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INTEGRAL REPRESENTATION RESULTS IN BV × L^p

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Abstract. Integral representation results are obtained for the relaxation of some classes of energy functionals depending on two vector fields with different behaviors, which may appear in the context of image decomposition and thermochemical equilibrium problems.

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1. Introduction

Minimization of energies depending on two independent vector fields have been introduced to model several phenomena. Namely, when u is a Sobolev function in $W^{1,q}, q > 1$, and v is in L^p, the study of these energies (see (1.1)) was motivated by the analysis of coherent thermochemical equilibria in a multiphase multicomponent system, with ∇u representing the elastic strain and v the chemical composition of the material. In the theory of linear magnetostriction, the stored energy depends on the linearized strain and the direction of magnetization, we refer to [14, 15] and the references therein for more details. Moreover, when $p = q$ this type of energies is used to model Cosserat theory and bending phenomena in nonlinear elasticity and also for the description of thin structures, see [11, 20]. Here v takes into account either Cosserat vectors or bending moments and ∇u is the elastic strain. When u is a function of bounded variation, functionals similar to (1.1) enter into image decomposition models, *i.e.*, in order to denoise and restore a given image f, it is required to minimize a functional which is the sum of a "total variation" term (*i.e.* a "norm" of Du) and a penalization term, *i.e.* a norm in a suitable functional space of $f - u - v$. Essentially f can be decomposed into the sum of two components u and v. The first component (cartoon), u, is well structured and it describes the homogeneous objects which are present in the image, while the second component, v, contains the oscillating pattern (both textures and noise). We refer to [4, 21, 25, 27, 28] among the extensive literature in this field.

In order to cover a wide class of applications we start from the functional setting $W^{1,1} \times L^p$, with anisotropic energies with linear growth in the gradient variable ∇u. Indeed, let Ω be a bounded open set in $\mathbb{R}^N$ and $1 < p \leq \infty$, for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathbb{R}^m)$ define the functional

$$J(u, v) := \int_\Omega f(x, u, v, \nabla u)\mathrm{d}x \tag{1.1}$$

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where $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty)$ is a continuous function satisfying standard coercivity and growth conditions that will be precised below. We discuss separately the cases $1 < p < \infty$ and $p = \infty$. Thus we introduce for $1 < p < \infty$ the functional

$$\overline{J}_p(u, v) := \inf\{\liminf_{n\to+\infty} J(u_n, v_n) : u_n \in W^{1,1}(\Omega; \mathbb{R}^d),\ v_n \in L^p(\Omega; \mathbb{R}^m),\ u_n \to u \text{ in } L^1,\ v_n \rightharpoonup v \text{ in } L^p\}, \quad (1.2)$$

for any pair $(u, v) \in BV(\Omega; \mathbb{R}^d) \times L^p(\Omega; \mathbb{R}^m)$ and, for $p = \infty$ the functional

$$\overline{J}_\infty(u, v) := \inf\{\liminf_{n\to\infty} J(u_n, v_n) : u_n \in W^{1,1}(\Omega; \mathbb{R}^d), v_n \in L^\infty(\Omega; \mathbb{R}^m), u_n \to u \text{ in } L^1, v_n \overset{*}{\rightharpoonup} v \text{ in } L^\infty\}, \quad (1.3)$$

for any pair $(u, v) \in BV(\Omega; \mathbb{R}^d) \times L^\infty(\Omega; \mathbb{R}^m)$.

Since bounded sequences $\{u_n\}$ in $W^{1,1}(\Omega; \mathbb{R}^d)$ converge in L^1 to a BV function u and bounded sequences $\{v_n\}$ in $L^p(\Omega; \mathbb{R}^m)$ if $1 < p \le \infty$, weakly converge to a function $v \in L^p(\Omega; \mathbb{R}^m)$, (weakly $*$ in L^∞), the relaxed functionals $\overline{J}_p$ and $\overline{J}_\infty$ will be composed by a Lebesgue part, a jump part concentrated on the jump set of $u \in BV(\Omega; \mathbb{R}^d)$ and a Cantor part, absolutely continuous with respect to the Cantor part of the distributional gradient Du. On the other hand, as already emphasized in [15], it is crucial to observe that v is not pointwise defined on the jump and the 'Cantor' parts sets of u, thus specific features of the density f will come into play to ensure a proper integral representation. The one of (1.2) is obtained in Theorem 1.1 below, via the blow-up method introduced in [18], under the following hypotheses:

(H_0) $f(x, u, \cdot, \cdot)$ is convex-quasiconvex for every $(x, u) \in \Omega \times \mathbb{R}^d$;

$(H_1)_p$ there exists a positive constant C such that

$$\frac{1}{C}(|b|^p + |\xi|) - C \le f(x, u, b, \xi) \le C(1 + |b|^p + |\xi|) \text{ for every } (x, u, b, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N};$$

$(H_2)_p$ for every compact set $K \subset \Omega \times \mathbb{R}^d$ there exists a continuous function $\omega_K : \mathbb{R} \to [0, +\infty)$ with $\omega_K(0) = 0$ such that

(1) $|f(x, u, b, \xi) - f(x', u', b, \xi)| \le \omega_K(|x - x'| + |u - u'|)(1 + |b|^p + |\xi|)$ for every (x, u, b, ξ) and $(x', u', b, \xi) \in K \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$;

(2) Moreover, given $x_0 \in \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then

$$f(x, u, b, \xi) - f(x_0, u, b, \xi) \ge -\varepsilon(1 + |b|^p + |\xi|), \text{ for every } (u, b, \xi) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N};$$

$(H_3)_p$ there exist $c' > 0,\ L > 0,\ 0 < \tau \le 1$, such that

$$t > 0,\ \xi \in \mathbb{R}^{d\times N},\ \text{with } t|\xi| + t|b|^p > L \Longrightarrow \left|\frac{f(x, u, t^{\frac{1}{p}} b, t\xi)}{t} - f_p^\infty(x, u, b, \xi)\right| \le c' \frac{|b|^{(1-\tau)p} + |\xi|^{1-\tau}}{t^\tau}$$

for every $(x, u) \in \Omega \times \mathbb{R}^d$, where f_p^∞ is the $(p, 1)-$ recession function of f defined for every $(x, u, b, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$ as

$$f_p^\infty(x, u, b, \xi) := \limsup_{t\to+\infty} \frac{f(x, u, t^{\frac{1}{p}} b, t\xi)}{t}\cdot \quad (1.4)$$

Theorem 1.1. *Let J and $\overline{J}_p$ be given by (1.1) and (1.2), respectively, with f satisfying (H_0), $(H_1)_p - (H_3)_p$ then*

$$\overline{J}_p(u, v) = \int_\Omega f(x, u, v, \nabla u)\mathrm{d}x + \int_{J_u} K_p(x, 0, u^+, u^-, \nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega f_p^\infty(x, u, 0, \tfrac{\mathrm{d}Du}{\mathrm{d}|D^c u|})\mathrm{d}|D^c u|, \quad (1.5)$$

for every $(u,v)\in BV(\Omega;\mathbb{R}^d)\times L^p(\Omega;\mathbb{R}^m)$*, where* $K_p:\Omega\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}\to[0,+\infty)$ *is defined as*

$$K_p(x,b,c,d,\nu):=\inf\left\{\int_{Q_\nu} f_p^\infty(x,w(y),\eta(y),\nabla w(y))\mathrm{d}y : w\in\mathcal{A}(c,d,\nu),\eta\in L^\infty(Q_\nu;\mathbb{R}^m),\int_{Q_\nu}\eta\,\mathrm{d}y=b\right\}, \tag{1.6}$$

with

$$\begin{aligned}\mathcal{A}(c,d,\nu):=\{w\in W^{1,1}(Q_\nu;\mathbb{R}^d): w(y)=c \text{ if } y\cdot\nu=\tfrac12, w(y)=d \text{ if } y\cdot\nu=-\tfrac12,\\ w \text{ is } 1-\text{periodic in } \nu_1,\dots,\nu_{N-1} \text{ directions}\},\end{aligned} \tag{1.7}$$

and Q_ν *the unit cube in* $\mathbb{R}^N$*, centered at the origin with faces parallel to the orthonormal basis* $\{\nu,\nu_1,\dots,\nu_{N-1}\}$*.*

In order to provide an integral description of the functional $\overline{J}_\infty$, introduced in (1.3), we prove Theorem 1.2 replacing assumptions $(H_1)_p-(H_3)_p$ by the following ones:

$(H_1)_\infty$ Given $M>0$, there exists $C_M>0$ such that, if $|b|\le M$ then

$$\tfrac{1}{C_M}|\xi|-C_M\le f(x,u,b,\xi)\le C_M(1+|\xi|), \text{ for every } (x,u,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}.$$

$(H_2)_\infty$ For every $M>0$, and for every compact set $K\subset\Omega\times\mathbb{R}^d$ there exists a continuous function $\omega_{M,K}(0)=0$ such that if $|b|\le M$ then

$$|f(x,u,b,\xi)-f(x',u',b,\xi)|\le\omega_{M,K}(|x-x'|+|u-u'|)(1+|\xi|)$$

for every $(x,u,\xi),(x',u',\xi)\in K\times\mathbb{R}^{d\times N}$. Moreover, given $M>0$, $x_0\in\Omega$, and $\varepsilon>0$ there exists $\delta>0$ such that if $|b|\le M$ and $|x-x_0|\le\delta$ then

$$f(x,u,b,\xi)-f(x_0,u,b,\xi)\ge-\varepsilon(1+|\xi|) \text{ for every } (u,\xi)\in\mathbb{R}^d\times\mathbb{R}^{d\times N}.$$

$(H_3)_\infty$ Given $M>0$, there exist $c'_M>0,\ L>0,\ 0<\tau\le1$ such that

$$|b|\le M,\ t>0,\ \xi\in\mathbb{R}^{d\times N}, \text{ with } t|\xi|>L\Longrightarrow\left|\tfrac{f(x,u,b,t\xi)}{t}-f^\infty(x,u,b,\xi)\right|\le c'_M\tfrac{|\xi|^{1-\tau}}{t^\tau}$$

for every $(x,u)\in\Omega\times\mathbb{R}^d$, where $f^\infty(b,\xi)$ is the $(\infty-1)$−recession function, *i.e.* the 'standard' recession function in the last variable, defined for every $(x,u,b,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}$ as

$$f^\infty(x,u,b,\xi):=\limsup_{t\to+\infty}\frac{f(x,u,b,t\xi)}{t}. \tag{1.8}$$

Theorem 1.2. *Let* J *and* $\overline{J}_\infty$ *be given by* (1.1) *and* (1.3)*, respectively, with* f *satisfying* (H_0)*,* $(H_1)_\infty-(H_3)_\infty$ *then*

$$\overline{J}_\infty(u,v)=\int_\Omega f(x,u,v,\nabla u)\mathrm{d}x+\int_{J_u}K_\infty(x,0,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1}+\int_\Omega f^\infty(x,u,0,\tfrac{\mathrm{d}Du}{\mathrm{d}|D^cu|})\mathrm{d}|D^cu|,$$

for every $(u,v)\in BV(\Omega;\mathbb{R}^d)\times L^\infty(\Omega;\mathbb{R}^m)$*, where* $K_\infty:\Omega\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}\to[0,+\infty)$ *is defined by*

$$K_\infty(x,b,c,d,\nu):=\inf\left\{\int_{Q_\nu} f^\infty(x,w(y),\eta(y),\nabla w(y))\mathrm{d}y : w\in\mathcal{A}(c,d,\nu),\eta\in L^\infty(Q_\nu;\mathbb{R}^m),\int_{Q_\nu}\eta\,\mathrm{d}y=b\right\}, \tag{1.9}$$

where $\mathcal{A}(c,d,\nu)$ *is as in* (1.7)*.*

It is worth to observe that assumption (H_0) can be removed in Theorems 1.1 and 1.2, thus replacing f by its convex-quasiconvex envelope in the above integral representations, and in (1.4), (1.6), (1.8) and (1.9).

We stress the fact that Theorems 1.1 and 1.2 generalize the result contained in ([22], Thm. 1.1) where an energy density $f(x,u,b,\xi) := W(x,u,\xi) + \varphi(x,u,v)$ has been considered. On the other hand we observe that the density K_p (respectively K_∞) in the latter case reduces to the density K introduced in [19], appearing in ([18], Thm. 2.16), relative to W, and f_p^∞ (respectively f^∞) coincides with W^∞, the latter being defined as $W^\infty(x,u,\xi) := \limsup_{t\to+\infty} \frac{W(x,u,t\xi)}{t}$.

It is worth to observe that if f does not depend on u, the energy densities involved in the representations of $\overline{J}_p$ and $\overline{J}_\infty$ coincide, see Remark 3.9, (1.4) and (1.8).

The paper is organized as follows. Section 2 is devoted to notations, preliminaries and auxiliary results. Section 3 contains the properties of the energy densities. The proofs of main theorems are in Sections 4 and 5. The Appendix is devoted to remove (H_0) in the proof of Theorems 1.1 and 1.2.

2. Notations and auxiliary results

In this section we establish notations and present some preliminary results on measure theory and functions of bounded variation that will be useful through the paper. An auxiliary lemma, crucial to obtain the lower bound inequality is also proven.

All over the paper Ω will represent a bounded open set of $\mathbb{R}^N$ and $\mathcal{A}(\Omega)$ will be the family of all open subsets of Ω. We denote by $Q := (-1/2, 1/2)^N$ the unit cube in $\mathbb{R}^N$ and if $\nu \in S^{N-1}$ and $(\nu_1, \ldots, \nu_{N-1}, \nu)$ is an orthonormal basis of $\mathbb{R}^N$, Q_ν denotes the unit cube centered at the origin with its faces parallel to $\nu_1, \ldots, \nu_{N-1}, \nu$. If $x \in \mathbb{R}^N$ and $\varepsilon > 0$, we set $Q(x,\varepsilon) := x + \varepsilon\, Q$ and $Q_\nu(x,\varepsilon) := x + \varepsilon\, Q_\nu$, and $B(x_0,\varepsilon) \subset \mathbb{R}^N$ is the ball centered at x_0 with radius ε. By $\mathcal{M}(\Omega)$ we represent the space of all signed Radon measures in Ω with bounded total variation. By the Riesz Representation Theorem, $\mathcal{M}(\Omega)$ can be identified to the dual of the separable space $C_0(\Omega)$ of continuous functions on Ω vanishing on the boundary $\partial\Omega$. The N-dimensional Lebesgue measure in $\mathbb{R}^N$ is designated as $\mathcal{L}^N$, while $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. If $\mu \in \mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ the Radon–Nikodým derivative of μ with respect to λ. By a generalization of the Besicovitch Differentiation Theorem (see [16], Thm. 1.153 and related results in Sects. 1.2.1 and 1.2.2), it can be proved that there exists a Borel set $E \subset \Omega$ such that $\lambda(E) = 0$ and

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = \lim_{\varepsilon\to 0^+} \frac{\mu(x+\varepsilon\, C)}{\lambda(x+\varepsilon\, C)} \tag{2.1}$$

for all $x \in \operatorname{Supp}\mu \setminus E$ and any open convex set C containing the origin. We recall that the exceptional set E does not depend on C. The theorem below will be exploited in the sequel, besides not explicitly mentioned. for all $x \in \operatorname{Supp}\mu \setminus E$ and any open convex set C containing the origin. We recall that the exceptional set E does not depend on C. The theorem below will be exploited in the sequel, besides not explicitly mentioned.

Theorem 2.1. *If μ is a nonnegative Radon measure and if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mu)$ then*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\mu(x+\varepsilon C)} \int_{x+\varepsilon C} |f(y) - f(x)| \mathrm{d}\mu(y) = 0$$

for $\mu-$ a.e. $x \in \mathbb{R}^N$ and for every bounded, convex, open set C containing the origin.

Definition 2.2. A function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^d)$, if all its first distributional derivatives $D_j u_i$ belong to $\mathcal{M}(\Omega)$ for $1 \le i \le d$ and $1 \le j \le N$.

The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du and $|Du|$ stands for its total variation. We observe that if $u \in BV(\Omega;\mathbb{R}^d)$ then $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega;\mathbb{R}^d)$ with respect to the $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ topology. A set $E\subset\Omega$ has finite perimeter in Ω if $\mathrm{Per}(E;\Omega):=|D\chi_E|(\Omega)<+\infty$, where χ_E denotes the characteristic function of E.

We briefly recall some facts about functions of bounded variation and we refer the reader to [5] for details.

Definition 2.3. Given $u\in BV(\Omega;\mathbb{R}^d)$ the *approximate upper limit* and the *approximate lower limit* of each component u^i, $i=1,\dots,d$, are defined by

$$(u^i)^+(x):=\inf\left\{t\in\mathbb{R}:\ \lim_{\varepsilon\to0^+}\frac{\mathcal{L}^N(\{y\in\Omega\cap Q(x,\varepsilon):\ u^i(y)>t\})}{\varepsilon^N}=0\right\}$$

and

$$(u^i)^-(x):=\sup\left\{t\in\mathbb{R}:\ \lim_{\varepsilon\to0^+}\frac{\mathcal{L}^N(\{y\in\Omega\cap Q(x,\varepsilon):\ u^i(y)<t\})}{\varepsilon^N}=0\right\},$$

respectively. The *jump set* of u is given by

$$J_u:=\bigcup_{i=1}^d\{x\in\Omega:\ (u^i)^-(x)<(u^i)^+(x)\}.$$

Proposition 2.4. *If $u\in BV(\Omega;\mathbb{R}^d)$ then*

(i) *for $\mathcal{L}^N-$a.e. $x_0\in\Omega$,* $\displaystyle\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}\left\{\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}|u(x)-u(x_0)-\nabla u(x_0)\cdot(x-x_0)|^{\frac{N}{N-1}}\mathrm{d}x\right\}^{\frac{N-1}{N}}=0;$

(ii) *for $\mathcal{H}^{N-1}-$a.e. $x_0\in J_u$ there exist $u^+(x_0)$, $u^-(x_0)\in\mathbb{R}^d$ and $\nu\in S^{N-1}$ normal to J_u at x_0, such that*

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}\int_{Q^+_\nu(x_0,\varepsilon)}|u(x)-u^+(x_0)|\mathrm{d}x=0,\qquad \lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}\int_{Q^-_\nu(x_0,\varepsilon)}|u(x)-u^-(x_0)|\mathrm{d}x=0,\tag{2.2}$$

where $Q^+_\nu(x_0,\varepsilon):=\{y\in Q_\nu(x_0,\varepsilon):\langle x-x_0,\nu\rangle>0\}$ and $Q^-_\nu(x_0,\varepsilon):=\{x\in Q_\nu(x_0,\varepsilon):\langle x-x_0,\nu\rangle<0\}$;

(iii) *for $\mathcal{H}^{N-1}-$a.e. $x_0\in\Omega\backslash J_u$* $\quad\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}|u(x)-u(x_0)|\mathrm{d}x=0.$

The next result, which will be exploited in the proof of the upper bound, can be found in ([26], Thm. 1, Chap. 4).

Theorem 2.5 (Whitney's covering theorem)**.** *Let $F\subset\mathbb{R}^N$ be a closed set. Then there exists a countable family of closed cubes of the form $Q_i:=a_i+\delta Q_\nu$, such that the following hold:*

(i) $\mathbb{R}^N\setminus F=\cup_{i=1}^\infty\overline{Q_i}$;
(ii) *the cubes Q_i have mutually disjoint interiors;*
(iii) $\mathrm{diam}\,Q_i\le\mathrm{dist}(Q_i,F)\le4\,\mathrm{diam}\,Q_i$.

The proof of the result below can be found in ([8], Lem. 3.1). With the aim of the applications below, we state it as in ([10], Thm. 2.7).

Proposition 2.6. *Let E be a subset of Ω such that* $\mathrm{Per}(E;\Omega)<+\infty$*. There exists a sequence of polyhedral sets $\{E_k\}$ (*i.e. *E_k are bounded, strongly Lipschitz domains) with $\partial E_k=H_1\cup H_2\cup\dots\cup H_p$ where each H_i is a closed subset of a hyperplane $\{x\in\mathbb{R}^N:x\cdot\nu_i=\alpha_i\}$ satisfying the following properties:*

(i) $\mathcal{L}^N(((E_k\cap\Omega)\setminus E)\cup(E\setminus(E_k\cap\Omega)))\to0$ *as $k\to+\infty$;*
(ii) $\mathrm{Per}(E_k;\Omega)\to\mathrm{Per}(E;\Omega)$ *as $k\to+\infty$;*
(iii) $\mathcal{H}^{N-1}(\partial E_k\cap\partial\Omega)=0$ *for any $k\in\mathbb{N}$;*
(iv) $\mathcal{L}^N(E_k)=\mathcal{L}^N(E)$ *for any $k\in\mathbb{N}$.*

The following Lemma that will be exploited for the lower bound inequality in Theorem 1.1 is very similar to ([18], Lem. 3.1).

