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STABILITY OF OBSERVATIONS OF PARTIAL DIFFERENTIAL EQUATIONS UNDER UNCERTAIN PERTURBATIONS

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Abstract. We demonstrate the stability of observability estimates for solutions to wave and Schrödinger equations subjected to additive perturbations. This work generalises recent averaged observability/control results by allowing for systems consisting of operators of different types. We also consider the simultaneous observability problem by which one tries to estimate the energy of each component of a system under consideration. Our analysis relies on microlocal defect tools, in particular on standard H-measures when the main system dynamic is governed by the wave operator, and parabolic H-measures in the case of the Schrödinger operator.

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1. Introduction

The notion of averaged control has been recently introduced in [12, 17], both for parameter-dependent ODEs and for systems of PDEs with variable coefficients. Its goal is to control the average (or more generally a suitable linear combination) of system components by a single control. The problem is relevant in practice when the control has to be chosen independently of the coefficient.

This notion is equivalent to averaged observability, by which the energy of the system is recorded by observing the average of solutions on a suitable subdomain.

In this paper, we investigate a more general problem based on a system whose first component represents its main dynamic, while the others correspond to perturbations. Assuming that the main component is observable, we explore conditions by which that property remains stable under additive perturbations.

In general, operators entering the system are not assumed to be of the same type. In the special case of a system consisting of operators of the same type, the result corresponds to the averaged control of the system, thus supporting results obtained in [12].

We apply these methods to the simultaneous observability problem as well, estimating the energy of all system components by observing their average. The corresponding dual problem consists of controlling individual components of the adjoint system by means of the same control (*cf.* [1]).

In both cases, the observability result is obtained in a standard manner consisting of two steps. First, the initial energy is estimated using the observation term plus lower order terms. In accordance with the literature, we refer to this result as a relaxed observability inequality (*cf.* [6,12]). Second, lower order terms are dropped by way of a standard unique continuation argument in the case of simultaneous observability, while a special link between the initial data of different system components is required for averaged observability.

Keywords and phrases. Averaged control, robust observability, parabolic H-measures.

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The study of this problem explores microlocal analysis tools, in particular H-measures and their variants. H-measures, introduced independently in [8,16], are defect tools measuring the deflection of weak from strong convergence of L^2 sequences. Since their introduction, they have been successfully applied in many domains, including the generalisation of compensated compactness results to equations with variable coefficients [8,16] and applications in control theory [5, 6, 12]. Most of these applications employ the so-called *localisation principle* providing constraints on the support of H-measures (*e.g.* [16]), and the proofs in this paper rely on it as well. More elaborated introduction to H-measures and the corresponding localisation principle is given in the Appendix.

The paper is organised as follows. In the next section, we provide an averaged observability result for a system whose main dynamic is governed by the wave operator. The finite system is analysed first, followed by generalisations to an infinite discrete setting. Application of the approach to simultaneous observability is provided in Section 2.3. Section 3 is devoted to observation of the Schrödinger equation under perturbations determined either by a hyperbolic or by a parabolic type operator. In the latter case, parabolic H-measures (generalisation of original H-measures to a parabolic setting) are explored. Their definition and basic properties used in the note are provided in the Appendix. The paper is closed with concluding remarks and pointers to open and related problems.

2. Observation of the wave equation under uncertain perturbations

2.1. Averaged observability

We analyse the problem of recovering the energy of the wave equation by observing an additive perturbation of the solution. The perturbation is determined by a differential operator P_2, in general different from the wave operator.

More precisely, we consider the following system of equations:

$$\begin{aligned} P_1 u_1 = \partial_{tt} u_1 - \mathsf{div}\,(\mathbf{A}_1(t,\mathbf{x})\nabla u_1) &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ P_2 u_2 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ u_1 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \partial\Omega \\ u_1(0,\cdot) &= \beta_0 \in \mathrm{L}^2(\Omega) \\ \partial_t u_1(0,\cdot) &= \beta_1 \in \mathrm{H}^{-1}(\Omega), \end{aligned} \tag{2.1}$$

where Ω is an open, bounded set in $\mathbf{R}^d$, $\mathbf{A}_1$ is a bounded, positive definite real matrix field, while P_2 is some nearly arbitrary, differential operator (precise conditions on it are given below). In the sequel we shall also use the notation $\mathcal{A}_1 = -\mathsf{div}\,(\mathbf{A}_1\nabla)$ for the elliptic part of P_1.

For the moment, we specify neither initial nor boundary conditions for the second equation; we assume that corresponding problem is well-posed and that it admits an L^2 solution (in both variables). We assume that the coefficients of both of the operators are bounded and continuous, and that P_2 allows for complex-valued coefficients as well.

Proposition 2.1. *Suppose that there is a constant $\tilde{C}$, time T and an open subdomain ω such that for any choice of initial conditions β_0, β_1 the solution u_1 of (2.1) satisfies*

$$E_1(0) := \|\beta_0\|^2_{\mathrm{L}^2} + \|\beta_1\|^2_{\mathrm{H}^{-1}} \le \tilde{C}\int_0^T\int_\omega |u_1|^2 \mathrm{d}\mathbf{x}\mathrm{d}t. \tag{2.2}$$

In addition, we assume that characteristic sets $\{p_i(t,\mathbf{x},\tau,\boldsymbol{\xi}) = 0\}, i = 1,2$ have no intersection for $(t,\mathbf{x}) \in \langle 0,T\rangle \times \omega, (\tau,\boldsymbol{\xi}) \in \mathrm{S}^d$, where p_i represents the principal symbol of the operator P_i.

Then for any $\theta_1, \theta_2 \in \mathbf{R}, \theta_1 \neq 0$ there exists a constant $\tilde{C}_{\theta_1}$ such that the relaxed observability inequality

$$E_1(0) \le \tilde{C}_{\theta_1}\left(\int_0^T\int_\omega |\theta_1 u_1 + \theta_2 u_2|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \|\beta_0\|^2_{\mathrm{H}^{-1}} + \|\beta_1\|^2_{\mathrm{H}^{-2}}\right) \tag{2.3}$$

holds for any pair of solutions (u_1, u_2) to (2.1).

Proof. We argue by contradiction. Assume there exists a sequence of solutions u_1^n, u_2^n such that

$$E_1^n(0) > n\left(\int_0^T \int_\omega |\theta_1 u_1^n + \theta_2 u_2^n|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \|\beta_0^n\|_{\mathrm{H}^{-1}}^2 + \|\beta_1^n\|_{\mathrm{H}^{-2}}^2\right), \tag{2.4}$$

where $E_1^n(0) = \|\beta_0^n\|_{\mathrm{L}^2}^2 + \|\beta_1^n\|_{\mathrm{H}^{-1}}^2$ is the initial energy of u_1^n.

As the problem under consideration is linear, without loss of generality we may assume that $E^n(0) = 1$. Thus (2.4) implies that $\|\beta_0^n\|_{\mathrm{H}^{-1}}^2 + \|\beta_1^n\|_{\mathrm{H}^{-2}}^2 \to 0$, resulting in the weak convergence $(\beta_0^n, \beta_1^n) \rightharpoonup (0,0)$ in $\mathrm{L}^2(\Omega) \times \mathrm{H}^{-1}(\Omega)$. Therefore, the solutions (u_1^n) converge weakly to zero in $\mathrm{L}^2(\langle 0,T\rangle \times \Omega)$ as well. In order to obtain a contradiction, we have to show that the last convergence is strong, at least on the observability region.

From the contradiction assumption (2.4) we have that the H-measure ν associated to a subsequence of $(\theta_1 u_1^n + \theta_2 u_2^n)$ vanishes on $\langle 0,T\rangle \times \omega$. Furthermore, it is of the form

$$\nu = \theta_1^2 \mu_1 + \theta_2^2 \mu_2 + \theta_1 \theta_2 2\Re \mu_{12},$$

where on the right hand side the elements of the matrix measure associated to the vector subsequence of (u_1^n, u_2^n) are listed, with μ_{12} denoting the off-diagonal element. Note that (u_2^n) is bounded in $\mathrm{L}^2(\langle 0,T\rangle \times \omega)$, since that is the case for (u_1^n) (by boundedness of the initial data), and for the linear combination $(\theta_1 u_1^n + \theta_2 u_2^n)$ (by the contradiction assumption (2.4)), which allows one to associate an H-measure to it.

