\left(E\right)}{\mathcal{H}^{n-1}\left(\partial^* E\right)}. \tag{1.3}$$

We note that (1.3) is equivalent to (1.2) when $f \equiv 0$ and $g \equiv -1$. Existence and uniqueness of Cheeger sets have been studied in Caselles−Chambolle−Novaga [11,12], Alter-Caselles [2] and the references therein. We also refer the interested reader to Figalli−Maggi−Pratelli [19], Alter−Caselles−Chambolle [3] and Cheeger [14]. Applications of Cheeger sets to landslide modeling can be found in Carlier−Comte−Peyre [13] and Ionescu−Lachand−Robert [23]. The case when $f \equiv 0$ and $g \in L^\infty(\Omega)$ has been considered in Butazzo-Carlier-Comte [10], where a numerical method to compute Cheeger sets was developed.

Even though in many cases the optimal value can not be attained, we obtain in this paper conditions under which we can prove the existence of minimizers (see Thms. 5.1, 5.3 and 5.4). In particular, these theorems imply the existence of Cheeger sets (see Cor. 5.2).

Our main results are proven under the assumption that $g \in L^n(\Omega)$. However, given the problem (1.2), it is natural to define the spaces $M_p(\Omega)$ (see Def. (6.1)), since $g \in M_n(\Omega)$ implies that the infimum in (1.2) is finite, which is a necessary condition for the minimizer of (1.2) to exist. Moreover, $M_p(\Omega)$ coincides with the weak L^p space, $L^{p,w}(\Omega)$, for $p > 1$, and $L^{p,w}(\Omega) \subset L^n(\Omega)$ for $p > n$. That is, Lemma 4.16 remains true if $g \in L^{p,w}(\Omega)$, $p > n$, and hence our main results in Sections 4 and 5 also remain true (see Rem. 6.2). This motivates our interest in the weak L^p spaces, and in particular the analysis of the critical case $g \in L^{n,w}(\Omega) \setminus L^n(\Omega)$.

The organization of this paper is as follows. In Section 2 we introduce the occupational measures, which are fundamental in our analysis, and present previous results that will be used in this paper. In Section 3 we give examples that illustrate the difficulties of (1.1). In Section 4 we introduce the *atomic value* of the problem (1.1) and show the main approximation results. We then extend these approximation results to the perturbed problem (1.2). In Section 5 we prove existence theorems for (1.2). Finally, Section 6 and the appendix discuss the minimization problem for the cases when g belongs to the critical spaces $L^{p,w}(\Omega),\ 1 < p \leq n$.

2. Sets of finite perimeter and occupational measures

In this section we first recall some properties of Radon measures, and sets of finite perimeter ([5, 18]). For the sake of completeness, we start with some basic notions and definitions. First, denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, and by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ (recall that $\mathcal{L}^n = \mathcal{H}^n$). We will use the notation $\mathcal{L}^n(E) = |E|$. For any set $E \subset \mathbb{R}^n$, we denote the topological interior of E as $\mathring{E}$, and the topological closure and boundary as $\overline{E}$ and ∂E, respectively. The complement of the set E is denoted by $E^c = \mathbb{R}^n \backslash E$. Also, we denote $B(x,r)$ as the open ball of radius r and center at x. Let w_{n-1} be the surface area of the n-dimensional unit ball.

Definition 2.1. For any open set $\Omega \subset \mathbb{R}^n$, the space $L^p(\Omega)$, $1 \leq p \leq \infty$, consists of all the functions f with the property that $|f|^p$ is Lebesgue integrable, and $\|f\|_p$ denotes its norm. For Ω bounded, we will work in this paper with the space $L^{p,w}(\Omega)$, $1 \leq p < \infty$, which is the weak L^p space. The measurable function g belongs to $L^{p,w}(\Omega)$ if there exists a constant C such that:

$$t^p |\{x \in \Omega : |g(x)| > t\}| \leq C, \text{ for every } t > 0. \tag{2.1}$$

Let X be a locally compact separable metric space, for example, a subset of the Euclidean space. We denote by $E \Subset X$ that the closure of E is compact and contained in X. Let $C_c(X)$ be the space of compactly supported continuous functions on X with $\|\varphi\|_{\infty;X} := \sup\{|\varphi(y)| : y \in X\}$, and we denote by $C_0(X)$ its completion.

Definition 2.2. A Radon measure on X is a signed regular Borel measure whose total variation on each compact set $K \Subset X$ is finite, *i.e.* $\|\mu\|(K) < \infty$. The space of finite Radon measures in X is denoted by $\mathcal{M}(X)$. If $\mu \in \mathcal{M}(X)$ does not take negative values, then we will refer to such μ as a non-negative Radon measure.

Let $\mu_k, \mu \in \mathcal{M}(X)$. We say that μ_k weakly* converges to μ if

$$\mu_k(\varphi) \to \mu(\varphi) \qquad \text{for each } \varphi \in C_0(X),$$

and this convergence is denoted as

$$\mu_k \overset{*}{\rightharpoonup} \mu \qquad \text{in } \mathcal{M}(X).$$

Next, we quote a familiar result concerning weak*-convergence (see Ambrosio−Fusco−Pallara [5], Prop. 1.62).

Lemma 2.3. *Let $\mu_k, \mu \in \mathcal{M}(X)$ such that $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(X)$. If $\|\mu_k\| \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{M}(X)$, then $\|\mu\| \leq \sigma$. In addition, if the μ-measurable set $E \Subset X$ satisfies $\sigma(\partial E) = 0$, then*

$$\mu(E) = \lim_{k \to \infty} \mu_k(E).$$

More generally, if f is a bounded Borel function with compact support in X such that the set of its discontinuity points is σ-negligible, then

$$\lim_{k\to\infty}\int_X f\,\mathrm{d}\mu_k = \int_X f\,\mathrm{d}\mu.$$

Remark 2.4. Let $\mathcal{P}(X)$ denote the subset of $\mathcal{M}(X)$ consisting of all probability measures in X. The weak* convergence of probability measures is characterized as follows (see Billingsley [9]):

$$\mu_k \overset{*}{\rightharpoonup} \mu \qquad \text{in } \mathcal{P}(X),$$

if and only if

$$\mu_k(\varphi) \to \mu(\varphi) \qquad \text{for each continuous and bounded } \varphi. \tag{2.2}$$

In this paper we consider the space $X = \mathbb{R}^n \times \mathbb{S}^{n-1}$. Thus, a sequence of measures $\mu_1, \mu_2, \ldots \in P\left(\mathbb{R}^n \times \mathbb{S}^{n-1}\right)$ weakly* converges to a measure $\mu_0 \in P\left(\mathbb{R}^n \times \mathbb{S}^{n-1}\right)$ if for every bounded continuous function $g \in C\left(\mathbb{R}^n \times \mathbb{S}^{n-1}\right)$,

$$\lim_{i\to\infty}\int_{\mathbb{R}^n\times\mathbb{S}^{n-1}} g\left(x,v\right)\mathrm{d}\mu_i\left(x,v\right) = \int_{\mathbb{R}^n\times\mathbb{S}^{n-1}} g\left(x,v\right)\mathrm{d}\mu_0\left(x,v\right). \tag{2.3}$$

The space $P\left(K \times \mathbb{S}^{n-1}\right)$ is compact in the weak* topology, whenever $K \subset \mathbb{R}^n$ is compact (see, Billingsley [9], p. 72).

Another tool we need for the next theorem is the disintegration of measures. Given a probability measure $\mu \in P\left(\mathbb{R}^n \times \mathbb{S}^{n-1}\right)$, we denote its disintegration by $\mu = p \circledast \mu^x$; the marginal measure is $p \in P\left(\mathbb{R}^n\right)$, which is the push forward of the projection map $\pi : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}^n$; that is $p = \pi_{\#}\mu$, and $p\left(A\right) = \mu\left(A \times \mathbb{S}^{n-1}\right)$ for every Borel set $A \subset \mathbb{R}^n$. The measure-valued function $\mu^x \in P\left(\mathbb{S}^{n-1}\right)$ is the disintegration with respect to p, for p-almost every x. With this notation, for every Borel sets $C \subset \mathbb{R}^n$ and $D \subset \mathbb{S}^{n-1}$, we have that $\mu\left(C \times D\right) = \int_C \mu^x\left(D\right)\mathrm{d}p\left(x\right)$.

Definition 2.5. We define the *occupational measure* $\mu \in P\left(\mathbb{R}^n \times \mathbb{S}^{n-1}\right)$ corresponding to a set of finite perimeter E by

$$\mu\left(U \times V\right) = \frac{1}{\mathcal{H}^{n-1}\left(\partial^* E\right)}\mathcal{H}^{n-1}\left(\left\{x \in \partial^* E : \left(x, \boldsymbol{\nu}_E\left(x\right)\right) \in U \times V\right\}\right),$$

for every measurable sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{S}^{n-1}$.

A useful property of occupational measures is that, for every continuous function $g \in C\left(\mathbb{R}^n \times \mathbb{S}^{n-1}\right)$,

$$\frac{1}{\mathcal{H}^{n-1}\left(\partial^* E\right)}\int_{\partial^* E} g\left(x, \boldsymbol{\nu}_E\left(x\right)\right)\mathrm{d}\mathcal{H}^{n-1}\left(x\right) = \int_{\mathbb{R}^n\times\mathbb{S}^{n-1}} g\left(x,v\right)\mathrm{d}\mu\left(x,v\right). \tag{2.4}$$

Note that when μ is the occupational measure of a set of finite perimeter, then the disintegration is a Dirac measure p-almost everywhere.

Definition 2.6. For every $\alpha \in [0,1]$ and every $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$, define

$$E^\alpha := \{y \in \mathbb{R}^n \,:\, D(E,y) = \alpha\}, \tag{2.5}$$

where

$$D(E,y) := \lim_{r\to 0}\frac{|E \cap B(y,r)|}{|(B(y,r)|}\cdot \tag{2.6}$$

Then E^α is the set of all *points with density* α. We define the *measure-theoretic boundary* of E, $\partial^m E$, as

$$\partial^m E := \mathbb{R}^n \setminus (E^0 \cup E^1). \tag{2.7}$$

Definition 2.7. Let $E \subset \mathbb{R}^n$. We say that E is a *set of finite perimeter* in the open set W if

$$P(E, W) := \sup \left\{ \int_E \operatorname{div} \varphi \mathrm{d}x : \varphi \in C_c^1(W), \|\varphi\|_\infty \leq 1 \right\} < \infty. \tag{2.8}$$

Condition (2.8) implies that the distributional gradient $D\chi_E$ is a finite vector measure in W. We denote the total variation as $\|D\chi_E\|$ and sometimes we use the notation $\|D\chi_E\|(W) = \int_W |D\chi_E|$.

Definition 2.8. Let E be a set of finite perimeter in $\mathbb{R}^n$. The *reduced boundary* of E, denoted as $\partial^* E$, is the set of all points $y \in \mathbb{R}^n$ such that

(1) $\|D\chi_{_E}\|(B(y,r)) > 0$ for all $r > 0$;
(2) The limit $\boldsymbol{\nu}_E(y) := \lim_{r\to 0} \frac{D\chi_{_E}(B(y,r))}{\|D\chi_{_E}\|(B(y,r))}$ exists and $|\boldsymbol{\nu}_E(y)| = 1$.

Remark 2.9. If E is a set of finite perimeter in $\mathbb{R}^n$ then

$$\|D\chi_{_E}\| = \mathcal{H}^{n-1} \llcorner \partial^* E, \tag{2.9}$$

Remark 2.10. Throughout the paper we use indistinctly the notation

$$P(E) = P(E, \mathbb{R}^n) = \mathcal{H}^{n-1}(\partial^* E)$$

to denote the perimeter of the set E.

The unit vector, $\boldsymbol{\nu}_{_E}(y)$, is called the *measure-theoretic interior unit normal* to E at y (we sometimes write $\boldsymbol{\nu}$ instead of $\boldsymbol{\nu}_E$ for notational simplicity). In view of the following, we see that $\boldsymbol{\nu}$ is aptly named because $\boldsymbol{\nu}$ is the interior unit normal to E provided that E (in the limit and in measure) lies in the appropriate half-space determined by the hyperplane orthogonal to $\boldsymbol{\nu}$; that is, $\boldsymbol{\nu}$ is the interior unit normal to E at x provided that

$$D(\{y : (y - x) \cdot \boldsymbol{\nu} > 0, y \notin E\} \cup \{y : (y - x) \cdot \boldsymbol{\nu} < 0, y \in E\}, x) = 0.$$

The following result is due to Federer (see also [28] Lem. 5.9.5. and [5], Thm. 3.61):

Theorem 2.11. *If E is a set of finite perimeter in $\mathbb{R}^n$, then*

$$\partial^* E \subset E^{\frac{1}{2}} \subset \partial^m E, \quad \mathcal{H}^{n-1}\left(\mathbb{R}^n \setminus \left(E^0 \cup \partial^* E \cup E^1\right)\right) = 0. \tag{2.10}$$

In particular, E has density either 0 or 1/2 or 1 at $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$ and $\mathcal{H}^{n-1}$-a.e. $x \in \partial^m E$ belongs to $\partial^* E$.

We will refer to the sets E^0 and E^1 as the *measure-theoretic exterior and interior* of E. We note that, in general, the sets E^0 and E^1 do not coincide with the topological exterior and interior of the set E. We note that (2.10) implies, for any set $E \Subset \mathbb{R}^n$ of finite perimeter,

$$\mathbb{R}^n = E^1 \cup \partial^* E \cup E^0 \cup \mathcal{N}$$

where $\mathcal{H}^{n-1}(\mathcal{N}) = 0$.