Lemma 2.7. *Le $f : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a continuous function such that*

$$0 \le f(u, b, \xi) \le C(1 + |b|^p + |\xi|), \tag{2.3}$$

for some $C > 0$ and for $(x, u, b, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$. Let $u_0(x) := \begin{cases} c \text{ if } x_N > 0 \\ d \text{ if } x_N \le 0 \end{cases}$, $\{w_n\} \subset W^{1,1}(Q; \mathbb{R}^d)$ and $\{v_n\} \subset L^p(Q; \mathbb{R}^m)$ be such that $w_n \to u_0$ in $L^1(Q; \mathbb{R}^d)$ and $v_n \rightharpoonup v$ in $L^p(Q; \mathbb{R}^m)$, with $v \in L^\infty(Q; \mathbb{R}^m)$ and $\int_Q v\mathrm{d}x = b$. If ϱ is a mollifier, $\varrho_n(x) := (\frac{1}{\varepsilon_n})^N \varrho(\frac{x}{\varepsilon_n})$, and $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \to 0^+$, then there exist two sequences of functions $\{\xi_n\} \subset \mathcal{A}(c, d, \nu)$ and $\{\bar{v}_n\} \subset L^p(Q; \mathbb{R}^m)$ such that

$$\begin{aligned} &\xi_n = \varrho_{i(n)} * u_0 \text{ on } \partial Q, \quad \xi_n \to u_0 \text{ in } L^1(Q; \mathbb{R}^d), \quad \bar{v}_n \rightharpoonup v \text{ in } L^p(Q; \mathbb{R}^m), \\ &\int_Q \bar{v}_n \mathrm{d}x = b, \ \liminf_{n\to+\infty} \int_Q f(w_n, v_n, \nabla w_n)\mathrm{d}x \ge \limsup_{n\to+\infty} \int_Q f(\xi_n, \bar{v}_n, \nabla \xi_n)\mathrm{d}x. \end{aligned} \tag{2.4}$$

Proof. Without loss of generality, assume that

$$\liminf_{n\to+\infty} \int_Q f(w_n, v_n, \nabla w_n)\mathrm{d}x = \lim_{n\to+\infty} \int_Q f(w_n, v_n, \nabla w_n)\mathrm{d}x < +\infty.$$

Define $z_n(x) := (\varrho_n * u_0)(x) = \int_{B(x,\varepsilon_n)} \varrho_n(x - y)u_0(y)\mathrm{d}y$. Since ϱ is a mollifier, we have $z_n(x + e_i) = z_n(x)$ for every $i = 1, \dots, N - 1$,

$$z_n = \begin{cases} c & \text{if } x_N > \varepsilon_n, \\ d & \text{if } x_N \le -\varepsilon_n, \end{cases} \quad \|\nabla z_n\|_{L^\infty(Q)} = O\left(\varepsilon_n\right), \quad z_n \in \mathcal{A}(c, d, e_N).$$

For $j \in \mathbb{N}$, define $L_j := \left\{x \in Q : \mathrm{dist}(x; \partial Q) < \frac{1}{j}\right\}$. Take $j = 2$, and divide L_2 into two layers S_2^1, S_2^2. It is clear that for every $n \in \mathbb{N}$, there exists $S \in \{S_2^1, S_2^2\}$ such that $\int_S(|\nabla w_n| + |v_n|^p)\mathrm{d}x \le \frac{C}{2}$, where C is the constant which uniformly bounds $\int_Q |\nabla w_n|\mathrm{d}x$ and $\int_Q |v_n|^p\mathrm{d}x$ in Q, since $w_n \to u_0$ in $L^1(Q; \mathbb{R}^d)$ and $v_n \rightharpoonup v$ in $L^p(Q; \mathbb{R}^m)$. Since there are only two layers and infinitely many indices, we can conclude that one of the two layers, defined as $S_2 := \{x \in Q : \alpha_2 < \mathrm{dist}(x, \partial Q) < \beta_2\}$, for $0 < \alpha_2 < \beta_2 < 1$ satisfies

$$\int_{S_2} |\nabla w_{n_2}| + |v_{n_2}|^p \mathrm{d}x \le \frac{C}{2},$$

for a subsequence $\{n_2\}$ of $\{n\}$. Let η_2 be a smooth cut-off function $0 \le \eta_2 \le 1$, such that $\eta_2 = 1$ in the complement of $\{x \in Q : \mathrm{dist}(x, \partial Q) < \beta_2\}$ and $\eta_2 = 0$ in $\{x \in Q : \mathrm{dist}(x, \partial Q) < \alpha_2\}$, and $\|\nabla(\eta_2)\|_{L^\infty} = O(\frac{1}{|S_2|})$. Clearly,

$$\lim_{n_2\to+\infty} \int_Q \eta_2 v_{n_2} \mathrm{d}x = \int_Q \eta_2 v \,\mathrm{d}x,$$

since $\eta_2 \in L^\infty(Q; \mathbb{R}^m)$ and $v_{n_2} \rightharpoonup v$ in $L^p(Q; \mathbb{R}^m)$. Also, for the same sequence $\{n_2\}$, we have

$$\lim_{n_2\to+\infty} \left|\int_Q (v - \eta_2 v_{n_2})\mathrm{d}x\right| = \left|\int_Q (1 - \eta_2)v \,\mathrm{d}x\right| \le \|v\|_{L^\infty} \left|\int_Q (1 - \eta_2) \,\mathrm{d}x\right|.$$

Moreover, we can find a number $n(2) \in \{n_2\}$ large enough so that

$$\frac{1}{|S_2|} \int_{S_2} |w_{n(2)} - z_{n(2)}|\mathrm{d}x < \frac{1}{2}, \quad \frac{|\int_Q v - \eta_2 v_{n(2)}\mathrm{d}x|}{\left|1 - \int_Q \eta_2 \mathrm{d}x\right|} < \|v\|_{L^\infty} + 1.$$

Next we divide L_3 into three layers S_3^1, S_3^2, S_3^3. For each n_2 there exists $S \in \{S_3^1, S_3^2, S_3^3\}$, such that $\int_S |\nabla w_{n_2}| + |v_{n_2}|^p \mathrm{d}x \leq \frac{C}{3}$. Since there are only three layers with infinitely many indices, we conclude that one of the layers $S_3 \in \{S_3^1, S_3^2, S_3^3\}$ satisfies

$$\int_{S_3} |\nabla w_{n_3}| + |v_{n_3}|^p \mathrm{d}x \leq \frac{C}{3},$$

for a subsequence $\{n_3\}$ of $\{n_2\}$. Let η_3 be a smooth cut off function, $0 \leq \eta_3 < 1$, $\eta_3 = 1$ in the complement of $\{x \in Q : \mathrm{dist}(x, \partial Q) < \beta_3\}$ and $\eta_3 = 0$ in $\{x \in Q : \mathrm{dist}(x, \partial Q) < \alpha_3\}$, and $\|\nabla \eta_3\|_{L^\infty} = O\left(\frac{1}{|S_3|}\right)$, and

$$\lim_{n_3 \to +\infty} \left| \int_Q (v - \eta_3 v_{n_3}) \mathrm{d}x \right| = \left| \int_Q (1 - \eta_3) v \, \mathrm{d}x \right| \leq \|v\|_{L^\infty} \left| \int_Q (1 - \eta_3) \, \mathrm{d}x \right|.$$

The convergence of $w_{n_3} \to u_0$ in L^1, allows us to choose $n(3) \in \{n_3\}$, $n(3) > n(2)$ large enough, such that

$$\frac{1}{|S_3|} \int_{S_3} |w_{n(3)} - z_{n(3)}| \mathrm{d}x \leq \frac{1}{3} \text{ and } \frac{|\int_Q (v - \eta_3 v_{n(3)}) \mathrm{d}x|}{\left|1 - \int_Q \eta_3 \mathrm{d}x\right|} < \|v\|_{L^\infty} + 1.$$

Precisely, in this way, we construct the sequence $n(j)$ such that

$$\int_{S_j} |\nabla w_{n(j)}| + |v_{n(j)}|^p \mathrm{d}x \leq \tfrac{C}{j}, \frac{1}{|S_j|} \int_{S_j} |w_{n(j)} - z_{n(j)}| \mathrm{d}x \leq \tfrac{1}{j} \text{ and } \frac{|\int_Q (v - \eta_j v_{n(j)}) \mathrm{d}x|}{\left|1 - \int_Q \eta_j \mathrm{d}x\right|} < \|v\|_{L^\infty} + 1. \tag{2.5}$$

Let us define $\overline{w}_j(x) := (1 - \eta_j(x)) z_{n(j)}(x) + \eta_j(x) w_{n(j)}(x)$, and

$$\overline{v}_j(x) := (1 - \eta_j(x)) \frac{\int_Q (v - \eta_j v_{n(j)}) \mathrm{d}x}{1 - \int_Q \eta_j \mathrm{d}x} + \eta_j(x) v_{n(j)}(x).$$

Then

$$\frac{1}{|Q|} \int_Q \overline{v}_j \mathrm{d}x = b, \quad \|\overline{v}_j\|_{L^p(Q)} \leq C, \text{ and } \overline{w}_j \lfloor_{\partial Q} = \overline{w}_j \lfloor_{\partial Q} = u_0.$$

In particular, $\overline{w}_j \to u_0$ in $L^1(Q; \mathbb{R}^d)$ and $\overline{v}_j \rightharpoonup b$ in $L^p(Q; \mathbb{R}^m)$. The first convergence is trivial, the second one can be proven first observing that it is enough to consider test functions $\varphi \in C_0(Q)$. Then the bounds in (2.5) entail that

$$\lim_{j \to +\infty} \int_Q ((1 - \eta_j) \tfrac{\int_Q (v - \eta_j v_{n(j)}) \mathrm{d}x}{1 - \int_Q \eta_j \mathrm{d}x} + \eta_j v_{n(j)} - b) \varphi \mathrm{d}x = \lim_{j \to +\infty} \int_Q (\eta_j v_{n(j)} - v) \varphi \mathrm{d}x.$$

Then

$$\lim_{j \to +\infty} \int_Q (\eta_j v_{n(j)} - v) \varphi \mathrm{d}x = \lim_{j \to +\infty} \int_Q (v_{n(j)} - v) \varphi \mathrm{d}x + \lim_{j \to +\infty} \int_Q (-1 + \eta_j) v_{n(j)} \varphi \mathrm{d}x = 0,$$

The first limit in the right hand side is 0 since $v_{n(j)} \rightharpoonup v$ in $L^p(Q; \mathbb{R}^m)$ and the second is 0 since $v_{n(j)}$ is s-equi-integrable for every $1 \leq s < p$ and $\eta_j \to 1$. Hence we have

$$\begin{aligned} \lim_{n \to +\infty} \int_Q f(w_n, v_n, \nabla w_n) \mathrm{d}x &= \lim_{j \to +\infty} \int_Q f(w_j, v_j, \nabla w_j) \mathrm{d}x \\ &\geq \lim_{j \to +\infty} \int_Q f(\overline{w}_j, \overline{v}_j, \nabla \overline{w}_j) \mathrm{d}x - \limsup_{j \to +\infty} \int_{\{x \in Q : \mathrm{dist}(x, \partial Q) < \alpha_j\}} f(\overline{w}_j, \overline{v}_j, \nabla \overline{w}_j) \mathrm{d}x \\ &\quad - \limsup_{j \to +\infty} \int_{S_j} f(\overline{w}_j, \overline{v}_j, \nabla \overline{w}_j) \mathrm{d}x. \end{aligned}$$

Thus it results

$$\lim_{n\to+\infty}\int_Q f(w_n,v_n,\nabla w_n)\mathrm{d}x \geq \lim_{j\to+\infty}\int_Q f(\overline{w}_j,\overline{v}_j,\nabla\overline{w}_j)\mathrm{d}x - \limsup_{j\to+\infty}\int_{L_j} C(1+\|v\|_{L^\infty})^p\mathrm{d}x$$
$$-\limsup_{j\to+\infty}\int_{S_j}(1+|\nabla z_{n(j)}|+|\nabla w_{n(j)}|+|v_{n(j)}|^p+\|v\|^p_{L^\infty}+\frac{1}{|S_j|}|w_{n(j)}-z_{n(j)}|)\mathrm{d}x,$$

where we have used the fact that $\nabla z_{n(j)}=0$ in L_j. Observing that, using co-area formula, $\int_{S_j}|\nabla z_{n(j)}|\mathrm{d}x\to 0$ as $j\to+\infty$, we obtain the desired result. □

Remark 2.8.

(i) For every $v\in L^p(\Omega;\mathbb{R}^m)$, under the same assumptions of Lemma 2.7 we can prove (2.4) without keeping the average.

(ii) We observe that the same type of arguments can be exploited to prove a similar result for the $BV\times L^\infty$ case. Namely, if $f:\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}\to[0,+\infty)$ is a continuous function such that for every $b\in\mathbb{R}^m$, with $|b|\leq M$ there exists a constant C_M for which $0\leq f(u,b,\xi)\leq C_M(1+|\xi|)$, for $(u,b,\xi)\in\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}$, then (2.4) holds considering the sequence $\{v_n\}\subset L^\infty(Q;\mathbb{R}^m)$ and finding a correspondent sequence $\{\overline{v}_n\}\subset L^\infty(Q;\mathbb{R}^m)$ such that $\overline{v}_n \overset{*}{\rightharpoonup} v$ in $L^\infty(Q;\mathbb{R}^m)$, and $\int_Q\overline{v}_n\mathrm{d}x=b$ The main differences in the proof are the use of the above growth condition in place of (2.3), and the fact that $\{\overline{v}_j\}$ and $\{v_{n(j)}\}$ are uniformly bounded in L^∞.

Next we recall the definition of Yosida transform that it will be useful in the proof of the upper bound.

Definition 2.9. For any function $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}\longrightarrow\mathbb{R}$ let, for any $\lambda>0$, the *Yosida transform* of f be defined as

$$f_\lambda(x,u,b,\xi):=\sup_{(x',u')\in\Omega\times\mathbb{R}^d}\left\{f(x',u',b,\xi)-\lambda C(|x-x'|+|u-u'|)(1+|b|+|\xi|)\right\},$$

for any $(x,u,b,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}$.

The proof of next proposition follows along the lines ([6], Prop. 4.6).

Proposition 2.10. *. Let $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}\longrightarrow\mathbb{R}$ be such that $f(\cdot,\cdot,b,\xi)$ is continuous for any $(b,\xi)\in\mathbb{R}^m\times\mathbb{R}^{d\times N}$. Then the* Yosida transform *of f satisfies the following properties:*

(i) *$f_\lambda(x,u,b,\xi)\geq f(x,u,b,\xi)$ and $f_\lambda(x,u,b,\xi)$ decreases to $f(x,u,b,\xi)$ as $\lambda\to+\infty$.*
(ii) *$f_\lambda(x,u,b,\xi)\geq f_\eta(x,u,b,\xi)$ if $\lambda\leq\eta$ for every $(x,u,b,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}$.*
(iii) *$|f_\lambda(x,u,b,\xi)-f_\lambda(x',u',b,\xi)|\leq\lambda(|x-x'|+|u-u'|)(1+|\xi|+|b|)$ for every (x,u,b,ξ), $(x',u',b,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}$.*
(iv) *The approximation is uniform on compact sets. Precisely, let K be a compact subset of $\Omega\times\mathbb{R}^d$ and let $\delta>0$. There exists $\lambda>0$ such that $f(x,u,b,\xi)\leq f_\lambda(x,u,b,\xi)\leq f(x,u,b,\xi)+\delta(1+|b|+|\xi|)$, for every $(x,u,b,\xi)\in K\times\mathbb{R}^m\times\mathbb{R}^{d\times N}$.*

3. Properties of the energy densities

3.1. Convex-quasiconvex functions

We start by recalling the notion of convex-quasiconvex function, presented in [14] (see also [13, 15, 20]).

Definition 3.1. A Borel measurable function $f:\mathbb{R}^m\times\mathbb{R}^{d\times N}\to\mathbb{R}$ is said to be convex-quasiconvex if, for every $(b,\xi)\in\mathbb{R}^m\times\mathbb{R}^{d\times N}$, there exists a bounded open set D of $\mathbb{R}^N$ such that

$$f(b,\xi)\leq\frac{1}{|D|}\int_D f(b+\eta(x),\xi+\nabla\varphi(x))\,\mathrm{d}x,$$

for every $\eta\in L^\infty(D;\mathbb{R}^m)$, with $\int_D\eta(x)\,\mathrm{d}x=0$, and for every $\varphi\in W^{1,\infty}_0(D;\mathbb{R}^d)$.

Remark 3.2.

(i) If f is convex-quasiconvex then the inequality above is true for any bounded open set $D \subset \mathbb{R}^N$.

(ii) A convex-quasiconvex function is separately convex, in each entry of the $m \times d \times N$ vector (b, ξ), since it turns out to be separately convex in b and quasiconvex in ξ.

(iii) Throughout this paper we will work with functions f defined in $\Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$ and when saying that f is convex-quasiconvex, we consider the previous definition with respect to the last two variables of f.

(iv) If f satisfies $(H_1)_p$, Proposition 2.11 ii in [12] entails that f is $(p, 1)-$ *Lipschitz continuous*, namely there exists $\gamma > 0$ such that

$$|f(x,u,b,\xi) - f(x,u,b',\xi')| \leq \gamma(|\xi - \xi'| + (1 + |b|^{p-1} + |b'|^{p-1} + |\xi|^{\frac{1}{p'}} + |\xi'|^{\frac{1}{p'}})|b - b'|) \tag{3.1}$$

for every $b,\ b' \in \mathbb{R}^m$, $\xi,\ \xi' \in \mathbb{R}^{d\times N}$ and $(x, u) \in \Omega \times \mathbb{R}^d$, where p' is the conjugate exponent of p.

(v) If f satisfies $(H_1)_\infty$, ([23], Prop. 4) guarantees that f is $(\infty, 1)$-*Lipschitz continuous*, *i.e.* given $M > 0$ there exists a constant $\beta(M) > 0$ such that

$$|f(x,u,b,\xi) - f(x,u,b',\xi')| \leq \beta(1 + |\xi| + |\xi'|)|b - b'| + \beta|\xi - \xi'| \tag{3.2}$$

for every $b,\ b' \in \mathbb{R}^m$, such that $|b| \leq M$ and $|b'| \leq M$, for every $\xi,\ \xi' \in \mathbb{R}^{d\times N}$ and for every $(x, u) \in \Omega \times \mathbb{R}^d$.

3.2. The recession functions

Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty[$, and let $f_p^\infty : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty[$, be its $(p, 1)-$ recession function, defined in (1.4). We observe that f_p^∞ satisfies the following homogeneity property,

$$f_p^\infty(x, u, t^{\frac{1}{p}} b, t\xi) = t f_p^\infty(x, u, b, \xi) \text{ for every } t \in \mathbb{R}^+, x \in \Omega, u \in \mathbb{R}^d, b \in \mathbb{R}^m, \xi \in \mathbb{R}^{d\times N}. \tag{3.3}$$

Notice that, under growth condition $(H_1)_p$ on f, we could consider both f_p^∞ and f^∞ (*i.e.* $(\infty, 1)-$ recession function of f as in (1.8)), and the latter one turns out to be independent on b, *i.e.* $f^\infty(x, u, b, \xi) = f^\infty(x, u, 0, \xi)$ for every $(x, u, b, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$, provided f is separately convex as in (ii) of Rem. 3.2). Moreover, it results that in general $f_p^\infty(x, u, b, \xi) \neq f^\infty(x, u, b, \xi)$ but the equality holds if $b = 0$.

The following properties are an easy consequence of the definition of $(p, 1)-$ recession function and of properties (H_0), $(H_1)_p$, $(H_2)_p$, when $1 < p < \infty$.

Proposition 3.3. *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty[$, and let f_p^∞ defined by* (1.4)*, provided f satisfies (H_0), $(H_1)_p$, $(H_2)_p$, then*

(i) *f_p^∞ is convex-quasiconvex;*

(ii) *there exists $C > 0$ such that*

$$\tfrac{1}{C}(|b|^p + |\xi|) \leq f_p^\infty(x, u, b, \xi) \leq C(|b|^p + |\xi|); \tag{3.4}$$

(iii) *for every $K \subset\subset \Omega \times \mathbb{R}^d$ there exists a continuous function ω_K with $\omega_K(0) = 0$ such that*

$$|f_p^\infty(x, u, b, \xi) - f_p^\infty(x', u', b, \xi)| \leq \omega_K(|x - x'| + |u - u'|)(|b|^p + |\xi|) \tag{3.5}$$

for every (x, u, b, ξ) and (x', u', b, ξ) in $K \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$.
Moreover, given $x_0 \in \Omega$, and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then

$$f_p^\infty(x, u, b, \xi) - f_p^\infty(x_0, u, b, \xi) \geq -\varepsilon(|b|^p + |\xi|)$$

for every $(u, b, \xi) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$;

(iv) *in particular, f_p^∞ is continuous.*

Proof.

(i) The convexity-quasiconvexity of f_p^∞ can be proven exactly as in ([15], Lem. 2.1).

(ii) By definition (1.4) we may find a subsequence $\{t_k\}$ such that

$$f_p^\infty(x,u,b,\xi) = \lim_{k\to+\infty} \frac{f(x,u,t_k^{\frac{1}{p}}b, t_k\xi)}{t_k}\cdot$$

By $(H_1)_p$ one has

$$f_p^\infty(x,u,b,\xi) \leq \lim_{k\to+\infty} \frac{C(1+t_k|b|^p + t_k|\xi|)}{t_k} = C(|b|^p+|\xi|)$$

and

$$f_p^\infty(x,u,b,\xi) \geq \lim_{k\to+\infty} \frac{\frac{1}{C}(t_k|b|^p + t_k|\xi|)-C}{t_k} \geq \frac{1}{C}(|b|^p+|\xi|).$$

Hence $(H_1)_p$ holds for f^∞.

(iii) Again (1.4) entails that for every $(x,u),(x',u') \in \Omega\times\mathbb{R}^d$ and $(b,\xi)\in\mathbb{R}^m\times\mathbb{R}^{d\times N}$ that, up to a subsequence not relabeled,

$$f_p^\infty(x,u,b,\xi) - f_p^\infty(x',u',b,\xi) \leq \lim_{k\to+\infty} \frac{f(x,u,t_k^{\frac{1}{p}}b,t_k\xi) - f(x',u',t_k^{\frac{1}{p}}b,t_k\xi)}{t_k}\cdot$$

By $(H_2)_p$, for every $K\subset\Omega\times\mathbb{R}^d$ there exists $\omega_K:\mathbb{R}\to[0,+\infty)$ continuous with $\omega_K(0)=0$ such that if $(x,u),(x',u')\in K$, for every $(b,\xi)\in\mathbb{R}^m\times\mathbb{R}^{d\times N}$ it results

$$\begin{aligned}\lim_{k\to+\infty} \frac{f(x,u,t_k^{\frac{1}{p}}b,t_k\xi) - f(x',u',t_k^{\frac{1}{p}}b,t_k\xi)}{t_k} &\leq \lim_{k\to+\infty}\frac{\omega_K(|x-x'|+|u-u'|)(1+t_k|b|^p+t_k|\xi|)}{t_k}\\ &= \omega_K(|x-x'|+|u-u'|)(|b|^p+|\xi|).\end{aligned}$$

Changing the role of $f_p^\infty(x,u,b,\xi)$ with $f_p^\infty(x',u',b,\xi)$, (3.5) follows.
For what concerns the second inequality in (iii), by (1.4) and (2) of $(H_2)_p$ and, up to a subsequence not relabeled, we have for every $x,x_0\in\Omega$ such that $|x-x_0|<\delta$, and every $(u,b,\xi)\in\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}$

$$\begin{aligned}f_p^\infty(x,u,b,\xi) - f_p^\infty(x_0,u,b,\xi) &\geq \lim_{k\to+\infty}\frac{f(x,u,t_k^{\frac{1}{p}}b,t_k\xi) - f(x_0,u,t_k^{\frac{1}{p}}b,t_k\xi)}{t_k}\\ &\geq -\varepsilon\lim_{k\to+\infty}\frac{1+t_k|b|^p+|t_k\xi|}{t_k} = -\varepsilon(|b|^p+|\xi|).\end{aligned}$$

(iv) The convexity-quasiconvexity and (3.4) guarantee that f_p^∞ is continuous with respect to (b,ξ), in particular it is $(p,1)$-Lipschitz continuous in b and ξ uniformly with respect to (x,u). Thus (3.5), (3.1) and the triangular inequality entail that

$$\begin{aligned}&|f_p^\infty(x,u,b,\xi) - f_p^\infty(x',u',b',\xi')|\\ &\leq \omega_k(|x-x'|+|u-u'|)(|\xi|+|b|^p) + \gamma|\xi-\xi'| + \gamma(1+|b|^{p-1}+|b'|^{p-1}+|\xi|^{\frac{1}{p'}}+|\xi'|^{\frac{1}{p'}})|b-b'| \leq \varepsilon\end{aligned}$$

provided that $|x-x'|, |u-u'|$, $|b-b'|$ and $|\xi-\xi'|$ are small. □

Remark 3.4. We emphasize that not all the assumptions on f in Proposition 3.3 are necessary to prove the items above. In particular, one has that the proof of (ii) uses only the fact that f verifies $(H_1)_p$. Moreover, (iii) follows from (1.4) and $(H_2)_p$ (i) and (ii).