According to the localisation property for H-measures (Thm. A.3), each μ_j is supported within the corresponding characteristic set $\{p_i(t, \mathbf{x}, \tau, \boldsymbol{\xi}) = 0\}$, $i = 1, 2$, which, by assumption, are disjoint on the observability region. On the other hand, from the very definition of matrix H-measures it follows that off-diagonal entries are dominated by the corresponding diagonal elements (Cor. A.4). More precisely, it holds that $\operatorname{supp} \mu_{12} \subseteq \operatorname{supp} \mu_1 \cap \operatorname{supp} \mu_2$, implying that $\mu_{12} = 0$ on the observability region.

Thus, we obtain

$$\nu = \theta_1^2 \mu_1 + \theta_2^2 \mu_2 = 0 \qquad \text{on} \quad \langle 0,T\rangle \times \omega.$$

As μ_1 and μ_2 are positive measures and $\theta_1 \neq 0$, it follows that μ_1 vanishes on $\langle 0,T\rangle \times \omega$ as well. Thus we obtain strong convergence of (u_1^n) in $\mathrm{L}^2(\langle 0,T\rangle \times \omega)$, which together with the assumption of constant, non-zero initial energy contradicts the observability estimate (2.2). □

Remark 2.2. The last result provides surprising stability of the observability estimate (2.2) under uncertain perturbations, up to compact reminders. Essentially, the only requirement for the perturbation is separation of the characteristic sets. This implies that the wave component can be observed robustly when adding unknown perturbations, up to a finite number of low frequencies.

In the next step we would like to obtain the observability inequality for initial energy $E_1(0)$ by removing compact terms in (2.3). To this effect, we have to specify some additional constraints on the problem for the perturbation u_2.

We take P_2 to be an evolution operator of the form

$$P_2 = (\partial_t)^k + c_2(\mathbf{x})\mathcal{A}_1, \quad k \in \mathbf{N}, \tag{2.5}$$

where $\mathcal{A}_1$ is the elliptic part of the wave operator P_1, while c_2 is a bounded and continuous complex-valued function. In addition, the problem for the perturbation u_2 is accompanied by a series of initial conditions

$$\left((\partial_t)^j u_2\right)(0) = \gamma_j \in \mathrm{H}^{-j}(\Omega), \quad j = 0, \dots, k-1. \tag{2.6}$$

Then the following result holds.

Theorem 2.3. *Assume that the coefficients of the operator P_1 are bounded and continuous, and that the observability inequality (2.2) holds for any solution u_1 to (2.1).*

In addition, consider the perturbation operator P_2 of the form (2.5). In the case $k = 2$ the separation of coefficients $c_2(\mathbf{x}) - 1 \neq 0$ is supposed on ω.

For given $\theta_1, \theta_2 \in \mathbf{R}, \theta_1 \neq 0$ assume that the initial value of a solution u_2 to (2.1), (2.5)–(2.6) is related by a linear operator to $u_1(0)$ in such a way that whenever $((\theta_1 u_1(0) + \theta_2 u_2(0))|_\omega = 0)$ then $\big(u_1(0)|_\omega = u_2(0)||_\omega = 0\big)$, and analogously for the initial first order time derivatives (for $k \geq 2$).

Then there is a positive constant C_{θ_1} such that for any $(\beta_0, \beta_1) \in \mathrm{L}^2(\Omega) \times \mathrm{H}^{-1}(\Omega)$ the following observability inequality holds

$$E_1(0) \leq C_{\theta_1} \int_0^T \int_\omega |\theta_1 u_1 + \theta_2 u_2|^2 \mathrm{d}\mathbf{x}\mathrm{d}t. \tag{2.7}$$

Remark 2.4. Note that the above assumptions directly imply that characteristic sets of P_1 and P_2 are disjoint on the observability region. Indeed, for P_2 being an evolution operator of order $k \neq 2$, its principal symbol equals 0 only in poles $\boldsymbol{\xi} = \mathsf{0}$ (case $k = 1$), or on the equator $\tau = 0$ (case $k > 2$) of the unit sphere in the dual space, where $p_1 = \tau^2 - \mathbf{A}_1(t, \mathbf{x})\boldsymbol{\xi}^2$ differs from zero.

In the case $k = 2$ separation of the characteristic sets is provided by the assumption $c_2(\mathbf{x}) \neq 1$ on ω.

Proof. As in the proof of Proposition 2.1, let us suppose the contrary. Then there exists a sequence of solutions u_1^n, u_2^n to (2.1) such that $E_1^n(0) = 1$ and

$$\int_0^T \int_\omega |\theta_1 u_1^n + \theta_2 u_2^n|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \longrightarrow 0. \tag{2.8}$$

Thus the corresponding weak limits satisfy both the equation $P_i u_i = 0$, as well as the relation

$$\theta_1 u_1 + \theta_2 u_2 = 0 \tag{2.9}$$

on the observability region. As solutions u_i are continuous with respect to time, it follows that $(\theta_1 u_1(0) + \theta_2 u_2(0))|_\omega = 0$. Taking into account assumptions on the initial data, one gets $u_1(0) = 0$ on ω. Similarly, one deduces the same conclusion for $\partial_t u_1(0)$.

Assumptions on the operators $P_i, i = 1, 2$ ensure that corresponding characteristic sets do not intersect. By applying the localisation property of H-measures as in the proof of Proposition 2.1 we get that u_1^n converges to u_1 strongly on $\langle 0, T\rangle \times \omega$.

It remains to show that the limit u_1 vanishes on the observability region, which, together with the assumption of the constant non-zero initial energy, will contradict the observability assumption (2.2).

We split the rest of the proof into several cases.

(a) $(\mathbf{k = 2})$. Due to the relation (2.9) it follows that

$$(c_2 - 1)\partial_{tt} u_1 = 0, \qquad (t, \mathbf{x}) \in \langle 0, T\rangle \times \omega.$$

As $|c_2 - 1| > 0$, and the initial data are 0 on ω, it implies $u_1 = 0$ on the observability region.

(b) $(\mathbf{k = 1})$. Relation (2.9) implies

$$\partial_t u_1 = c_2(\mathbf{x})\partial_{tt} u_1,$$

which together with $u_1(0, \cdot) = \partial_t u_1(0, \cdot) = 0$ on ω provides the claim.

(c) $(\mathbf{k > 2})$. Similarly as above we obtain

$$\partial_t^{k-2}(\partial_{tt} u_1) = c_2(\mathbf{x})\partial_{tt} u_1.$$

As $\partial_{tt} u_1(0, \cdot) = -\mathcal{A}_1 u_1(0, \cdot) = 0$ on ω, and similarly for the higher order derivatives, the claim follows. $\square$

Remark 2.5. Several remarks are in order.

- Under some additional assumptions (a smooth boundary $\partial\Omega$ and time independent coefficient $\mathbf{A}_1$), the observability estimate (2.2) on a solution of the wave equation is equivalent to the Geometric Control Condition (GCC, [4]), stating that projection of each bicharacteristic ray on a physical space has to enter the observability region in time T.
- An example of an operator linking the initial data in the above statement is the operator of multiplication by a bounded function α satisfying $\alpha(\mathbf{x}) \neq -\theta_1/\theta_2$ for $\mathbf{x} \in \omega$.
- The constants $\tilde{C}_{\theta_1}$ and C_{θ_1} from (2.3) and (2.7), respectively, are θ_2-independent and can be taken uniformly for θ_1 ranging within a compact set excluding the origin.
- Note that the observability result (2.7) is weaker than the one required in the simultaneous control problem, where one has to estimate the initial energy of all components entering the system. Of course, the assumptions in the latter case are stronger, as one has to assume that the observability set satisfies the GCC for the second component as well.