Remark 2.12. From the definition of set of finite perimeter in (2.8) it follows that if E is altered by a set of $\mathcal{L}^n$-measure zero to obtain the set $\tilde{E}$, then both sets have the same reduced boundary $\partial^* E$. We remark that, since $E \subset \overline{\Omega}$ implies that $|E \Delta (E \cap \Omega)| \leq |\partial \Omega| = 0$, then E and $E \cap \Omega$ determine the same reduced boundary. Therefore, the condition $E \subset \overline{\Omega}$ can be replaced by $E \subset \Omega$ in (1.1).

Remark 2.13. We will refer to an open set with polyhedral boundary as polytope. A polyhedral boundary is piecewise affine.

In this paper, we will frequently use the isoperimetric inequality which states that, if E is a set of finite perimeter in $\mathbb{R}^n$, then there exist a universal constant $C(n)$ such that

$$|E|^{\frac{n-1}{n}} \leq C(n)P(E), \tag{2.11}$$

and the equality holds if and only if E is Lebesgue equivalent to a ball (see Maggi [24], Chap. 14).

We now present some results that will be used in this paper.

Theorem 2.14 ([8], Thm. 3.2)**.** *Let $E_1, E_2 \subset \mathbb{R}^n$ be sets of finite perimeter, then for $F = E_2, E_2^1, E_2^0$ or $E_2^0 \cup \partial^m E_2$,*

$$\left| \int_{\partial^* E_1 \cap F} \boldsymbol{\nu}_{E_1}(x) \mathrm{d}\mathcal{H}^{n-1}(x) \right| \leq \frac{\mathcal{H}^{n-1}(\partial^* E_2)}{2}. \tag{2.12}$$

The relevance of the inequality (2.12) is that the bound depends only on E_2. We now recall that if E_i is a sequence of sets of finite perimeter with uniformly bounded perimeter then, up-to a subsequence, the sequence converges in L^1 to a set of finite perimeter E_0 and the following lower semicontinuity property holds:

$$\mathcal{H}^{n-1}\left(\partial^* E_0\right) \leq \liminf_{i\to\infty} \mathcal{H}^{n-1}\left(\partial^* E_i\right). \tag{2.13}$$

Two degenerate cases can be considered. The first when the perimeters of the sets E_i grow to infinity, and the second when the Lebesgue measure of the sets E_i converges to zero. Using the estimate (2.12), these degenerate cases were studied in Ido−Torres [8], by means of occupational measures. We now state these results:

Theorem 2.15 ([8], Thms. 5.2 and 5.3)**.** *Let $E_1, E_2, \ldots \subset \mathbb{R}^n$ be sets of finite perimeter satisfying $\lim_{i\to\infty} \mathcal{H}^{n-1}\left(\partial^* E_i\right) = \infty$ or $\lim_{i\to\infty} |E_i| = 0$. If the corresponding sequence of occupational measures $\mu_1, \mu_2, \ldots$ weakly* converges to $\mu_0 \in P\left(\mathbb{R}^n \times \mathbb{S}^{n-1}\right)$ then*

$$\int_{\mathbb{S}^{n-1}} v \mathrm{d}\mu_0^x(v) = 0,$$

for p_0-almost every x, where $\mu_0 = p_0 \circledast \mu_0^x$ is the disintegration of μ_0 with respect to its projection, p_0.

3. The averaged shape optimization problem

In this section we consider the minimization of averaged surface integrals of the type

$$\inf_{E\subset\bar{\Omega}} V_1(E), \quad V_1(E) = \frac{1}{P(E)} \int_{\partial^* E} f\left(x, \boldsymbol{\nu}_E(x)\right) \mathrm{d}\mathcal{H}^{n-1}(x), \tag{3.1}$$

where $f(x,v) \in C\left(\bar{\Omega} \times \mathbb{S}^{n-1}\right)$. We will use the following notation

$$v_1^* = \inf_{E\subset\bar{\Omega}} V_1(E).$$

Definition 3.1. We say that the minimization problem $v_1^* = \inf_{E\subset\bar{\Omega}} V_1(E)$ is attained if there exists a set $E \subset \bar{\Omega}$ such that $v_1^* = V_1(E)$.

Since in this paper we are dealing with averaged minimization problems, the standard techniques from calculus of variations do not apply. In general, the optimal value $v_!^*$ does not need to be attained. The following example shows that, even if f depends only on v, the optimal value v_1^* may not be attained.

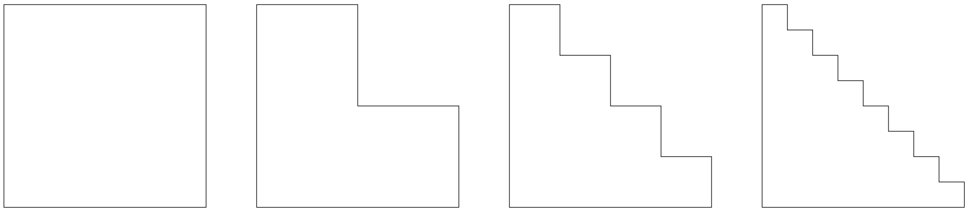

FIGURE 1. This picture shows the first four sets of the minimizing sequence $E_1, E_2, \ldots$ in Example 3.5. Note that for every i, $P(E_i) = 4$, $|\boldsymbol{\nu}_{E_i}(x) \cdot e_1| + |\boldsymbol{\nu}_{E_i}(x) \cdot e_2| = 1$ for all $x \in \partial^* E_i$, and that E_i converges to a triangle R satisfying that $d(x_i, \partial R) \to 0$ uniformly for $x_i \in \partial E_i$.

Example 3.2 (Nonexistence of a minimizer)**.**

Suppose $h : \mathbb{R} \to \mathbb{R}$ satisfies $h(x) > 0$, $x \neq 0$, and $h(0) = 0$. Let $e_1 = (1,0), e_2 = (0,1), f(x,v) = h(v \cdot e_1)$ and $\Omega = (0,1) \times (0,1)$. Clearly, $v_1^* \geq 0$. Choose $E_i = [0, 2^{-i}] \times [0, 2^{-2i}]$, then $V_1(E_i) = \frac{2^{-2i}(h(1)+h(-1))}{2(2^{-2i}+2^{-i})} = \frac{h(1)+h(-1)}{2(1+2^i)} \to 0$, and hence $v_1^* = 0$. Suppose that the minimization problem is attained. Then, there exists a set of finite perimeter $E \subset [0,1] \times [0,1]$ with $|E| > 0$ such that $V_1(E) = v_1^*$. Hence $\frac{1}{\mathcal{H}^1(\partial^* E)} \int_{\partial^* E} h(\boldsymbol{\nu}_E(x) \cdot e_1) \mathrm{d}\mathcal{H}^1(x) = 0$, and thus $h(\boldsymbol{\nu}_E(x) \cdot e_1) = 0$ for $\mathcal{H}^1$-a.e. $x \in \partial^* E$. Therefore, by the definition of h, $\boldsymbol{\nu}_E(x) \cdot e_1 = 0$ for $\mathcal{H}^1$-a.e. $x \in \partial^* E$. This implies that $\boldsymbol{\nu}_E(x) = \pm e_2$ for $\mathcal{H}^1$-a.e. $x \in \partial^* E$, which implies, by Lemma 3.3 below, that $|E| = 0$, which is a contradiction.

Lemma 3.3. *If E is a set of finite perimeter in $\mathbb{R}^n$ with $|E| < \infty$ and $\boldsymbol{\nu}_E(x) = \pm \boldsymbol{\nu}$, for some $\boldsymbol{\nu} \in \mathbb{S}^{n-1}$, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$, then $|E| = 0$.*

Proof. Up to a rotation of the set, we can assume that $\boldsymbol{\nu} = e_n$. We consider the vector field $\boldsymbol{F} = (x_1, 0, \ldots, 0)$ and use the Gauss–Green formula to obtain

$$|E| = \int_E \operatorname{div} \boldsymbol{F} \, \mathrm{d}x = -\int_{\partial^* E} \boldsymbol{F}(x) \cdot \boldsymbol{\nu}_E(x) \, \mathrm{d}\mathcal{H}^{n-1}(x) = 0. \tag{3.2}$$

□

Remark 3.4. If the condition $|E| < \infty$ is removed from the Lemma 3.3 and all the other conditions remain, then E could have positive measure. Indeed, consider the disjoint union of infinite strips (see also [24], P. 182, Exercise 15.18).

The following example shows that even if the minimizing sequence is uniformly bounded and converging to a set of positive measure, the limit set is not a minimizer.

Example 3.5 (Nonexistence of minimizer)**.** In this example we show that, even if there exists a minimizing sequence with uniformly bounded perimeter converging to a set of positive measure, the minimization might not be attained (compare with Exp. 3.2). Let $\Omega = (-2,2) \times (-2,2)$ and R be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$, and let $e_1 = (0,1), e_2 = (1,0)$. Let $f(x,v) = d(x, \partial R) + |v \cdot e_1| + |v \cdot e_2|$, which is a one-homogeneous continuous convex function with respect to v. By the elementary inequality $|\cos \alpha| + |\sin \alpha| \geq 1$, $\alpha \in [0, 2\pi]$, we have

$$|v \cdot e_1| + |v \cdot e_2| \geq 1, \quad \text{and "=" holds if and only if } v = \pm e_1, \pm e_2 \tag{3.3}$$

So $f \geq 1$, and thus $v_1^* \geq 1$. Actually, $v_1^* = 1$, by choosing a minimizing sequence as shown in the picture 1.

If E is a minimizer of (3.1), then by (3.3) and the definition of f, $d(x, \partial R) = 0$, $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$, hence up to a set of $\mathcal{H}^{n-1}$-measure zero, $\partial^* E \subset \partial R$. Since $(\overline{R})^c$ and $\mathring{R}$ are both connected, by ([24], Lem. 7.5) and the definition of set of finite perimeter, $\chi_E = C_1$ a.e. on $(\overline{R})^c$ and $\chi_E = C_2$ a.e. in $\mathring{R}$. However, since $E \subset \Omega$,

$\chi_E = 0$ on Ω^c a.e., thus $C_1 = 0$ in $(\overline{R})^c$. Hence C_2 has to be equal to 1 for otherwise $|E| = 0$, which is not a candidate of our minimizer. Therefore, $E = R$ up to a set of Lebesgue measure zero. However in this case $V_1(E) = V_1(R) > 1 = v_1^*$ because of (3.3). Hence E is not a minimizer and thus we have shown that v_1^* cannot be attained in this example.

The added complexity of this optimization problem is depicted in the following example, where one can see that the optimal solution can be approximated by sequences which are substantially different in their nature.

Example 3.6. Let Ω be the open unit ball, $B\left((0,0),1\right) \subset \mathbb{R}^2$. Consider the minimization problem (3.1) with $f\left(x,v\right) = |x|^2$. Clearly, the infimum is $v_1^* = 0$ and it can be realized by a sequence of balls shrinking to the origin. This sequence is not unique; an alternative sequence is obtained by sets with perimeter increasing to infinity. Indeed, fix a sequence $\epsilon_i \to 0$. Suppose that the sets E_i are obtained by applying a finite number of iterations of the Koch snowflake construction. Assume that each set is centered at the origin, contained in $B\left((0,0),\epsilon_i\right)$, and has perimeter larger than i. The perimeter of the sets E_i diverges to infinity, however, the boundary is concentrated at the origin, and the sequence approximates the optimal value.

The preceding example shows how the averaging allows local non-optimal behavior to diminish as the perimeter increases to infinity. Approximations with increasing perimeter are not desirable. In the main result of the next section we will show that if the optimal value is approximated by a sequence of sets with perimeter increasing to infinity, then it can always be approximated by a sequence of convex polytopes shrinking to a point (see Thm. 4.14).

4. The atomic value and the optimal value of the problem

In this section we introduce the concept of atomic value for the problem (3.1). The main result of this section is an approximation theorem (see Thm. 4.14) that shows that if v_1^* can be approximated with a sequence of sets E_i satisfying $P(E_i) \to \infty$ or $|E_i| \to 0$, then v_1^* can be approximated with a sequence of convex polytopes with $n+1$ faces.