Regarding the recession function for $p=\infty$ in (1.8), a result analogous to Proposition 3.3 holds, but the proof is omitted for the sake of brevity.

Proposition 3.5. *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0,+\infty[$, and let f^∞ be defined by (1.8), provided f satisfies (H_0), $(H_1)_\infty$, $(H_2)_\infty$, then*

(i) *f^∞ is convex-quasiconvex.*
(ii) *For every $M>0$, there exists $C_M>0$ such that $\frac{1}{C_M}|\xi| \leq f^\infty(x,u,b,\xi) \leq C_M|\xi|$, for every $b \in \mathbb{R}^m$ such that $|b| \leq M$.*
(iii) *For every $M>0$, and for every compact set $K \subset \Omega \times \mathbb{R}^d$ there exists a continuous function $\omega_{M,K}$: $\mathbb{R} \to [0,+\infty)$ with $\omega_{M,K}(0)=0$ such that if $|b|\leq M$ then*

$$|f^\infty(x,u,b,\xi) - f^\infty(x',u',b,\xi)| \leq \omega_{M,K}(|x-x'|+|u-u'|)|\xi|$$

for every $(x,u,\xi),(x',u',\xi) \in K \times \mathbb{R}^{d\times N}$.
Moreover, given $x_0 \in \Omega$, and $\varepsilon>0$ there exists $\delta>0$ such that if $|x-x_0|<\delta$ then

$$f^\infty(x,u,b,\xi) - f^\infty(x_0,u,b,\xi) \geq -\varepsilon|\xi|$$

for every $(u,b,\xi) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$.
(iv) *In particular, f^∞ is continuous.*

3.3. The surface energy densities

For any convex-quasiconvex function $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0,+\infty)$, and $1<p\leq\infty$, we define the following surface energy densities $K_p : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \times S^{N-1} \to [0,+\infty)$ by (1.6) if $1<p<\infty$ and by (1.9) if $p=\infty$.

A density argument guarantees that the family $\mathcal{A}$ in formulas (1.7) can be constituted by functions in $W^{1,\infty}$, as quoted in [5]. Analogously, in (1.6) the set L^∞ can be replaced by L^p.

The following result provides some properties of the density K_p and develops along the lines of Lemma 2.15 in [18].

Proposition 3.6. *Assume $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \longrightarrow [0,+\infty)$ is a convex-quasiconvex function satisfying $(H_1)_p$, $(H_2)_p$ and $(H_3)_p$. Then*

(a) *there exists a constant C such that*

$$|K_p(x,0,c,d,\nu) - K_p(x,0,c',d',\nu)| \leq C(|c-c'|+|d-d'|)$$

for every (x,c,d,ν) and (x,c',d',ν) in $\Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$;
(b) *$(x,b,\nu) \mapsto K_p(x,b,c,d,\nu)$ is upper semicontinuous for every $c,d \in \mathbb{R}^d$;*
(c) *$K_p(\cdot,\cdot,\cdot,0,\cdot)$ is upper semicontinuous in $\Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$;*
(d) *there exists a constant $C>0$ such that*

$$0 \leq K_p(x,b,c,d,\nu) \leq C(|c-d|+|b|^p),\ \forall\ (x,b,c,d,\nu) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1};$$

(e) *for all $x_0 \in \Omega$ and for all $\varepsilon>0$ there exists $\delta>0$ such that $|x-x_0|<\delta$ implies*

$$|K_p(x,b,c,d,\nu) - K_p(x_0,b,c,d,\nu)| \leq \varepsilon C(1+|b|^p+|d-c|).$$

Proof. Condition (c) is a consequence of (a) and (b). To prove (a) we construct an admissible field $w^* \in \mathcal{A}(c',d',\nu)$ as in Lemma 2.15 in [18] and we define $\eta^* \in L^\infty(Q;\mathbb{R}^m)$ with 0 average in Q_ν as follows

$$\eta^*(y) := \begin{cases} 2^{\frac{1}{p}}\eta(2y) & \text{if } |y\cdot\nu| \leq \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} \leq |y\cdot\nu| \leq \frac{1}{2}. \end{cases}$$

where η has been extended by periodicity to all $\mathbb{R}^N$ and still denoted by η. Using conditions (3.3), (3.4) and the periodicity of w and η one obtains

$$K_p(x, 0, c', d', \nu) \leq \int_{Q_\nu} f_p^\infty(x, w(z), \eta(z), \nabla w(z))\,\mathrm{d}z + C(|c - c'| + |d - d'|).$$

Taking the infimum over all $w \in \mathcal{A}(c, d, \nu)$ and $\eta \in L^\infty(Q; \mathbb{R}^m)$ we conclude that

$$K_p(x, 0, c', d', \nu) \leq K_p(x, 0, c, d, \nu) + C(|c - c'| + |d - d'|).$$

The reverse inequality is obtained by letting $w \in \mathcal{A}(c', d', \nu)$ and building $w^* \in \mathcal{A}(c, d, \nu)$.

To prove (b), we start noticing that

$$K_p(x, b, c, d, \nu) := \inf \left\{ \int_Q f_p^\infty(x, w(y), \eta(y), \nabla w(y) R^T)\,\mathrm{d}y : \; w \in \mathcal{A}(c, d, e_N), \eta \in L^\infty(Q; \mathbb{R}^m), \; \int_Q \eta\,\mathrm{d}y = b \right\},$$

where $R \in SO(N)$ is such that $Re_N = \nu$ and $RQ = Q_\nu$. Also, due to the growth conditions, by density arguments, it suffices to choose smooth functions w.

Let $(x_n, b_n, \nu_n) \to (x, b, \nu)$, given $\varepsilon > 0$ let $w \in \mathcal{A}(c, d, e_N)$ be a smooth function and $\eta \in L^\infty(Q; \mathbb{R}^m)$ with $\int_Q \eta\,\mathrm{d}y = b$ such that

$$\left| K_p(x, b, c, d, \nu) - \int_Q f_p^\infty(x, w(y), \eta(y), \nabla w(y) R^T)\,\mathrm{d}y \right| < \varepsilon.$$

Consider $\eta_n \in L^\infty(Q; \mathbb{R}^m)$ such that $\int_Q \eta_n\,\mathrm{d}y = b_n$ and $\eta_n \to \eta$ in $L^p(Q; \mathbb{R}^m)$. For example $\eta_n := \eta + b_n - b$. Let X be a compact subset of $\Omega \times \mathbb{R}^d$ containing a neighborhood of $\{(x, w(y)) : \; y \in Q\}$. By condition (3.5), there exists a continuous function ω_X, with $\omega_X(0) = 0$ such that

$$\left| f_p^\infty(y, u, b, \xi) - f_p^\infty(y', u', b, \xi) \right| \leq \omega_X(|y - y'| + |u - u'|)(|b|^p + |\xi|) \tag{3.6}$$

for every $(y, u, \xi), (y', u', \xi) \in X \times \mathbb{R}^{d \times N}$ and $b \in \mathbb{R}^m$. As already noticed in Proposition 3.3, the recession function f_p^∞ is convex-quasiconvex and we have the following $(p, 1)$-Lipschitz condition for f_p^∞:

$$\left| f_p^\infty(x, u, b, \xi) - f_p^\infty(x, u, b', \xi') \right| \leq \gamma(|\xi - \xi'| + (1 + |b|^{p-1} + |b'|^{p-1} + |\xi|^{\frac{1}{p}} + |\xi'|^{\frac{1}{p}})|b - b'|), \tag{3.7}$$

for every (x, u, b, ξ) and (x, u, b', ξ') in $\Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d \times N}$.

As in ([18], Lem. 2.15), consider orthogonal transformations R_n such that $R_n e_N = \nu_n$ and $R_n \to R$. By virtue of the preceding estimates, and standard arguments, for n large enough we have

$$K_p(x, b_n, c, d, \nu_n) \leq \varepsilon + \int_Q f_p^\infty(x, w(y), \eta(y), \nabla w(y) R^T)\,\mathrm{d}y \leq 2\varepsilon + K_p(x, b, c, d, \nu).$$

Letting $\varepsilon \to 0^+$ we conclude that $\limsup_{n \to +\infty} K_p(x, b_n, c, d, \nu_n) \leq K_p(x, b, c, d, \nu)$.

The proof of (d) is identical to the proof of ([18], Lem. 2.15 d)).

The proof of (e) develops along the lines of ([10], Prop. 2.9 (ii)). □

Proposition 3.7. *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d \times N} \longrightarrow [0, +\infty)$ be a continuous function, f^∞ be its recession function given by (1.8) and let K_∞ be defined as in (1.9). Then $K_\infty(x, \cdot, c, d, \nu)$ is a constant function for any fixed $(x, c, d, \nu) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$.*

Proof. Let (x,c,d,ν) be arbitrary in $\Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$. Let $b, \overline{b} \in \mathbb{R}^m$ such that $b \neq \overline{b}$. We claim that $K_\infty(x,\overline{b},c,d,\nu) \leq K_\infty(x,b,c,d,\nu)$. Let $w \in \mathcal{A}(c,d,\nu)$ and $\eta \in L^\infty(Q_\nu;\mathbb{R}^m)$ such that $\int_{Q_\nu} \eta \,\mathrm{d}y = b$ be arbitrary and extend them by Q_ν-periodicity to all $\mathbb{R}^N$. Then define in Q_ν

$$\overline{w}(y) := \begin{cases} c & \text{if } -\frac{1}{2} \leq y\cdot\nu < -\frac{1}{4}, \\ w(2y) & \text{if } |y\cdot\nu| \leq \frac{1}{4}, \\ d & \text{if } \frac{1}{4} < y\cdot\nu \leq \frac{1}{2}, \end{cases} \qquad \overline{\eta}(y) := \begin{cases} \eta(2y) & \text{if } |y\cdot\nu| \leq \frac{1}{4}, \\ k & \text{if } \frac{1}{4} < |y\cdot\nu| \leq \frac{1}{2}, \end{cases}$$

where k is the constant such that $\int_{Q_\nu} \overline{\eta}\,\mathrm{d}y = \overline{b}$, $k = 2\overline{b} - b$. Notice that $\overline{w} \in \mathcal{A}(c,d,\nu)$, thus

$$\begin{aligned} K_\infty(x,\overline{b},c,d,\nu) &\leq \int_{Q_\nu} f^\infty(x,\overline{w}(y),\overline{\eta}(y),\nabla\overline{w}(y))\,\mathrm{d}y = \int_{\{y\in Q_\nu:\ |y\cdot\nu|\leq 1/4\}} f^\infty(x,w(2y),\eta(2y),2\nabla w(2y))\,\mathrm{d}y \\ &= \frac{2}{2^N}\int_{\{z\in\mathbb{R}^N:|z\cdot\nu_i|\leq 1, i=1,\ldots,N-1, |z\cdot\nu|\leq 1/2\}} f^\infty(x,w(z),\eta(z),\nabla w(z))\,\mathrm{d}z \\ &= \int_{Q_\nu} f^\infty(x,w(z),\eta(z),\nabla w(z))\,\mathrm{d}z, \end{aligned}$$

where we have used in the second identity the fact that $f^\infty(x,u,b,\cdot)$ is a positively 1-homogeneous function so, in particular, $f^\infty(x,u,b,0) = 0$. The last identity follows from the periodicity of w and η. The claim is achieved by taking the infimum on w and η on the right hand side.

The reverse inequality follows by interchanging the roles of b and $\overline{b}$. □

Proposition 3.8. *Assume that* $f : \Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N} \longrightarrow [0,+\infty)$ *is a convex-quasiconvex function satisfying* (H_0), $(H_1)_\infty$, $(H_2)_\infty$ *and* $(H_3)_\infty$. *Then*

(a) *there exists a constant* $C > 0$ *such that*

$$|K_\infty(x,b,c,d,\nu) - K_\infty(x,b',c',d',\nu)| \leq C(|c-c'| + |d-d'|)$$

for every (x,b,c,d,ν) *and* (x,b',c',d',ν) *in* $\Omega\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$*;*

(b) $(x,b,\nu) \mapsto K_\infty(x,b,c,d,\nu)$ *is upper semicontinuous for every* $c,d \in \mathbb{R}^d$*;*

(c) K_∞ *is upper semicontinuous in* $\Omega\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$*;*

(d) *there exists a constant* $C > 0$ *such that* $K_\infty(x,b,c,d,\nu) \leq C|c-d|$, *for every* $(x,b,c,d,\nu) \in \Omega\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$.

Proof. The proof is very similar to Proposition 3.6. We just emphasize the main differences. To prove a) we start by noticing that by Proposition 3.7, $K_\infty(x,b,c,d,\nu) = K_\infty(x,0,c,d,\nu)$ and $K_\infty(x,b',c',d',\nu) = K_\infty(x,0,c',d',\nu)$. So we fix $w \in \mathcal{A}(c,d,\nu)$, $\eta \in L^\infty(Q_\nu;\mathbb{R}^m)$ with $\int_{Q_\nu}\eta\,\mathrm{d}y = 0$ and construct $w^* \in \mathcal{A}(c',d',\nu)$ similarly as in Lemma 2.15 in [18] and let $\eta^* \in L^\infty(Q_\nu;\mathbb{R}^m)$ with average 0 in Q_ν be given by

$$\eta^*(y) := \begin{cases} \eta(2y) & \text{if } |y\cdot\nu| \leq \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} \leq |y\cdot\nu| \leq \frac{1}{2}. \end{cases}$$

The proof of b) follows directly from Proposition 3.6 (b) using again Proposition 3.7, replacing (3.6) by

$$|f^\infty(y,u,b,\xi) - f^\infty(y',u',b,\xi)| \leq \omega_{X,M}(|y-y'| + |u-u'|)|\xi|$$

for every $(y,u,\xi),(y',u',\xi) \in X\times\mathbb{R}^{d\times N}$ and $b \in \mathbb{R}^m$ with $|b| \leq M$, where $M := \|\eta\|_{L^\infty}$. And the p-Lipschitz continuity (3.7) should be replaced by the condition

$$|f^\infty(x,u,b,\xi) - f^\infty(x,u,b',\xi')| \leq \beta(M,n,m,N)\left((1+|\xi|+|\xi'|)\,|b-b'| + |\xi-\xi'|\right),$$

for every (x,u,ξ) and (x,u,ξ') in $\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}$ and $b,b' \in \mathbb{R}^m$ with $|b|,|b'| \leq M$. □

Remark 3.9. If f does not depend on u, *i.e.* $f \equiv f(x, v, \nabla u)$, then K_p and K_∞ coincide with the recession functions f_p^∞ and f^∞, respectively. Namely, for every $(x, b, c, d, \nu) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \times S^{N-1}$,

$$K_p(x, b, c, d, \nu) = f_p^\infty(x, b, (c - d) \otimes \nu),$$

and

$$K_\infty(x, b, c, d, \nu) = f^\infty(x, b, (c - d) \otimes \nu). \tag{3.8}$$

To obtain the above formulas, we refer to the arguments used to prove ([5], formula (5.83)). From Proposition 3.7, (3.8) becomes

$$K_\infty(x, b, c, d, \nu) = f^\infty(x, b, (c - d) \otimes \nu) = f^\infty(x, 0, (c - d) \otimes \nu).$$

We observe that the latter equality, in the above formula, was already proven in [14].

We underline that, besides we are not able to prove equality between K_p and K_∞ in general, there might be cases, including those when f exhibits explicit dependence on u, in which there is coincidence between K_p and K_∞, for example consider the cases $f(x, u, b, \xi) := g(x, u)\sqrt{|b|^{2p} + |\xi|^2}$, with g suitably chosen in order to satisfy assumptions $(H_1)_p - (H_3)_p$, or $f(x, u, b, \xi) := \sqrt{|b|^{2p} + |(u, \xi)|^2}$.

The following approximation result will be used to prove the upper bound inequality in Theorem 1.2.

Proposition 3.10. *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \longrightarrow [0, +\infty)$ be a continuous function, and let f^∞ be as in (1.8). Fix $r \geq 0$ and let $K_r : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \longrightarrow [0, +\infty)$ be such that*

$$K_r(x, b, c, d, \nu) := \inf\left\{\int_{Q_\nu} f^\infty(x, w(y), \eta(y), \nabla w(y))\,\mathrm{d}y : \ w \in \mathcal{A}(c, d, \nu),\ \eta \in L^\infty(Q_\nu; \mathbb{R}^m), \|\eta\|_{L^\infty(Q_\nu)} \leq |b| + r,\ \int_{Q_\nu} \eta\,\mathrm{d}y = b\right\}.$$

Then, for each (x, b, c, d, ν),

$$K_\infty(x, b, c, d, \nu) = \lim_{r\to+\infty} K_r(x, b, c, d, \nu) = \inf_{r\geq 0} K_r(x, b, c, d, \nu). \tag{3.9}$$

Proof. The fact that, $K_r(x, b, c, d, \nu)$ is decreasing in r gives the last identity. Moreover, $K_\infty(x, b, c, d, \nu) \leq K_r(x, b, c, d, \nu)$ for any r, therefore it is enough to find r_n such that $\lim_{n\to+\infty} K_{r_n}(x, b, c, d, \nu) = K_\infty(x, b, c, d, \nu)$. By definition of K_∞, given $n \in \mathbb{N}$ we can get $\omega_n \in \mathcal{A}(c, d, \nu)$, and $\eta_n \in L^\infty(Q_\nu; \mathbb{R}^m)$ with $\int_{Q_\nu} \eta_n(y)\,\mathrm{d}y = b$ and such that

$$K_\infty(x, b, c, d, \nu) + \frac{1}{n} > \int_{Q_\nu} f^\infty(x, \omega_n(y), \eta_n(y), \nabla\omega_n(y))\,\mathrm{d}y.$$

Setting $r_n := \|\eta_n\|_{L^\infty} - |v|$ we get

$$K_\infty(x, b, c, d, \nu) + \frac{1}{n} \geq K_{r_n}(x, b, c, d, \nu) \geq K_\infty(x, b, c, d, \nu)$$

which yields the desired condition by letting $n \to +\infty$. □

Remark 3.11. Notice that, for all $x_0 \in \Omega$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies the existence of a suitable constant $C_{|b|+r}$ for which

$$|K_r(x, b, c, d, \nu) - K_r(x_0, b, c, d, \nu)| \leq \varepsilon C_{|b|+r}(1 + |d - c|), \tag{3.10}$$

for every $b \in \mathbb{R}^m$, $c, d, \in \mathbb{R}^d, \nu \in S^{N-1}$.

We also observe that arguments entirely similar to those in Proposition 3.8 guarantee that

$$K_r(x, b, c, d, \nu) \leq C_{|b|+r}|c - d|,$$

for every $(x, b, c, d, \nu) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$.

4. Main Results: $BV \times L^p, 1 < p < \infty$

In this section we prove Theorem 1.1.

4.1. Lower semicontinuity in $BV \times L^p$

Theorem 4.1. *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a continuous function satisfying (H_0), $(H_1)_p-(H_3)_p$. Then*

$$\liminf_{n\to+\infty} \int_\Omega f(x, u_n, v_n, \nabla u_n)\mathrm{d}x \geq \int_\Omega f(x, u, v, \nabla u)\mathrm{d}x + \int_{J_u} K_p(x, 0, u^+, u^-, \nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega f_p^\infty(x, u, 0, \tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|})\mathrm{d}|D^c u| \tag{4.1}$$

in $BV(\Omega;\mathbb{R}^d)\times L^p(\Omega;\mathbb{R}^m)$ with respect to the (L^1-strong $\times$ L^p-weak)$-$convergence, where K_p is given by (1.6) and f_p^∞ is the $(p, 1)-$ recession function *given by (1.4).*

Proof. Using the same arguments as in ([1], Thm. II.4 and [17], Prop. 2.4) we may reduce to $u_n \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^d)$ and $v_n \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^m)$. Due to $(H_1)_p$ we may assume, without loss of generality, that

$$\liminf_{n\to+\infty} \int_\Omega f(x, u_n, v_n, \nabla u_n)\mathrm{d}x = \lim_{n\to+\infty} \int_\Omega f(x, u_n, v_n, \nabla u_n)\mathrm{d}x < +\infty.$$

Hence, up to a subsequence, $\mu_n := f(x, u_n, v_n, \nabla u_n)\mathcal{L}^N \overset{*}{\rightharpoonup} \mu$ in the sense of measures for some positive Radon measure μ. By the Radon–Nikodým theorem we can decompose μ as a sum of four mutually nonnegative measures, namely, $\mu = \mu_a\mathcal{L}^N + \mu_j\mathcal{H}^{N-1}\lfloor J_u + \mu_c|D^c u| + \mu_s$.

By Besicovitch derivation theorem

$$\mu_a(x_0) = \lim_{\varepsilon\to0^+} \frac{\mu(B(x_0, \varepsilon))}{\mathcal{L}^N(B(x_0, \varepsilon))} < +\infty, \text{ for } \mathcal{L}^N - \text{a.e. } x_0 \in \Omega,$$

$$\mu_j(x_0) = \lim_{\varepsilon\to0^+} \frac{\mu(Q_\nu(x_0, \varepsilon))}{\mathcal{H}^{N-1}(Q_\nu(x_0, \varepsilon) \cap J_u)} < +\infty, \text{ for } \mathcal{H}^{N-1} - \text{a.e. } x_0 \in J_u \cap \Omega,$$

$$\mu_c(x_0) = \lim_{\varepsilon\to0^+} \frac{\mu(Q(x_0, \varepsilon))}{|Du|(Q(x_0, \varepsilon))} < +\infty, \text{ for } |D^c u| - \text{a.e. } x_0 \in \Omega.$$

We claim that

$$\mu_a(x_0) \geq f(x_0, u(x_0), v(x_0), \nabla u(x_0)), \qquad \text{for } \mathcal{L}^N - \text{a.e. } x_0 \in \Omega, \tag{4.2}$$

$$\mu_j(x_0) \geq K_p(x_0, 0, u^+(x_0), u^-(x_0), \nu_u(x_0)), \quad \text{for } \mathcal{H}^{N-1} - \text{a.e. } x_0 \in J_u \cap \Omega, \tag{4.3}$$

$$\mu_c(x_0) \geq f_p^\infty(x_0, u(x_0), 0, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x_0)), \text{ for } |D^c u| - \text{a.e. } x_0 \in \Omega. \tag{4.4}$$

If (4.2)–(4.4) hold then (4.1) follows immediately. Indeed, since $\mu_n \overset{*}{\rightharpoonup} \mu$ in the sense of measures then

$$\begin{aligned}\liminf_{n\to+\infty} \int_\Omega f(x, u_n, v_n, \nabla u_n)\mathrm{d}x &\geq \liminf_{n\to+\infty} \mu_n(\Omega) \geq \mu(\Omega) \geq \int_\Omega \mu_a \mathrm{d}x + \int_{J_u} \mu_j \mathrm{d}\mathcal{H}^{N-1} + \int_\Omega \mu_c \mathrm{d}|D^c u| \\ &\geq \int_\Omega f(x, u, v, \nabla u)\mathrm{d}x + \int_{J_u} K_p(x, 0, u^+, u^-, \nu_u)\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \int_\Omega f_p^\infty\left(x, u, 0, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\right)\mathrm{d}|D^c u|,\end{aligned}$$

where we have used the fact that μ_s is nonnegative.