If P_2 is a wave operator accompanied by a homogeneous boundary condition, then the obtained observability result (2.7) corresponds to the notion of average controllability. The equivalence between two notions, of averaged observability and averaged controllability, is described into details in [18] for finite dimensional systems, while it has been studied in [12] for the case of the wave equation. In particular, if the operator linking the initial data is of the form $\partial_t^j u_2(0) = \alpha\, \partial_t^j u_1(0), j = 0, 1$ for $\alpha \in \mathbf{R}$ and $\theta_1 + \alpha\theta_2 \neq 0$, then the result of Theorem 2.3 is equivalent to the controllability of the combination $v_1 + \alpha v_2$ of solutions to the adjoint system under a single control. Meanwhile, the relaxed observability result (2.3) in that case corresponds to the average controllability of the adjoint system up to a finite number of low frequencies.

The last theorem generalises the results of [12] by allowing for a general evolution operator P_2 which does not have to be the wave operator. In addition, it allows for an arbitrary linear combination of system components, while in [12] just their (weighted) average is explored. Specially, if the difference $u_1 - u_2$ is considered, the result corresponds to the synchronisation problem (*e.g.* [14]) in which all components are driven to the same state by employing the null controllability of their differences.

Furthermore, unlike in [12], the proof of the relaxed observability inequality (2.3) does not rely on the propagation property of H-measures, which allows for the system's coefficients to be merely continuous. On the other hand, this approach avoids technical issues related to the reflection of H-measures on the domain boundary.

The theorem to some extent also generalises the results of [18] in which a similar result is provided for the system (2.1) consisting of a wave and a heat operator with constant coefficients (or more generally with a common elliptic part). However, although allowing for a more general perturbation operator, it requires the initial data of two components of the system (2.1) to be related, while in [18] no assumptions on initial data for the second component is assumed.

Some generalisations of the obtained results are discussed in the following remark.

Remark 2.6.

- The last theorem also holds if, instead of initial data of two components being linked by an operator, we assume a cone condition $\|u_2(0)\|_{\mathrm{L}^2(\omega)} \leq c\|u_1(0)\|_{\mathrm{L}^2(\omega)}$ ($\|u_2(0)\|_{\mathrm{L}^2(\omega)} \geq c\|u_1(0)\|_{\mathrm{L}^2(\omega)}$), with a constant $c < |\theta_1/\theta_2|$ ($c > |\theta_1/\theta_2|$). The latter condition is stable under passing to a limit, and also ensures the implication $((\theta_1 u_1(0) + \theta_2 u_2(0))|_\omega = 0) \Longrightarrow \big(u_1(0)|_\omega = u_2(0)||_\omega = 0\big)$, which suffices for the proof.
- The result (2.7) can be generalised to a more general perturbation operator P_2 by assuming that coefficients of both of the operators are real and analytic. In that case the separation of characteristic sets implies the separation of corresponding analytic wave front sets ([11], Thm. 9.5.1). Together with (2.9) it provides that u_1 is analytic on the observability region. Constraints on the initial data and finite velocity of propagation imply $u_1 = 0$ on an open set near $t = 0$, and as the solution is analytic it vanishes on the whole observability region which contradicts the observability assumption (2.2).
- The relaxed observability result (2.3) is easily generalised to a system with a finite number of components, under the assumption that the characteristic set of the leading operator P_1 is separated from the characteristic sets of all the other operators, while the latter ones can be arbitrary related.

However, the generalisation of the result to an infinite dimensional setting is not straightforward. It requires study of the localisation property for H-measures determined by a sequence of function series, and is the subject of the next subsection.

On the other hand, the generalisation of the observability result (2.7) to a system consisting of more than two components has still not been obtained, and is the subject of current investigation.

- The result (2.3) also holds if the observability region is not cylindrical, but a more general set satisfying the observability inequality (2.2). This generalisation corresponds to a moving control (*cf.* [13]). However, possible derivation of the corresponding observability result (without compact terms) remains an open problem, due to the last part of the proof of Theorem 2.3, in which the special (cylindrical) shape of the observability region is used.

2.2. Infinite discrete setting

In this subsection we analyse the stability of the observability estimates for the wave equation when the perturbation is given as a superposition of infinitely many components, each determined by a differential operator P_i, which, in general, does not have to be the wave operator. Thus the system of interest reads as:

$$\begin{aligned} P_1 u_1 = \partial_{tt} u_1 - \mathsf{div}\,(\mathbf{A}_1(t,\mathbf{x})\nabla u_1) &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ P_i u_i &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega, i \geq 2 \\ u_1 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \partial\Omega \\ u_1(0,\cdot) &= \beta_0 \in \mathrm{L}^2(\Omega) \\ \partial_t u_1(0,\cdot) &= \beta_1 \in \mathrm{H}^{-1}(\Omega), \end{aligned} \tag{2.10}$$

with the same assumptions on the domain Ω and the operator P_1 assumed for system (2.1). For the other equations, neither initial nor boundary conditions are specified. For the moment, we just assume the corresponding coefficients are bounded and continuous, complex-valued in general, and that the corresponding problems are well-defined with solutions in $\mathrm{L}^2(\Omega)$.

In this setting, the same microlocal analysis tool as in the finite case, in particular the localisation property of H-measures, is applied in the study of the stability of the observability estimates. However, as perturbations are determined by a superposition of infinitely many solutions, this requires special analysis of the mentioned property for a sequence of function series. This analysis, presented in the Appendix (Lem. A.6), provides a constraint on support of an H-measure determined by superposition of infinitely many sequences. As Lemma A.6 requires sequences under consideration to be uniformly bounded, in order to apply the localisation property within the analysis of observability estimates for solutions to (2.10), uniform boundedness on the solutions has to be assumed.

Proposition 2.7. *Suppose the observability inequality* (2.2) *holds for any solution* u_1 *to* (2.10).

As for the system (2.10), *suppose that the* $\mathrm{L}^2(\mathbf{R}^+ \times \Omega)$ *norm of all the solutions* u_i *is dominated (up to a multiplicative constant, independent of a choice of initial data) by the energy norm of* u_1*. In addition assume that the characteristic set* $\{p_1(t,\mathbf{x},\tau,\boldsymbol{\xi}) = 0\}$ *has no intersection with* $\mathrm{Cl}(\cup_{i\geq 2}\{p_i(t,\mathbf{x},\tau,\boldsymbol{\xi}) = 0\})$ *for* $(t,\mathbf{x}) \in \langle 0,T\rangle \times \omega, (\tau,\boldsymbol{\xi}) \in \mathrm{S}^d$, *where* p_i *is the principal symbol of the operator* P_i.

Then for any averaging sequence (θ_i) *of nonnegative numbers summing to 1, with* $\theta_1 > 0$, *there exists a constant* $\tilde{C}_\theta$ *such that the relaxed observability inequality*

$$E_1(0) \leq \tilde{C}_\theta \left(\int_0^T \int_\omega |\sum \theta_i u_i|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \|\beta_0\|^2_{\mathrm{H}^{-1}} + \|\beta_1\|^2_{\mathrm{H}^{-2}} \right) \tag{2.11}$$

holds for any family of solutions (u_i) *to* (2.10).

Proof. Assume the contrary. Then there exist sequences of initial conditions $(\beta_0^n), (\beta_1^n)$, and of associated solutions (u_i^n), such that

$$1 = E_1^n(0) := \|\beta_0^n\|^2_{\mathrm{L}^2} + \|\beta_1^n\|^2_{\mathrm{H}^{-1}} > n \left(\int_0^T \int_\omega \left| \sum_{i=1}^\infty \theta_i u_i^n \right|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \|\beta_0^n\|^2_{\mathrm{H}^{-1}} + \|\beta_1^n\|^2_{\mathrm{H}^{-2}} \right). \tag{2.12}$$

Let ν be an H-measure associated to a (sub)sequence of $\sum_{i=1}^{\infty}\theta_i u_i^n$. Due to the inequality (2.12), its value is equal to zero on $\langle 0,T\rangle\times\omega$.