Definition 4.1. We define the atomic value of the minimization problem at the point $x_0 \in \bar{\Omega}$ by

$$f_{\text{atom}}\left(x_0\right) = \inf_{\mu \in P_0(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} f\left(x_0, v\right) \mathrm{d}\mu\left(v\right), \tag{4.1}$$

where

$$P_0\left(\mathbb{S}^{n-1}\right) = \left\{ \mu \in P\left(\mathbb{S}^{n-1}\right) : \int_{\mathbb{S}^{n-1}} v \mathrm{d}\mu\left(v\right) = \mathbf{0} \in \mathbb{R}^n \right\}. \tag{4.2}$$

Lemma 4.2. *Let*

$$A = \left\{ \int_{\mathbb{S}^{n-1}} f\left(x_0, v\right) \mathrm{d}\mu\left(v\right) : \mu \in P_0\left(\mathbb{S}^{n-1}\right) \right\} \subset \mathbb{R}. \tag{4.3}$$

Then

$$A = \left\{ \sum_{j=1}^{n+2} \lambda_j f\left(x_0, v_j\right) : v_j \in \mathbb{S}^{n-1}, \ \sum_{j=1}^{n+2} \lambda_j v_j = 0, \quad \sum_{j=1}^{n+2} \lambda_j = 1, \ \lambda_j \in [0,1] \right\}. \tag{4.4}$$

Proof. The set

$$\hat{A} = \left\{ \int_{\mathbb{S}^{n-1}} [f\left(x_0, v\right), v] \mathrm{d}\mu\left(v\right) : \mu \in P\left(\mathbb{S}^{n-1}\right) \right\} \subset \mathbb{R}^{n+1},$$

is convex since it is the image of the convex set $P\left(\mathbb{S}^{n-1}\right)$ under the linear map $\mu \to \int_{\mathbb{S}^{n-1}} [f\left(x_0, v\right), v] \mathrm{d}\mu(v)$. By Caratheodory's theorem (see Rockafeller [25], p. 153), every element in $\hat{A}$ can be written as a convex combination of $n+2$ extreme points of $\hat{A}$. For every extreme point in $\hat{A}$, its pre-image contains an extreme point of $P(\mathbb{S}^{n-1})$. Since the extreme points of $P\left(\mathbb{S}^{n-1}\right)$ are Dirac measures, we have:

$$\hat{A}=\left\{\sum_{j=1}^{n+2}\lambda_j[f\left(x_0,v_j\right),v_j]:v_j\in\mathbb{S}^{n-1},\sum_{j=1}^{n+2}\lambda_j=1,\ \lambda_j\in[0,1]\right\}\subset\mathbb{R}^{n+1}. \tag{4.5}$$

We now define the set

$$\hat{B}:=\left\{\int_{\mathbb{S}^{n-1}}[f\left(x_0,v\right),v]\mathrm{d}\mu\left(v\right):\mu\in P_0\left(\mathbb{S}^{n-1}\right)\right\}\subset\mathbb{R}^{n+1}$$

and

$$\tilde{B}:=\left\{\left[\sum_{j=1}^{n+2}\lambda_j f\left(x_0,v_j\right),\mathbf{0}\right]:v_j\in\mathbb{S}^{n-1},\sum_{j=1}^{n+2}\lambda_j=1,\ \lambda_j\in[0,1],\ \sum_{j=1}^{n+2}\lambda_j v_j=0\right\}\subset\mathbb{R}^{n+1}$$

We claim that $\hat{B}=\tilde{B}$. Indeed, for any $w\in\hat{B}$, since $\hat{B}\subset\hat{A}$, w can be written as $\sum_{j=1}^{n+2}\lambda_j[f(x_0,v_j),v_j]$, where $v_j\in\mathbb{S}^{n-1},\sum_{j=1}^{n+2}\lambda_j=1,\lambda_j\in[0,1]$. By the definition of $\hat{B}$ and comparing the second component of w, we find that $\sum_{j=1}^{n+2}\lambda_j v_j=0$, hence $w\in\tilde{B}$, thus $\hat{B}\subset\tilde{B}$. If $\tilde{w}\in\tilde{B}$, then $\tilde{w}$ can be written as $[\sum_{j=1}^{n+2}\lambda_j f\left(x_0,v_j\right),\mathbf{0}]$, where $v_j\in\mathbb{S}^{n-1},\sum_{j=1}^{n+2}\lambda_j=1,\ \lambda_j\in[0,1],\ \sum_{j=1}^{n+2}\lambda_j v_j=0$. Let $\tilde{\mu}=\sum_{j=1}^{n+2}\lambda_j\delta_{v_j}$ where δ_{v_j} is the Dirac measure at v_j, then clearly $\tilde{\mu}\in P_0(\mathbb{S}^{n-1})$, and clearly $\tilde{w}=\int_{\mathbb{S}^{n-1}}[f(x,v),v]d\tilde{\mu}$, hence $\tilde{w}\in\hat{B}$, and thus $\tilde{B}\subset\hat{B}$. Therefore, $\hat{B}=\tilde{B}$.

Notice that A is the projection onto the first variable of $\hat{B}$. Hence,

$$A=\left\{\sum_{j=1}^{n+2}\lambda_j f\left(x_0,v_j\right):v_j\in\mathbb{S}^{n-1},\quad\sum_{j=1}^{n+2}\lambda_j=1,\ \lambda_j\in[0,1],\ \sum_{j=1}^{n+2}\lambda_j v_j=0\right\}\subset\mathbb{R}.\qquad\square$$

Corollary 4.3. *The infimum value of A is attained at an element of*

$$C:=\left\{\sum_{j=1}^{n+1}\lambda_j f\left(x_0,v_j\right):v_j\in\mathbb{S}^{n-1},\ \sum_{j=1}^{n+1}\lambda_j v_j=0,\quad\sum_{j=1}^{n+1}\lambda_j=1,\ \lambda_j\in[0,1]\right\}. \tag{4.6}$$

In particular, $f_{\mathrm{atom}}(x_0)=\min C$.

Proof. Since f is continuous, the infimum of A is attained, say $\inf A=\sum_{j=1}^{n+2}\mu_j f(x_0,\overline{v}_j)$ where $\overline{v}_j\in S^{n-1},\sum_{j=1}^{n+2}\mu_j=1,\mu_j\in[0,1]$ and $\sum_{j=1}^{n+2}\mu_j\overline{v}_j=0$. With the fixed vectors $\overline{v}_j$ we define

$$\Lambda=\left\{(\lambda_1,\ldots,\lambda_{n+2}):\sum_{j=1}^{n+2}\lambda_j=1,\lambda_j\in[0,1]\,\text{and}\,\sum_{j=1}^{n+2}\lambda_j\overline{v}_j=0\right\}.$$

Since $(\mu_1,\ldots,\mu_{n+2})\in\Lambda$ we have that $\Lambda\neq\emptyset$. We define the linear map:

$$L:\Lambda\to\mathbb{R},(\lambda_1,\ldots,\lambda_{n+2})\mapsto\sum_{j=1}^{n+2}\lambda_j f(x_0,\overline{v}_j).$$

Clearly, $\sum_{j=1}^{n+2}\mu_j f(x_0,\overline{v}_j)\geq\inf L(\Lambda)$, and thus $\inf A\geq\inf L(\Lambda)$. On the other hand, it is clear that $\inf A\leq\inf L(\Lambda)$, and hence $\inf A=\inf L(\Lambda)$. One can easily check that Λ is convex and compact. The set Λ is the intersection of $B=\{(\lambda_1,\ldots,\lambda_{n+2}):\sum_{j=1}^{n+2}\lambda_i=1,\lambda_i\in[0,1]\}$ and n hyperplanes. Since L is a linear map, the minimum of $L(\Lambda)$ is attained at an extreme point of Λ, say $(\mu_1^*,\ldots,\mu_{n+2}^*)$, and thus $\inf A=\sum_{j=1}^{n+2}\mu_j^* f(x_0,\overline{v}_j)$,

$\sum_{j=1}^{n+2} \mu_j^* = 1$, $\mu_j^* \in [0,1]$ and $\sum_{j=1}^{n+2} \mu_j^* \overline{v}_j = 0$. The extreme point of Λ are on the relative boundary of B. Since B is the convex hull of the $n+2$ standard unit vectors in $\mathbb{R}^{n+2}$, then any point in the relative boundary of B is in the convex hull of only $n+1$ of the unit vectors. Since $(\mu_1^*, \ldots, \mu_{n+2}^*)$ is an extreme point of Λ, we conclude that $\mu_j^* = 0$, for some $j \in \{1, 2, \ldots, n+2\}$. We have shown that $\min C = \min A$, and thus $f_{\text{atom}}(x_0) = \min A = \min C$. □

Definition 4.4. We define the atomic value of the problem by

$$f_{\text{atom}} = \inf_{x_0 \in \bar{\Omega}} f_{\text{atom}}(x_0) \tag{4.7}$$

Lemma 4.5. $f_{\text{atom}}(x) = \inf \left\{ \int_{S^{n-1}} f(x,v) \mathrm{d}\mu(v) : \mu \in P_0 \right\}$ *is a continuous function in* $\bar{\Omega}$.

Proof. By Corollary 4.3,

$$f_{\text{atom}}(x) = \min \left\{ \sum_{j=1}^{n+1} \lambda_i f(x, v_j) : v_j \in S^{n-1}, \sum_{j=1}^{n+1} \lambda_j = 1, \lambda_j \in [0,1], \sum_{j=1}^{n+1} \lambda_j v_j = 0 \right\}.$$

Let

$$K = \left\{ (\lambda_1, \ldots, \lambda_{n+1}, v_1, \ldots, v_{n+1}) : v_j \in S^{n-1}, \sum_{j=1}^{n+1} \lambda_j = 1, \lambda_j \in [0,1], \sum_{j=1}^{n+1} \lambda_j v_j = 0, j = 1, \ldots, n+1 \right\},$$

and define

$$F(x,y) = \sum_{j=1}^{n+1} \lambda_i f(x, v_j) \quad \text{ on } \quad \overline{\Omega} \times K, \; y = (\lambda_1, \ldots, \lambda_{n+1}, v_1, \ldots, v_{n+1}).$$

We have that K is a compact subset of $\mathbb{R}^{2(n+1)}$. Hence $f_{\text{atom}}(x) = \min\{F(x,y) : y \in K\}$. By the following Lemma 4.6, we conclude that $f_{\text{atom}}(x)$ is a continuous function. □

Lemma 4.6. *Let* $F(x,y)$ *be a real-valued continuous function defined in* $A \times B$*, where* A*,* B *are compact sets in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively. Let* $G(x) = \min_{y \in B} F(x,y)$*. Then* G *is a continuous function.*

Proof. Since F is continuous for every $x \in A$, there exits $y_x \in B$ such that $G(x) = F(x, y_x)$. We now prove the Lemma by contradiction. We assume that for some $x_0 \in A$, there exits $\epsilon_0 > 0$ and a sequence $x_n \to x_0$ as such that $G(x_0) < G(x_n) - \epsilon_0$, *i.e.* $F(x_0, y_{x_0}) < F(x_n, y_{x_n}) - \epsilon_0$. For such ϵ_0, there exits $\delta > 0$ such that $|F(a_1, b) - F(a_2, b)| < \epsilon_0/2$ if $|a_1 - a_2| < \delta$. Therefore, for n large enough, $|x_n - x_0| < \delta$, and thus $F(x_n, y_{x_n}) < F(x_0, y_{x_n}) + \epsilon_0/2$, hence $F(x_n, y_{x_n}) > F(x_0, y_{x_0}) + \epsilon_0 > F(x_n, y_{x_0}) - \epsilon_0/2 + \epsilon_0 = F(x_n, y_{x_0}) + \epsilon_0/2$, which contradicts the fact that $F(x_n, y_{x_n}) = \min_{b \in B} F(x_n, b)$. We now assume that for some $x_0 \in A$, there exits $\epsilon_0 > 0$ and a sequence $x_n \to x_0$ such that $G(x_0) > G(x_n) + \epsilon_0$, *i.e.* $F(x_0, y_{x_0}) > F(x_n, y_{x_n}) + \epsilon_0$. For such ϵ_0, there exits $\delta > 0$ such that $|F(a_1, b) - F(a_2, b)| < \epsilon_0/2$ if $|a_1 - a_2| < \delta$. Therefore, for n large enough, $|x_n - x_0| < \delta$, and thus $F(x_n, y_{x_n}) > F(x_0, y_{x_n}) - \epsilon_0/2$, hence $F(x_0, y_{x_0}) > F(x_n, y_{x_n}) + \epsilon_0 > F(x_0, y_{x_n}) - \epsilon_0/2 + \epsilon_0 = F(x_0, y_{x_n}) + \epsilon_0/2$, which contradicts the fact that $F(x_0, y_{x_0}) = \min_{b \in B} F(x_0, b)$. □

Corollary 4.7. *By Lemma* 4.5*, the infimum value in* (4.7) *is attained and hence we can write*

$$f_{\text{atom}} = \min_{x_0 \in \overline{\Omega}} f_{\text{atom}}(x_0).$$

We now show that the atomic value can be realized by a sequence of convex polytopes with $n+1$ faces. For that we need the following classical result due to Minkowski (see, Alexandrov [1], Chap. 7, p. 311).

Theorem 4.8. *Suppose $\alpha_1, \ldots, \alpha_N > 0$ and the unit vectors $v_1, \ldots, v_N \in \mathbb{R}^n$ span $\mathbb{R}^n$. If $\sum_{i=1}^N \alpha_i v_i = 0$ then there exists a convex polytope with N faces, where the i'th face has area α_i and normal v_i. Moreover, this polytope is unique up to translations.*

Proposition 4.9. *For every point $x_0 \in \bar{\Omega}$, the atomic value at x_0, $f_{\text{atom}}(x_0)$, can be realized by a sequence of convex polytopes $\Delta_i \subset \Omega$ with $n+1$ faces shrinking to x_0, in the sense that*

$$\lim_{i\to\infty} \sup_{y\in\Delta_i} |y - x_0| = 0,$$

and such that

$$\lim_{i\to\infty} V_1(\Delta_i) = f_{\text{atom}}(x_0).$$

Remark 4.10. Clearly $\lim_{i\to\infty} \sup_{y\in\Delta_i} |y - x_0| = 0$ implies $|\Delta_i| \to 0$.

Proof. From Corollary 4.3 it follows that

$$f_{\text{atom}}(x_0) = \sum_{j=1}^{n+1} \lambda_j f(x_0, v_j),$$

for some $\lambda_j \in [0,1]$, $v_j \in S^{n-1}$, $\sum_{j=1}^{n+1} \lambda_j = 1$, and $\sum_{j=1}^{n+1} \lambda_j v_j = 0$.