We prove (4.2)–(4.4) using the blow up method introduced in [17].

Bulk part. Inequality (4.2) is obtained as in [23] Section 3 and [24].

Jump part. Consider $x_0 \in J_u$, then there exist $u^-(x_0), u^+(x_0) \in \mathbb{R}^d$ and $\nu := \nu_u(x_0) \in S^{N-1}$ such that (2.2) holds,

$$\mu_j(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q_\nu(x_0,\varepsilon))}{|u^+ - u^-|\mathcal{H}^{N-1}\lfloor J_u(Q_\nu(x_0,\varepsilon))} \in \mathbb{R}$$

and assume $\mu(\partial Q_\nu(x_0,\varepsilon_k)) = 0$ for $\{\varepsilon_k\} \searrow 0^+$. Moreover, for $\mathcal{H}^{N-1}\lfloor J_u-$ a.e. x_0, we may assume

$$\frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to+\infty} \frac{1}{\varepsilon_k^{N-1}} \int_{Q_\nu(x_0+\varepsilon_k)} |v(x)|^p \mathrm{d}x = 0. \tag{4.5}$$

Then

$$\begin{aligned}
\mu_j(x_0) &= \lim_{k\to+\infty} \frac{\mu(Q_\nu(x_0,\varepsilon_k))}{|u^+ - u^-|\mathcal{H}^{N-1}\lfloor J_u(Q_\nu(x_0,\varepsilon_k))} \\
&\geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to+\infty} \lim_{n\to+\infty} \frac{1}{\varepsilon_k^{N-1}} \int_{Q_\nu(x_0+\varepsilon_k)} f(x, u_n(x), v_n(x), \nabla u_n(x))\mathrm{d}x \\
&= \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to+\infty} \lim_{n\to+\infty} \int_{Q_\nu} \varepsilon_k f(x_0+\varepsilon_k y, u_n(x_0+\varepsilon_k y), v_n(x_0+\varepsilon_k y), \nabla u_n(x_0+\varepsilon_k y))\mathrm{d}y \\
&= \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to+\infty} \lim_{n\to+\infty} \int_{Q_\nu} \varepsilon_k f(x_0+\varepsilon_k y, u_{n,k}(y), \varepsilon_k^{-\frac{1}{p}} v_{n,k}(y), \frac{1}{\varepsilon_k}\nabla u_{n,k}(y))\mathrm{d}y,
\end{aligned} \tag{4.6}$$

where

$$u_{n,k}(y) := u_n(x_0+\varepsilon_k y), \qquad v_{n,k}(y) := \varepsilon_k^{\frac{1}{p}} v_n(x_0+\varepsilon_k y).$$

We observe that,

$$\lim_{k\to+\infty} \lim_{n\to+\infty} \|u_{n,k}(y) - u_0\|_{L^1(Q;\mathbb{R}^d)} = 0, \tag{4.7}$$

with

$$u_0(y) := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu > 0, \\ u^-(x_0) & \text{if } y\cdot\nu \leq 0, \end{cases} \tag{4.8}$$

and for every $\varphi \in L^q(Q;\mathbb{R}^m)$,

$$\lim_{k\to+\infty} \lim_{n\to+\infty} \int_Q v_{n,k}(y)\varphi(y)\mathrm{d}y = \lim_{k\to+\infty} \varepsilon_k^{\frac{1}{p}} \int_Q v(x_0+\varepsilon_k y)\varphi(y)\mathrm{d}y = 0, \tag{4.9}$$

where the latter equality is obtained from (4.5).

Using the separability of $L^q(Q;\mathbb{R}^m)$, together with a diagonalization argument, from (4.10), (4.7) and (4.9), we obtain the existence of sequences $\bar{u}_k := u_{n(k),k}$ and $\bar{v}_k := v_{n(k),k}$ such that $\bar{u}_k \to u_0$ in $L^1(Q;\mathbb{R}^d)$, $\bar{v}_k \rightharpoonup 0$ in $L^p(Q;\mathbb{R}^m)$, and we obtain the following estimation for μ_j in terms of f_p^∞

$$\begin{aligned}
\mu_j(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to+\infty} \Bigg\{ &\int_Q f_p^\infty(x_0, \bar{u}_k, \bar{v}_k, \nabla\bar{u}_k)\mathrm{d}y \\
&+ \int_Q f_p^\infty(x_0+\varepsilon_k y, \bar{u}_k, \bar{v}_k, \nabla\bar{u}_k) - f_p^\infty(x_0, \bar{u}_k, \bar{v}_k, \nabla\bar{u}_k)\mathrm{d}y \\
&+ \int_Q \varepsilon_k f(x_0+\varepsilon_k y, \bar{u}_k, \varepsilon_k^{-\frac{1}{p}}\bar{v}_k, \tfrac{1}{\varepsilon_k}\nabla\bar{u}_k) - f_p^\infty(x_0+\varepsilon_k y, \bar{u}_k, \bar{v}_k, \nabla\bar{u}_k)\mathrm{d}y \Bigg\}.
\end{aligned} \tag{4.10}$$

From Proposition 3.3 iii we get that for any $\varepsilon > 0$, if k is sufficiently large

$$\int_Q f_p^\infty(x_0+\varepsilon_k y,\bar{u}_k,\bar{v}_k,\nabla\bar{u}_k) - f_p^\infty(x_0,\bar{u}_k,\bar{v}_k,\nabla\bar{u}_k)\mathrm{d}y$$
$$\geq -\varepsilon\int_Q |\bar{v}_k|^p+|\nabla\bar{u}_k|\mathrm{d}y = -\varepsilon\int_Q \varepsilon_k(|v_k(x_0+\varepsilon_k y)|^p+|\nabla u_k(x_0+\varepsilon_k y)|)\mathrm{d}y \geq O(\varepsilon).$$

On the other hand, using $(H_3)_p$ and Hölder inequality we get

$$\int_Q \varepsilon_k f(x_0+\varepsilon_k y,\bar{u}_k,\varepsilon_k^{-\frac{1}{p}}\bar{v}_k,\tfrac{1}{\varepsilon_k}\nabla\bar{u}_k) - f_p^\infty(x_0+\varepsilon_k y,\bar{u}_k,\bar{v}_k,\nabla\bar{u}_k)\mathrm{d}y$$
$$\leq c'\int_{\{y\in Q:\frac{|\nabla\bar{u}_k|}{\varepsilon_k}+\frac{|\bar{v}_k|^p}{\varepsilon_k}\geq L\}}(\varepsilon_k^\tau|\bar{v}_k|^{(1-\tau)p}+|\nabla\bar{u}_k|^{1-\tau}\varepsilon_k^\tau)\mathrm{d}y + C\int_{\{y\in Q:\frac{|\nabla\bar{u}_k|}{\varepsilon_k}+\frac{|\bar{v}_k|^p}{\varepsilon_k}<L\}}(|\bar{v}_k|^p+|\nabla\bar{u}_k|)\mathrm{d}y$$
$$\leq O(\varepsilon)+c'\varepsilon_k^\tau\left(\int_Q|\nabla\bar{u}_k|\mathrm{d}y\right)^{1-\tau}+c'\varepsilon_k\int_Q|v_k(x_0+\varepsilon_k y)|^{(1-\tau)p}\mathrm{d}y+O(\varepsilon)$$
$$\leq O(\varepsilon)+c'\varepsilon_k^\tau\left(\int_Q|\nabla\bar{u}_k|\mathrm{d}y\right)^{1-\tau}+c'\varepsilon_k^\tau\left(\varepsilon_k\int_Q|v_k(x_0+\varepsilon_k y)|^p\mathrm{d}y\right)^{1-\tau}+O(\varepsilon)=O(\varepsilon),$$

where we have used in last equality (4.9). Thus we are led to

$$\mu_j(x_0)\geq\frac{1}{|u^+(x_0)-u^-(x_0)|}\lim_{k\to+\infty}\int_Q f_p^\infty(x_0,\bar{u}_k,\bar{v}_k,\nabla\bar{u}_k)\mathrm{d}y+O(\varepsilon). \tag{4.11}$$

Next we apply Lemma 2.7 to $f_p^\infty(x_0,\cdot,\cdot,\cdot)$, obtaining

$$\lim_{k\to+\infty}\int_Q f_p^\infty(x_0,\bar{u}_k,\bar{v}_k,\nabla\bar{u}_k)\mathrm{d}y\geq\limsup_{k\to+\infty}\int_Q f_p^\infty(x_0,\xi_k,\zeta_k,\nabla\xi_k)\mathrm{d}y, \tag{4.12}$$

where $\xi_k\to u_0$ in $L^1(Q;\mathbb{R}^d)$ and $\xi_k\in\mathcal{A}(u^+(x_0),u^-(x_0),\nu_u(x_0))$, $\zeta_k\rightharpoonup 0$ in $L^p(Q;\mathbb{R}^m)$ with $\int_Q\zeta_k\,dy=0$. In particular, by (1.6) we have

$$\mu_j(x_0)\geq K_p(x,0,u^+(x_0),u^-(x_0),\nu_u(x_0))\text{ for }\mathcal{H}^{N-1}-\text{a.e.}\,x_0\in J_u\cap\Omega.$$

Cantor Part. By definition,

$$\mu^c(x)=\lim_{\varepsilon\to0^+}\frac{\mu(x+\varepsilon Q)}{|Du|(x+\varepsilon Q)},\ |D^cu|-\text{a.e. }x\in\Omega. \tag{4.13}$$

We start recalling that, by Alberti's rank-one theorem (see [2]), together with (2.1),

$$\lim_{\varepsilon\to0^+}\frac{Du(x+\varepsilon Q)}{|Du|(x+\varepsilon Q)}=\lim_{\varepsilon\to0^+}\frac{D^cu(x+\varepsilon Q)}{|D^cu|(x+\varepsilon Q)}=A(x),\ |D^cu|-\text{a.e. }x\in\Omega \tag{4.14}$$

for some rank-one matrix $A(x)$ with $|A(x)|=1$.

Since $|D^cu|(J_u)=0$ and still denoting by u the approximate limit of u, which is defined in $\Omega\setminus J_u$, we have (cf. Def. 3.63 in [5])

$$\lim_{\varepsilon\to0^+}\frac{1}{|x+\varepsilon Q|}\int_{x+\varepsilon Q}|u(y)-u(x)|\mathrm{d}y=0,|D^cu|-\text{a.e. }x\in\Omega. \tag{4.15}$$

Finally, by Besicovitch Derivation theorem ([5], Thm. 5.52),

$$\lim_{\varepsilon\to0^+}\frac{|Du|(x+\varepsilon Q)}{\varepsilon^{N-1}}=0,\quad\lim_{\varepsilon\to0^+}\frac{|Du|(x+\varepsilon Q)}{\varepsilon^N}=+\infty,\ |D^cu|-\text{a.e. }x\in\Omega. \tag{4.16}$$

Let $x_0 \in \Omega$ be such that (4.13)−(4.16) hold. Notice that, as in Lemma 2.13 in [18], we can also assume that

$$\lim_{t\to 1^-} \liminf_{\varepsilon\to 0^+} \frac{|Du|((x_0+\varepsilon Q)\setminus(x_0+t\varepsilon Q))}{|Du|(x_0+\varepsilon Q)} = 0,$$

so, we can write, for some continuous function $\omega : [0,1] \to \mathbb{R}$ with $\omega(0) = 0$,

$$\liminf_{\varepsilon\to 0^+} \frac{|Du|((x_0+\varepsilon Q)\setminus(x_0+t\varepsilon Q))}{|Du|(x_0+\varepsilon Q)} \le \omega(1-t). \tag{4.17}$$

In the sequel, without loss of generality, we assume $A := A(x_0) = a \otimes e_N$ with $|a| = 1$ and, as in [18], we divide the proof in several steps.

Step 1. For each $0 < t < 1$ and $\gamma \in (t, 1)$ consider $\varepsilon_k \to 0^+$ such that

$$\lim_{k\to+\infty} \frac{\int_{Q_k} |v(x)|\mathrm{d}x}{|Du|(Q_k)} = \lim_{k\to+\infty} \frac{\int_{Q_k} |v(x)|^p\mathrm{d}x}{|Du|(Q_k)} = 0,\ |D^c u| - \text{a.e.}, \tag{4.18}$$

where $Q_k := x_0 + \varepsilon_k Q$. We also observe that, from (4.14)

$$\lim_{k\to+\infty} \frac{Du(Q_k)}{|Du|(Q_k)} = \lim_{k\to+\infty} \frac{D^c u(Q_k)}{|D^c u|(Q_k)} = A. \tag{4.19}$$

Arguing as in ([18], Sect. 4), conditions (4.13) and (4.15) imply the existence of subsequences $\{\bar u_k\} \subseteq \{u_n\}$, and $\{\bar v_k\} \subseteq \{v_n\}$, defined in Ω, such that

a) $\mu^c(x_0) \ge \limsup_{k\to+\infty} \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x, \bar u_k(x), \bar v_k(x), \nabla \bar u_k(x))\,\mathrm{d}x$;

b) $\lim_{k\to+\infty} \frac{1}{|Q_k|} \int_{Q_k} |\bar u_k(x) - u(x_0)|\,\mathrm{d}x = 0$;

c) $\lim_{k\to+\infty} \frac{1}{\varepsilon_k |Du|(Q_k)} \int_{Q_k} \left|\bar u_k(x) - u(x) - \frac{1}{|Q_k|}\int_{Q_k} (\bar u_k(z) - u(z))\,\mathrm{d}z\right| \mathrm{d}x = 0$;

d) $\frac{(\varepsilon_k^N)^{\frac{1}{p}} \bar v_k(x_0+\varepsilon_k\cdot)}{|D^c u|^{\frac{1}{p}}(Q_k)} \rightharpoonup 0$ in $L^p(Q;\mathbb{R}^m)$ as $k \to +\infty$, which follows from Hölder inequality and (4.18).

Step 2. In this step we will obtain an estimate for $\mu^c(x_0)$ similar to condition a), fixing on f the value of x and u. Precisely, we prove that there is $n_0 \in \mathbb{N}$ such that, for each $n \ge n_0$ there exist $\{\tilde u_k\} \subset W^{1,1}(\Omega;\mathbb{R}^d)$, $\{\tilde v_k\} \subset L^p(\Omega;\mathbb{R}^m)$, and $\{a_k\} \subset \mathbb{R}$ such that $a_k \to u(x_0)$, $\tilde v_k \rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^m)$ as $k \to +\infty$, $\|\tilde u_k - u(x_0)\|_{L^\infty} \le 1/n$ and

$$(1 + \omega_K(\tfrac{1}{n}))\,\mu^c(x_0) \ge \limsup_{k\to+\infty} \frac{1}{|Du|(Q_k)} \int_{\gamma Q_k} f(x_0, u(x_0), \tilde v_k(y), \nabla \tilde u_k(y))\,\mathrm{d}y, \tag{4.20}$$

where ω_K is the function in $(H_2)_p$, related to a compact set $K \subset \Omega \times \mathbb{R}^d$ containing $(x_0, u(x_0))$, and the estimate does not depend on k. We also prove that

$$\lim_{k\to+\infty} \frac{1}{\varepsilon_k |Du|(Q_k)} \int_{Q_k} \left|\tilde u_k(y) - a_k - \left(u(y) - \frac{1}{|Q_k|}\int_{Q_k} u(z)\,\mathrm{d}z\right)\right| \mathrm{d}y = 0. \tag{4.21}$$

Observe that by (4.15) and condition b) above, we can assume

$$\frac{1}{|Q_k|}\int_{Q_k} |u(y) - u(x_0)|\,\mathrm{d}y \le \frac{1}{n^2} \quad \text{and} \quad \frac{1}{|Q_k|}\int_{Q_k} |\bar u_k(y) - u(x_0)|\,\mathrm{d}y \le \frac{1}{n^2}.$$

Then let $a_k := \frac{1}{|Q_k|}\int_{Q_k} \bar{u}_k(y)\,\mathrm{d}y$. Clearly, from condition b) above, $a_k \to u(x_0)$. To define $\tilde{u}_k$ we start by considering a family of smooth cut-off functions $\varphi_{r,s} : \mathbb{R} \to [0,1]$ such that

$$\varphi_{r,s}(t) := \begin{cases} 1 & \text{if } t \le r, \\ 0 & \text{if } t \ge s, \end{cases} \tag{4.22}$$

and $\|\varphi'_{r,s}\|_{L^\infty} \le \frac{c}{s-r}$ for $\frac{2}{n^2} \le r < s \le \frac{1}{2n}$. Consider the sequence $\{\tau_{L_k}(\bar{v}_k)\}$ of p-equi-integrable functions derived from $\{\bar{v}_k\}$, as in ([16], Lem. 8.13). Then, for every $\lambda \in (0,+\infty)$ define two families of sequences

$$\tilde{u}_k^{r,s,\lambda} := a_k + \varphi_{r,s}\left(|\bar{u}_k - a_k| + \tfrac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}\right)(\bar{u}_k - a_k),$$

$$\tilde{v}_k^{r,s,\lambda} := \tau_{L_k}(\bar{v}_k) + \varphi_{r,s}\left(|\bar{u}_k - a_k| + \tfrac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}\right)(\bar{v}_k - \tau_{L_k}(\bar{v}_k)).$$

Notice that, since $a_k \to u(x_0)$, for sufficiently large k and independently of r, s and λ, $\|\tilde{u}_k^{r,s} - u(x_0)\|_{L^\infty} \le 1/n$, $v_k^{r,s,\lambda} \rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^m)$ and satisfies $d)$ as $k \to +\infty$. The sequences $\{\tilde{u}_k\}$ and $\{\bar{v}_k\}$ will be chosen among the sequences of the previous family for convenient r, s and λ. In order to make that choice we start doing some estimates. Using hypothesis $(H_2)_p$, for some compact set K containing $(x_0, u(x_0))$ and $(y, \tilde{u}_k^{r,s,\lambda})$ for $y \in \gamma Q_k$,

$$\begin{aligned}
&\int_{\gamma Q_k} f(x_0, u(x_0), \tilde{v}_k^{r,s,\lambda}, \nabla\tilde{u}_k^{r,s,\lambda})\,\mathrm{d}y \\
&= \int_{\gamma Q_k} f(x_0, u(x_0), \tilde{v}_k^{r,s,\lambda}, \nabla\tilde{u}_k^{r,s,\lambda}) - f(y, \tilde{u}_k^{r,s,\lambda}, \tilde{v}_k^{r,s,\lambda}, \nabla\tilde{u}_k^{r,s,\lambda})\mathrm{d}y + \int_{\gamma Q_k} f(y, \tilde{u}_k^{r,s,\lambda}, \tilde{v}_k^{r,s,\lambda}, \nabla\tilde{u}_k^{r,s,\lambda})\,\mathrm{d}y \\
&\le \int_{\gamma Q_k} \omega(|y-x_0| + |\tilde{u}_k^{r,s,\lambda} - u(x_0)|)(1 + |\nabla\tilde{u}_k^{r,s,\lambda}| + |\tilde{v}_k^{r,s,\lambda}|^p)\,\mathrm{d}y + \int_{\gamma Q_k} f(y, \tilde{u}_k^{r,s,\lambda}, \tilde{v}_k^{r,s,\lambda}, \nabla\tilde{u}_k^{r,s,\lambda})\,\mathrm{d}y \\
&\le \int_{\gamma Q_k} \omega(\gamma\varepsilon_k + 1/n)(1 + |\nabla\tilde{u}_k^{r,s,\lambda}| + |\tilde{v}_k^{r,s,\lambda}|^p)\,\mathrm{d}y + \int_{\gamma Q_k} f(y, \tilde{u}_k^{r,s,\lambda}, \tilde{v}_k^{r,s,\lambda}, \nabla\tilde{u}_k^{r,s,\lambda})\,\mathrm{d}y.
\end{aligned} \tag{4.23}$$

Using hypothesis $(H_1)_p$ and for sufficiently large n we can get the estimate

$$\int_{\gamma Q_k} (|\nabla\tilde{u}_k^{r,s,\lambda}| + |\tilde{v}_k^{r,s,\lambda}|^p)\,\mathrm{d}y \le c\int_{\gamma Q_k} f(y, \tilde{u}_k^{r,s,\lambda}, \tilde{v}_k^{r,s,\lambda}, \nabla\tilde{u}_k^{r,s,\lambda})\,\mathrm{d}y.$$

Recalling that $\frac{\varepsilon_k^N}{|Du|(Q_k)} \to 0$ (see (4.16)), to estimate (4.23) we are left with

$$\begin{aligned}
&\frac{1}{|Du|(Q_k)}\int_{\gamma Q_k} f\left(y, \tilde{u}_k^{r,s,\lambda}, \tilde{v}_k^{r,s,\lambda}, \nabla\tilde{u}_k^{r,s,\lambda}\right)\,\mathrm{d}y \\
&\le \frac{1}{|Du|(Q_k)}\Bigg\{\int_{\gamma Q_k} f(y, \bar{u}_k, \bar{v}_k, \nabla\bar{u}_k)\,\mathrm{d}y + \int_{\gamma Q_k \cap \left\{|\bar{u}_k - a_k| + \frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda} \ge s\right\}} f(y, a_k, \tau_{L_k}(\bar{v}_k), 0)\,\mathrm{d}y \\
&\quad + C\int_{\gamma Q_k \cap \left\{r < |\bar{u}_k - a_k| + \frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda} < s\right\}} \left(1 + \tfrac{1}{s-r}|\bar{u}_k - a_k|\left|\nabla\left(|\bar{u}_k - a_k| + |\tfrac{\tau_{L_k}(\bar{v}_k)-\bar{v}_k}{\lambda}|\right)\right|\right)\mathrm{d}y \\
&\quad + C\int_{\gamma Q_k \cap \left\{r < |\bar{u}_k - a_k| + \frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda} < s\right\}} (|\nabla(\bar{u}_k - a_k)||\tau_{L_k}(\bar{v}_k) - \bar{v}_k|^p + |\tau_{L_k}(\bar{v}_k)|^p)\,\mathrm{d}y\Bigg\},
\end{aligned}$$

where we have used $(H_1)_p$, co-area formula and exploited the fact that $\bar{v}_k$ is regular.