We split the last sum into two parts $\theta_1 u_1^n + \sum_{i=2}^{\infty}\theta_i u_i^n$, and we rewrite ν in the form

$$\nu = \nu_1 + \nu_2 + 2\Re\nu_{12},$$

where ν_1 and ν_2 are H-measures associated to (sub)sequences (of) $(\theta_1 u_1^n)$ and $(\sum_2^{\infty}\theta_i u_i^n)$, respectively, while ν_{12} is a measure corresponding to their product. In addition, $\nu_1 = \theta_1^2\mu_1$, where by μ_i we denote a measure associated to a (sub)sequence (of) the ith component u_i^n.

From here the statement of the theorem is obtained easily (following similarly as the proof in finite discrete case, Prop. 2.1), once we show that ν_1 and ν_2 have disjoint supports.

By the localisation property for H-measures, each measure μ_i is supported within the set $\{p_i(t,\mathbf{x},\tau,\boldsymbol{\xi}) = 0\}$.

The assumption on the domination of solutions to (2.10) by an energy norm of u_1, together with the constant initial energy $E_1^n(0)$ implies a uniform bound on solutions u_i^n, both with respect to i and n. Thus we can apply Lemma A.6 to conclude that ν_2 is supported within the set

$$\mathrm{Cl}\Big(\cup_{i\geq 2}\{p_i(t,\mathbf{x},\tau,\boldsymbol{\xi})=0\}\Big),$$

which, due to the assumption on separation of the characteristic set, does not intersect the support of $\nu_1 = \theta_1^2\mu_1$. As θ_1 is strictly positive, we obtain that u_1^n converges to 0 strongly in $\mathrm{L}^2(\langle 0,T\rangle\times\omega)$, which contradicts the observability estimate (2.2). $\square$

Remark 2.8.

- The assumption of the last proposition requiring solutions u_i of (2.10) to be dominated by the energy norm of u_1 occurs, for example, in case of a system consisting of the operators of the same form, $P_i = \partial_{tt} - \mathsf{div}\,(\mathbf{A}_i(t,\mathbf{x})\nabla)$, with uniformly bounded (both from below and above) coefficients and initial energies. The assumption on separation of characteristics sets in that case can be stated as

  $$\mathbf{A}_1(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \;>(<)\; \sup_{i\geq 2}\,(\inf_{i\geq 2})\;\mathbf{A}_i(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi}, \qquad (t,\mathbf{x})\in\langle 0,T\rangle\times\omega,\ \boldsymbol{\xi}\neq\mathsf{0},$$

  i.e. the fastest (or the slowest) velocity is strictly separated from all the others.
  Additionally assuming that all the operators P_i are accompanied by Dirichlet boundary conditions, the relaxed observability result (2.11) in that case is equivalent to the averaged controllability of the adjoint system up to a finite number of low frequencies.
  Of course, one can construct more general systems, including operators of different types, that satisfy the required boundedness assumption.
- As mentioned in the previous subsection, obtaining a corresponding observability result without compact terms in this setting remains an open problem.
- The constant C_θ appearing in (2.11) can be taken uniformly for a family of averaging sequences, each satisfying:
  (i) $\theta_1 \geq \theta_*$,
  (ii) $\sum_k^{\infty}\theta_i \leq \epsilon_k$,
  where $\theta_* \in \langle 0,1]$ and (ε_k) is a null sequence, both independent of a choice of a particular sequence (θ_i).

2.3. Simultaneous observability

This subsection deals with the problem of recovering the energy of a system by observing an average of solutions on a suitable subdomain. For this purpose one has to estimate the initial energies of all system components, unlike the case of the average observability where this was required just for the first component. We first analyse a two component system, and generalisation to a higher-dimensional case is discussed at the end.

Reconsider the system (2.1) assuming that P_2 is an evolution operator of the form

$$P_2 = (\partial_t)^k + \mathcal{A}_2, \quad k \in \mathbf{N}, \tag{2.13}$$

where $\mathcal{A}_2$ is an (uniformly) elliptic operator (in general different from $\mathcal{A}_1$) with homogeneous Dirichlet boundary condition, and the problem for the perturbation u_2 is accompanied by a series of initial conditions

$$\left((\partial_t)^j u_2\right)(0) = \gamma_j \in \mathrm{H}^{-j}(\Omega), \quad j = 0, \ldots, k-1. \tag{2.14}$$

Its initial energy is given by

$$E_2(0) = \sum_0^{k-1} \|\gamma_j\|_{\mathrm{H}^{-j}(\Omega)}.$$

As in the previous subsection, we start with a relaxed observability result.

Proposition 2.9. *Suppose that there is a constant $\tilde{C}$, time T and an open subdomain ω such that for any choice of initial conditions the solutions to (2.1), (2.13), (2.14) satisfy*

$$E_i(0) \leq \tilde{C} \int_0^T \int_\omega |u_i|^2 \mathrm{d}\mathbf{x}\mathrm{d}t, \quad i = 1, 2. \tag{2.15}$$

In addition assume that characteristic sets $\{p_i(t, \mathbf{x}, \tau, \boldsymbol{\xi}) = 0\}, i = 1, 2$ have no intersection for $(t, \mathbf{x}) \in \langle 0, T\rangle \times \omega, (\tau, \boldsymbol{\xi}) \in \mathrm{S}^d$, where p_i stands for the principal symbol of the operator P_i.

Then for any $\theta_1, \theta_2 \in \mathbf{R} \setminus \{0\}$ there exists a constant $\tilde{C}_\theta$ such that the relaxed observability inequality

$$E_1(0) + E_2(0) \leq \tilde{C}_\theta \left(\int_0^T \int_\omega |\theta_1 u_1 + \theta_2 u_2|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \|\beta_0\|^2_{\mathrm{H}^{-1}} + \|\beta_1\|^2_{\mathrm{H}^{-2}} + \|\gamma_0\|^2_{\mathrm{H}^{-1}} + \ldots + \|\gamma_{k-1}\|^2_{\mathrm{H}^{-k}} \right) \tag{2.16}$$

holds for any pair of solutions (u_1, u_2).

The result is obtained easily by following the steps of the proof presented above in the averaged observability setting. Assuming the contrary and employing microlocal analysis tools, one shows that both components u_i^n converge to 0 strongly on the observability region, thus obtaining a contradiction.

However, a different approach is required in order to obtain the observability inequality for initial energy without compact terms from (2.16). It is based on a standard compactness-uniqueness procedure of reducing the observability for low frequencies to an elliptic unique continuation result [4, 6].

We introduce a subspace $N(T)$ of $H = \mathrm{L}^2(\Omega) \times \mathrm{H}^{-1}(\Omega) \times \mathrm{L}^2(\Omega) \times \cdots \times \mathrm{H}^{1-k}(\Omega)$, consisting of initial data for which the average of solutions to (2.1) vanishes on the observability region

$$N(T) := \{(\beta_0, \beta_1, \gamma_0, \ldots, \gamma_{1-k}) \in H |\, \theta_1 u_1 + \theta_2 u_2 = 0 \text{ on } \langle 0, T\rangle \times \omega\}.$$

Based on the relaxed observability inequality (2.16) it follows that $N(T)$ is a finite-dimensional space. Furthermore, the following characterisation holds.

Lemma 2.10. *We assume that the coefficients of both the operators P_1 and P_2 are time independent and that one of the following statements holds:*

(a) *The order k of time derivative in (2.13) satisfies $k \not\equiv 2 \pmod 4$. Coefficients of both the operators P_1 and P_2 are of class $\mathrm{C}^{1,1}$,*
(b) *The time derivative order k is of the form $k = 4l + 2$ for some non-negative integer l, and $\mathcal{A}_1^{k/2} - \mathcal{A}_2$ (or $-(\mathcal{A}_1^{k/2} - \mathcal{A}_2)$) is an uniformly elliptic operator. Coefficients of both operators P_1 and P_2 are analytic.*

Then $N(T) = \{0\}$.