Case 1. $x_0 \in \Omega$. In this case, we assume $B(x_0, \delta_i) \subset \Omega$ and $\delta_i \to 0$. If the set of vectors v_j span $\mathbb{R}^n$, then by Theorem 4.8, we set Δ to be a polytope with $n+1$ faces, such that the j^{th} face has area λ_j and normal v_j, and $0 \in \Delta$. For every i we scale and translate Δ so that it is contained in $B(x_0, \delta_i)$, and set Δ_i accordingly. Indeed, $\Delta_i = \delta_i \Delta + x_0$. We have,

$$\begin{aligned}
\lim_{i\to\infty} V_1(\Delta_i) &= \lim_{i\to\infty} \frac{1}{P(\Delta_i)} \left[\int_{\partial^*\Delta_i} f(x, \boldsymbol{\nu}_{\Delta_i}(x)) \mathrm{d}\mathcal{H}^{n-1}(x) \right], \\
&= \lim_{i\to\infty} \frac{\sum_{j=1}^{n+1} (\delta_i^{n-1}\lambda_j) f(x_0, v_j)}{\sum_{j=1}^{n+1} (\delta_i^{n-1}\lambda_j)}, \text{ by the continuity of } f, \\
&= \frac{\sum_{j=1}^{n+1} \lambda_j f(x_0, v_j)}{\sum_{j=1}^{n+1} \lambda_j} = \sum_{j=1}^{n+1} \lambda_j f(x_0, v_j) = f_{\text{atom}}(x_0).
\end{aligned} \tag{4.8}$$

If $\{v_j\}_{j=1}^{n+1}$ do not span $\mathbb{R}^n$ then, for every i, we perturb (see Cor. A.3 in the appendix) the original λ_j and v_j, $j = 1, 2, \ldots, n+1$, by choosing $\lambda_{i,j}, v_{i,j}$ that satisfy the assumption of Minkowski's theorem, $|\lambda_{i,j} - \lambda_j| \leq 1/i, |v_{i,j} - v_j| \leq 1/i$ and the corresponding Δ_i, by scaling, are still contained in $B(x_0, \delta_i)$. Then, by the continuity of f we obtain

$$\lim_{i\to\infty} V_1(\Delta_i) = \lim_{i\to\infty} \frac{\sum_{j=1}^{n+1} \lambda_{i,j} f(x_0, v_{i,j})}{\sum_{j=1}^{n+1} \lambda_{i,j}} = \sum_{j=1}^{n+1} \lambda_j f(x_0, v_j) = f_{\text{atom}}(x_0).$$

Case 2. $x_0 \in \partial\Omega$. In this case, by the continuity of $f_{\text{atom}}(x)$ proved in Lemma 4.5, we can choose $x_k \in \Omega, x_k \to x_0$ such that $|f_{\text{atom}}(x_0) - f_{\text{atom}}(x_k)| < 1/k$. For each k, by **Case 1**, there exists Δ_k and $\delta'_k \to 0$ such that $|f_{\text{atom}}(x_k) - V_1(\Delta_k)| < 1/k$ and $\Delta_k \subset B(x_k, \delta'_k)$, thus $|f_{\text{atom}}(x_0) - V_1(\Delta_k)| < 2/k$ and $|y - x_0| \leq \delta'_k + |x_k - x_0|$, for all $y \in \Delta_k$. Hence $\lim_{k\to\infty} \sup_{y\in\Delta_k} |y - x_0| = 0$ and $\lim_{k\to\infty} V_1(\Delta_k) = f_{\text{atom}}(x_0)$. ◻

We now have the following:

Corollary 4.11. $v_1^* \leq f_{\text{atom}}$.

Proof. Since $f_{\text{atom}}(x)$ is continuous function on $\bar{\Omega}$, there exists $x_0 \in \bar{\Omega}$ such that $f_{\text{atom}} = f_{\text{atom}}(x_0)$. Then by Proposition 4.9 there exists $\Delta_i \subset \Omega$ such that $\lim_{i\to\infty} V_1(\Delta_i) = f_{\text{atom}}(x_0)$. By the definition of v_1^*, $v_1^* \leq V_1(\Delta_i)$, hence $v_1^* \leq f_{\text{atom}}$. □

Remark 4.12. If f depends only on x, then the property $v_1^* \leq f_{\text{atom}}$ follows by choosing any sequence of sets of finite perimeter E_i, $E_i \subset \overline{\Omega}$, such that $\lim_{i\to\infty} \sup_{y\in E_i} |y - x_0| = 0$. Here, x_0 is the point where f attains its minimum. Indeed, by the continuity of f and since $v_1^* \leq V_1(E_i)$, we have $v_1^* \leq f(x_0) = f_{\text{atom}}$.

Lemma 4.13. *If there exists a minimizing sequence E_i such that $P(E_i) \to \infty$ or $|E_i| \to 0$, then $v_1^* \geq f_{\text{atom}}$.*

Proof. Let $E_1, E_2, \ldots \subset \bar{\Omega}$ be a sequence of sets of finite perimeter, such that $\lim_{i\to\infty} V_1(E_i) = v_1^*$ and $\lim_{i\to\infty} P(E_i) = \infty$ or $\lim_{i\to} |E_i| = 0$. Let $\mu_1, \mu_2, \ldots \in P\left(\bar{\Omega} \times \mathbb{S}^{n-1}\right)$ be the corresponding sequence of occupational measures. By compactness there exists a subsequence, denoted again as the full sequence, such that $\mu_i \xrightharpoonup{*} \mu_0 \in P\left(\bar{\Omega} \times \mathbb{S}^{n-1}\right)$. Note that μ_0 is not necessarily an occupational measure corresponding to a set of finite perimeter.

Hence,

$$\begin{aligned} v_1^* &= \lim_{i\to\infty} V_1(E_i) \\ &= \lim_{i\to\infty} \frac{1}{P(E_i)} \int_{\partial^* E_i} f(x, \boldsymbol{\nu}_{E_i}(x)) d\mathcal{H}^{n-1}(x), \\ &= \lim_{i\to\infty} \int_{\bar{\Omega}\times\mathbb{S}^{n-1}} f(x, v)\, \mathrm{d}\mu_i, \text{ from (2.4)}, \\ &= \int_{\bar{\Omega}\times\mathbb{S}^{n-1}} f(x, v)\, \mathrm{d}\mu_0, \text{ from (2.3)}, \\ &= \int_{\bar{\Omega}} \left(\int_{\mathbb{S}^{n-1}} f(x, v)\, \mathrm{d}\mu_0^x \right) \mathrm{d}p_0, \end{aligned} \tag{4.9}$$

where $\mu_0 = p_0 \circledast \mu_0^x$ is the disintegration of the measure μ_0. Since the conditions of Theorem 2.15 are satisfied, then $\mu_0^x \in P_0\left(\mathbb{S}^{n-1}\right)$, for p_0-almost every x. Then, Definition 4.4 implies that the inner integral is bounded from below by f_{atom}, and, since $p_0\left(\bar{\Omega}\right) = \mu_0\left(\bar{\Omega} \times \mathbb{S}^{n-1}\right) = 1$, $f_{\text{atom}} \leq v_1^*$. □

Proposition 4.9, Corollary 4.11 and Lemma 4.13 are crucial to study the average shape optimization (3.1). They give an estimate of the optimal value as well as information about minimizing sequences. In particular, we get the following:

Theorem 4.14 (Approximation)**.** *Consider the minimization problem $v_1^* = \inf_{E\subset\bar{\Omega}} V_1(E)$ given by*

$$V_1(E) = \frac{1}{P(E)} \left[\int_{\partial^* E} f(x, \boldsymbol{\nu}_E(x)) \mathrm{d}\mathcal{H}^{n-1}(x) \right]$$

where $f \in C(\bar{\Omega} \times \mathbb{S}^{n-1})$. If there exists a minimizing sequence E_i such that $P(E_i) \to \infty$ or $|E_i| \to 0$, then $v_1^* = f_{\text{atom}}$, and the optimal value can be approximated by convex polytopes Δ_i with $n+1$ faces shrinking to a point x_0, in the sense that $\lim_{i\to\infty} \sup_{y\in\Delta_i} |y - x_0| = 0$.

Proof. This is an immediate consequence of Proposition 4.9, Corollary 4.11 and Lemma 4.13. □

Corollary 4.15 (Approximation)**.** *Assume f depends only on the variable v. We minimize $v_1^* = \inf_{E\subset\bar{\Omega}} V_1(E)$ with*

$$V_1(E) = \frac{1}{P(E)}\left[\int_{\partial^* E} f(\boldsymbol{\nu}_E(x))\mathrm{d}\mathcal{H}^{n-1}(x)\right]$$

where $f \in C(\mathbb{S}^{n-1})$. Then $v_1^* = f_{\text{atom}}$, and the optimal value can be approximated by convex polytopes Δ_i with $n+1$ faces shrinking to a point x_0, in the sense that $\lim_{i\to\infty}\sup_{y\in\Delta_i}|y-x_0| = 0$.

Proof. We claim that for any set of finite perimeter $E \subset \overline{\Omega}$, there exists a sequence of sets E_r such that $\lim_{r\to 0}|E_r| = 0$ and $V_1(E_r) = V_1(E)$. Indeed, since $V_1(E)$ is translation invariant, without loss of generality we can assume that $0 \in E \subset \bar{\Omega}$. For any $0 < r < 1$, we have $rE \subset \bar{\Omega}$. Since $P(rE) = r^{n-1}P(E)$, and $\boldsymbol{\nu}_{rE}(y) = \lim_{\rho\to 0}\frac{\int_{B(y,\rho)} D\chi_{rE}}{\int_{B(y,\rho)}|D\chi_{rE}|} = \lim_{\rho\to 0}\frac{\int_{B(y/r,\rho/r)} D\chi_E}{\int_{B(y/r,\rho/r)}|D\chi_E|} = \boldsymbol{\nu}_E(y/r)$, for every $y \in \partial^*(rE)$, we have

$$\begin{aligned}
V_1(rE) &= \frac{1}{P(rE)}\int_{\partial^*(rE)} \boldsymbol{\nu}_{rE}(y)\mathrm{d}\mathcal{H}^{n-1}(y)\\
&= \frac{1}{r^{n-1}P(E)}\int_{\partial^*(rE)} \boldsymbol{\nu}_E(y/r)\mathrm{d}\mathcal{H}^{n-1}(y)\\
&= \frac{1}{r^{n-1}P(E)}\int_{\partial^* E} \boldsymbol{\nu}_E(rx/r)r^{n-1}\mathrm{d}\mathcal{H}^{n-1}(x),\ x = y/r,\\
&= \frac{1}{P(E)}\int_{\partial^* E} \boldsymbol{\nu}_E(x)\mathrm{d}\mathcal{H}^{n-1}(x) = V_1(E).
\end{aligned}\tag{4.10}$$

Let E_i be any minimizing sequence and let $r_i > 0$ with $r_i \to 0$. We consider the sequence of sets r_iE_i. From (4.10) it follows that

$$V_1(E_i) = V_1(r_iE_i). \tag{4.11}$$

Also, since each E_i is contained in the bounded set $\overline{\Omega}$ we have that

$$|r_iE_i| \to 0. \tag{4.12}$$

We note that r_iE_i is also a minimizing sequence since $\lim_{i\to\infty} V_1(r_iE_i) = \lim_{i\to\infty} V_1(E_i) = v_1^*$. Moreover, since $|r_iE_i| \to 0$, the desired result follows from Theorem 4.14. ☐

As explained in the introduction, the minimization of the averaged surface integral can be perturbed with a Cheeger type term. Cheeger sets maximize the ratio $\frac{\mathcal{L}^n(E)}{P(E)}$ over sets of finite perimeter contained in some domain $\Omega \Subset \mathbb{R}^n$. The Cheeger constant is one over the maximal ratio. These sets appear in the study of partial differential equations (see, *e.g.*, [14]). Thus, we consider in this section averaged optimization problems of the form

$$\inf_{E\subset\overline{\Omega}} V(E), \quad V(E) = \frac{1}{P(E)}\left[\int_{\partial^* E} f(x, \boldsymbol{\nu}_E(x))\,\mathrm{d}\mathcal{H}^{n-1}(x) + \int_E g(x)\,\mathrm{d}x\right], \tag{4.13}$$

where $f(x,v) \in C\left(\bar{\Omega}\times\mathbb{S}^{n-1}\right)$ and $g \in L^n(\Omega)$. We will use the following notation

$$v^* = \inf_{E\subset\bar{\Omega}} V(E). \tag{4.14}$$

We have the following

Lemma 4.16. *If $g \in L^n(\Omega)$, E_i are sets of finite perimeter in $\overline{\Omega}$, and $|E_i| \to 0$, then $\frac{\int_{E_i} g(x)\mathrm{d}x}{P(E_i)} \to 0$.*

Proof. We have,

$$\begin{aligned}
\frac{\int_{E_i} g(x)\mathrm{d}x}{P(E_i)} &\leq \frac{||g||_{L^n(E_i)}|E_i|^{1-1/n}}{P(E_i)}, \text{ by Holder inequality,}\\
&\leq \frac{C(n)\,\|g\|_{L^n(E_i)}\,|E|^{1-1/n}}{|E_i|^{1-1/n}}, \text{ by the isoperimetric inequality (2.11),}\\
&= C(n)\,\|g\|_{L^n(E_i)} \qquad (4.15)\\
&\to 0, \text{ since } |E_i| \to 0 \text{ and the absolute continuity property of the integral.} \qquad \square
\end{aligned}$$

Remark 4.17. We note that if $g \in L^n(\Omega)$, then $v^* > -\infty$. Indeed, for every set of finite perimeter $E \subset \overline{\Omega}$,

$$\begin{aligned}
V(E) &\geq \min_{(x,v)\in\overline{\Omega}\times\mathbb{S}^{n-1}} f(x,v) - \frac{\int_E |g(x)|\mathrm{d}x}{P(E)}\\
&\geq \min_{(x,v)\in\overline{\Omega}\times\mathbb{S}^{n-1}} f(x,v) - \frac{C(n)\int_E |g(x)|\mathrm{d}x}{|E|^{1-1/n}}, \text{ by the isoperimetric inequality (2.11),}\\
&\geq \min_{(x,v)\in\overline{\Omega}\times\mathbb{S}^{n-1}} f(x,v) - C(n)\,\|g\|_{L^n(\Omega)}\,, \text{ by Holder's inequality,}
\end{aligned}$$

which implies $v^* > -\infty$.