Thus

$$
\begin{aligned}
&\frac{1}{|Du|(Q_k)}\int_{\gamma Q_k} f(y,\tilde{u}_k^{r,s,\lambda},\tilde{v}_k^{r,s,\lambda},\nabla\tilde{u}_k^{r,s,\lambda})\,\mathrm{d}y\\
&\leq \frac{1}{|Du|(Q_k)}\Bigg\{\int_{\gamma Q_k} f(y,\bar{u}_k,\bar{v}_k,\nabla\bar{u}_k)\,\mathrm{d}y + C|\gamma Q_k| + C\int_{\gamma Q_k\cap\left\{|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}\geq s\right\}}|\tau_{L_k}(\bar{v}_k)|^p\mathrm{d}y\\
&\quad + C\frac{s}{s-r}\int_{\gamma Q_k\cap\left\{r<|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}<s\right\}}\left|\nabla\left(|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}\right)\right| + |\nabla\bar{u}_k|\mathrm{d}y\\
&\quad + C\int_{\gamma Q_k\cap\left\{r<|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}<s\right\}}[|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|^p + |\tau_{L_k}(\bar{v}_k)|^p]\,\mathrm{d}y\Bigg\}\\
&\leq \frac{1}{|Du|(Q_k)}\Bigg\{\int_{\gamma Q_k} f(y,\bar{u}_k,\bar{v}_k,\nabla\bar{u}_k)\,\mathrm{d}y + C|\gamma Q_k| + \int_{\gamma Q_k\cap\left\{|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}\geq s\right\}}|\tau_{L_k}(\bar{v}_k)|^p\mathrm{d}y\\
&\quad + C\frac{s}{s-r}\int_r^s\mathcal{H}^{N-1}\left(\gamma Q_k\cap\left\{|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}=t\right\}\right)\,\mathrm{d}t + C\int_{\gamma Q_k\cap\left\{r<|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}<s\right\}}|\nabla\bar{u}_k|\,\mathrm{d}y\\
&\quad + C\int_{\gamma Q_k\cap\left\{r<|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}<s\right\}}|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|^p + |\tau_{L_k}(\bar{v}_k)|^p,\mathrm{d}y\Bigg\}.
\end{aligned}
$$

By condition a) above

$$\limsup_{k\to+\infty}\frac{1}{|Du|(Q_k)}\int_{\gamma Q_k} f(y,\bar{u}_k,\bar{v}_k,\nabla\bar{u}_k)\,\mathrm{d}y \leq \mu^c(x_0).$$

Moreover, for fixed k, for every λ, and for almost every s,

$$\lim_{r\to s}\frac{s}{s-r}\int_r^s\mathcal{H}^{N-1}\left(\gamma Q_k)\cap\left\{|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)|}{\lambda}=t\right\}\right)\,\mathrm{d}t = s\mathcal{H}^{N-1}\left(\gamma Q_k\cap\left\{|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}=s\right\}\right)$$

and

$$
\begin{aligned}
&\lim_{r\to s}\int_{\gamma Q_k\cap\left\{r<|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}<s\right\}}|\nabla\bar{u}_k|\,\mathrm{d}y\\
&=\lim_{r\to s}\int_{\gamma Q_k\cap\left\{r<|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}<s\right\}}(|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|^p+|\tau_{L_k}(\bar{v}_k)|^p)\,\mathrm{d}y = 0.
\end{aligned}
$$

Then, for each k, we can choose $\frac{2}{n^2}\leq r_k<s_k\leq\frac{1}{2n}$ such that

$$\frac{1}{|Du|(Q_k)}\int_{\gamma Q_k\cap\left\{r_k<|\bar{u}_k-a_k|+\frac{|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|}{\lambda}<s_k\right\}}|\nabla\bar{u}_k|\,\mathrm{d}y\leq\frac{\varepsilon_k^N}{|Du|(Q_k)},$$

and we choose λ_k such that

$$\int_Q\frac{\nabla(|\tau_{L_k}(\bar{v}_k)-\bar{v}_k|)}{\lambda_k}\mathrm{d}y\leq C \tag{4.24}$$

for a fixed constant C and, making use of Lemma 2.12 in [18], we observe that

$$\frac{1}{|Du|(Q_k)}\frac{s}{s-r}\int_r^s \mathcal{H}^{N-1}\left(\gamma Q_k\cap\left\{|\bar u_k-a_k|+\tfrac{|\tau_{L_k}(\bar v_k)-\bar v_k|}{\lambda_k}=t\right\}\right)\,\mathrm{d}t$$
$$\leq \frac{1}{|Du|(Q_k)}\frac{c}{\ln(n)}\int_{\gamma Q_k\cap\left\{|\bar u_k-a_k|+\frac{|\tau_{L_k}(\bar v_k)-\bar v_k|}{\lambda_k}\leq\frac{1}{2n}\right\}}\nabla\left(|\bar u_k|+\tfrac{|\tau_{L_k}(\bar v_k)-\bar v_k|}{\lambda_k}\right)\,\mathrm{d}y.$$

We can estimate the last expression by (4.24) and arguing as in ([18], (4.17))

$$\frac{1}{|Du|(Q_k)}\frac{c}{\ln(n)}\int_{\gamma Q_k\cap\left\{|\bar u_k-a_k|+\frac{|\tau_{L_k}(\bar v_k)-\bar v_k|}{\lambda_k}\leq\frac{1}{n}\right\}}|\nabla\bar u_k|\,\mathrm{d}y$$
$$\leq \frac{1}{|Du|(Q_k)}\frac{c}{\ln(n)}\int_{\gamma Q_k\cap\left\{|\bar u_k-a_k|+\frac{|\tau_{L_k}(\bar v_k)-\bar v_k|}{\lambda_k}\leq\frac{1}{n}\right\}}f(y,\bar u_k,\bar v_k,\nabla\bar u_k)\,\mathrm{d}y.$$

Moreover, the p-equiintegrability of $\{\tau_{L_k}(\bar v_k)\}$ guarantees that

$$\int_{\gamma Q_k\cap\{|\bar u_k-a_k|+\frac{|\tau_{L_k}(\bar v_k)-\bar v_k|}{\lambda_k}\geq s_k\}}|\tau_{L_k}(\bar v_k)|^p\,\mathrm{d}y=O\left(\frac{1}{n}\right),$$

and from condition a) in Step 1 it follows (4.20) as we claimed.

To achieve Step 2 it remains to prove (4.21). By a change of variables this is equivalent to prove

$$\lim_{k\to+\infty}\frac{\varepsilon_k^{N-1}}{|Du|(Q_k)}\int_Q\left|\tilde u_k(x_0+\varepsilon_k z)-a_k-\left(u(x_0+\varepsilon_k z)-\frac{1}{|Q_k|}\int_{Q_k}u\,\mathrm{d}y\right)\right|\,\mathrm{d}z=0$$

which can be written $\|\hat u_k-\bar w_k\|_{L^1(Q)}\to 0$ if we introduce the functions

$$\hat u_k(z):=\frac{\varepsilon_k^{N-1}}{|Du|(Q_k)}\left[u(x_0+\varepsilon_k z)-\frac{1}{|Q_k|}\int_{Q_k}u(y)\,\mathrm{d}y\right],$$

$$\bar w_k(z):=\tfrac{\varepsilon_k^{N-1}}{|Du|(Q_k)}\left(\tilde u_k(x_0+\varepsilon_k z)-a_k\right),\ w_k(z):=\tfrac{\varepsilon_k^{N-1}}{|Du|(Q_k)}\left(\bar u_k(x_0+\varepsilon_k z)-a_k\right).\tag{4.25}$$

Thus we have

$$\|\hat u_k-\bar w_k\|_{L^1}\leq\|\hat u_k-w_k\|_{L^1}+\tfrac{\varepsilon_k^{N-1}}{|Du|(Q_k)}\int_Q|\bar u_k(x_0+\varepsilon_k z)-\tilde u_k(x_0+\varepsilon_k z)|\,\mathrm{d}z$$
$$=\|\hat u_k-w_k\|_{L^1}+\tfrac{\varepsilon_k^{N-1}}{|Du|(Q_k)}$$
$$\times\int_Q\left|\bar u_k(x_0+\varepsilon_k z)-a_k)(1-\varphi_k\left(|\bar u_k(x_0+\varepsilon_k z)-a_k|+\tfrac{|\tau_{L_k}\bar v_k(x_0+\varepsilon_k z)-\bar v_k(x_0+\varepsilon_k z)|}{\lambda_k}\right)\right|\,\mathrm{d}z$$
$$\leq\|\hat u_k-w_k\|_{L^1}+\frac{\varepsilon_k^{N-1}}{|Du|(Q_k)}$$
$$\times\int_{\left\{y\in Q:\ |\bar u_k(x_0+\varepsilon_k y)-a_k|+\frac{|\tau_{L_k}\circ\bar v_k(x_0+\varepsilon_k z)-\bar v_k(x_0+\varepsilon_k z)|}{\lambda_k}\geq r_k\right\}}|\bar u_k(x_0+\varepsilon_k z)-a_k|\,\mathrm{d}z$$
$$\leq\|\hat u_k-w_k\|_{L^1}+\int_{\left\{y\in Q:\ |\bar u_k(x_0+\varepsilon_k y)-a_k|+\frac{|\tau_{L_k}\circ\bar v_k(x_0+\varepsilon_k z)-\bar v_k(x_0+\varepsilon_k z)|}{\lambda_k}\geq r_k\right\}}|w_k(z)|\,\mathrm{d}z.$$

Observe that $\|\hat{u}_k - w_k\|_{L^1} \to 0$. Indeed it is exactly condition c) in Step 1, if we make the evident change of variables.

For the second term, we start by proving that $\{w_k\}$ is equi-integrable. Indeed, by the definition of total variation of the BV function $\hat{u}_k$, it is clear that $|D\hat{u}_k|(Q) = 1$. Moreover, since $\int_Q \hat{u}_k \, \mathrm{d}z = 0$, using Poincaré inequality (*cf.* [5], Thm. 3.44) we deduce that $\{\hat{u}_k\}$ is bounded in L^1. Therefore the compactness of BV in L^1 (*cf.* [5], Thm. 3.23) implies that $\{\hat{u}_k\}$ is equi-integrable. Then adding the fact that $\|\hat{u}_k - w_k\|_{L^1} \to 0$ as $k \to +\infty$, we get that $\{w_k\}$ is equi-integrable as desired. It remains to prove that $\left|\left\{y \in Q : \ |\bar{u}_k(x_0 + \varepsilon_k y) - a_k| + \frac{|\tau_{L_k} \circ \bar{v}_k(x_0+\varepsilon_k y) - \bar{v}_k(x_0+\varepsilon_k y)|}{\lambda_k} \geq r_k\right\}\right| \to 0$ as $k \to +\infty$ to obtain $\int_{\{y\in Q:\ |\bar{u}_k(x_0+\varepsilon_k y)-a_k| + \frac{|\tau_{L_k}\circ\bar{v}_k(x_0+\varepsilon_k y)-\bar{v}_k(x_0+\varepsilon_k y)|}{\lambda_k} \geq r_k\}} |w_k(y)| \, \mathrm{d}y \to 0$ and thus (4.21). Indeed, since $a_k \to u(x_0)$, and $\frac{|\tau_{L_k}\circ\bar{v}_k(x_0+\varepsilon_k y)-\bar{v}_k(x_0+\varepsilon_k y)|}{\lambda_k} \to 0$, for a.e. $y \in Q$ for sufficiently large k

$$\begin{aligned}
&\left|\left\{y \in Q : \ |\bar{u}_k(x_0 + \varepsilon_k y) - a_k| + \tfrac{|\tau_{L_k}\circ\bar{v}_k(x_0+\varepsilon_k y)-\bar{v}_k(x_0+\varepsilon_k y)|}{\lambda_k} \geq r_k\right\}\right| \\
&\leq \left|\left\{y \in Q : \ |\bar{u}_k(x_0 + \varepsilon_k y) - u(x_0)| + \tfrac{|\tau_{L_k}\circ\bar{v}_k(x_0+\varepsilon_k y)-\bar{v}_k(x_0+\varepsilon_k y)|}{\lambda_k} \geq \tfrac{1}{n^2}\right\}\right| \\
&\leq \int_Q n^2 |\bar{u}_k(x_0 + \varepsilon_k y) - u(x_0)| + \tfrac{|\tau_{L_k}\circ\bar{v}_k(x_0+\varepsilon_k y)-\bar{v}_k(x_0+\varepsilon_k y)|}{\lambda_k} \, \mathrm{d}y \\
&= \frac{n^2}{|Q_k|} \int_{Q_k} \left(|\bar{u}_k(z) - u(x_0)| + \tfrac{|\tau_{L_k}\circ\bar{v}_k(z)-\bar{v}_k(z)|}{\lambda_k}\right) \mathrm{d}z,
\end{aligned}$$

the result following from condition b) and the arbitrariness of λ_k and ([16], Lem. 8.13).

Step 3. Notice that, defining

$$\theta_k := \frac{|Du|(Q_k)}{\varepsilon_k^N}, \tag{4.26}$$

and recalling the definition of $\bar{w}_k$, (4.20) can be written as

$$\left(1 + \omega\left(\tfrac{1}{n}\right)\right) \mu^c(x_0) \geq \limsup_{k\to+\infty} \frac{1}{\mu_k} \int_{\gamma Q} f(x_0, u(x_0), \tilde{v}_k(x_0 + \varepsilon_k z), \theta_k \nabla \bar{w}_k(z)) \, \mathrm{d}z. \tag{4.27}$$

Let $V_k(z) := \theta_k^{-\frac{1}{p}} \tilde{v}_k(x_0 + \varepsilon_k z)$. By *d)* it results that $V_k \rightharpoonup 0$ in $L^p(Q;\mathbb{R}^m)$, and

$$\left(1 + \omega\left(\tfrac{1}{n}\right)\right) \mu^c(x_0) \geq \limsup_{k\to+\infty} \frac{1}{\theta_k} \int_{\gamma Q} f(x_0, u(x_0), \theta_k^{\frac{1}{p}} V_k(z), \theta_k \nabla \bar{w}_k(z)) \, \mathrm{d}z.$$

Then, modifying $\{V_k\}$ and $\{\bar{w}_k\}$, we get new sequences $\{\tilde{V}_k\}$ and $\{\tilde{w}_k\}$ in order apply the convexity-quasiconvexity of f. In fact we will need to work on the boundary of an inner cube τQ, $\tau \in (t, \gamma)$, and the sequences will be modified in a layer $\tau Q \setminus \tau(1-\delta)Q$.

We claim that it is possible to define $\tilde{V}_k \rightharpoonup 0$ in $L^p(\tau Q;\mathbb{R}^m)$, $\int_{\tau Q} \tilde{V}_k \mathrm{d}z = 0$, and $\tilde{w}_k(x) = A + \varphi_k(x)$ for some $\varphi_k \in W^{1,\infty}_{per}(\tau Q;\mathbb{R}^d)$ and such that

$$\left(1 + \omega\left(\tfrac{1}{n}\right)\right) \mu^c(x_0) \geq \lim_{k\to+\infty} \frac{1}{\theta_k} \int_{\tau Q} f(x_0, u(x_0), \theta_k^{\frac{1}{p}} \tilde{V}_k(z), \theta_k \nabla \tilde{w}_k(z)) \, \mathrm{d}z + \Lambda(1-t),$$

for some continuous function $\Lambda : [0,1] \to \mathbb{R}$ with $\Lambda(0) = 0$.

We start choosing the function into which $\bar{w}_k$ will be modified. We will show that there is a sequence $\{\xi_k\}$ of smooth functions depending only on x_N such that

$$\|\xi_k - \hat{u}_k\|_{L^1} \to 0 \quad \text{and} \quad \nabla\xi_k(\tau Q) - D\hat{u}_k(\tau Q) \to 0 \text{ a.e. } \tau \in (0,1) \text{ as } k \to +\infty. \tag{4.28}$$

The procedure is to average $\hat{u}_k$ in $x_1, \dots, x_{N-1}$ and regularize the function obtained as follows. Let $\eta_k(x_N) := \int_{Q'} \hat{u}_k(x', x_N)\,\mathrm{d}x'$ where $Q' := (-1/2, 1/2)^{N-1}$ and $x' := (x_1, \dots, x_{N-1})$. Define $\zeta_k(x_N) := (\eta_k * \rho_k)(x_N)$ for some mollifying function ρ_k such that $\|\zeta_k - \eta_k\|_{L^1((-\frac{1}{2},\frac{1}{2}))} \le \frac{1}{k}$ and define $\xi_k(x) = \zeta_k(x_N)$. Then

$$\|\xi_k - \hat{u}_k\|_{L^1(Q)} \le \|\zeta_k - \eta_k\|_{L^1((-\frac{1}{2},\frac{1}{2}))} + \|\eta_k - \hat{u}_k\|_{L^1(Q)},$$

where we have identified η_k with its natural extension to Q. By the choice of ζ_k, $\lim_{k\to+\infty} \|\zeta_k - \eta_k\|_{L^1} = 0$ and for the other term we have, using Poincaré inequality,

$$\|\eta_k - \hat{u}_k\|_{L^1((-\frac{1}{2},\frac{1}{2}))} \le \int_Q \left| \int_{Q'} \hat{u}_k(x', z_N)\,\mathrm{d}x' - \hat{u}_k(z) \right| \mathrm{d}z \le \int_{-1/2}^{1/2} c\,|D_{x'}\hat{u}_k(\cdot, z_N)|\,(Q')\,\mathrm{d}z_N.$$

By definition of $\hat{u}_k$, doing the natural change of variables one get

$$D\hat{u}_k(Q) = \frac{Du(Q_k)}{|Du|(Q_k)}, \tag{4.29}$$

which, accordingly to (4.19) converges to the matrix $A = a \otimes e_N$. Thus we are in conditions to apply Proposition A.1 in [18] and obtain $|D\hat{u}_k - (D\hat{u}_k \cdot A)A| \to 0$. In particular, for $i = 1, \dots, N-1$, $|D\hat{u}_k\, e_i|(Q) = |(D\hat{u}_k - (D\hat{u}_k \cdot A)A)e_i + (D\hat{u}_k \cdot A)A\, e_i|(Q) \to 0$.

Now we choose the layer where we will change the sequences. Let $\tau \in (t, \gamma)$ be such that $\nabla\xi_k(\tau Q) - D\hat{u}_k(\tau Q) \to 0$, choose $\delta > 0$ such that $(1-\delta)\tau > t$ and

$$|\nabla\xi_k|(\tau Q \setminus \tau(1-\delta)Q) \le |D\hat{u}_k|(Q \setminus tQ) = \frac{|Du|((Q_k) \setminus (tQ_k))}{|Du|(Q_k)}. \tag{4.30}$$

Notice that $\nabla\bar{w}_k(z) = \frac{1}{\theta_k}\nabla\tilde{u}(x_0 + \varepsilon_k z)$. Then, (4.20), $(H_1)_p$, and the second limit in (4.16) imply that $\nabla\bar{w}_k$ is bounded in $L^1(\gamma Q)$. In particular we can say

$$\int_{\tau Q \setminus \tau(1-\delta)Q} (|V_k|^p + |\nabla\bar{w}_k|)\,\mathrm{d}z \le C, \ \forall\ k. \tag{4.31}$$

Then we use the slicing method as in the proof of Lemma 2.7, replacing the cube Q therein by τQ. Thus we divide for every $j \in \mathbb{N}$, $\tau Q \setminus \tau(1-\delta)Q$ into j layers, getting recursively a sequence $k(j)$, layers $S_j := \{z \in \tau Q \setminus \tau(1-\delta)Q :\ \alpha_j < \mathrm{dist}(z, \partial(\tau Q)) < \beta_j\}$ and cut-off functions η_j on τQ such that

$$\int_{S_j} (|V_{k(j)}|^p + |\nabla\bar{w}_{k(j)}|)\,\mathrm{d}z \le \frac{C}{j},\ \frac{1}{|S_j|}\int_{S_j} |\bar{w}_{k(j)} - \xi_{k(j)}|\,\mathrm{d}z \le \frac{1}{j},\ \text{and}\ \frac{\left|\frac{1}{|\tau Q|}\int_{\tau Q} -\eta_j V_{k(j)}\,\mathrm{d}z\right|}{\left|\frac{1}{|\tau Q|}\int_{\tau Q} (1-\eta_j)\,\mathrm{d}z\right|} \le 1.$$

Now, define

$$\tilde{V}_j(z) := (1-\eta_j(z)) \frac{\frac{1}{|\tau Q|}\int_{\tau Q} -\eta_j V_{k(j)}\,\mathrm{d}z}{\frac{1}{|\tau Q|}\int_{\tau Q} (1-\eta_j)\,\mathrm{d}x} + \eta_j(z) V_{k(j)}(z)$$

and

$$\tilde{w}_j(z) := (1-\eta_j(z))\xi_{k(j)}(z) + \eta_j(z)\bar{w}_{k(j)}(z). \tag{4.32}$$

By (4.27), adding and subtracting $f(x_0, u(x_0), \theta_{k(j)}^{\frac{1}{p}} \tilde{V}_j(z), \theta_{k(j)} \nabla \tilde{w}_j(z))$ inside the integral, having in mind the definition of η_j and using $(H_1)_p$, we get

$$\begin{aligned}
&\left(1+\omega\left(\tfrac{1}{n}\right)\right)\, \mu^c(x_0) \geq \limsup_{j\to+\infty} \frac{1}{\theta_{k(j)}} \int_{\tau Q} f(x_0, u(x_0), \theta_{k(j)}^{\frac{1}{p}} V_{k(j)}(z), \theta_{k(j)} \nabla \bar{w}_{k(j)}(z))\,\mathrm{d}z \\
&\geq \limsup_{j\to+\infty} \frac{1}{\theta_{k(j)}} \left\{ \int_{\tau Q} f(x_0, u(x_0), \theta_{k(j)}^{\frac{1}{p}} \tilde{V}_j, \theta_{k(j)} \nabla \tilde{w}_j)\,\mathrm{d}z \right. \\
&\quad \left. - \int_{\{x\in\tau Q:\mathrm{dist}(x,\partial(\tau Q))\leq\beta_j\}} f(x_0, u(x_0), \theta_{k(j)}^{\frac{1}{p}} \tilde{V}_j, \theta_{k(j)} \nabla \tilde{w}_j)\,\mathrm{d}z \right\} \\
&\geq \limsup_{j\to+\infty} \frac{1}{\theta_{k(j)}} \int_{\tau Q} f(x_0, u(x_0), \theta_{k(j)}^{\frac{1}{p}} \tilde{V}_j, \theta_{k(j)} \nabla \tilde{w}_j)\,\mathrm{d}z - \int_{S_j} C(|\nabla \bar{w}_{k(j)}| + |\nabla \eta_j|\, |\bar{w}_{k(j)} - \xi_{k(j)}|)\mathrm{d}z \\
&\quad - \int_{S_j} C + |V_{k(j)}|^p\,\mathrm{d}z - \int_{\tau Q\setminus\tau(1-\delta)Q} c(1+|\nabla \xi_{k(j)}|)\,\mathrm{d}z \\
&\geq \limsup_{j\to+\infty} \frac{1}{\theta_{k(j)}} \int_{\tau Q} f(x_0, u(x_0), \theta_{k(j)}^{\frac{1}{p}} \tilde{V}_j, \theta_{k(j)} \nabla \tilde{w}_j)\,\mathrm{d}z - \frac{c}{j} - \int_{\tau Q\setminus\tau(1-\delta)Q} c(1+|\nabla \xi_{k(j)}|)\,\mathrm{d}z.
\end{aligned}$$

By (4.30) and (4.17), $\int_{\tau Q\setminus\tau(1-\delta)Q} c(1+|\nabla \xi_{k(j)}|)\,\mathrm{d}z \leq \Lambda(1-t)$ for some continuous $\Lambda : [0,1] \to \mathbb{R}$ with $\Lambda(0) = 0$. Therefore we have

$$\left(1+\omega\left(\tfrac{1}{n}\right)\right)\, \mu^c(x_0) \geq \limsup_{j\to+\infty} \frac{1}{\theta_{k(j)}} \int_{\tau Q} f(x_0, u(x_0), \theta_{k(j)}^{\frac{1}{p}} \tilde{V}_j(z), \theta_{k(j)} \nabla \tilde{w}_j(z))\,\mathrm{d}z - \Lambda(1-t),$$

which proves our claim up to a relabeling of the sequence.