Proof. By using a standard procedure (*e.g.* [6], Lem. 3.4) one shows that $N(T)$ is an A-invariant, where A is an unbounded operator on H:

$$A = \left(\begin{array}{cc|cc} \mathbf{0} & -1 & & \\ \mathcal{A}_1 & \mathbf{0} & \multicolumn{2}{c}{\mathbf{0}} \\ \hline & & \mathbf{0} & -\mathbf{I}_{k-1} \\ \multicolumn{2}{c|}{\mathbf{0}} & \mathcal{A}_2 & \mathbf{0} \end{array}\right), \tag{2.17}$$

with the domain $D(A) = \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega) \times \mathrm{H}^1_0(\Omega) \times \cdots \times \mathrm{H}^{2-k}(\Omega)$.

Being A-invariant and finite-dimensional, it contains an eigenfunction of A. Thus there is a $\lambda \in \mathbf{C}$ and $(\beta_0, \beta_1, \gamma_0, \ldots, \gamma_{1-k}) \in N(T)$ such that

$$\begin{aligned} \mathcal{A}_1\beta_0 &= -\lambda^2\beta_0 \\ \mathcal{A}_2\gamma_0 &= (-1)^{k-1}\lambda^k\gamma_0 \\ \beta_1 &= -\lambda\beta_0 \\ \gamma_j &= (-1)^j\lambda^j\gamma_0, \quad j = 1, \ldots, k-1. \end{aligned} \tag{2.18}$$

By the definition of $N(T)$ it follows that $\theta_1 u_1 + \theta_2 u_2 = 0$ on $\langle 0, T\rangle \times \omega$, and in particular

$$\theta_1\beta_0 + \theta_2\gamma_0 = 0 \quad \text{on} \quad \omega. \tag{2.19}$$

At this point, we want to show that each assumption of the lemma implies $\beta_0 = \gamma_0 = 0$.

(a) Due to the choice of k, numbers $-\lambda^2$ and $(-1)^{k-1}\lambda^k$ can not be both positive. As $\mathcal{A}_i, i = 1, 2$ are positive operators, from (2.18) it follows that one of functions β_0, γ_0 is trivial. By relation (2.19) it follows that the other also equals zero on ω. Being eigenfunctions of elliptic operators, the unique continuation argument (*e.g.* [9], Thm. 3) implies both β_0 and γ_0 are zero everywhere.

(b) Analyticity assumption on coefficients of the operators $\mathcal{A}_i, i = 1, 2$ implies analyticity of the eigenfunctions β_0 and γ_0 (*cf.* [10], Thm. 7.5.1). In particular, it follows that $\theta_1\beta_0 + \theta_2\gamma_0 = 0$ everywhere, and the relations (2.18) imply

$$(\mathcal{A}_1^{k/2} - \mathcal{A}_2)\beta_0 = 0.$$

Assumptions on the operator $\mathcal{A}_1^{k/2} - \mathcal{A}_2$ imply $\beta_0 = 0$ on Ω. □

Remark 2.11. In the special case $\mathcal{A}_2 = -\mathsf{div}\,(c_2(\mathbf{x})\nabla)$ one easily proves that the last Lemma holds with analytic coefficients c_1, c_2 separated only on an arbitrary non-empty open set, and not on the whole Ω.

Theorem 2.12. *Under the assumptions of Proposition 2.9 and Lemma 2.10, for any $\theta_1, \theta_2 \in \mathbf{R} \setminus \{0\}$ there exists a positive constant C_θ such that the observability inequality*

$$E_1(0) + E_2(0) \le C_\theta \int_0^T \int_\omega |\theta_1 u_1 + \theta_2 u_2|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \tag{2.20}$$

holds for any pair of solutions (u_1, u_2) to (2.1), (2.13), (2.14).

Proof. As in the proof of Proposition 2.1, let us suppose the contrary. Then there exists a sequence of solutions u_1^n, u_2^n to (2.1) such that $E_1^n(0) = 1$ and

$$\int_0^T \int_\omega |\theta_1 u_1^n + \theta_2 u_2^n|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \longrightarrow 0.$$

Thus for weak limits (u_1, u_2) of solutions on the observability region we have $\theta_1 u_1 + \theta_2 u_2 = 0$, implying $\big(u_1(0), \partial_t u_1(0), u_2(0), \ldots, \big((\partial_t)^{k-1} u_2\big)(0)\big) \in N(T)$. By means of the above lemma and taking into account the relaxed observability inequality, it follows that

$$1 \le \tilde{C}_\theta \left(\int_0^T \int_\omega |\theta_1 u_1^n + \theta_2 u_2^n|^2 \mathrm{d}\mathbf{x}\mathrm{d}t + \|\beta_0^n\|^2_{\mathrm{H}^{-1}} + \|\beta_1^n\|^2_{\mathrm{H}^{-2}} + \|\gamma_0^n\|^2_{\mathrm{H}^{-1}} + \ldots + \|\gamma_{k-1}^n\|^2_{\mathrm{H}^{-k}} \right) \longrightarrow 0,$$

thus obtaining a contradiction. □

We close this subsection with the following remarks.

Remark 2.13.

- If P_2 is a second order evolution operator, the observability inequality (2.20) is equivalent to the simultaneous controllability of the adjoint system, also studied in [12], by which one controls each component individually (and not just their average).
  The relaxed observability result (2.16) in that case corresponds to the simultaneous controllability of the adjoint system up to a finite number of low frequencies.
- The notion of simultaneous observability is stronger than average observability, as it estimates the energy of all system components, whose initial data, in this case, are not related. Consequently, it requires the stronger assumption of the observability inequality (2.15) satisfied by each component.
- The applied compactness-uniqueness procedure for removing compact terms from the relaxed observability estimate allows the perturbation P_2 to be an evolution operator with an arbitrary elliptic part. However, this approach is not feasible in the averaged observability setting. Namely, in order for subspace $N(T)$ to be finite-dimensional, one has to relate the initial data of two components by a bounded linear operator. But this constraint would not be preserved under action of the operator A given by (2.17), and as a consequence $N(T)$ would not be A-invariant.
- The relaxed observability result (2.16) is easily generalised to a system with a finite number of components, under the assumption that the characteristic sets of all operators are mutually disjoint.
- As in the averaged observability case, the generalisation of the observability result (2.20) to a system consisting of more than two components has still not been obtained, and is the subject of the current work.

3. Observation of the Schrödinger equation under uncertain perturbations

In this section we consider a system in which the first component, the one whose energy is observed, satisfies the Schrödinger equation, while the second component, corresponding to a perturbation, is governed by an evolution operator P_2:

$$\begin{aligned} P_1u_1 = i\partial_t u_1 + \mathsf{div}\,(\mathbf{A}_1(t,\mathbf{x})\nabla u_1) &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ P_2u_2 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega \\ u_1 &= 0, \qquad (t,\mathbf{x}) \in \mathbf{R}^+ \times \partial\Omega \\ u_1(0,\cdot) &= \beta_0 \in \mathrm{L}^2(\Omega). \end{aligned} \tag{3.1}$$

As in the study of perturbations of wave dynamics in Section 2, we specify no initial or boundary conditions for the second operators, assuming only that the corresponding problem is well posed and that it admits an L^2 solution (in both variables). As for the system coefficients, as before we impose merely boundedness and continuous assumptions, and suppose that $\mathbf{A}_1$ is a positive definite matrix field. In addition, we allow for complex-valued coefficients of both the operators.

3.1. Averaged observability under non-parabolic perturbations

For the reasons explained below, in this subsection we restrict our analysis to evolution operators P_2 of order strictly larger than one. In this case the stability of the Schrödinger observability estimate is given by the next theorem.

Theorem 3.1. *Consider a constant $\tilde{C}$, time T and an open subdomain ω such that for any choice of initial datum β_0 the solution u_1 of* (3.1) *satisfies*

$$E_1(0) := \|\beta_0\|^2_{\mathrm{L}^2} \leq \tilde{C} \int_0^T \int_\omega |u_1|^2 \mathrm{d}\mathbf{x}\mathrm{d}t. \tag{3.2}$$

In addition, for the system (3.1) *we assume the following*:

(a) *The perturbation operator is an evolution operator of the form* (2.5) *and order* $k > 1$.
(b) *The initial values of solutions* $u_i, i = 1, 2$ *are related by a linear operator such that whenever* $((\theta_1 u_1(0) + \theta_2 u_2(0))|_\omega = 0)$ *then* $\big(u_1(0)|_\omega = u_2(0)|_{|\omega} = 0\big)$.