As a consequence of Lemma 4.16, the approximation theorems proved in the previous section also hold for (4.13). We have

Theorem 4.18 (Approximation)**.** *Consider the minimization problem $v^* = \inf_{E\subset\bar{\Omega}} V(E)$ given by*

$$V(E) = \frac{1}{P(E)}\left[\int_{\partial^* E} f(x,\boldsymbol{\nu}(x))\mathrm{d}\mathcal{H}^{n-1}(x) + \int_E g(x)\mathrm{d}x\right] = V_1(E) + V_2(E), V_2(E) = \frac{\int_E g(x)\mathrm{d}x}{P(E)},$$

where $f \in C(\bar{\Omega}\times\mathbb{S}^{n-1})$ and $g \in L^n(\Omega)$. If there exists a minimizing sequence E_i such that $P(E_i) \to \infty$ or $|E_i| \to 0$, then $v^* = f_{\text{atom}}$, and the optimal value can be approximated by convex polytopes Δ_i with $n+1$ faces shrinking to a point x_0, in the sense that $\lim_{i\to\infty}\sup_{y\in\Delta_i}|y - x_0| = 0$.

Proof. Let $x_0 \in \overline{\Omega}$ such that $f_{\text{atom}} = f_{\text{atom}}(x_0)$ and let Δ_i be the sequence of convex polytopes constructed in Proposition 4.9. Then

$$V_1(\Delta_i) \to f_{\text{atom}}(x_0). \qquad (4.16)$$

Since $|\Delta_i| \to 0$, Lemma 4.16 yields

$$\lim_{i\to\infty} V(\Delta_i) = f_{\text{atom}}(x_0) = f_{\text{atom}} \Longrightarrow v^* \leq f_{\text{atom}}. \qquad (4.17)$$

In order to see the reverse inequality we note that, for the minimizing sequence E_i, if $\lim_{i\to\infty} P(E_i) = \infty$ then clearly

$$V_2(E_i) \to 0. \qquad (4.18)$$

Moreover, (4.18) also holds by Lemma 4.16 when $\lim_{i\to\infty}|E_i| = 0$. Let $\mu_1, \mu_2, \ldots \in P\left(\bar{\Omega} \times \mathbb{S}^{n-1}\right)$ be the corresponding sequence of occupational measures associated to the minimizing sequence E_i. Proceeding as in Therem 4.14 and using the same notation,

$$\begin{aligned} v^* &= \lim_{i\to\infty} \frac{1}{P(E_i)} \int_{\partial^* E_i} f(x, \boldsymbol{\nu}_{E_i}(x)) d\mathcal{H}^{n-1}(x) + 0, \text{ since } V_2(E_i) \to 0, \\ &= \int_{\bar{\Omega}} \left(\int_{S^{n-1}} f(x, v)\, \mathrm{d}\mu_0^x \right) \mathrm{d}p_0 \geq f_{\text{atom}}. \end{aligned} \tag{4.19}$$

Hence, $v^* = f_{\text{atom}}$ and we conclude

$$v^* = \lim_{i\to\infty} V(\Delta_i), \tag{4.20}$$

that is, the optimal value can also be approximated by convex polytopes Δ_i with $n+1$ faces shrinking to a point x_0. ☐

Corollary 4.19 (Approximation)**.** *Assume f depends only on the variable v. We minimize $v^* = \inf_{E\subset\bar{\Omega}} V(E)$ with*

$$V(E) = \frac{1}{P(E)} \left[\int_{\partial^* E} f(\boldsymbol{\nu}_E(x)) \mathrm{d}\mathcal{H}^{n-1}(x) + \int_E g(x) \mathrm{d}x \right], \tag{4.21}$$

where $f \in C(S^{n-1})$ and $g \in L^n(\Omega)$, $g \geq 0$. Then $v^* = f_{\text{atom}}$, and the optimal value can be approximated by convex polytopes Δ_i with $n+1$ faces shrinking to a point x_0, in the sense that $\lim_{i\to\infty} \sup_{y\in\Delta_i} |y - x_0| = 0$.

Proof. Let E_i be any minimizing sequence of (4.21) and let $r_i > 0$ with $r_i \to 0$. We consider the sequence of sets $r_i E_i$. Proceeding as in Corollary 4.15 it follows that

$$V_1(E_i) = V_1(r_i E_i) \tag{4.22}$$

and, since each E_i is contained in the bounded set $\overline{\Omega}$,

$$|r_i E_i| \to 0. \tag{4.23}$$

We now show that $r_i E_i$ is also minimizing sequence of (4.21). Indeed, we have

$$\begin{aligned} \limsup_{i\to\infty} V(r_i E_i) &\leq \limsup_{i\to\infty} V_1(r_i E_i) + \limsup_{i\to\infty} V_2(r_i E_i) \\ &= \limsup_{i\to\infty} V_1(r_i E_i), \text{ by (4.23) and Lemma 4.16}, \\ &\leq \limsup_{i\to\infty} V_1(E_i) + \liminf_{i\to\infty} V_2(E_i), \text{ by (4.22) and since } g \geq 0, \\ &\leq \lim_{i\to\infty} (V_1(E_i) + V_2(E_i)) = \lim_{i\to\infty} V(E_i) = v^*. \end{aligned} \tag{4.24}$$

Therefore, up to a subsequence, we have $V(r_i E_i) \to v^*$, and hence we have constructed a minimizing sequence satisfying $|r_i E_i| \to 0$. The desired result follows from Theorem 4.18. ☐

5. Existence of minimizers

The map $E \mapsto P(E)$ is lower semicontinuous under L^1 convergence. However, even if $E \mapsto \int_{\partial^* E} f(x, \boldsymbol{\nu}_E(x)) \mathrm{d}\mathcal{H}^{n-1}(x)$ is lower semicontinuous, we can not expect the map $E \mapsto V(E)$ to be lower semicontinuous, since the ratio does not preserve in general the lower semicontinuity property (see Exp. 3.2). However, we will show next that we can impose conditions on f to guarantee that $E \mapsto \int_{\partial^* E} f(x, \boldsymbol{\nu}_E(x)) \mathrm{d}\mathcal{H}^{n-1}(x)$ is lower semicontinuous and that the minimizer exists. We have the following:

Theorem 5.1 (Existence)**.** *Consider the minimization problem $v^* = \inf_{E\subset\bar{\Omega}} V(E)$ given by*

$$V(E) = \frac{1}{P(E)}\left[\int_{\partial^* E} f(x,\boldsymbol{\nu}_E(x))\mathrm{d}\mathcal{H}^{n-1}(x) + \int_E g(x)\mathrm{d}x\right]$$

where $f\in C(\bar{\Omega}\times\mathbb{S}^{n-1})$ and $g\in L^n(\Omega)$. If $f(x,v)$ is both convex and positive homogeneous of order 1 in v, $v^* < f_{\mathrm{atom}}$ and $v^* < 0$, then v^* is attained.

Proof. By (4.19), if $v^* < f_{\mathrm{atom}}$, then for any minimizing sequence $\{E_i\}$ we have that $\{P(E_i)\}$ is uniformly bounded and $\inf_i |E_i| > 0$. Therefore, by the compactness of sets of finite perimeter we have that, up to a further subsequence, there exists a set of finite perimeter E_0 such that $E_i\to E_0$ in $L^1(\Omega)$ and

$$D\chi_{E_i} \xrightharpoonup{*} D\chi_{E_0},\quad \|D\chi_{E_i}\| \xrightharpoonup{*} \sigma. \tag{5.1}$$

Moreover, by Lemma 2.3, we have

$$\|D\chi_{E_0}\| \le \sigma. \tag{5.2}$$

In particular, $\lim_{i\to\infty} P(E_i) = \sigma(\bar{\Omega}) = P_\infty$. We note that $P_\infty > 0$. Indeed, if $P_\infty = 0$ then the isoperimetric inequality implies that $|E_i|\to 0$ which violates the assumption $\inf_i |E_i| > 0$. Now, by the the lower semicontinuity of the perimeter stated in (2.13) (or by (5.2)) it follows that $P_\infty \ge P(E_0)$.

Also, the conditions on f imply that $E\mapsto \int_{\partial^* E} f(x,\boldsymbol{\nu}_E(x))\mathrm{d}\mathcal{H}^{n-1}(x)$ is lower semicontinuous (see [5], Thm. 2.38 (Reshetnyak lower semicontinuity theorem)). Therefore,

$$\int_{\partial^* E_0} f(x,\boldsymbol{\nu}_{E_0}(x))\mathrm{d}\mathcal{H}^{n-1}(x) \le \liminf_{i\to\infty}\int_{\partial^* E_i} f(x,\boldsymbol{\nu}_{E_i}(x))\mathrm{d}\mathcal{H}^{n-1}(x). \tag{5.3}$$

Hence

$$\begin{aligned} v^* &= \lim_{i\to\infty} V(E_i)\\ &\ge \frac{1}{P_\infty}\left[\int_{\partial^* E_0} f(x,\boldsymbol{\nu}_{E_0}(x))\mathrm{d}\mathcal{H}^{n-1}(x) + \int_{E_0} g(x)\mathrm{d}x\right],\ \text{since } g\in L^1(\Omega) \text{ and by (5.3)},\\ &= \frac{P(E_0)}{P_\infty}\frac{1}{P(E_0)}\left[\int_{\partial^* E_0} f(x,\boldsymbol{\nu}_{E_0}(x))\mathrm{d}\mathcal{H}^{n-1}(x) + \int_{E_0} g(x)\mathrm{d}x\right]\\ &= \frac{P(E_0)}{P_\infty} V(E_0). \end{aligned}$$

Since $v^* < 0$ and $\frac{P_\infty}{P(E_0)} \ge 1$, then $V(E_0) \le \frac{P_\infty}{P(E_0)} v^* \le v^*$. Therefore by the definition of v^*, $v^* = V(E_0)$. □

Corollary 5.2 (Existence of Cheeger sets)**.** *Let $h\in L^n(\Omega)$, $h\ge 0$. A bounded Lipschitz domain Ω contains a set maximizing*

$$\sup_{E\subset\bar{\Omega}} \frac{\int_E h(x)\mathrm{d}x}{P(E)}\cdot \tag{5.4}$$

In particular, Ω contains a Cheeger set maximizing

$$\sup_{E\subset\bar{\Omega}} \frac{|E|}{P(E)}\cdot \tag{5.5}$$

Proof. If $h=0$ almost everywhere then the sup is zero and attained at any admissible set E. Otherwise, we have that the sup is positive. Now, to maximize (5.4) is equivalent to minimize (4.13) when $f=0$, $g=-h$. Clearly $v^* < 0$, and such f and g satisfy the conditions in Theorem 5.1. Therefore, v^* is attained in the minimization (4.13), which implies that the maximization (5.4) is attained. □

Theorem 5.3. *Consider the minimization problem $v^* = \inf_{E\subset\bar{\Omega}} V(E)$ given by*

$$V(E) = \frac{1}{P(E)}\left[\int_{\partial^* E} f(x)\mathrm{d}\mathcal{H}^{n-1}(x) + \int_E g(x)\mathrm{d}x\right]$$

where $f \in C(\bar{\Omega})$ and $g \in L^n(\Omega)$. Then either $v^* = \min_{x\in\bar{\Omega}} f(x)$ or $v^* < \min_{x\in\bar{\Omega}} f(x)$ (and v^* is attained). In the first case, v^* can be approximated by any sequence E_i shrinking to a point $x_0 \in \bar{\Omega}$, in the sense that $\lim_{i\to\infty}\sup_{y\in E_i}|y - x_0| = 0$.