Step 4. Using the convexity-quasiconvexity of f we will achieve in this step the desired conclusion. Indeed, as remarked above, the functions $\tilde{V}_k$ have 0 average in τQ. On the other hand, we can always construct ξ_k such that $\xi_k(x) - (\frac{\zeta_k(\frac{\tau}{2})-\zeta_k(-\frac{\tau}{2})}{\tau} \otimes e_N)\, x$ is a τQ-periodic function. This, together with the fact that $\tilde{w}_j = \xi_{k(j)}$ on $\partial(\tau Q)$, yields that $\tilde{w}_j \in (\frac{\zeta_{k(j)}(\frac{\tau}{2})-\zeta_{k(j)}(-\frac{\tau}{2})}{\tau} \otimes e_N)\, x + W^{1,\infty}_{per}(\tau Q;\mathbb{R}^d)$. Therefore

$$\left(1+\omega\left(\tfrac{1}{n}\right)\right)\, \mu^c(x_0) \geq O(1-t) + \limsup_{j\to+\infty} \frac{|\tau Q|}{\theta_{k(j)}} f\left(x_0, u(x_0), 0, \theta_{k(j)} \tfrac{\zeta_k(\frac{\tau}{2})-\zeta_k(-\frac{\tau}{2})}{\tau} \otimes e_N\right).$$

If we add and subtract in the previous limit the quantity $\frac{|\tau Q|}{\theta_{k(j)}} f(x_0, u(x_0), 0, \frac{\theta_{k(j)}}{|\tau Q|} A)$ we get two terms. One gives, by definition, the expected value of the f_p^∞ function, *i.e.*

$$\lim_{j\to+\infty} \tfrac{|\tau Q|}{\theta_{k(j)}} f\left(x_0, u(x_0), 0, \tfrac{\theta_{k(j)}}{|\tau Q|} A\right) = f_p^\infty(x_0, u(x_0), 0, A).$$

The other term can be estimated using the Lipschitz continuity of $f(x_0, u(x_0), 0, \cdot)$, *i.e.* (3.1), and (4.29).

After passing to the limit on k, and using (4.28), (4.17) and (4.19), we get

$$\left(1+\omega\left(\tfrac{1}{n}\right)\right)\, \mu^c(x_0) \geq O(1-t) + f_p^\infty(x_0, u(x_0), 0, A) + \Lambda(1-t)$$

where Λ is a continuous function with $\Lambda(0) = 0$. We finally obtain the desired estimate letting $n \to +\infty$ and $t \to 1^-$. □

4.2. Upper bound in BV × L^p

In order to achieve the representation in Theorem 1.1, we localize our functionals. We define for open sets $A \subset \Omega$ and for any $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$,

$$\mathcal{J}_p(u,v;A) := \inf\left\{ \liminf_{n\to+\infty} J(u_n, v_n; A) :\ u_n \in BV(\Omega;\mathbb{R}^d),\ v_n \in L^p(\Omega;\mathbb{R}^m),\ u_n \to u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^p \right\}$$

where, with an abuse of notation,

$$J(u,v;A) := \begin{cases} \displaystyle\int_A f(x,u,v,\nabla u)\,\mathrm{d}x, & \text{if } (u,v)\in W^{1,1}(\Omega;\mathbb{R}^d)\times L^p(\Omega;\mathbb{R}^m), \\ +\infty, & \text{otherwise.}\end{cases} \tag{4.33}$$

We start by observing that $(H_1)_p$ implies that for every $u\in BV(\Omega;\mathbb{R}^d)$ and for every $v\in L^p(\Omega;\mathbb{R}^m)$, it results

$$\mathcal{J}_p(u,v;A)\le C\left(|A|+|Du|(A)+\int_A |v|^p\mathrm{d}x\right).$$

We observe that, arguing as in ([12], Lem. 3.5), $\mathcal{J}_p$ is a variational functional. This means that the following conditions hold:

1. $\mathcal{J}_p$ is local, that is $\mathcal{J}_p(u,v;A)=\mathcal{J}_p(u',v';A)$, for every $A\in\mathcal{A}(\Omega)$ and every $(u,v),(u',v')\in BV(A;\mathbb{R}^d)\times L^p(A;\mathbb{R}^m)$ such that $u=u'$ and $v=v'\mathcal{L}^N$ a.e. in A;
2. $\mathcal{J}_p$ is sequentially lower semicontinuous, that is

$$\mathcal{J}_p(u,v;A)\le\liminf_{n\to+\infty}\mathcal{J}_p(u_n,v_n;A),\ \forall\, A\subset\Omega \text{ open},\ \ u_n\to u \text{ in } L^1(A;\mathbb{R}^d) \text{ and } v_n\rightharpoonup v \text{ in } L^p(A;\mathbb{R}^m);$$

3. $\mathcal{J}_p(u,v;\cdot)$ is the trace of a Radon measure restricted to the family $\mathcal{A}(\Omega)$.

The following result is devoted to prove the upper bound in $BV\times L^p$, $1<p<+\infty$.

Theorem 4.2. *Let $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N}\to[0,+\infty)$ be a continuous function satisfying (H_0), $(H_1)_p-(H_3)_p$, and $\overline{J}_p$ be defined in (1.2). Then for every $(u,v)\in BV(\Omega;\mathbb{R}^d)\times L^p(\Omega;\mathbb{R}^m)$*

$$\overline{J}_p(u,v;\Omega)\le\int_\Omega f(x,u,v,\nabla u)\mathrm{d}x+\int_{J_u\cap\Omega}K_p(x,0,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1}+\int_\Omega f_p^\infty(x,u,0,\tfrac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|})\mathrm{d}|D^cu|. \tag{4.34}$$

Proof. The representation (4.34) is achieved first for $(u,v)\in BV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega\,\mathbb{R}^d)\times L^\infty(\Omega;\mathbb{R}^m)$, then, *via* an approximation argument as in [6], the result will be obtained in $BV(\Omega;\mathbb{R}^d)\times L^\infty(\Omega;\mathbb{R}^m)$. Then a standard truncation argument (see [23], Thm. 14) leads us to $BV(\Omega;\mathbb{R}^d)\times L^p(\Omega;\mathbb{R}^m)$.

Part 1. Let $(u,v)\in BV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)\times L^\infty(\Omega;\mathbb{R}^m)$. Since $\overline{J}_p(u,v)=\mathcal{J}_p(u,v;\Omega)$ and $\mathcal{J}_p(u,v;\cdot)$ is the trace of a Radon measure on the open subsets of Ω, absolutely continuous with respect to $|Du|+\mathcal{L}^N$, it will be enough to prove the following inequalities

$$\frac{\mathrm{d}\mathcal{J}_p(u,v;\cdot)}{\mathrm{d}\mathcal{L}^N}(x)\le f(x,u(x),v(x),\nabla u(x)),\ \ \mathcal{L}^N-\text{a.e. } x\in\Omega, \tag{4.35}$$

$$\frac{\mathrm{d}\mathcal{J}_p(u,v;\cdot)}{\mathrm{d}|D^cu|}(x)\le f_p^\infty\left(x,u(x),0,\tfrac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}(x)\right),\ \ |D^cu|-\text{a.e. } x\in\Omega. \tag{4.36}$$

$$\mathcal{J}_p(u,v;J_u\cap\Omega)\le\int_{J_u\cap\Omega}K_p(x,0,u^-(x),u^+(x),\nu_u(x))\mathrm{d}\mathcal{H}^{N-1}. \tag{4.37}$$

The proof of these inequalities exploits results proven in [6].
Bulk part. The inequality (4.35) is an immediate consequence of ([23], Thms. 12 and 14), observing that the same arguments therein can be applied when u is a function of bounded variation.
Cantor part. Let $u\in BV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$ and $v\in L^\infty(\Omega;\mathbb{R}^m)$. We follow [14, 18], identifying u with its approximate limit defined in $\Omega\setminus J_u$.

Let $u_n := u*\rho_n$, where ρ_n be a sequence of mollifiers, then by ([18], Lem. 2.5),

$$u_n(x)\to u(x),\ \ |D^cu|-\text{a.e. } x\in\Omega. \tag{4.38}$$

Therefore u is $|D^c u|-$measurable. We write $|Du| = |D^c u| + \eta$, where η and $|D^c u|$ are mutually singular Radon measures. Let $x_0 \in \Omega$ be such that

$$\frac{\mathrm{d}\mathcal{J}_p(u,v;\cdot)}{\mathrm{d}|D^c u|}(x_0) \text{ exists and is finite,}$$

$$\lim_{\varepsilon\to 0^+} \frac{\eta(B(x_0,\varepsilon))}{|D^c u|(B(x_0,\varepsilon))} = 0, \qquad \lim_{\varepsilon\to 0^+} \frac{|Du|(B(x_0,\varepsilon))}{|D^c u|(B(x_0,\varepsilon))} \text{ exists and is finite,} \tag{4.39}$$

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon^N}{|D^c u|(B(x_0,\varepsilon))} = 0, \tag{4.40}$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{|D^c u|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |v(x)|\,\mathrm{d}x = 0, \quad \lim_{\varepsilon\to 0^+} \frac{1}{|D^c u|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |v(x)|^p\,\mathrm{d}x = 0, \tag{4.41}$$

$$A(x_0) = \lim_{\varepsilon\to 0} \frac{D^c u(B(x_0,\varepsilon))}{|D^c u|(B(x_0,\varepsilon))} \text{ exists and is a rank one matrix of norm one,} \tag{4.42}$$

$$\lim_{\varepsilon\to 0} \frac{1}{|D^c u|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f_p^\infty(x_0,u(x_0),0,A(x))\mathrm{d}|D^c u| = f_p^\infty(x_0,u(x_0),0,A(x_0)). \tag{4.43}$$

Fix $\delta > 0$. Using the Yosida transform of f introduced in Definition 2.9 and the properties in Proposition 2.10, we get

$$\begin{aligned}
\mathcal{J}_p(u,v;B(x_0,\varepsilon)) &\le \liminf_{n\to+\infty} \int_{B(x_0,\varepsilon)} f(x,u_n,v,\nabla u_n)\,\mathrm{d}x \le \liminf_{n\to+\infty} \left\{ \int_{B(x_0,\varepsilon)} f(x_0,u(x_0),v,\nabla u_n)\,\mathrm{d}x \right.\\
&\quad \left. + \int_{B(x_0,\varepsilon)} \delta(1+|v|+|\nabla u_n|) + \lambda(\varepsilon + |u_n - u(x_0)|)(1+|v|+|\nabla u_n|)\,\mathrm{d}x \right\}\\
&\le \liminf_{n\to+\infty} \left\{ \int_{B(x_0,\varepsilon)} f\left(x_0,u(x_0),v,(Du*\rho_n)\right)\,\mathrm{d}x + (\delta+\lambda\varepsilon)(1+\|v\|_{L^\infty})|B(x_0,\varepsilon)| \right.\\
&\quad \left. +(\lambda\varepsilon+\delta)\int_{B(x_0,\varepsilon)} |\nabla u_n|\,\mathrm{d}x) + \lambda C \int_{B(x_0,\varepsilon)} |u_n - u(x_0)|(1+\|v\|_{L^\infty}|\nabla u_n|)\,\mathrm{d}x \right\}.
\end{aligned}$$

An argument entirely similar to ([18], Sect. 5, steps 1 and 2, p. 37), allows us to write

$$\mathcal{J}_p(u,v;B(x_0,\varepsilon)) \le \liminf_{\varepsilon\to 0^+} \liminf_{n\to+\infty} \frac{1}{|D^c u|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f(x_0,u(x_0),v,Du*\varrho_n)\mathrm{d}x + O(\delta).$$

Let $h : \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0,\infty)$ given by $h(b,\xi) := \sup_{t\ge 0} \frac{f(x_0,u(x_0),t^{\frac{1}{p}}b,t\xi) - f(x_0,u(x_0),0,0)}{t}$. Then, h is positively homogeneous of degree $(p,1)$ and satisfies (3.1). The convexity of $f_p^\infty(x_0,u(x_0),\cdot,\cdot)$ when ξ is at most a rank-one matrix entails $f_p^\infty(x_0,u(x_0),b,\xi) = h(b,\xi)$, for every $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$, with rank$\xi \le 1$. Thus,

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{J}_p(u,v;B(x_0,\varepsilon))}{\mathrm{d}|D^c u|(B(x_0,\varepsilon))} &= \liminf_{\varepsilon\to 0^+} \liminf_{n\to+\infty} \frac{1}{|D^c u|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} h(v,Du*\varrho_n)\mathrm{d}x\\
&\quad + \limsup_{\varepsilon\to 0^+} \frac{1}{|D^c u|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f(x_0,u(x_0),0,0)\mathrm{d}x + O(\delta).
\end{aligned} \tag{4.44}$$

Observe that, (3.1) gives

$$\liminf_{\varepsilon\to 0^+}\liminf_{n\to+\infty}\int_{B(x_0,\varepsilon)} h\left(\frac{v}{(|D^c u|(B(x_0,\varepsilon)))^{\frac{1}{p}}}, \frac{Du*\varrho_n}{|D^c u|(B(x_0,\varepsilon))}\right)\mathrm{d}x$$

$$\leq \limsup_{\varepsilon\to 0^+}\limsup_{n\to+\infty}\int_{B(x_0,\varepsilon)} h\left(0, \tfrac{Du*\varrho_n}{|D^c u|(B(x_0,\varepsilon))}\right)\mathrm{d}x + \lim_{\varepsilon\to 0^+}\int_{B(x_0,\varepsilon)} \tfrac{|v|^p}{|D^c u|(B(x_0,\varepsilon))}\mathrm{d}x$$
$$+ \lim_{\varepsilon\to 0^+}\lim_{n\to+\infty}\int_{B(x_0,\varepsilon)}\left|\tfrac{v}{(|D^c u|(B(x_0,\varepsilon)))^{\frac{1}{p}}}\right|\left|\tfrac{Du*\varrho_n}{|D^c u|(B(x_0,\varepsilon))}\right|^{\frac{1}{p'}}.$$

and (4.41) guarantees that the second limit from below is 0. The last term can be estimated via Hölder inequality, leading to

$$\lim_{\varepsilon\to 0^+}\lim_{n\to+\infty}\left(\int_{B(x_0,\varepsilon)}\frac{|v|^p}{|D^c u|(B(x_0,\varepsilon))}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B(x_0,\varepsilon)}\frac{|Du*\varrho_n|}{|D^c u|(B(x_0,\varepsilon))}\mathrm{d}x\right)^{\frac{1}{p'}}.$$

The first term of the above product is null by (4.41), while the latter, exploiting ([6], Lem. 4.5, see also [18], Lem. 2.5) becomes

$$\left(\lim_{\varepsilon\to 0^+}\int_{B(x_0,\varepsilon)}\frac{|Du|}{|D^c u|(B(x_0,\varepsilon))}\right)^{\frac{1}{p'}}$$

which is finite by (4.39). Thus, from, (4.44) we can conclude that

$$\frac{\mathrm{d}\mathcal{J}_p(u,v;B(x_0,\varepsilon))}{\mathrm{d}|D^c u|(B(x_0,\varepsilon))} \leq \limsup_{\varepsilon\to 0^+}\limsup_{n\to+\infty}\frac{1}{|D^c u|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} h(0,Du*\varrho_n)\mathrm{d}x + O(\delta).$$

Then the thesis is achieved via the same arguments in [18], (4.43) and letting $\delta\to 0^+$.
Jump part. We show that

$$\mathcal{J}_p(u,v;J_u\cap\Omega)\leq\int_{J_u\cap\Omega} K_p(x,0,u^-,u^+,\nu_u))\mathrm{d}\mathcal{H}^{N-1}, \tag{4.45}$$

for every $u\in BV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)\times L^\infty(\Omega;\mathbb{R}^m)$.

The proof of (4.45) develops exploiting the arguments in ([6], Prop. 4.8; [19], Lem. 4.2; and [10], Prop. 4.1) and it is divided into three parts according to the limit function u.

Case 1. $u(x)=c\chi_E+d(1-\chi_E)$ with $\mathrm{Per}(E;\Omega)<\infty$.
Case 2. $u(x)=\sum c_i\chi_{E_i}(x)$, where $\{E_i\}_{i=1}^\infty$ forms a partition of Ω into sets of finite perimeter.
Case 3. $u\in BV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$.

Proof of Case 1. We start to consider $u:=c\chi_E+d(1-\chi_E)$, with $\mathrm{Per}(E;\Omega)<+\infty$, and $v\in L^\infty(\Omega;\mathbb{R}^m)$ and we aim to prove that

$$\mathcal{J}_p(u,v;A)\leq\int_A f(x,u,v,0)\mathrm{d}x+\int_{J_u\cap A} K_p(x,0,c,d,\nu_u)\mathrm{d}\mathcal{H}^{N-1}, \text{ for every } A\in\mathcal{A}(\Omega). \tag{4.46}$$

This proof is divided into several steps.

Step 1. First we assume that u has a planar interface, *i.e.* let $\nu\in S^{N-1}$, $a_0\in\mathbb{R}^N$, consider $A=a_0+\lambda Q_\nu$, an open cube centered at a_0, with two faces orthogonal to ν, with side length λ, and let

$$u(x):=\begin{cases} c & \text{if } (x-a_0)\cdot\nu>0,\\ d & \text{if } (x-a_0)\cdot\nu\leq 0.\end{cases}$$

We start to consider the case where f does not depend on x and we claim that there exist a sequence $\{u_n\} \subset W^{1,1}(a_0+\lambda Q_\nu;\mathbb{R}^d)$ such that

$$u_n = \begin{cases} c & \text{if } x\cdot\nu = -\frac{\lambda}{2},\\ d & \text{if } x\cdot\nu = \frac{\lambda}{2},\end{cases}$$

$u_n(x) = u_n(x+k\lambda\nu_i), i = 1,\dots N-1, k\in\mathbb{Z}$, where $\{\nu_1,\dots,\nu_{N-1},\nu\}$ is an orthonormal basis of $\mathbb{R}^N$, and a sequence $\{v_n\} \subset L^p(a_0+\lambda Q_\nu;\mathbb{R}^m)$, such that $v_n(x) = v(x)$ if $|(x-a_0)\cdot\nu| > \frac{\lambda}{2(2n+1)}$, with $u_n \to u$ in $L^1(a_0+\lambda Q_\nu;\mathbb{R}^d)$, $v_n \rightharpoonup v$ in $L^p(a_0+\lambda Q_\nu;\mathbb{R}^m)$ and

$$\lim_{n\to+\infty}\int_{a_0+\lambda Q_\nu} f(u_n,v_n,\nabla u_n)\mathrm{d}x = \int_{a_0+\lambda Q_\nu} f(u,v,0)\mathrm{d}x + \lambda^{N-1}K_p(0,c,d,\nu). \tag{4.47}$$

Step 1 a). We first consider the case $a_0 = 0$ and $\lambda = 1$ and, without loss of generality, we assume that $\nu = e_N$. We claim that for all $\xi \in \mathcal{A}(c,d,e_N)$ and for all $\varphi \in L^p(Q;\mathbb{R}^m)$, with $\int_Q \varphi\mathrm{d}x = 0$, there exists $\xi_n \in \mathcal{A}(c,d,e_N)$ and $v_n \in L^p(Q;\mathbb{R}^m)$ such that $v_n(x) = v(x)$ if $|x_N| > \frac{1}{2(2n+1)}$

$$\|\xi_n - u\|_{L^1(Q;\mathbb{R}^d)} \to 0, \quad v_n \rightharpoonup v \text{ in } L^p(Q;\mathbb{R}^m) \text{ as } n\to+\infty, \tag{4.48}$$

and

$$\lim_{n\to+\infty}\int_Q f(\xi_n,v_n,\nabla\xi_n)\mathrm{d}x = \int_Q f(u,v,0)\mathrm{d}x + \int_Q f_p^\infty(\xi,\varphi,\nabla\xi)\mathrm{d}x. \tag{4.49}$$

Let $\Sigma := \{x\in Q : x_N = 0\}$. For $k\in\mathbb{N}$, we label the elements of $(\mathbb{Z}\cap[-k,k]^N)\times\{0\}$ by $\{a_i\}_{i=1}^{2k+1^{N-1}}$ and we observe

$$(2k+1)\overline{\Sigma} = \bigcup_{i=1}^{(2k+1)^{N-1}} (a_i + \overline{\Sigma}),$$

with $(a_i+\Sigma)\cap(a_j+\Sigma) = \emptyset$ if $i\neq j$. Extending $\xi(\cdot,x_N)\to\mathbb{R}^{N-1}$ by periodicity we define

$$\xi_{2k+1}(x) := \begin{cases} c & \text{if } x_N > \frac{1}{2(2k+1)},\\ \xi((2k+1)x) & \text{if } |x_N| \le \frac{1}{2(2k+1)},\\ d & \text{if } x_N < -\frac{1}{2(2k+1)}.\end{cases} \tag{4.50}$$

Clearly $\xi_{2k+1}\in\mathcal{A}(c,d,e_N)$ and $\|\xi_{2k+1}-u\|_{L^1(Q;\mathbb{R}^d)} \to 0$ as $k\to+\infty$ (see proof of [18], Lem. 4.2). Extending $\varphi(\cdot,x_N)$ to $\mathbb{R}^{N-1}$ by periodicity define

$$v_{2k+1}(x) := \begin{cases} v(x) & \text{if } |x_N| > \frac{1}{2(2k+1)},\\ (2k+1)^{\frac{1}{p}}\varphi((2k+1)x) & \text{if } |x_N| \le \frac{1}{2(2k+1)}.\end{cases}$$

We observe that $v_{2k+1}\rightharpoonup v$ in $L^p(Q;\mathbb{R}^m)$. Indeed, there exists $C>0$ such that for every $k\in\mathbb{N}$,

$$\begin{aligned}\int_Q |v_{2k+1}|^p\mathrm{d}x &\le \int_\Sigma\int_{|x_N|\ge\frac{1}{2(2k+1)}} |v|^p\mathrm{d}x + \int_\Sigma\int_{|x_N|\le\frac{1}{2(2k+1)}} (2k+1)|\varphi((2k+1)x)|^p\mathrm{d}x\\ &\le C + \int_\Sigma\int_{-\frac12}^{\frac12} |\varphi((2k+1)x',x_N)|^p\mathrm{d}x = C + \int_\Sigma\int_{-\frac12}^{\frac12}|\varphi(x',x_N)|^p\mathrm{d}x \le C,\end{aligned}$$

where the periodicity of φ has been exploited. In order to achieve the weak convergence of $\{v_{2k+1}\}$ to v it is enough to prove that $\lim_{k\to+\infty}\int_E v_{2k+1}\mathrm{d}x = \int_E v\mathrm{d}x$ for every $E\subseteq Q$ (see [16], Cor. 2.49). In fact,

$$\int_E (v_{2k+1}-v)\mathrm{d}x = -\int_{\{x\in E:|x_N|<\frac{1}{2(2k+1)}\}} v\,\mathrm{d}x + \int_{\{x\in E:|x_N|<\frac{1}{2(2k+1)}\}} (2k+1)^{\frac1p}\varphi((2k+1)x)\mathrm{d}x.$$

The first integral trivially converges to 0 as $k \to +\infty$, while, concerning the second one,

$$\left|\int_{\{x\in E:|x_N|<\frac{1}{2(2k+1)}\}}(2k+1)^{\frac{1}{p}}\varphi((2k+1)x)\mathrm{d}x\right| \le \frac{(2k+1)^{\frac{1}{p}}}{2k+1}\int_{\Sigma}\int_{-\frac{1}{2}}^{\frac{1}{2}}|\varphi(x',x_N)|\mathrm{d}x,$$

which, using periodicity of φ and letting $k \to +\infty$ converges to 0.
Consider

$$\int_Q f(\xi_{2k+1}, v_{2k+1}, \nabla\xi_{2k+1})\mathrm{d}x = \int_{\Sigma}\int_{-\frac{1}{2}}^{-\frac{1}{2(2k+1)}} f(c, v(x), 0)\mathrm{d}x + \int_{\Sigma}\int_{\frac{1}{2(2k+1)}}^{\frac{1}{2}} f(d, v(x), 0)\mathrm{d}x$$
$$+\int_{\Sigma}\int_{|x_N|<\frac{1}{2(2k+1)}} f(\xi((2k+1)x), (2k+1)^{\frac{1}{p}}\varphi((2k+1)x), (2k+1)\nabla\xi((2k+1)x)\mathrm{d}x.$$

The first two integrals in the right hand side, converge as $k \to +\infty$, to $\int_Q f(u(x), v(x), 0)\mathrm{d}x$. The latter integral, after a change of variables becomes

$$\int_{\Sigma}\int_{|x_N|<\frac{1}{2(2k+1)}} f(\xi((2k+1)x), (2k+1)^{\frac{1}{p}}\varphi((2k+1)x), (2k+1)\nabla\xi((2k+1)x))\mathrm{d}x$$
$$= \frac{1}{2k+1}\int_Q f(\xi(y), (2k+1)^{\frac{1}{p}}\varphi(y), (2k+1)\nabla\xi(y))\mathrm{d}y \to \int_Q f_p^{\infty}(\xi(y), \varphi(y), \nabla\xi(y))\mathrm{d}y$$

as $k \to +\infty$. Putting together the last two limits we obtain (4.49).