Then for any $\theta_1, \theta_2 \in \mathbf{R}, \theta_1 \neq 0$ *there exists a constant* C_{θ_1} *such that the observability inequality*

$$E_1(0) \leq C_{\theta_1} \int_0^T \int_\omega |\theta_1 u_1 + \theta_2 u_2|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \tag{3.3}$$

holds for any pair of solutions (u_1, u_2) *to* (3.1), (2.5), (2.6).

The proof follows similarly as for the observations of the wave equation. Conditions a) and b) are necessary for obtaining the observability inequality (3.3) including no compact terms.

On the other hand, in order to obtain a relaxed inequality with lower order terms from (3.2), no assumptions are required. Namely, the assumption on the separation of characteristic sets $\{p_i(t, \mathbf{x}, \tau, \boldsymbol{\xi}) = 0\}$ required in Proposition 2.1 becomes superfluous in this setting, as it is directly satisfied by an arbitrary evolution operator P_2 of order k strictly larger than 1. Namely, its characteristic set does not contain the poles $\boldsymbol{\xi} = \mathsf{0}$ of the unit sphere in the dual space, which constitute the characteristic set of the Schrödinger operator P_1.

However, every Schrödinger or heat operator P_2 fails to satisfy the assumption. Namely, no matter the coefficients entering the equation, both have characteristic set within the poles $\boldsymbol{\xi} = \mathsf{0}$, as does P_1, and the localisation principle fails to distinguish corresponding H-measures. To analyse this type of system one needs a microlocal tool better adopted to the study of parabolic problems. Namely, original H-measures were constructed with the aim of analysing hyperbolic problems and are not capable of distinguishing differences between time and space variables that are intrinsic to parabolic equations. Their variant, parabolic H-measures, were recently introduced with the purpose of overcoming the mentioned constraint.

3.2. Averaged observability under parabolic perturbations

We reconsider the system (3.1) with the aim of obtaining the stability of the observability estimate for the Schrödinger operator under perturbation determined by the operator P_2 under minimal conditions on the latter. In particular, we want allow it to be the Schrödinger, or some other parabolic-type operator, such as P_1.

To this effect we have to explore parabolic H-measures presented in the Appendix, in particular the corresponding (localisation) Theorem A.8. The localisation result allows us to separate supports of measures determined by two Schrödinger operators with separated coefficients, as demonstrated by the following example.

Example 3.2 (Applications of the localisation principle for parabolic H-measures to various equations)**.**

In all examples we assume that the equation coefficients satisfy the assumptions of the localisation Theorem A.8, *i.e.* are continuous and bounded.

- **The Schrödinger equation**
  Let (u_n) be a sequence of solutions to the Schrödinger equation

$$i\partial_t u_n + \mathsf{div}\,(\mathbf{A}(t, \mathbf{x})\nabla u_n) = 0.$$

  If (u_n) is bounded in $\mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$ then the associated parabolic H-measure μ satisfies

$$(2\pi\tau + 4\pi^2 \mathbf{A}(t, \mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi})\mu = 0,$$

  implying μ is supported in points of the form $2\pi\tau = -4\pi^2 \mathbf{A}(t, \mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi}$.

- **The heat equation**
  Let (u_n) be a sequence of solutions to the heat equation

$$\partial_t u_n - \mathsf{div}\,(\mathbf{A}(t,\mathbf{x})\nabla u_n) = 0$$

  where $\mathbf{A}$ is a bounded, positive definite matrix field. If (u_n) is bounded in $\mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$ then the associated parabolic H-measure μ satisfies

$$(2\pi i\tau + 4\pi^2\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi})\mu = 0.$$

  As (parabolic) H-measures live on a hypersurface in the dual space excluding the origin, and $\mathbf{A}$ is positive definite, the term in braces above never equals zero, implying μ is a trivial (null) measure.

- **The wave equation**
  Let (u_n) be a sequence of solutions to the wave equation

$$\partial_{tt} u_n - \mathsf{div}\,(\mathbf{A}(t,\mathbf{x})\nabla u_n) = 0.$$

  If (u_n) is bounded in $\mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$ then the associated parabolic H-measure μ satisfies

$$4\pi^2\tau^2\mu = 0,$$

  implying μ is supported on the equator $\tau = 0$ of the hypersurface P^d.

From the given examples it is clear that by taking two Schrödinger operators with separated coefficients we are able to distinguish corresponding parabolic H-measures. Of course, this distinction is also possible if the considered operators are of different type (*e.g.* Schrödinger and wave), as was the case with respect to the original H-measures. This enables the following generalisation of Theorem 3.1.

Theorem 3.3. *In addition to the assumptions of Theorem 3.1, we allow P_2 to be an evolution operator of any integer order $k \geq 1$, and assume the separation of coefficients $c_2(\mathbf{x}) \neq -i$ holds on ω in the case $k = 1$.*

Then for any $\theta_1, \theta_2 \in \mathbf{R}, \theta_1 \neq 0$ there exists a constant C_{θ_1} such that the observability inequality

$$E_1(0) \leq C_{\theta_1} \int_0^T \int_\omega |\theta_1 u_1 + \theta_2 u_2|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \tag{3.4}$$

holds for any pair of solutions (u_1, u_2) to (3.1), (2.5)–(2.6).

The proof follows similarly as that of Theorem 2.3, considering that parabolic H-measures share basic properties (positive definiteness, diagonal domination) with their original (hyperbolic) counterparts. Essential to the proof is the separation of corresponding parabolic characteristic sets determined by (A.6). For an evolution operator P_2 of order $k \geq 2$, that set consists of the equator $\tau = 0$ of the hypersurface P^d given by (A.5), while the symbol $p_1 = 2\pi\tau + 4\pi^2\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi}$ of the Schrödinger operator differs from zero there (*cf.* Example 3.2). For $k = 1$, separation follows from the assumption $c_2(\mathbf{x}) \neq -i$.

The novelty obtained by the application of parabolic measures is that both the operators entering the system (3.1) are allowed to be of the same type (*e.g.* two Schrödinger operators with coefficients separated on the observability region $\langle 0, T\rangle \times \omega$). In that case, and additionally assuming that the problem for the second component is accompanied by a homogeneous boundary condition, the observability estimate (3.4) is equivalent to the averaged control of the adjoint system.

Of course, operators of different types are admissible as well, thus the last theorem incorporates the results of Section 3.1.

Finally, let us note that as in Section 2 one can obtain the analogous result for robust observability of the Schrödinger equation in the simultaneous and infinite discrete settings.

4. Conclusion

One of the most interesting aspects of the results detailed in this work is the robustness of observability estimates for the wave (2.2) and the Schrödinger equation (3.2). We show that these estimates are stable, up to lower order terms, under additive perturbations. The required assumptions are optimised: the coefficients are merely continuous and separated on the observability region.

If a system under consideration is determined by two operators of the same type - either Schrödinger or wave operators, the obtained relaxed observability estimates correspond to the average control of solutions to the adjoint system up to a finite number of low frequencies. Averaged observability results (2.7) and (3.4), corresponding to the exact averaged controllability of all frequencies, assume in addition that the initial data of the two components are related in an appropriate manner. Their proofs, unlike in [12], employ neither the propagation property of H-measures nor a unique continuation procedure, therefore not requiring additional smoothness assumptions apart from continuity.

This paper restricts itself to decoupled systems, but one could analyse more general systems with coupling in lower order terms (*e.g.* [1,14]). These terms do not effect microlocal properties of solutions, thus enabling a generalisation of obtained estimates.

Surely of interest would be a generalisation of the results to a parameter dependent system with parameter ranging over a continuous set. Previous work [15] has explored heat and Schrödinger equations with a randomly distributed parameter. Constant coefficient operators are considered, with eigenfunctions of corresponding elliptic parts being independent of the parameter value. Thus explored averages are presented as a solution (or a superposition of two solutions) to a similar evolution problem(s), which is crucial to the proof.