Proof. Clearly, $f_{\text{atom}} = \min_{x\in\bar{\Omega}} f(x)$, so by Remark 4.12 and Lemma 4.16, we have $v^* \le \min_{x\in\bar{\Omega}} f(x)$. If $v^* = \min_{x\in\bar{\Omega}} f(x)$, then any sequence E_i as in Remark 4.12 is actually a minimizing sequence. If $v^* < \min_{x\in\bar{\Omega}} f(x)$, therefore, for any minimizing sequence E_i, (4.19) implies that $\{P(E_i)\}$ is uniformly bounded and $\inf_i |E_i| > 0$. Hence, up to a further subsequence, there exists a set of finite perimeter E_0 such that $E_i \to E_0$ in $L^1(\Omega)$ and

$$D\chi_{E_i} \overset{*}{\rightharpoonup} D\chi_{E_0}, \quad \|D\chi_{E_i}\| \overset{*}{\rightharpoonup} \sigma, \tag{5.6}$$

and Lemma 2.3 yields

$$\|D\chi_{E_0}\| \le \sigma. \tag{5.7}$$

In particular, $\lim_{i\to\infty} P(E_i) = \sigma(\bar{\Omega}) = P_\infty$. Again, the same argument in the proof of Theorem 5.1 implies $P_\infty > 0$. Let $\lambda = \frac{P(E_0)}{P_\infty}$, then $\lambda \in (0, 1]$ by (5.7). We now show that the infimum is attained at the set E_0. By (5.7), for every $A \subset \bar{\Omega}$, $\sigma(A) \ge \|D\chi_{E_0}\|(A)$, thus, by the weak* convergence (5.6) and since f is continuous,

$$\begin{aligned}\lim_{i\to\infty}\int_{\bar{\Omega}} f(x)\,\|D\chi_{E_i}\|(x) &= \int_{\bar{\Omega}} f(x)\mathrm{d}\sigma(x) \\ &= \int_{\bar{\Omega}} f(x)\,\|D\chi_{E_0}\|(x) + \int_{\bar{\Omega}} f(x)\mathrm{d}\tau(x),\end{aligned} \tag{5.8}$$

where $\tau = \sigma - \|D\chi_{E_0}\|$ is a non-negative measure. This implies that

$$\begin{aligned} v^* &= \lim_{i\to\infty} V(E_i) = \lim_{i\to\infty}\frac{1}{P(E_i)}\left[\int_{\partial^* E_i} f(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) + \int_{E_i} g(x)\,\mathrm{d}x\right] \\ &= \frac{1}{P_\infty}\lim_{i\to\infty}\int_{\bar{\Omega}} f(x)\,\|D\chi_{E_i}\| + \frac{1}{P_\infty}\int_{E_0} g(x)\mathrm{d}x \\ &= \frac{1}{P_\infty}\int_{\bar{\Omega}} f(x)\mathrm{d}\tau + \frac{1}{P_\infty}\int_{\bar{\Omega}} f(x)\,\|D\chi_{E_0}\| + \frac{1}{P_\infty}\int_{E_0} g(x)\,\mathrm{d}x, \text{ by (5.8)}, \\ &= \frac{1}{P_\infty}\left[\int_{\bar{\Omega}} f(x)\mathrm{d}\tau + P(E_0)\,V(E_0)\right] \\ &\ge \frac{1}{P_\infty}\left[P(E_0)\,V(E_0) + (P_\infty - P(E_0))\min_{x\in\bar{\Omega}} f(x)\right], \text{ since } \tau\left(\bar{\Omega}\right) = P_\infty - P(E_0) \\ &= \lambda V(E_0) + (1-\lambda)\min_{x\in\bar{\Omega}} f(x)\end{aligned}$$

Thus, if $0 < \lambda < 1$, we have $v^* \ge \lambda V(E_0) + (1-\lambda)\min_{x\in\bar{\Omega}} f(x) > \lambda V(E_0) + (1-\lambda)\,v^*$, and hence $v^* > V(E_0)$, which is a contradiction to the definition of v^*. Hence, we must have $\lambda = 1$. In this case, $v^* \ge V(E_0)$ and, by the minimality of v^*, $V(E_0) = v^*$, and the minimum is attained. ☐

If the function f depends only on the space variable x and if we only assume that $g^- \in L^n(\Omega)$, then we cannot argue as in Theorem 5.3 to conclude $v^* \le f_{\text{atom}}$. However, we will show next that $v^* \le f_{\text{atom}}$ is still true and that a similar result to Theorem 5.3 holds, but in this case we can not guarantee that v^* can be approximated with sets with bounded perimeter. We have the following:

Theorem 5.4. *Consider the minimization problem* $v^* = \inf_{E \subset \bar{\Omega}} V(E)$ *given by*

$$V(E) = \frac{1}{P(E)} \left[\int_{\partial^* E} f(x) \mathrm{d}\mathcal{H}^{n-1}(x) + \int_E g(x) \mathrm{d}x \right]$$

where $f \in C(\bar{\Omega})$*,* $g^+ \in L^1(\Omega)$ *and* $g^- \in L^n(\Omega)$*. Then either* $v^* = \min_{x \in \bar{\Omega}} f(x)$ *or* $v^* < \min_{x \in \bar{\Omega}} f(x)$ *(and* v^* *can be attained).*

Proof. We claim that $v^* \leq \min_{x \in \bar{\Omega}} f(x) = f_{\text{atom}}$. Indeed, let x_0 be the point at which f achieves its minimum. We consider a sequence of sets $F_i \subset \overline{\Omega}$ such that $\lim_{i \to \infty} \sup_{y \in F_i} |y - x_0| = 0$, $P(F_i) \to \infty$ and $|F_i| \to 0$. We note that $g \in L^1(\Omega)$ and therefore $\frac{\int_{F_i} g(x) \mathrm{d}x}{P(F_i)} \to 0$. Hence, by the continuity of f we have that $v^* \leq \lim_{i \to 0} \frac{\int_{\partial^* F_i} f(x) \mathrm{d}\mathcal{H}^{n-1}}{P(F_i)} = f(x_0) = f_{\text{atom}}$, which proves our claim.

We have shown that $v^* \leq \min_{x \in \bar{\Omega}} f(x)$. Then either $v^* = \min_{x \in \bar{\Omega}} f(x)$ or $v^* < \min_{x \in \bar{\Omega}} f(x)$. We now assume that $v^* < \min_{x \in \bar{\Omega}} f(x)$. Then, if there exists a minimizing sequence E_i such that $P(E_i) \to \infty$, then $v^* = \lim_{i \to \infty} V(E_i) = \lim_{i \to \infty} \frac{\int_{\partial^* E} f(x) \mathrm{d}\mathcal{H}^{n-1}(x)}{P(E)} \geq \min_{x \in \overline{\Omega}} f(x) = f(x_0)$, contradicting that $v^* < \min_{x \in \bar{\Omega}} f(x)$. Similarly, if there exists E_i such that $|E_i| \to 0$, then $v^* \geq f(x_0) + \liminf_{i \to \infty} \frac{\int_{E_i} g^+(x) \mathrm{d}x}{P(E_i)} \geq f(x_0)$, contradicting that $v^* < \min_{x \in \bar{\Omega}} f(x)$. Hence, for any minimizing sequence E_i, $P(E_i)$ is uniformly bounded and $\inf_i |E_i| > 0$. Therefore, up to a subsequence, we have $E_i \to E_0$ in $L^1(\Omega)$, $\|D\chi_{E_i}\| \xrightarrow{*} \sigma$ and $\lim_{i \to \infty} P(E_i) = \sigma(\bar{\Omega}) = P_\infty > 0$. Following the exact argument in the proof of Theorem 5.3 we conclude that $v^* = V(E_0)$, and thus v^* is attained. $\square$

6. The cases $g \in L^{p,w}(\Omega)$

We recall that $L^{p,w}$ denotes the weak L^p space defined in (2.1). In order to motivate our interest in the weak L^p spaces, we first define the space of functions $M_p(\Omega)$, $p > 1$. We will show below that this space coincides with $L^{p,w}(\Omega)$. This provides a characterization of the space $L^{p,w}(\Omega)$ that will be used in the construction of Example A.1.

Definition 6.1. For $p > 1$, let

$$M_p(\Omega) := \left\{ g \text{ Lebesgue measurable } : \sup_{A \subset \Omega, A \text{ measurable}} \frac{\int_A |g| \mathrm{d}x}{|A|^{1-1/p}} < +\infty \right\}. \tag{6.1}$$

Remark 6.2. We immediately see from Definition 6.1 and the isoperimetric inequality that if $g \in M_n(\Omega)$ then $v^* > -\infty$, which is a necessary condition for the minimizer of (4.13) to exist. This motivates our analysis in this section. We also note that $M_p(\Omega) \subset M_q(\Omega)$, if $1 < q < p$. We will show in Remark 6.3 and Lemma 6.5 below that $M_p(\Omega) = L^{p,w}(\Omega)$, $p > 1$, and that $L^{p,w}(\Omega) \subset L^n(\Omega)$ for $p > n$. That is, our results in Sections 4 and 5 remain true if $g \in L^{p,w}$, $p > n$. However, we now ask the question whether our results remain true for the critical cases when $g \in L^{n,w}(\Omega) \setminus L^n(\Omega)$ or $g \in L^{p,w}(\Omega) \setminus L^{n,w}(\Omega), 1 \leq p < n$. In general, this is not true, as the two examples presented in this section will show.

We note that $M_p(\Omega) \subset L^1(\Omega)$ by choosing $A = \Omega$ in the definition above. If $g \in M_p(\Omega)$, we define

$$||g||_{M_p(\Omega)} := \sup_{A \subset \Omega, A \text{ measurable}} \frac{\int_A |g| \mathrm{d}x}{|A|^{1-1/p}}. \tag{6.2}$$

Clearly, $L^p(\Omega) \subset L^{p,w}(\Omega)$ for every $p \geq 1$, while the converse is not true. However, we have the following

Remark 6.3. $L^{p,w}(\Omega) \subset L^q(\Omega), \quad 1 \le q < p$. Indeed, given $g \in L^{w,p}(\Omega)$ and $1 \le q < p$ we have

$$\begin{aligned}\int_\Omega |g|^q &= \int_0^\infty qt^{q-1}|\{|g| > t\}|\mathrm{d}t \\ &= \int_0^1 qt^{q-1}|\{|g| > t\}|\mathrm{d}t + \int_1^\infty qt^{q-1}|\{|g| > t\}|\mathrm{d}t \\ &\le q|\Omega| + C\int_1^\infty qt^{q-p-1}\mathrm{d}t < \infty.\end{aligned}$$

Remark 6.4. If $1 \le q < p$ then $L^{p,w}(\Omega) \subset L^{q,w}(\Omega)$.

We now proceed to present a characterization of the weak L^p space:

Lemma 6.5. *$L^{p,w}(\Omega) = M_p(\Omega)$, $p > 1$.*

Proof. Let $g \in M_p(\Omega)$ and define $A := \{|g| > \lambda\}$. We have

$$\lambda|A| \le \int_A |g| \le \|g\|_{M_p(\Omega)} |A|^{1-1/p}.$$

Hence $\lambda|A|^{1/p} \le \|G\|_{M_p(\Omega)}$; that is, $|A| \le \frac{\|G\|^p_{M_p(\Omega)}}{\lambda^p}$ which yields $g \in L^{p,w}(\Omega)$. Conversely, if $g \in L^{p,w}(\Omega)$ then, for fixed $\lambda > 0$, we have that

$$\int_A |g| = \int_0^\infty |A \cap \{|g| > t\}|\mathrm{d}t \le \lambda|A| + \int_\lambda^\infty \frac{C}{t^p}\mathrm{d}t = \lambda|A| + \frac{C}{p-1}\lambda^{1-p}.$$

Hence the fact that $g \in M_p(\Omega)$ follows by letting $\lambda = |A|^{-1/p}$. □

For general $f(x, \boldsymbol{\nu})$, we showed in Theorem 4.18 that we can always approximate v^* with a sequence of sets with bounded perimeter (and in particular with convex polytopes with $n+1$ faces shrinking to a point if there exists a minimizing sequence E_i with unbounded perimeter or with $\liminf_{i\to\infty} |E_i| = 0$). Moreover, when f depends only on x, we showed in Theorem 5.3 that either the optimal value is attained (when $v^* < \min f$) or it can be approximated with a sequence of convex polytopes with $n+1$ faces shrinking to a point (actually, any sequence of sets shrinking to a point is also a minimizing sequence).

The next Example 6.6 gives a function $g \in L^{n,w}(\Omega) \setminus L^n(\Omega)$, for which Theorems 5.3 and 4.18 fail.

6.1. The case $g \in L^{n,w}(\Omega) \setminus L^n(\Omega)$

The following Example 6.6 shows that $v^* < \min_{x\in\overline{\Omega}} f(x)$ but v^* cannot be attained. Therefore, Theorem 5.3 fails for this example. Moreover, v^* can be approximated with a minimizing sequence of balls shrinking to the origin, but it can not be realized by a sequence of convex polytopes with $n+1$ faces, and hence Theorem 4.18 also fails.

Example 6.6. Let $f(x) = |x|$, and $g(x) = -\frac{n-1}{|x|}$, and assume $0 \in \Omega$, and $\bar{\Omega} \subset \mathbb{R}^n$. Note that $g \in L^{n,w}(\Omega)$. Indeed, for any $t > 0$, $t^n|\{x \in \Omega : \frac{1}{|x|} > t\}| = t^n\left|\left\{x \in \Omega : |x| < \frac{1}{t}\right\}\right| \le t^n|B(0, \frac{1}{t})| = c(n)$.

We note now that $g \notin L^n(\Omega)$. Therefore

$$g \in L^{n,w}(\Omega) \setminus L^n(\Omega).$$

We now proceed to show that $v^* = -1 < 0 = \min_{x\in\bar{\Omega}} f(x)$, but v^* cannot be attained. Furthermore, v^* can be approximated by balls shrinking to 0, but it cannot be approximated by polytopes with $n+1$ faces as Theorem 4.18 shows. Indeed, let $\Omega \subset B_R$, and choose $\gamma_\epsilon(x) \in C_c^\infty(B(0, R+\epsilon)\setminus\overline{B(0, \frac{\epsilon}{2})})$ such that $0 \le \gamma_\epsilon(x) \le 1$, $\gamma_\epsilon(x) = 1$ on $\{x : \epsilon \le |x| \le R\}$. Also, we can choose $\gamma_\epsilon(x)$ so that $|\nabla\gamma_\epsilon(x)| \le 4/\varepsilon$ if $\frac{\epsilon}{2} < |x| < \epsilon$.