Step 1 b). Let $\{(\eta_n, \varphi_n)\} \subset \mathcal{A}(c, d, e_N) \times L^p(Q;\mathbb{R}^m)$ with $\int_Q \varphi_n\mathrm{d}y = 0$ be a minimizing sequence for $K_p(0, c, d, \varepsilon_N)$. Observe that since $K_p(0, c, d, e_N)$ is finite and f_p^{∞} satisfies (3.4), then we can assume that $\{\varphi_n\}$ is bounded in $L^p(Q;\mathbb{R}^m)$.

By (4.49), for every $n \in \mathbb{N}$ we can find $k_n \in N$, $u_n \in \mathcal{A}(c, d, e_N)$ and $v_n \in L^p(Q;\mathbb{R}^m)$ such that $\|u_n - u\|_{L^1(Q;\mathbb{R}^d)} < \frac{1}{n}$, $\left|\int_Q (v_n - v)\psi_l\mathrm{d}x\right| < \frac{1}{n}$, (for $l = 1, \dots, n$) and $\{\psi_l\}$ a dense sequence of functions in $L^q(Q;\mathbb{R}^m)$, with

$$v_n(x) := \begin{cases} v(x) & \text{if } |x_N| > \frac{1}{2(2k_n+1)},\\ (2k_n+1)^{\frac{1}{p}}\varphi_n((2k_n+1)x) & \text{if } |x_N| \le \frac{1}{2(2k_n+1)}, \end{cases}$$

and

$$\left|\int_Q f(u_n, v_n, \nabla u_n)\mathrm{d}x - \int_Q f(u(x), v(x), 0)\mathrm{d}x - \int_Q f_p^{\infty}(\eta_n, \varphi_n, \nabla\eta_n)\mathrm{d}x\right| < \frac{1}{n}\cdot$$

By the lower bound inequality and the last estimate we have (4.47), up to a relabeling of the sequences $\{u_n\}$ and $\{v_n\}$ with the same indices k_n, when $\lambda = 1$ and $a_0 = 0$.

Now we consider the case of $A := \lambda Q$, for $\lambda > 0$. Define

$$f_\lambda(u, v, \xi) := f\left(u, v, \frac{\xi}{\lambda}\right),\ u_0 := \begin{cases} c & \text{if } x_N > 0,\\ d & \text{if } x_N \le 0, \end{cases} \text{ and } v_0(x) := v(\lambda x) \quad \text{for every } x \in Q. \tag{4.51}$$

By (4.47) when $a_0 = 0$ and $\lambda = 1$, there exists $(u_n, v_n) \in \mathcal{A}(c, d, e_N) \times L^p(Q;\mathbb{R}^m)$ such that $u_n \to u_0$ in $L^1(Q;\mathbb{R}^d)$, $v_n \rightharpoonup v_0$ in $L^p(Q;\mathbb{R}^m)$ and

$$\int_Q f_\lambda(u_n, v_n, \nabla u_n)\mathrm{d}x \to \int_Q f_\lambda(u_0(x), v_0(x), 0)\mathrm{d}x + (K_p)_\lambda(0, c, d, e_N), \tag{4.52}$$

where $(K_p)_\lambda$ is the function defined in (1.6), with f replaced by f_λ above. Consider any $a_0 \in \mathbb{R}^N$ and set

$$\bar{u}_n(x) := u_n\left(\tfrac{x-a_0}{\lambda}\right), \quad \bar{v}_n(x) := v_n\left(\tfrac{x-a_0}{\lambda}\right), \quad x \in a_0 + \lambda Q. \tag{4.53}$$

Clearly $\{\bar{u}_n\}$ meets the boundary conditions, and is periodic in the $e_1, \dots, e_{N-1}$ directions with period λ. Moreover, $\|\bar{u}_n - u\|_{L^1(a_0+\lambda Q)} \to 0$ and $\bar{v}_n \rightharpoonup v$ in $L^p(a_0 + \lambda Q; \mathbb{R}^m)$ and

$$\begin{aligned}&\int_{a_0+\lambda Q} f(\bar{u}_n, \bar{v}_n, \nabla \bar{u}_n)\mathrm{d}x = \int_{a_0+\lambda Q} f(u_n\left(\tfrac{x-a_0}{\lambda}\right), v_n\left(\tfrac{x-a_0}{\lambda}\right), \tfrac{1}{\lambda}\nabla u_n\left(\tfrac{x-a_0}{\lambda}\right))\mathrm{d}x \\ &= \lambda^N \int_Q f_\lambda(u_n(y), v_n(y), \nabla u_n(y))\mathrm{d}y \to \lambda^N \int_Q f_\lambda(u_0(y), v_0(y), 0)\mathrm{d}y + \lambda^N (K_p)_\lambda(0, c, d, e_N),\end{aligned} \tag{4.54}$$

as $n \to +\infty$. Moreover,

$$\lambda^N \int_Q f_\lambda(u_0(y), v_0(y), 0)\mathrm{d}y = \int_{a_0+\lambda Q} f(u(x), v(x), 0)\mathrm{d}x, \text{ and } (K_p)_\lambda(0, c, d, e_N) = \frac{1}{\lambda} K_p(0, c, d, e_N). \tag{4.55}$$

Hence we obtain (4.47).

Step 1 c). We allow f to have explicit x-dependence. Let A be an on open subset of Ω and $A^* := \alpha + \lambda Q_\nu \subset\subset A$ for some $\alpha \in \mathbb{R}^N, \lambda > 0$. Without loss of generality, we may assume that $a_0 = 0$ and $\nu = e_N$. We denote Q_ν by Q and we let $A' := \{x \in A^* : x_N = 0\}$ and $Q' := \{x \in Q : x_N = 0\}$. Since A^* is compactly included in A, fixing $\varepsilon > 0$ it is possible to find $\delta > 0$ such that $(H_2)_p$ and Proposition 3.6 (e) hold uniformly in A^*, *i.e.*

$$x, y \in A^*, |x - y| < \delta \Rightarrow |f(x, u, b, \xi) - f(y, u, b, \xi)| \leq \varepsilon C(1 + |\xi| + |b|^p) \tag{4.56}$$

and

$$x, y \in A^*, |x - y| < \delta \Rightarrow |K_p(x, b, c, d, \nu) - K_p(y, b, c, d, \nu)| \leq \varepsilon C(1 + |d - c| + |b|^p). \tag{4.57}$$

Let $h \in \mathbb{N}$ be such that

$$\eta := \frac{\lambda}{h} < \delta \tag{4.58}$$

and partition A' into h^{N-1} $(N-1)$-dimensional cubes, aligned according to the coordinate axes and with mutually disjoint interiors. Namely,

$$A' = \bigcup_{i=1}^{h^{N-1}} (a_i + \eta \overline{Q}'). \tag{4.59}$$

Denoting $Q'_i := a_i + \eta Q'$ and $Q_i := a_i + \eta Q$, we claim that there exist $\{u_k\} \subset W^{1,1}(A^*; \mathbb{R}^d)$ and $\{v_k\} \subset L^p(A^*; \mathbb{R}^m)$ such that $u_k \to u$ in $L^1(A; \mathbb{R}^d)$, $v_k \rightharpoonup v$ in $L^p(A; \mathbb{R}^m)$ and

$$\lim_{k \to +\infty} \int_{A^*} f(x, u_k, v_k, \nabla u_k)\mathrm{d}x = \int_{J_u \cap A^*} K_p(x, 0, c, d, e_N)\mathrm{d}\mathcal{H}^{N-1} + \int_{A^*} f(x, u(x), v(x), 0)\mathrm{d}x. \tag{4.60}$$

By Step 1 b), there exist sequences $\{u_k^{(1)}\} \subset \mathcal{A}(c, d, e_N)$, related to the cube Q_1 and $\{v_k^{(1)}\} \subset L^p(Q_1; \mathbb{R}^m)$, such that

$$\lim_{k \to +\infty} \int_{Q_1} f(a_1, u_k^{(1)}, v_k^{(1)}, \nabla u_k^{(1)})\mathrm{d}x = \eta^{N-1} K_p(a_1, 0, c, d, e_N) + \int_{Q_1} f(a_1, u(x), v(x), 0)\mathrm{d}x.$$

By Remark 2.8 i), there exist subsequences, not relabeled, $\{\xi_k^{(1)}\} \subset W^{1,1}(Q_1; \mathbb{R}^d)$ and $\{\overline{v}_k^{(1)}\} \subset L^p(Q_1; \mathbb{R}^m)$ such that $\xi_k^{(1)} \to u$ in $L^1(Q_1; \mathbb{R}^d)$, with $\xi_k^{(1)}(x) = U_k^{(1)}((x - a_1)/\eta)$ on ∂Q_1, ($U_k^{(1)}$ is a mollification of u), $\overline{v}_k^{(1)} \rightharpoonup v$ in $L^p(Q_1; \mathbb{R}^m)$ and

$$\begin{aligned}\limsup_{k \to +\infty} \int_{Q_1} f(a_1, \xi_k^{(1)}, \overline{v}_k^{(1)}, \nabla \xi_k^{(1)})\mathrm{d}x &\leq \liminf_{k \to +\infty} \int_{Q_1} f(a_1, u_k^{(1)}, v_k^{(1)}, \nabla u_k^{(1)})\mathrm{d}x \\ &= \eta^{N-1} K_p(a_1, 0, c, d, e_N) + \int_{Q_1} f(a_1, u(x), v(x), 0)\mathrm{d}x.\end{aligned}$$

By the lower bound inequality proved in the previous section and the above estimate we have

$$\lim_{k\to+\infty}\int_{Q_1} f(a_1,\xi_k^{(1)},\overline{v}_k^{(1)},\nabla\xi_k^{(1)})\mathrm{d}x=\eta^{N-1}K_p(a_1,0,c,d,e_N)+\int_{Q_1} f(a_1,u(x),v(x),0)\mathrm{d}x.$$

By induction we can repeat the same argument, obtaining h^{N-1} further subsequences $\{k\}$ and corresponding sequences $\{\xi_k^{(j)}\}\subset\mathcal{A}(c,d,e_N)$ related to the cube Q_j, with $\xi_k^{(j)}\to u$ in $L^1(Q_j;\mathbb{R}^d)$, $\xi_k^{(j)}=U_k^{(j)}$ on ∂Q_j and $\{\overline{v}_k^{(j)}\}\subset L^p(Q_j;\mathbb{R}^m)$, with $\overline{v}_k^{(j)}\rightharpoonup v$ in $L^p(Q_j;\mathbb{R}^m)$ and for every $j=1,\dots,h^{N-1}$,

$$\lim_{k\to+\infty}\int_{Q_j} f(a_j,\xi_k^{(j)},\overline{v}_k^{(j)},\nabla\xi_k^{(j)})\mathrm{d}x=\eta^{N-1}K_p(a_j,0,c,d,e_N)+\int_{Q_j} f(a_j,u(x),v(x),0)\mathrm{d}x.$$

Next we take the h^{N-1} subsequence and for all $j=1,\dots,h^{N-1}$ we consider sequences $\{\zeta_k\}$ and $\{\tilde{v}_k\}$, defined in $\cup_{j=1}^{h^{N-1}}Q_j$ with $\zeta_k=\xi_k^{(j)}$, $\tilde{v}_k=\overline{v}_k^{(j)}$ on Q_j, such that for every $j=1,\dots,h^{N-1}$.

$$\lim_{k\to+\infty}\int_{Q_j} f(a_j,\zeta_k,\tilde{v}_k,\nabla\zeta_k)\mathrm{d}x=\eta^{N-1}K_p(a_j,0,c,d,e_N)+\int_{Q_j} f(a_j,u,v,0)\mathrm{d}x. \tag{4.61}$$

Define the sequences $\{u_{k,\varepsilon}\}$ and $\{v_{k,\varepsilon}\}$ almost everywhere on A^*, as follows

$$u_{k,\varepsilon}(x):=\begin{cases}\zeta_k(x) & \text{if } x\in\cup_{j=1}^{h^{N-1}}Q_j,\\ d & \text{if } x_N>\eta/2,\\ c & \text{if } x_N<-\eta/2,\end{cases}\qquad v_{k,\varepsilon}(x):=\begin{cases}\tilde{v}_k(x) & \text{if } x\in\cup_{j=1}^{h^{N-1}}Q_j,\\ v(x) & \text{if } |x_N|>\eta/2.\end{cases} \tag{4.62}$$

Clearly, since $\zeta_k=U_k^{(j)}$ on ∂Q_j and $U_k^{(j)}(x)=d$ (respectively c) for $x_N=\eta/2$ (respectively for $x_N=-\eta/2$), $u_{k,\varepsilon}\in W^{1,1}(A^*;\mathbb{R}^d)$. Also $\tilde{v}_k\in L^p(A^*;\mathbb{R}^m)$ and moreover it coincides with $v(x)$ if $|x_N|>\eta/2$. Furthermore, we have

$$\lim_{\varepsilon\to0^+}\lim_{k\to+\infty}\|u_{k,\varepsilon}-u\|_{L^1(A^*;\mathbb{R}^d)}=0 \tag{4.63}$$

and $v_{k,\varepsilon}\rightharpoonup v$ in $L^p(A^*;\mathbb{R}^m)$ as $k\to+\infty$ and as $\varepsilon\to0$. Also,

$$\int_A f(x,u_{k,\varepsilon},v_{k,\varepsilon},\nabla u_{k,\varepsilon})\mathrm{d}x=\sum_{i=1}^{h^{N-1}}\int_{Q_i} f(a_i,\zeta_k,\tilde{v}_k,\nabla\zeta_k)\mathrm{d}x+\sum_{i=1}^{h^{N-1}}\int_{Q_i}(f(x,\zeta_k,\tilde{v}_k,\nabla\zeta_k)-f(a_i,\zeta_k,\tilde{v}_k,\nabla\zeta_k))\mathrm{d}x$$
$$+\int_{A^*\cap\{x_N>\eta/2\}} f(x,d,v(x),0)\mathrm{d}x+\int_{A^*\cap\{x_N<-\eta/2\}} f(x,c,v(x),0)\mathrm{d}x=:I_1+I_2+I_3+I_4. \tag{4.64}$$

Then it is easily seen that by (4.61), we have

$$\lim_{k\to+\infty}I_1=\sum_{i=1}^{h^{N-1}}\left(\eta^{N-1}K_p(a_i,0,c,d,e_N)+\int_{Q_i} f(a_i,u(x),v(x),0)\mathrm{d}x\right).$$

Moreover,

$$\lim_{k\to+\infty}(I_3+I_4)=\int_{A^*\cap\{x_N>\eta/2\}} f(x,d,v(x),0)\mathrm{d}x+\int_{A^*\cap\{x_N<-\eta/2\}} f(x,c,v(x),0)\mathrm{d}x. \tag{4.65}$$

Regarding I_2, by (4.56), $(H_1)_p$ and since, by construction, the sequences $\{\tilde{v}_k\}$ and $\{\zeta_k\}$ are bounded in $L^p(\cup_{i=1}^{h^{N-1}}Q_i;\mathbb{R}^m)$ and in $W^{1,1}(\cup_{i=1}^{h^{N-1}}Q_i;\mathbb{R}^d)$ respectively, we have

$$\limsup_{k\to\infty}I_2\le\limsup_{k\to+\infty}\sum_{i=1}^{h^{N-1}}\int_{Q_i}\varepsilon C(1+|\tilde{v}_k|^p+|\nabla\zeta_k|)\mathrm{d}x=O(\varepsilon).$$

By (4.57) we have

$$\left|\int_{A^*\cap J_u} K_p(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1} - \eta^{N-1}\sum_{i=1}^{h^{N-1}} K_p(a_i,0,c,d,e_N)\right| \leq \sum_{i=1}^{h^{N-1}}\int_{Q_i}|K_p(x,0,c,d,e_N)-K_p(a_i,0,c,d,e_N)|\,\mathrm{d}\mathcal{H}^{N-1} = O(\varepsilon).$$

Finally, putting together, this estimate, the limits of I_2, I_3, I_4 and estimating $\sum_{i=1}^{h^{N-1}}\int_{Q_i} f(a_i,u,v,0)\mathrm{d}x$ in I_1 *via* (4.56), we obtain the desired approximating sequence, just letting $\varepsilon \to 0^+$ and using a diagonalization procedure. Thus we have proved (4.46) when u has a planar interface and A^* is a cube.

Step 1 d). Now let A^* be an open subset of Ω such that

$$\lim_{j\to+\infty}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\mathcal{L}^N\left(\left\{x\in A^* : \mathrm{dist}(x,\partial^* E\cap\overline{A^*})<\varepsilon, \mathrm{dist}(x,\partial A^*)<\tfrac{1}{j}\right\}\right)=0. \tag{4.66}$$

With u defined as in Step 1 c), we claim that, given any sequence $\varepsilon_n \to 0^+$, there exists a subsequence $\{\varepsilon_{n_k}\}$, a sequence $\{u_k\}\subset W^{1,1}(A^*;\mathbb{R}^d)$ and $\{v_k\}\subset L^p(A^*;\mathbb{R}^m)$ such that $u_k \to u$ in $L^1(A^*;\mathbb{R}^m)$ and $v_k \rightharpoonup v$ in $L^p(A^*;\mathbb{R}^m)$ and

$$\lim_{k\to+\infty}\int_{A^*} f(x,u_k,v_k,\nabla u_k)\mathrm{d}x = \int_{A^*} f(x,u(x),v(x),0)\mathrm{d}x + \int_{\partial^* E\cap A^*} K_p(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1}.$$

By Whitney covering theorem we may write

$$A^* = \bigcup_{i=1}^{\infty}(a_i+\delta_i\overline{Q}) =: \bigcup_{i=1}^{\infty}\overline{Q_i},$$

where $\delta_i, \mathrm{diam}Q_i, \mathrm{dist}(Q_i,\partial A^*)$, $\mathrm{dist}(a_i,\partial A^*)$ satisfy ([10], (4.11) and (4.12)). We choose $L>0$ as in ([10], p. 552) and introduce

$$\Omega_j := \left\{x\in A^* : \mathrm{dist}(x,\partial A^*)\geq \tfrac{1}{j}\right\}, \; \mathcal{G}_j := \left\{Q_i : \mathrm{dist}(a_i,\partial A^*)<\tfrac{1}{Lj}\right\}, \; \mathcal{F}_j := \left\{Q_i : \mathrm{dist}(a_i,\partial A^*)\geq \tfrac{1}{Lj}\right\}.$$

For every j, $\mathcal{F}_j$ is a finite family of cubes, the choice of L, provides that if $Q_i\in\mathcal{G}_j$ then $Q_i\cap\Omega_j=\emptyset$, so that $\Omega_j\subset\cup\mathcal{F}_j$ and thus Ω_j is covered by a finite number of cubes Q_i (see [10] for details).