As the next step in that direction, one might consider a system of equations determined by evolution operators whose elliptic parts coincide up to a scalar function:

$$P(\nu) = \partial_t^k + c(\mathbf{x}, \nu)\mathcal{A},$$

with ν the parameter, and $\mathcal{A}$ the elliptic part of P. The variable dependence of the coefficient c would require other techniques than those applied in [15]. In this paper, we have obtained a corresponding result for a parameter ranging over an infinite discrete set, but only at the level of a relaxed observability estimate with compact terms.

By using transmutation techniques developed in [7], the simultaneous observability result of Section 2.3 can be employed in order to obtain controllability and observability properties for a system of heat equations (*cf.* [12]). A similar transmutation procedure can be constructed, translating Schrödinger into wave type problems, and *vice versa*. Its application will result in observability estimates for a system of Schrödinger equations, derived from the corresponding results for wave equations. It would be interesting to compare those results with results obtained directly in Section 3 by means of parabolic H-measures, and to compare the efficiency of methods applied by each approach.

Appendix A.

In this section we present a brief introduction and main results on original and parabolic H-measures used in the paper.

A.1. H-measures

Original H-measures, introduced a quarter of century ago by L. Tartar [16] and (independently) P. Gérard [8] are kind of a microlocal defect tool, measuring deflection of weak from strong L^2 convergence. They are Radon measures on the cospherical bundle $\Omega \times \mathrm{S}^{d-1}$ over an open domain $\Omega \subseteq \mathbf{R}^d$ (by S^{d-1} we denote the unit sphere in the dual space). Their existence is given by the following theorem (*cf.* [8,16]).

Theorem A.1 (Existence of H-measures). *If (u_n) is a sequence in $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, such that $\mathsf{u}_n \xrightarrow{\mathrm{L}^2} \mathsf{0}$ (weakly), then there exists a subsequence $(\mathsf{u}_{n'})$ and a complex $r \times r$ matrix Radon measure $\boldsymbol{\mu}$ on $\Omega \times S^{d-1}$ such that* *for all* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(S^{d-1})$*:*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) \mathrm{d}\boldsymbol{\xi} = \int_{\Omega \times S^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}), \tag{A.1}$$

where $\otimes$ *stands for the vector tensor product on* $\mathbf{C}^r$*.*

Remark A.2. In order to apply the Fourier transform in (A.1), functions defined on the entire $\mathbf{R}^d$ should be considered and this can be achieved by extending the functions by zero outside of the domain.

The above matrix Radon measure $\boldsymbol{\mu}$ is hermitian, positive semi-definite and it is called *H-measure*.

As a direct consequence we have that any H-measure associated to a strongly convergent sequence is necessarily zero, and *vice versa*, if the H-measure is trivial, then the corresponding (sub)sequence converges strongly in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

If an H-measure is associated to solutions of an equation $Pu_n = 0$ (accompanied by a series of initial/boundary conditions), one can explore their basic properties: the propagation and the localisation one. The former states that the measure (as well as concentration and oscillation effects) propagates along bicharacteristics of P. The latter constrains the support of the H-measure within the characteristic set of the (pseudo) differential operator P and is closely related to the generalisation of compactness by compensation method to variable coefficients. More precisely, the following theorem can be stated (*cf.* [8, 16]).

Theorem A.3 (Localisation principle for H-measures)**.** *Let* $\mathsf{u}_n \rightharpoonup \mathsf{0}$ *in* $\mathrm{L}^2(\Omega; \mathbf{C}^r)$*, and let for a given* $m \in \mathbf{N}$

$$\sum_{|\boldsymbol{\alpha}| \le m} \partial^{\boldsymbol{\alpha}}_{\mathbf{x}}(\mathbf{A}_{\boldsymbol{\alpha}} \mathsf{u}_n) \longrightarrow \mathsf{0} \quad \textit{strongly in} \quad \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^q), \tag{A.2}$$

where $\mathbf{A}_{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_{q \times r}(\mathbf{C}))$*, while* $\boldsymbol{\alpha} \in \mathbf{N}_0^d$*. Then for the associated H-measure* $\boldsymbol{\mu}$ *we have*

$$\mathbf{p}\boldsymbol{\mu}^\top = \mathbf{0},$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}| = m} (2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}} \mathbf{A}_{\boldsymbol{\alpha}}(\mathbf{x})$$

is the principal symbol of the differential operator in (A.2)*.*

In particular, the principle implies that the support of $\boldsymbol{\mu}$ is contained within the set

$$\Sigma_{\mathbf{p}} := \left\{ (\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathrm{S}^{d-1} : \operatorname{rank} \mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) < r \right\}$$

of points where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})$ is not left invertible.

The following relation between supports of diagonal and non-diagonal elements is a direct consequence of the definition (A.1) and the Cauchy−Schwartz−Bunjakovskij inequality.

Corollary A.4. *Let* $\boldsymbol{\mu}$ *be an H-measure determined by the sequence* (u_n)*. The support of* μ_{ij} *is contained in the intersection of supports of the corresponding diagonal elements* μ_{ii} *and* μ_{jj}*, i.e.*

$$\operatorname{supp} \mu_{ij} \subseteq \operatorname{supp} \mu_{ii} \cap \operatorname{supp} \mu_{jj}.$$

The corollary implies that an H-measure associated to a linear combination of two sequences is supported within the union of supports of measures determined by each component, and the same property holds for any finite linear combination. However, in general it fails when considering superposition of infinitely many sequences, as shown in the next example.

Example A.5. Let (u^n) and (f^n) be $\mathrm{L}^2(\mathbf{R}^d)$ sequences, whose corresponding H-measures μ_u and μ_f have disjoint supports, and let (θ_i) be a sequence of nonnegative numbers summing to 1.

Define the following sequences

$$v_i^n = \begin{cases} \theta_i u^n & i \neq n \\ f^i & i = n. \end{cases}$$

Thus for each i an H-measure ν_i associated to v_i^n equals $\theta_i^2 \mu_u$.

On the other hand we have that $\sum_i v_i^n = (1-\theta_n)u^n + f^n$, and the corresponding measure equals $\mu_u + \mu_f$.

Thus in order to constrain the support of an H-measure by supports of corresponding components we have to impose additional assumptions on constituting sequences. More precisely, the following result holds.

Lemma A.6. *Let (θ_i) be an averaging sequence of nonnegative numbers summing to 1, and let $(u_i^n)_n, i \in \mathbf{N}$ be a family of uniformly bounded sequences in $\mathrm{L}^2(\mathbf{R}^d)$, i.e. we assume there exists a constant C_u such that $\|u_i^n\|_{\mathrm{L}^2} \leq C_u, i, n \in \mathbf{N}$.*

Define the linear combination $v_n = \sum_i \theta_i u_i^n$, and denote by μ_i and ν H-measures associated to (sub)sequences (of) $(u_i^n)_n$ and (v_n), respectively. Then

$$\operatorname{supp} \nu \subseteq \mathrm{Cl}\Big(\cup_i \operatorname{supp} \mu_i \Big). \tag{A.3}$$

Proof. Consider an arbitrary pseudodifferential operator of order zero, $P \in \Psi_c^0$, with a symbol $p(\mathbf{x}, \boldsymbol{\xi})$ compactly supported within the complement of the closure of $\cup_i \operatorname{supp} \mu_i$.

By the definition of H-measures we have

$$\langle \nu, p \rangle = \lim_n \int_{\mathbf{R}^d} P\left(\sum_1^\infty \theta_i u_i^n \right)(\mathbf{x}) \left(\sum_1^\infty \theta_j u_j^n \right)(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{A.4}$$

As P is a continuous operator on $\mathrm{L}^2(\mathbf{R}^d)$, whose bound we denote by C_P, it follows that

$$\begin{aligned} \lim_n \left| \int_{\mathbf{R}^d} P\left(\sum_k^\infty \theta_i u_i^n \right)(\mathbf{x}) \left(\sum_1^\infty \theta_j u_j^n \right)(\mathbf{x}) \mathrm{d}\mathbf{x} \right| &\leq \limsup_n C_P \left\| \sum_k^\infty \theta_i u_i^n \right\|_{\mathrm{L}^2} \left\| \sum_1^\infty \theta_i u_i^n \right\|_{\mathrm{L}^2} \\ &\leq \limsup_n C_P\, C_u^2 \left(\sum_k^\infty \theta_i \right) \xrightarrow{k} 0. \end{aligned}$$

The last sum is the remainder of a convergent series, and the above limit converges to zero uniformly with respect to n.