Since $\gamma_\epsilon(x)=0$ in a $\frac{\epsilon}{2}$-neighbourhood of the origin, $\frac{x}{|x|}\gamma_\epsilon(x)$ is a smooth vector field with compact support in $\mathbb{R}^n$, thus by the divergence theorem for sets of finite perimeter, for any set $E\subset\overline{\Omega}$,

$$\int_{\partial^* E}\frac{x}{|x|}\gamma_\epsilon(x)\cdot\boldsymbol{\nu}_E(x)\mathrm{d}\mathcal{H}^{n-1}(x)=\int_E \operatorname{div}\left(\frac{x}{|x|}\gamma_\epsilon(x)\right)\mathrm{d}x=\int_E\left(\frac{n-1}{|x|}\gamma_\epsilon(x)+\frac{x}{|x|}\cdot\nabla\gamma_\epsilon(x)\right)\mathrm{d}x. \tag{6.3}$$

Since $|\nabla\gamma_\epsilon(x)|\le 4/\varepsilon$ when $\frac{\epsilon}{2}<|x|<\epsilon$, and $\nabla\gamma_\epsilon(x)=0$ on $(\Omega\setminus B(0,\epsilon))\cup B(0,\frac{\epsilon}{2})$,

$$\int_E\left|\frac{x}{|x|}\cdot\nabla\gamma_\epsilon(x)\mathrm{d}x\right|\le\int_{\frac{\epsilon}{2}<|x|<\epsilon}|\nabla\gamma_\epsilon(x)|\mathrm{d}x\to 0\quad\text{as}\quad\epsilon\to 0.$$

And since $\lim_{\epsilon\to 0}\gamma_\epsilon(x)=\chi_{B(0,R)}(x)$, $\mathcal{H}^{n-1}$-almost everywhere, we now let $\epsilon\to 0$ on both sides of (6.3), and use the dominated convergence theorem to obtain

$$\int_{\partial^* E\cap B_R}\frac{x}{|x|}\cdot\boldsymbol{\nu}_E(x)\mathrm{d}\mathcal{H}^{n-1}=\int_{E\cap B_R}\frac{n-1}{|x|}\mathrm{d}x$$

Since $E\subset\bar{\Omega}\subset B_R$, the last equality implies

$$\int_{\partial^* E}\frac{x}{|x|}\cdot\boldsymbol{\nu}_E(x)\mathrm{d}\mathcal{H}^{n-1}=\int_E\frac{n-1}{|x|}\mathrm{d}x \tag{6.4}$$

By (6.4),

$$\frac{\int_E\frac{n-1}{|x|}\mathrm{d}x}{P(E)}=\frac{\int_{\partial^* E}\frac{x}{|x|}\cdot\boldsymbol{\nu}_E(x)\mathrm{d}\mathcal{H}^{n-1}(x)}{P(E)}\le 1, \tag{6.5}$$

where equality holds if and only if $\frac{x}{|x|}\cdot\boldsymbol{\nu}_E(x)=1$, for $\mathcal{H}^{n-1}$-a.e. $x\in\partial^* E$, and thus if and only if E is equivalent to a ball contained in Ω centered at the origin, (see [24], Exercise 15.19). Let $V_1(E)=\frac{\int_{\partial^* E}|x|\mathrm{d}\mathcal{H}^{n-1}(x)}{P(E)}$ and $V_2(E)=\frac{\int_E\frac{n-1}{|x|}\mathrm{d}x}{P(E)}$. Thus, $V(E)=V_1(E)-V_2(E)$. Note that (6.5) implies that $V_2(E)\le 1$, and since $V_1(E)>0$ for every set E with positive measure, we conclude that $V(E)>-1$ for every $E\subset\overline{\Omega}$. Hence $v^*\ge -1$.

Actually, $v^*=-1$, since it is clear that $B_{1/i}(0)$ is a minimizing sequence. Note that v^* can not be attained because, for every $E\subset\overline{\Omega}$ with positive measure, $V_1(E)>0$ and $V_2(E)\ge -1$. Hence $V(E)>-1$, and therefore E can not be a minimizer.

We now claim that there exists a universal constant $\alpha(n)>0$ depending only on n such that $V(E)\ge -1+\alpha(n)$ holds for any convex polytope E with $n+1$ faces. Thus, convex polytopes with $n+1$ faces can not form a minimizing sequence. Indeed, it suffices to show there exists $\alpha(n)>0$ such that, for any convex polytope E with $n+1$ faces,

$$W(E):=\frac{\int_E\frac{n-1}{|x|}\mathrm{d}x}{P(E)}\le 1-\alpha(n). \tag{6.6}$$

If (6.6) is not true, then there exists a sequence $\{E_i\}$ of convex polytopes with $n+1$ faces such that $\lim_{i\to\infty}W(E_i)=1$. By (6.4) and the change of variables formula, $W(E_i)$ does not change up to a homothetic transformation, and thus we may assume $\inf_{i\ge 1}|E_i|>0$. Moreover, $P(E_i)$ has to be uniformly bounded for otherwise $W(E_i)\to 0$. Hence, by the compactness theorem for sets of finite perimeter, we may assume that $E_i\to E_0$ in L^1, $D\chi_{E_i}\overset{*}{\rightharpoonup}D\chi_{E_0}, |E_0|>0$, and $\|D\chi_{E_i}\|\overset{*}{\rightharpoonup}\sigma$. By the lower semi-continuity and since $|E_0|>0$, we have $\lim_{i\to\infty}P(E_i)=\sigma(\overline{\Omega})=P_\infty\ge P(E_0)>0$. Therefore,

$$\begin{aligned}1=\lim_{i\to\infty}W(E_i)=\lim_{i\to\infty}\frac{\int_{E_i}\frac{n-1}{|x|}\mathrm{d}x}{P(E_i)}&=\frac{\int_{E_0}\frac{n-1}{|x|}\mathrm{d}x}{P_\infty},\ \text{by the dominated convergence theorem,}\\ &\le\frac{\int_{E_0}\frac{n-1}{|x|}\mathrm{d}x}{P(E_0)}\le 1.\end{aligned} \tag{6.7}$$

Therefore, $P(E_0) = P_\infty$ and again, by ([24], Exercise 15.19), E_0 is a ball centered at origin, denoted as B. Therefore, we have found convex polytopes E_i with $n+1$ facets such that $|E_i| \to |B|$ and $P(E_i) \to P(B)$. Hence,

$$\lim_{i\to\infty} \frac{P(E_i)^n}{|E_i|^{n-1}} = \frac{P(B)^n}{|B|^{n-1}}\cdot \tag{6.8}$$

Now, by ([22], Cor. 18.2), among all proper convex polytopes in $\mathbb{R}^n$ with a given number of faces, there exist polytopes with minimum isoperimetric quotient. Thus, there exists a convex polytope E with $n+1$ faces such that

$$\limsup_{i\to\infty} \frac{P(E_i)^n}{|E_i|^{n-1}} \geq \frac{P(E)^n}{|E|^{n-1}} > \frac{P(B)^n}{|B|^{n-1}},$$

which contradicts (6.8). Therefore, we have shown that polytopes with $n+1$ faces can not form a minimizing sequence.

Remark 6.7. Example 6.6 shows the nonexistence of minimizer when $f = |x|$ and $g = -\frac{n-1}{|x|}$. However, if we let $f \equiv 0$ and g remains the same, then a similar argument shows that any ball $B(0,r) \subset \overline{\Omega}$ is a minimizer with $v^* = -1$. This says that, in the critical case $g \in L^{n,w}(\Omega) \setminus L^n(\Omega)$, one can not give a definite conclusion even for the existence of the optimization problem.

6.2. The case $g \in L^{p,w}(\Omega) \setminus L^{n,w}(\Omega)$, $1 \leq p < n$

We assume, without loss of generality, that Ω contains the origin. If g is nonnegative, then we we can directly apply Theorem 5.4 since $g^- = 0$. We now fix $1 \leq p < n$ and choose s such that $1 < s \leq \frac{n}{p}$. We consider the nonpositive function

$$g(x) = -\frac{1}{|x|^s}\cdot$$

Note that $\{x \in \Omega : |g| > t\} = \{x \in \Omega : \frac{1}{|x|^s} > t\} = \{x \in \Omega : |x| < \frac{1}{t^{\frac{1}{s}}}\} = B(0, t^{\frac{1}{s}})$. Thus, if $t > 1$, then $t^p|\{|g| > t\}| \leq \frac{w_{n-1}}{n} t^p \frac{1}{t^{n/s}} = \frac{w_{n-1}}{n} t^{p-\frac{n}{s}} \leq \frac{w_{n-1}}{n}$, and if $t \leq 1$, then $t^p|\{|g| > t\}| \leq |\Omega|$. Therefore $g \in L^{p,w}(\Omega)$. We now show that $g \notin L^{n,w}(\Omega)$. For t large enough, $B(0, t^{\frac{1}{s}}) \subset \Omega$. Hence, $t^n|\{|g| > t\}| = t^n \frac{\omega_{n-1}}{n} (\frac{1}{t^{\frac{1}{s}}})^n = \frac{\omega_{n-1}}{n} t^{n-\frac{n}{s}} \to \infty$ as $t \to \infty$. We have proved that

$$g \in L^{p,w}(\Omega) \setminus L^{n,w}(\Omega).$$

Let $B(0,r)$ be the ball with radius r contained in Ω centered at the origin. Since

$$\int_{B(0,r)} g = -\int_0^r \frac{1}{\rho^s} w_{n-1} \rho^{n-1} \mathrm{d}\rho = \frac{w_{n-1}}{s-n} r^{n-s},$$

we obtain

$$\frac{\int_{B(0,r)} g}{P(B(0,r))} = \frac{1}{s-n} r^{1-s} \to -\infty,$$

hence for the g chosen above we have $v^* = -\infty$.

Remark 6.8. We now ask the question whether it is true that if g is negative and $g \notin L^{n,w}(\Omega)$, then v^* is always $-\infty$. The answer is no. For example, if $n = 2$, the Example A.1 presented in the appendix shows that, for any $1 < p < 2$, we can find g such that $g \notin L^{p,w}(\Omega)$ (and hence $g \notin L^{n,w}(\Omega)$ since these weak spaces get larger and larger as p converges to 1) but $v^* > -\infty$. However, even though the infimum in (1.2) is finite for the functions g constructed in Example A.1, we can only prove our main theorems in Sections 4 and 5 under the assumption $g \in L^n(\Omega)$. Thus, the examples in this section show that the conditions imposed on g in this paper are appropriate.

Appendix A.

In the first part of this appendix we construct the example discussed in the previous Remark 6.8. In the second part we prove a result of linear algebra that is used in Proposition 4.9.

Example A.1. We let $n = 2$ and $\Omega = (0,1) \times (0,1)$. Fix $1 < p < 2$. We now show that there exists $g \leq 0$ satisfying $|g| \notin M_p(\Omega) = L^{p,w}(\Omega)$, but $v^* > -\infty$. Let $x_k = k^{-\alpha}$, $k = 1, 2, \ldots$, where $\alpha > 0$ which will be specified later. We will use the notation $a_k \sim b_k$ which means that there exist constants $C_1(\alpha), C_2(\alpha)$ such that $C_1(\alpha)a_k \leq b_k \leq C_2(\alpha)a_k$.

We let $Q_k = [x_{k+1}, x_k) \times [x_{k+1}, x_k)$, and h is a function defined as $h \equiv k^{1+\alpha}$, on Q_k, and zero otherwise. We let $g = -h$ and $E_K = \bigcup_{k=K}^{\infty} Q_k$. Since $x_k = k^{-\alpha}$ and $x_k - x_{k+1} = k^{-\alpha} - (k+1)^{-\alpha} = \int_k^{k+1} \alpha s^{-\alpha-1} ds \sim k^{-\alpha-1}$, then we have the following:

$$|Q_k| \sim k^{-2\alpha-2},\ P(Q_k) \sim k^{-1-\alpha},\ \int_{Q_k} h \sim k^{-1-\alpha}, \tag{A.1}$$

and thus

$$|E_K| = \sum_{k=K}^{\infty} k^{-2\alpha-2} \sim \int_K^{\infty} s^{-2\alpha-2} \mathrm{d}s \sim K^{-1-2\alpha}, \text{and} \int_{E_K} h = \sum_{k=K}^{\infty} k^{-1-\alpha} \sim \int_K^{\infty} s^{-1-\alpha} \mathrm{d}s \sim K^{-\alpha}.$$

Define $t := 1 - \frac{1}{p}$ and note that $t \in (0, 1/2)$. If we now choose $0 < \alpha < \frac{t}{1-2t}$, then

$$\frac{\int_{E_K} h}{|E_K|^t} \sim \frac{K^{-\alpha}}{(K^{-1-2\alpha})^t} = K^{t-\alpha(1-2t)} \to \infty. \tag{A.2}$$

Therefore,

$$|g| \notin M_p(\Omega).$$

Now suppose $E \subset \Omega$ is a polytope. Let $\Omega \setminus \overline{\cup_{k=1}^{\infty} Q_k} = A_1 \cup A_2$, where A_1 is the connected component in Ω above $\cup_{k=1}^{\infty} Q_k$, and A_2 is the connected component in Ω below $\cup_{k=1}^{\infty} Q_k$. Let $C_{k,1}, C_{k,2}, C_{k,3}, C_{k,4}$ be the left, the top, the right, and the bottom side of each Q_k respectively, and let $C_i = \cup_{k=1}^{\infty} C_{k,i}$, $i = 1, 2, 3, 4$. Let π_1 be the projection of $\partial E \cap A_1$ on C_1 defined as follows:

$$\pi_1(x_1, y_1) \text{ is the unique point } (x_2, y_2) \text{ on } C_1 \text{ such that } x_2 \geq x_1,\ y_1 = y_2.$$

Let π_2 be the projection of $\partial E \cap A_1$ on C_2 defined as follows:

$$\pi_2(x_1, y_1) \text{ is the unique point } (x_2, y_2) \text{ on } C_2 \text{ such that } y_2 \leq y_1,\ x_1 = x_2.$$

In a similar way we define π_3 to be the projection of $\partial E \cap A_2$ on C_3, and π_4 be the projection of $\partial E \cap A_2$ on C_4. That is,

$$\pi_3(x_1, y_1) \text{ is the unique point } (x_2, y_2) \text{ on } C_3 \text{ such that } x_2 \leq x_1,\ y_1 = y_2.$$

and

$$\pi_4(x_1, y_1) \text{ is the unique point } (x_2, y_2) \text{ on } C_4 \text{ such that } y_2 \geq y_1,\ x_1 = x_2.$$

Geometrically, π_1 is the projection to the right on the left sides C_1 of the Q_k's, π_2 is the projection on the top sides C_2 of the Q_k's, π_3 is the projection to the left on the right sides C_3 of the Q_k's, and π_4 is the projection on the bottom sides C_4 of the Q_k's.

Note that $\mathring{E} = E^1$, $\partial^m E = \partial E$, and $(\cup_{k=1}^{\infty} Q_k)^0 = (\overline{\cup_{k=1}^{\infty} Q_k})^c = A_1 \cup A_2$. For any $x \in C_1 \cap E^1 = C_1 \cap \mathring{E}$, the horizontal ray starting from x to the left must intersect $\partial E \cap A_1$, thus $\pi_1^{-1}(x) \in \partial E \cap A_1$. Therefore we can conclude that $\pi_1^{-1}(C_1 \cap E^1) \subset \partial E \cap A_1$, thus $C_1 \cap E^1 \subset \pi_1(\partial E \cap A_1)$. We now apply the inequality

$\mathcal{H}^s(f(S)) \leq \mathrm{Lip}(f)^s \mathcal{H}^s(S)$ (see [24], Prop. 3.5), for any Lipschitz function f, to the Lipschitz function π_1. Hence we have:

$$\mathcal{H}^1(\pi_1(\partial E \cap A_1)) \leq \mathcal{H}^1(\partial E \cap A_1), \tag{A.3}$$

and therefore

$$\mathcal{H}^1(C_1 \cap E^1) \leq \mathcal{H}^1(\pi_1(\partial E \cap A_1)) \leq \mathcal{H}^1(\partial E \cap A_1). \tag{A.4}$$

Similarly,

$$\mathcal{H}^1(C_2 \cap E^1) \leq \mathcal{H}^1(\partial E \cap A_1). \tag{A.5}$$

Hence

$$\mathcal{H}^1(\partial E \cap A_1) \geq \frac{1}{2}\left(\mathcal{H}^1(C_1 \cap E^1) + \mathcal{H}^1(C_2 \cap E^1)\right) \tag{A.6}$$

Also, the same reasoning implies

$$\mathcal{H}^1(C_3 \cap E^1) \leq \mathcal{H}^1(\partial E \cap A_2), \quad \mathcal{H}^1(C_4 \cap E^1) \leq \mathcal{H}^1(\partial E \cap A_2), \tag{A.7}$$

which implies

$$\mathcal{H}^1(\partial E \cap A_2) \geq \frac{1}{2}\left(\mathcal{H}^1(C_3 \cap E^1) + \mathcal{H}^1(C_4 \cap E^1)\right). \tag{A.8}$$

Therefore,

$$\begin{aligned} P\left(E; (\cup_{k=1}^{\infty} Q_k)^0\right) &= \mathcal{H}^1(\partial E \cap A_1) + \mathcal{H}^1(\partial E \cap A_2) \\ &\geq \frac{1}{2}\left(\mathcal{H}^1(C_1 \cap E^1) + \mathcal{H}^1(C_2 \cap E^1) + \mathcal{H}^1(C_3 \cap E^1) + \mathcal{H}^1(C_4 \cap E^1)\right), \text{ by (A.6) and (A.8)}, \\ &= \frac{1}{2}\mathcal{H}^1\left(\partial(\cup_{k=1}^{\infty} Q_k) \cap E^1\right) \\ &= \frac{1}{2} P\left(\cup_{k=1}^{\infty} Q_k; E^1\right), \text{ since } \mathcal{H}^1\left(\partial(\cup_{k=1}^{\infty} Q_k)\right) = \mathcal{H}^1\left(\partial^*(\cup_{k=1}^{\infty} Q_k)\right). \end{aligned} \tag{A.9}$$

Since

$$P(E) = P\left(E; (\cup_{k=1}^{\infty} Q_k)^0\right) + P\left(E; (\cup_{k=1}^{\infty} Q_k)^1\right) + \mathcal{H}^{n-1}(\partial^* E \bigcap \partial^*(\cup_{k=1}^{\infty} Q_k)) \tag{A.10}$$

and (see [24], Thm. 16.3):

$$P\left(E \bigcap (\cup_{k=1}^{\infty} Q_k)\right) \leq P\left(\cup_{k=1}^{\infty} Q_k; E^1\right) + P\left(E; (\cup_{k=1}^{\infty} Q_k)^1\right) + \mathcal{H}^{n-1}(\partial^* E \bigcap \partial^*(\cup_{k=1}^{\infty} Q_k)), \tag{A.11}$$

by comparing (A.10) and (A.11), and using (A.9), we have

$$\begin{aligned} 2P\,(E) &\geq P\left(E \bigcap (\cup_{k=1}^{\infty} Q_k)\right) \\ &= \mathcal{H}^{n-1}\left(\partial^m\left(\bigcup_{k=1}^{\infty} (E \cap Q_k)\right)\right), \text{ by Federer's theorem (see [24], Thm. 16.2)}, \\ &= \sum_{k=1}^{\infty} \mathcal{H}^{n-1}\left(\partial^m(E \cap Q_k)\right), \text{ since } \mathcal{H}^1(\overline{Q_i} \cap \overline{Q_j}) = 0, \\ &= \sum_{k=1}^{\infty} P(E \cap Q_k). \end{aligned}$$

Also, since h is supported in $\cup_{k=1}^{\infty} Q_k$, we have

$$\int_E h = \sum_{k=1}^{\infty} \int_{E \cap Q_k} h.$$

Therefore,

$$\begin{aligned}\frac{\int_E h}{P(E)} &\leq \frac{\sum_{k=1}^{\infty}\int_{E\cap Q_k} h}{\frac{1}{2}\sum_{k=1}^{\infty}P(E\cap Q_k)}\\ &\leq 2\sup_k \frac{\int_{E\cap Q_k} h}{P(E\cap Q_k)}\\ &\leq 2\sup\left\{\frac{\int_F h}{P(F)} : F\subset Q_k, k\,=\,1,2,\dots\right\}\\ &\leq 2\sup\left\{\frac{\int_F h}{|F|^{\frac{1}{2}}} : F\subset Q_k, k\,=\,1,2,\dots\right\},\ \text{by the isoperimetric inequality.}\end{aligned}$$

Note that for any $F \subset Q_k$,

$$\frac{\int_F h}{|F|^{\frac{1}{2}}} = k^{1+\alpha}|F|^{\frac{1}{2}} \leq k^{1+\alpha}|Q_k|^{\frac{1}{2}} \sim 1,$$

hence

$$\sup\left\{\frac{\int_E h}{P(E)} : E \subset \Omega,\, E \text{ is a polytope}\right\} < \infty.$$

Now for any set of finite perimeter $E \subset \Omega$, by an approximation theorem (see [24], Rem. 13.13), there exist a sequence of polytopes $E_j \subset \Omega$, such that $E_j \to E$ in L^1, and $P(E_j) \to P(E)$. Thus, by the dominated convergence theorem,

$$\sup\left\{\frac{\int_E h}{P(E)} : E \subset \Omega\right\} < \infty.$$

Therefore, $v^* > -\infty$.

The following results are proved in Proposition 4.9.

Lemma A.2. *Assume A is a real $n \times m$ matrix, $n < m$, and $\operatorname{rank}(A) < n$. For $b \in \mathbb{R}^n$, assume $AX = b$ has a solution $X = (x_1, \dots, x_m)^T$. Then for any $\epsilon > 0$, there exist a $n \times m$ matrix A_ϵ with $\operatorname{rank}(A_\epsilon) = n$ such that $\|A - A_\epsilon\| < \epsilon$ and $A_\epsilon X = b$ for the same X. Here $\|\cdot\|$ denotes the spectral matrix norm.*

Proof. We may first assume $\operatorname{rank}(A) = n-1$. Using the SVD decomposition, we write $A = U\Sigma V^T$, where U is a $n \times n$ orthogonal matrix, V is a $m \times m$ orthogonal matrix, and Σ is the rectangular diagonal matrix. Let $Y = V^T X = (y_1, \dots, y_m)^T$ and $c = U^{-1}b = (c_1, \dots, c_n)^T$, thus $\Sigma Y = c$, and $c_n = 0$.

If $y_j = 0$ for some $j \in \{n, n+1, \dots, m\}$, then replace the j-th column of Σ with $(0, \dots, 0, \epsilon)^T$ and denote the new matrix by Σ_ϵ. Clearly $\operatorname{rank}(\Sigma_\epsilon) = n$, $\|\Sigma - \Sigma_\epsilon\| < \epsilon$ and $\Sigma_\epsilon Y = c$. Let $A_\epsilon = U\Sigma_\epsilon V^T$, hence $\operatorname{rank}(A_\epsilon) = n$, $A_\epsilon X = b$ and $\|A - A_\epsilon\| \leq \|U\|\|\Sigma - \Sigma_\epsilon\|\|V\| = \epsilon$.

If on the other hand $y_j \neq 0$ for any $j \in \{n, n+1, \dots, m\}$, then replace the n-th column of Σ with $(0, \dots, 0, -\frac{y_{n+1}}{y_n + y_{n+1}}\epsilon)^T$ and the $(n+1)$-th column of Σ with $(0, \dots, 0, \frac{y_n}{y_n + y_{n+1}}\epsilon)^T$. Denote the new matrix by Σ_ϵ. One can easily verify that $\operatorname{rank}(\Sigma_\epsilon) = n$, $\|\Sigma - \Sigma_\epsilon\| < \epsilon$ and $\Sigma_\epsilon Y = c$. Let $A_\epsilon = U\Sigma_\epsilon V^T$. It is easy to see A_ϵ satisfies the claim of the lemma.

If $\operatorname{rank}(A) < n-1$, we can similarly construct A_ϵ such that $\operatorname{rank}(A_\epsilon) = \operatorname{rank}(A)+1$ with the desired property. If still $\operatorname{rank}(A_\epsilon) < n-1$, we can construct another perturbed matrix strictly improving the rank. After a finite steps, we conclude the claim of the lemma. □

Corollary A.3. *Assume $n < m$ and $\{v_j\}_{j=1}^m \subset \mathbb{S}^{n-1} \subset \mathbb{R}^n$. If there exist $\{\lambda_j\}_{j=1}^m$, $\lambda_j \geq 0$, such that $\sum_{j=1}^m \lambda_j v_j = 0$, then for any $\epsilon > 0$, there exist $\{v_j^\epsilon\}_{j=1}^m \subset S^{n-1}$ and $\{\lambda_j^\epsilon\}_{j=1}^m$, $\lambda_j^\epsilon \geq 0$, such that $\{v_j^\epsilon\}_{j=1}^m$ linearly span $\mathbb{R}^n$, $|\lambda_j - \lambda_j^\epsilon| < \epsilon$, $|v_j - v_j^\epsilon| < \epsilon$ and $\sum_{j=1}^m \lambda_j^\epsilon v_j^\epsilon = 0$.*

Proof. If $\{v_j\}_{j=1}^m$ span $\mathbb{R}^n$ the result is trivial. If not, let $X = (\lambda_1, \ldots, \lambda_m)^T$ and A be the $n \times m$ matrix $(v_1, \ldots, v_m)$. The hypothesis imply $AX = 0$. By Lemma A.2, there exists $\tilde{A} = (\tilde{v}_1, \ldots, \tilde{v}_m)$ such that $\operatorname{rank}(\tilde{A}) = n$, $\|A - \tilde{A}\| < \epsilon$ and $\tilde{A}X = 0$. Thus, $\{\tilde{v}_j\}_{j=1}^m$ linearly span $\mathbb{R}^n$ and $\sum_{j=1}^m \lambda_j \tilde{v}_j = 0$. Since the spectral matrix norm is equivalent to the Frobenius matrix norm, we derive $|\tilde{v}_j - v_j| < \epsilon$ up to a constant factor. Let $v_j^\epsilon = \frac{\tilde{v}_j}{|\tilde{v}_j|}$ and $\lambda_j^\epsilon = \lambda_j |\tilde{v}_j|$, then clearly $\{v_j^\epsilon\}_{j=1}^m$ also linearly span $\mathbb{R}^n$ and $\sum_{j=1}^m \lambda_j^\epsilon v_j^\epsilon = 0$. Since $v_j \in S^{n-1}$, one can easily verify that $|\lambda_j - \lambda_j^\epsilon| < \epsilon$ and $|v_j - v_j^\epsilon| < \epsilon$ up to a constant factor. Therefore we conclude the corollary. □

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