Similarly, one shows the same property holds for $\lim_n \left| \int_{\mathbf{R}^d} P\Big(\sum_1^\infty \theta_i u_i^n \Big)(\mathbf{x}) \Big(\sum_l^\infty \theta_j u_j^n \Big)(\mathbf{x}) \mathrm{d}\mathbf{x} \right|$. Thus we can exchange limits in (A.4), getting

$$\begin{aligned} \langle \nu, p \rangle &= \sum_{i=1}^\infty \sum_{j=1}^\infty \lim_n \int_{\mathbf{R}^d} P\left(\theta_i u_i^n \right)(\mathbf{x})\, \theta_j u_j^n(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= \sum_{i=1}^\infty \sum_{j=1}^\infty \theta_i \theta_j \langle \mu_{ij}, p \rangle = 0, \end{aligned}$$

where μ_{ij} are H-measures determined by sequences (u_i^n) and (u_j^n), supported within the closure given in (A.3), outside which p is supported. ☐

The last lemma is used in Section 2.2 for deriving observability estimates for a system whose main dynamics is perturbed by a superposition of infinitely many components.

A.2. Parabolic H-measures

Parabolic H-measures were first introduced in [2], while a more exhaustive introduction can be found in [3], elaborating in particular on their basic properties: localisation and propagation. The former is used in the Section 3.3 for proving the stability of the observability estimate for the Schrödinger operator under parabolic perturbation.

Here we present the basic result on parabolic H-measures used in the note.

Parabolic H-measures are designed in a manner that takes into account the difference between time and space variables, intrinsic to parabolic type problems. The main idea in their construction is to replace the projection along the straight rays in the dual space, determined by the term $\boldsymbol{\xi}/|\boldsymbol{\xi}|$ in the definition of the original H-measures (A.1), by the projection along the meridians of paraboloids $\tau = a\boldsymbol{\xi}^2$. The hypersurface on which the dual space (excluding the origin) is projected is the rotational ellipsoid

$$\mathrm{P}^d := \left\{(\tau, \boldsymbol{\xi}) \in \mathbf{R}^{1+d} \,|\, \tau^2 + \frac{\boldsymbol{\xi}^2}{2} = 1\right\}. \tag{A.5}$$

Theorem A.7 (Existence of parabolic H-measures, [3])**.** *If* (u_n) *is a sequence in* $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{C}^r)$*, such that* $\mathsf{u}_n \xrightarrow{\mathrm{L}^2} \mathsf{0}$ *(weakly), then there exists a subsequence* $(\mathsf{u}_{n'})$ *and a complex* $r \times r$ *matrix Radon measure* $\boldsymbol{\mu}$ *on* $\mathbf{R}^{1+d} \times \mathrm{P}^d$ *such that for all* $\phi_1, \phi_2 \in \mathrm{C}_c(\mathbf{R}^{1+d})$ *and* $\psi \in \mathrm{C}(\mathrm{P}^d)$*:*

$$\lim_{n'} \int_{\mathbf{R}^{1+d}} \mathcal{F}\big(\phi_1 \mathsf{u}_{n'}\big) \otimes \mathcal{F}\big(\phi_2 \mathsf{u}_{n'}\big)(\psi \circ p)\, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbf{R}^{1+d} \times \mathrm{P}^d} \phi_1(\mathbf{x}) \bar{\phi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}),$$

where p *a parabolic projection of the set* $\mathbf{R}^{1+d} \setminus \{\mathsf{0}\}$ *onto* P^d*, defined by the relation*

$$p(\tau, \boldsymbol{\xi}) := \left(\frac{\tau}{|\boldsymbol{\xi}/2|^2 + \sqrt{|\boldsymbol{\xi}/2|^4 + \tau^2}}, \frac{\boldsymbol{\xi}}{\sqrt{|\boldsymbol{\xi}/2|^2 + \sqrt{|\boldsymbol{\xi}/2|^4 + \tau^2}}}\right).$$

Measure $\boldsymbol{\mu}$ from the above theorem we call *the parabolic H-measure* associated to (a sub)sequence (of) (u_n).

Although the ellipsoid P^d might seem an unnatural choice of surface on which to construct parabolic H-measures, a crucial reason for its selection is that the curves along which the projections are taken intersect it in the normal direction, as in the classical case, where the rays radiating from origin are perpendicular to the unit sphere. The mentioned normality property enables the study of propagation properties of the measures.

Concerning applications, the most important is that the new tool is also a defect measure, in the sense that null parabolic H-measure is equivalent to strong convergence of (u_n) in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d})$.

In order to formulate the localisation principle, we first introduce special anisotropic (Sobolev) function spaces

$$\mathrm{H}^{\frac{s}{2}, s}(\mathbf{R}^{1+d}) := \left\{u \in \mathcal{S}' \,|\, k_p^s \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d})\right\}, s \in \mathbf{R},$$

where $k_p(\tau, \boldsymbol{\xi}) := \sqrt[4]{1 + (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}$ is the weight function. These are Hilbert spaces and are particular examples of more general Hörmander spaces $B_{p,k}$ described in ([11], Sect. 10.1).

We also define the fractional derivative: $\sqrt{\partial_t}$ as a pseudodifferential operator with a polyhomogeneous symbol $\sqrt{2\pi i\tau}$, *i.e.*

$$\sqrt{\partial_t} u = \overline{\mathcal{F}}\left(\sqrt{2\pi i\tau}\, \hat{u}(\tau)\right),$$

where $\overline{\mathcal{F}}$ stands for the inverse Fourier transform. Here we assume that one branch of the square root has been selected.

Theorem A.8 (Localisation principle for parabolic H-measures, [3]).

Let (u_n) be a sequence of functions uniformly compactly supported in t and converging weakly to zero in $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$, *and let for $s\in\mathbf{N}$*

$$\sqrt{\partial_t}^{\,s}(\mathbf{A}_0\mathsf{u}_n)+\sum_{|\boldsymbol{\alpha}|=s}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathbf{A}_{\boldsymbol{\alpha}}\mathsf{u}_n)\longrightarrow 0\quad \textit{strongly in}\quad \mathrm{H}_{\mathrm{loc}}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d}),$$

where $\mathbf{A}_0,\mathbf{A}_{\boldsymbol{\alpha}}$ are continuous and bounded complex-valued matrix coefficients, while $\boldsymbol{\alpha}\in\mathbf{N}_0^d$.

Then for the associated parabolic H-measure $\boldsymbol{\mu}$ we have

$$\left((\sqrt{2\pi i\tau})^s\mathbf{A}_0+\sum_{|\boldsymbol{\alpha}|=s}(2\pi i\boldsymbol{\xi})^{\boldsymbol{\alpha}}\mathbf{A}_{\boldsymbol{\alpha}}\right)\boldsymbol{\mu}^\top=\mathbf{0}.$$

In particular, in case of square matrices $\mathbf{A}_0,\mathbf{A}_{\boldsymbol{\alpha}}$ the principle implies that the measure $\boldsymbol{\mu}$ is supported within the *parabolic characteristic set*:

$$\det\left((\sqrt{2\pi i\tau})^s\mathbf{A}_0+\sum_{|\boldsymbol{\alpha}|=s}(2\pi i\boldsymbol{\xi})^{\boldsymbol{\alpha}}\mathbf{A}_{\boldsymbol{\alpha}}\right)=0,\quad (\tau,\boldsymbol{\xi})\in\mathrm{P}^d. \tag{A.6}$$

Due to the local character of (parabolic) H-measures, the above theorems remain valid if a sequence (u_n) is taken from $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, where Ω is an open subset in $\mathbf{R}^{1+d}$.

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