By Step 1 c) given the sequence $\varepsilon_n \to 0^+$, there exists a subsequence $\{\varepsilon_k^{(1)}\}$ and sequences $\{u_k^{(1)}\}$ in $W^{1,1}(Q_1;\mathbb{R}^d)$ and $\{v_k^{(1)}\}$ in $L^p(Q_1;\mathbb{R}^m)$ such that $u_k^{(1)}\to u$ in $L^1(Q_1:\mathbb{R}^d)$, $v_k^{(1)}\rightharpoonup v$ in $L^p(Q_1;\mathbb{R}^m)$ and

$$\lim_{k\to+\infty}\int_{Q_1} f(x,u_k^{(1)},v_k^{(1)},\nabla u_k^{(1)})\mathrm{d}x = \int_{\partial^* E\cap Q_1} K_p(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1} + \int_{Q_1} f(x,u(x),v(x),0)\mathrm{d}x.$$

By i) in Remark 2.8 there exists a subsequence, still denoted by k, $\{w_k^{(1)}\}\subset W^{1,1}(Q_1,\mathbb{R}^d)$, and $\{\tilde{v}_k^{(1)}\}\subset L^p(Q_1;\mathbb{R}^m)$, such that $w_k^{(1)}\to u$ in $L^1(Q_1;\mathbb{R}^d)$ and $w_k^{(1)}(x) = U_k^{(1)}((x-a_1)/\delta_1)$ for every $x\in\partial Q_1$, where the latter functions are mollifications of u, with $\tilde{v}_k^{(1)}\rightharpoonup u$ in $L^p(Q_1;\mathbb{R}^m)$ and

$$\limsup_{k\to+\infty}\int_{Q_1} f(x,w_k^{(1)},\tilde{v}_k^{(1)},\nabla w_k^{(1)})\mathrm{d}x \leq \liminf_{k\to+\infty}\int_{Q_1} f(x,u_k^{(1)},v_k^{(1)},\nabla u_k^{(1)})\mathrm{d}x = \int_{\partial^* E\cap Q_1} K_p(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1} + \int_{Q_1} f(x,u(x),v(x),0)\mathrm{d}x.$$

This together with the lower bound inequality proved in the previous subsection gives

$$\limsup_{k\to+\infty}\int_{Q_1} f(x,w_k^{(1)},\tilde v_k^{(1)},\nabla w_k^{(1)})\mathrm{d}x=\int_{\partial^* E\cap Q_1}K_p(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1}+\int_{Q_1}f(x,u(x),v(x),0)\mathrm{d}x.$$

By repeatedly taking subsequences and applying Step 1 c) and (i) in Remark 2.8, following the same arguments as above, since in $\mathcal{F}_j$ there are only finitely many cubes, it is possible to obtain a sequence $\{\varepsilon_k\}$ of $\{\varepsilon_n\}$ and sequences $\{\xi_k^{(i)}\}\subset W^{1,1}(Q_i;\mathbb{R}^d)$ and in $\mathcal{A}(c,d,e_N)$ relative to the cube Q_i, such that $\xi_k^{(i)}\to u$ in $L^1(Q_i;\mathbb{R}^d)$ and $\xi_k^{(i)}(x)=U_{\varepsilon_k}((x-a_i)/\delta_i)$ for $x\in\partial Q_i$, and $\{\overline{v}_k^{(i)}\}\subset L^p(Q_i;\mathbb{R}^m)$, such that $\overline{v}_k^{(i)}\rightharpoonup v$ in $L^p(Q_i;\mathbb{R}^m)$ such that

$$\lim_{k\to+\infty}\int_{Q_i} f(x,\xi_k^{(i)},\overline{v}_k^{(i)},\nabla\xi_k^{(i)})\mathrm{d}x=\int_{\partial^* E\cap Q_i}K_p(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1}+\int_{Q_i}f(x,u(x),v(x),0)\mathrm{d}x, \tag{4.67}$$

for every Q_i in $\mathcal{F}_j$. Denote by $\{\zeta_k\}$ and $\{\tilde v_k\}$ the sequences defined in $\cup_{\{i:Q_i\in\mathcal{F}_j\}}Q_i$, such that $\zeta_k:=\xi_k^{(i)}$ and $\tilde v_k:=\overline{v}_k^{(i)}$ in Q_i. Next we define the sequences

$$u_k(x):=\begin{cases}\zeta_k(x) & \text{if } x\in\cup_{\{i:Q_i\in\mathcal{F}_j\}}Q_i,\\ U_{\varepsilon_k}(x) & \text{otherwise,}\end{cases}\qquad \text{and } v_k(x):=\begin{cases}\tilde v_k(x) & \text{if } x\in\cup_{\{i:Q_i\in\mathcal{F}_j\}}Q_i,\\ v(x) & \text{otherwise.}\end{cases}$$

Clearly, $u_k\to u$ in $L^1(A^*;\mathbb{R}^d)$ and $v_k\rightharpoonup v$ in $L^p(A^*;\mathbb{R}^m)$. Moreover, recalling that $v\in L^\infty(\Omega;\mathbb{R}^m)$, $\|U_{\varepsilon_k}\|_{L^\infty}\le C$, $\|\nabla U_{\varepsilon_k}\|_{L^\infty}=O(1/\varepsilon_{n_k})$ and since $x\in A^*\setminus\{Q_i:Q_i\in\mathcal{F}_j\}$ implies that $\operatorname{dist}(x,\partial A^*)<\frac{1}{j}$, we have, by (4.67) and $(H_1)_p$,

$$\begin{aligned}\limsup_{k\to+\infty}\int_{A^*}f(x,u_k,v_k,\nabla u_k)\mathrm{d}x&\le C\limsup_{k\to+\infty}\left(\tfrac{1}{\varepsilon_k}+1\right)\mathcal{L}^N(\{x\in A^*:|x_N|<\varepsilon_k,\operatorname{dist}(x,\partial A^*)<\tfrac{1}{j}\})\\ &\quad+\sum_{Q_i\in\mathcal{F}_j}\left(\int_{\partial E^*\cap Q_i}K_p(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1}+\int_{Q_i}f(x,u(x),v(x),0)\mathrm{d}x\right)\\ &\le C\limsup_{k\to+\infty}\left(\tfrac{1}{\varepsilon_k}+1\right)\mathcal{L}^N\left(\left\{x\in A^*:|x_N|<\varepsilon_k,\operatorname{dist}(x,\partial A^*)<\tfrac{1}{j}\right\}\right)\\ &\quad+\int_{\partial E^*\cap A^*}K_p(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1}+\int_{A^*}f(x,u(x),v(x),0)\mathrm{d}x.\end{aligned}$$

Thus taking the limit as $j\to+\infty$ by (4.66) and the lower bound inequality we conclude that

$$\limsup_{k\to+\infty}\int_{A^*}f(x,u_k,v_k,\nabla u_k)\mathrm{d}x\le\int_{\partial E^*\cap A^*}K_p(x,0,c,d,e_N)\mathrm{d}x+\int_{A^*}f(x,u,v,0)\mathrm{d}x.$$

Step 2. Now we assume that u has a polygonal interface, *i.e.* $u:=\chi_E c+(1-\chi_E)d$, where $E\subset\Omega$ is of the form $E=E'\cap\Omega$, $\partial^*E\cap\Omega=\partial^*E'\cap\Omega$, with E' a polyhedral set.

As in Step 1 d), let A^* be an open subset of Ω such that (4.66) holds. We claim that there exists a sequence $\{u_k\}$ in $W^{1,1}(A^*;\mathbb{R}^d)$ and a sequence $\{v_k\}$ in $L^p(A^*;\mathbb{R}^m)$ such that $u_k\to u$ in $L^1(A^*;\mathbb{R}^d)$, $v_k\rightharpoonup v$ in $L^p(A^*;\mathbb{R}^m)$ and

$$\lim_{k\to+\infty}\int_{A^*}f(x,u_k,v_k,\nabla u_k)\mathrm{d}x=\int_{A^*}f(x,u,v,0)\mathrm{d}x+\int_{\partial E^*\cap A^*}K_p(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1}.$$

The claim is achieved following a proof entirely similar to ([18], Sect. 5, Step 3). It relies on an induction argument and on the application of Step 1 c) and on a slicing procedure similar to Lemma 2.7 in order to connect recovery sequences between two domains for the u and the v.

Step 3. Finally, let A be a Lipschitz subdomain of Ω and consider an arbitrary $u := \chi_E c + (1-\chi_E)d$ with $\mathrm{Per}(E;A) < +\infty$. Since ∂E is Lipschitz, by Theorem 2.6 there exist polyhedral sets E_k such that $\chi_{E_k} \to \chi_E$ in $L^1(A)$, $\mathrm{Per}(E_k;A) \to \mathrm{Per}(E;A)$, $\mathcal{L}^N(E_k) = \mathcal{L}^N(E)$ and $\mathcal{H}^{N-1}(\partial^* E_k \cap \partial\Omega) = 0$ so that

$$\lim_{j\to+\infty}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\mathcal{L}^N(\{x\in A : \mathrm{dist}(x,\partial^* E_k\cap\overline{A}) < \varepsilon, \mathrm{dist}(x,\partial A) < 1/j\}) = 0.$$

By Step 2, for every k there exist $\{u_n^{(k)}\} \subset L^1(A;\mathbb{R}^d)$, such that $u_n^{(k)} \to \chi_{E_k}c + (1-\chi_{E_k})d$ in $L^1(A;\mathbb{R}^d)$ as $n\to+\infty$, and $\{v_n^{(k)}\}\subset L^p(A;\mathbb{R}^m)$ such that $v_n^{(k)} \rightharpoonup v$ in $L^p(A;\mathbb{R}^m)$ and

$$\lim_{n\to+\infty}\int_A f(x,u_n^{(k)},v_n^{(k)},\nabla u_n^{(k)})\mathrm{d}x \le \int_A f(x,u,v,0)\mathrm{d}x + \int_{\partial^* E_k\cap A} K_p(x,0,c,d,\nu_k)\mathrm{d}\mathcal{H}^{N-1},$$

where ν_k is the measure theoretic unit normal to $\partial^* E_k$ at x.

Regarding the weak convergence of $v_n^{(k)}$ to v, observe that $L^{p'}(A;\mathbb{R}^m)$ is separable, hence we can take a dense sequence of functions $\{\psi_l\}\subset L^{p'}(\Omega;\mathbb{R}^m)$ such that $\lim_{n\to+\infty}\int_A (v_n^k - v)\psi_l \mathrm{d}x = 0$ for every $l\in\mathbb{N}$ Consider an increasing sequence $\{k\}$ such that

$$\|u_{n(k)}^{(k)} - \chi_{E_k}c + (1-\chi_{E_k})d\|_{L^1(A;\mathbb{R}^d)} < \frac{1}{k}, \text{ and } \left|\int_A (v_{n(k)}^{(k)} - v)\psi_l\mathrm{d}x\right| < \frac{1}{k} \text{ for every } l = 1,\dots,k,$$

and

$$\left|\int_{\partial^* E_k\cap A} K_p(x,0,c,d,\nu_k)\mathrm{d}\mathcal{H}^{N-1} - \left(\int_A f(x,u,v,0)\mathrm{d}x + \int_A f(x,u_{n(k)}^{(k)},v_{n(k)}^{(k)},\nabla u_{n(k)}^{(k)})\mathrm{d}x\right)\right| < \frac{1}{k}.$$

Set $\overline{u}_k = u_{n(k)}^{(k)}$ and $\overline{v}_k = \overline{v}_{n(k)}^{(k)}$, then $\overline{u}_k \to \chi_E c + (1-\chi_E)d$ in $L^1(A;\mathbb{R}^d)$. Moreover, by the growth condition on f and the bounds on K_p it results that $\|v_k\|_{L^p(A;\mathbb{R}^m)} \le \frac{1}{k} + C$ and the density of $\{\psi_l\}$ in $L^{p'}(A;\mathbb{R}^m)$ ensures that $v_k \rightharpoonup v$ in $L^p(A;\mathbb{R}^m)$. Furthermore, recall that for every continuous function $g : A\times\mathbb{R}^d \to [0,+\infty)$ we have (see [5])

$$\int_{\partial^* E_k\cap A} g(x,\nu_k(x))\mathrm{d}\mathcal{H}^{N-1} \to \int_{\partial^* E\cap A} g(x,\nu(x))\mathrm{d}\mathcal{H}^{N-1}.$$

Since by (b) in Proposition 3.6 $K_p(\cdot,0,c,d,\cdot)$ is upper semicontinuous, there exist continuous functions g_m: $A\times\mathbb{R}^N\to[0,+\infty)$ such that

$$K_p(x,0,c,d,\xi) \le g_m(x,\xi) \le C|\xi| \text{ and } K_p(x,0,c,d,\xi) = \inf_m g_m(x,\xi)$$

for every $(x,\xi)\in A\times\mathbb{R}^N$, where $K_p(x,0,c,d,\cdot)$ has been extended as a positively one homogeneous function to $\mathbb{R}^N$. Thus for all $m\in\mathbb{N}$, it results

$$\limsup_{k\to+\infty}\int_A f(x,\overline{u}_k,\overline{v}_k,\nabla\overline{u}_k)\mathrm{d}x \le \int_A f(x,u,v,0)\mathrm{d}x + \int_{\partial^* E\cap\Omega} g_m(x,\nu(x))\mathrm{d}\mathcal{H}^{N-1}.$$

Taking the limit when $m\to+\infty$, using Lebesgue's monotone convergence theorem and the lower bound inequality we obtain

$$\lim_{k\to+\infty}\int_\Omega f(x,\overline{u}_k,\overline{v}_k,\nabla\overline{u}_k)\mathrm{d}x = \int_A f(x,u,v,0)\mathrm{d}x + \int_{\partial^* E\cap A} K_p(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1}.$$

Step 4. Let A be any domain in Ω. For any K compact subset of A we can find a Lipschitz domain A' such that $K\subset A'\subset A$, (see [9], Rem. 5.5 and [26], Chap. 6) and

$$\begin{aligned}\mathcal{J}_p(u,v;K) \le \mathcal{J}_p(u,v,A') &\le \int_{A'} f(x,u,v,0)\mathrm{d}x + \int_{A'\cap\partial^* E} K_p(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1}\\ &\le \int_A f(x,u,v,0)\mathrm{d}x + \int_{A\cap\partial^* E} K_p(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

By the inner regularity of $\mathcal{J}_p(u,v;\cdot)$, it results

$$\mathcal{J}_p(u,v;A)=\sup\left\{\mathcal{J}_p(u,v;K):K\subset A, K \text{ compact}\right\}\leq \int_A f(x,u,v,0)\mathrm{d}x+\int_{A\cap\partial^* E} K_p(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1}.$$

The additivity of $\mathcal{J}_p(u,v;\cdot)$ allows us to consider any open subset of Ω, not necessarily connected. The lower bound inequality provides

$$\mathcal{J}_p(u,v;A)=\int_A f(x,u,v,0)\mathrm{d}x+\int_{\partial^* E\cap A} K_p(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1}.$$

Moreover,

$$\begin{aligned}\mathcal{J}_p(u,v;E)&\leq \inf\{\mathcal{J}_p(u,v;A):A\subset\Omega \text{ is open}, E\subset A\}\\ &\leq \inf\left\{\int_A f(x,u,v,0)\mathrm{d}x+\int_{\partial^* E\cap A} K_p(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1}:A\subset\Omega \text{ is open}, E\subset A\right\}\\ &=\int_E K_p(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1},\end{aligned}$$

where ν is the normal to E.

Case 2. Consider $u:=\sum c_i\chi_{E_i}$, where $\{E_i\}_{i=1}^\infty$ forms a partitions of Ω into sets of finite perimeter. The proof follows along the lines of ([6], Prop. 4.8, Step 1), since the representation for the surface term is independent on the the target function v. Thus (4.34) follows for every $u\in BV(\Omega;T)$ with T a finite subset of $\mathbb{R}^d$ and $v\in L^\infty(\Omega;\mathbb{R}^m)$.
Case 3. Let $u\in BV(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$. As in Case 2 the fact that the integral representation is in terms of K_p evaluated at $v=0$, allows us to follow the same arguments in ([6], Prop. 4.8, Step 2), exploiting property (c) in Proposition 3.6. Hence, by (4.35) and (4.36), (1.5) holds.

Part 2. Let $u\in BV(\Omega;\mathbb{R}^d)$ and $v\in L^\infty(\Omega;\mathbb{R}^m)$. As in ([18], Sect. 5, Step 4), the lower semicontinuity of $\mathcal{J}_p$ and the result achieved in Part 1 provides

$$\begin{aligned}\mathcal{J}_p(u,v;\Omega)\leq \liminf_{i\to+\infty}\mathcal{J}_p(\phi_i(u),v;\Omega)=\liminf_{i\to+\infty}\Bigg\{&\int_\Omega f(x,(\phi_i(u)),v,(\nabla\phi_i(u)))\mathrm{d}x\\ &+\int_{J_{\phi_i(u)}} K_p(x,0,(\phi_i(u))^+,(\phi_i(u))^-,\nu_{\phi_i})\mathrm{d}\mathcal{H}^{N-1}+\int_\Omega f_p^\infty\left(x,\phi_i(u),0,\tfrac{\mathrm{d}D^c\phi_i(u)}{\mathrm{d}|D^c\phi_i(u)|}\right)\mathrm{d}x\Bigg\},\end{aligned}$$

where $\phi_i\in W_0^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\phi_i(\xi)=\begin{cases}x, & |\xi|<a_i,\\ 0, & |\xi|\geq a_i,\end{cases}$ with the sequence $\{a_i\}\subset\mathbb{R}^+$ such that $a_i\to+\infty$ as $i\to+\infty$, and $\|\nabla\phi_i\|_\infty\leq 1$. Then (1.5) holds for every $u\in BV(\Omega;\mathbb{R}^d)$, passing to the limit as $i\to+\infty$, exploiting (c) in Proposition 3.6, $(H_1)_p$, (3.4), and the lower bound inequality.
Part 3. Concerning the case $BV(\Omega;\mathbb{R}^d)\times L^p(\Omega;\mathbb{R}^m)$ we follow a standard truncation argument, defining for every positive real number L, $\tau_L:[0,+\infty)\to[0,+\infty)$, as $\tau_L(t):=\begin{cases}t & \text{if } 0\leq t<L,\\ 0 & \text{if } t\geq L.\end{cases}$ For every $v\in L^p(\Omega;\mathbb{R}^m)$, define $v_L:=\tau_L(|v|)v$, thus $v_L\in L^\infty(\Omega;\mathbb{R}^m)$, $\int_\Omega|v_L|^p\mathrm{d}x\leq\int_\Omega|v|^p\mathrm{d}x$ and $v_L\to v$ in $L^p(\Omega;\mathbb{R}^m)$, as $L\to+\infty$. By the lower semicontinuity of $\overline{J}_p$ and Part 2, we have that

$$\overline{J}_p(u,v)\leq\liminf_{L\to+\infty}\left(\int_\Omega f(x,u,v_L,\nabla u)\mathrm{d}x+\int_{J_u}K_p(x,0,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1}+\int_\Omega f^\infty\left(x,u,0,\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\right)\mathrm{d}|D^c u|\right).$$

Lebesgue's dominated convergence Theorem provides (4.34) for every $(u,v)\in BV(\Omega;\mathbb{R}^d)\times L^p(\Omega;\mathbb{R}^m)$. ☐

Proof of Theorem 1.1. The proof follows by Theorems 4.1 and 4.2. □

Remark 4.3. It is worth to observe that in the upper bound for the jump term, the proof of Case 1, Steps 1 d), 2, 3, and 4, could be replaced by arguments more similar to the ones in ([14], Prop. 4.1), *i.e.* adopt a covering $\{Q_i\}$ of the type ([14], (4.5)), placing together with the sequences $\{\xi_n^{(i)}\}$ in ([14], (4.6)), sequences $\{v_n^{(i)}\}$, coinciding with $v(x)$ in a layer (depending on n and (i)) of the sets $\{x \in Q_i : (x-a_i)\cdot e_N = -\frac{\eta}{2}\}$ and $\{x \in Q_i : (x-a_i)\cdot e_N = \frac{\eta}{2}\}$, and then exploiting diagonal arguments and the reasonings in [18].

5. Main Results: $BV \times L^\infty$

This section is devoted to the proof of Theorem 1.2 and it is divided in two subsections. The first for the lower bound and the second for the upper bound.

5.1. Lower semicontinuity in $BV \times L^\infty$

Theorem 5.1. *Let $f : \Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N} \to [0,+\infty)$ be a continuous function satisfying (H_0), $(H_1)_\infty$–$(H_3)_\infty$. Then*

$$\liminf_{n\to+\infty}\int_\Omega f(x,u_n,v_n,\nabla u_n)\mathrm{d}x \geq \int_\Omega f(x,u,v,\nabla u)\mathrm{d}x + \int_{J_u\cap\Omega} K_\infty(x,0,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1}$$
$$+\int_\Omega f^\infty(x,u,0,\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|})\mathrm{d}|D^c u|,$$

in $BV(\Omega;\mathbb{R}^d)\times L^\infty(\Omega;\mathbb{R}^m)$ with respect to the (L^1-strong $\times$ L^p-weak)-convergence, where K_∞ is defined in (1.9) and f^∞ is the $(\infty,1)-$ recession function defined in (1.8).

Proof. Up to a subsequence, denote by μ the weak $*$ limit of the measures $\mu_n := f(x,u_n,v_n,\nabla u_n)\mathcal{L}^N$, where $\{u_n\}$ and $\{v_n\}$ can be assumed in C_0^∞. Via Besicovitch derivation theorem it is enough to prove the equivalent of (4.2)–(4.4) with f_p^∞ and K_p replaced by f^∞ and K_∞, respectively. To achieve this, we apply the blow up method.

Bulk part. The proof can be found in [23], with the obvious adaptations to the BV case.

Jump part. We just emphasize the main differences from the $BV\times L^p$ case. Let $x_0\in J_u$, be as in the jump part of Theorem 4.1. Then the equivalent of (4.6) becomes

$$\mu_j(x_0) \geq \frac{1}{|u^+(x_0)-u^-(x_0)|}\lim_{k\to+\infty}\lim_{n\to+\infty}\int_Q \varepsilon_k f(x_0+\varepsilon_k y, u_{n,k}(y), v_{n,k}(y), \frac{1}{\varepsilon_k}\nabla u_{n,k}(y))\mathrm{d}y, \tag{5.1}$$

where $u_{n,k}(y) := u_{n,k}(x_0+\varepsilon_k y)$ and $v_{n,k}(y) := v_n(x_0+\varepsilon_k y)$. Moreover, (4.7) holds, with u_0 defined as in (4.8). Using the separability of $L^1(Q;\mathbb{R}^m)$, together with a diagonalization argument, from (5.1), (4.7) and (4.9), we obtain the existence of sequences $\bar u_k := u_{n(k),k}$ and $\bar v_k := v_{n(k),k}$ such that $\bar u_k \to u_0$ in $L^1(Q;\mathbb{R}^d)$, $\bar v_k \overset{*}{\rightharpoonup} \alpha$ in $L^\infty(Q;\mathbb{R}^m)$, where α is a function whose average in Q is y_0, which in turn is the limit, up to a subsequence, of $\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} v(y)\mathrm{d}y$.

We obtain an estimation for μ_j as in (4.10) replacing f_p^∞ therein by f^∞.

Using Proposition 3.5 iii) we get that for any $\varepsilon>0$, if k is sufficiently large

$$\int_Q f^\infty(x_0+\varepsilon_k y,\bar u_k,\bar v_k,\nabla\bar u_k) - f^\infty(x_0,\bar u_k,\bar v_k,\nabla \bar u_k)\mathrm{d}y \geq -\varepsilon C\int_Q|\nabla\bar u_k|\mathrm{d}y \geq O(\varepsilon).$$

On the other hand, by $(H_3)_\infty$ and Hölder inequality we conclude

$$\begin{aligned}
&\int_Q \varepsilon_k (f(x_0+\varepsilon_k y, \bar{u}_k, \bar{v}_k, \frac{1}{\varepsilon_k}\nabla \bar{u}_k) - f^\infty(x_0+\varepsilon_k y, \bar{u}_k, \bar{v}_k, \nabla \bar{u}_k))\mathrm{d}y \\
&\leq C_M \int_{\{y\in Q: \frac{|\nabla \bar{u}_k|}{\varepsilon_k}\geq L\}} (|\nabla \bar{u}_k|^{1-\tau}\varepsilon_k^\tau)\mathrm{d}y + C_M \int_{\{y\in Q: \frac{|\nabla \bar{u}_k|}{\varepsilon_k}< L\}} |\nabla \bar{u}_k|\mathrm{d}y \\
&\leq O(\varepsilon) + C_M \varepsilon_k^\tau \left(\int_Q |\nabla \bar{u}_k|\mathrm{d}y\right)^{1-\tau} = O(\varepsilon),
\end{aligned}$$

where the constants C_M vary from line to line but are all related to the L^∞ uniform bound on $\{\overline{v}_k\}$, M. Arguing as in [18] we are led to the existence of two, not relabeled, subsequences $\{\overline{u}_k\}$ and $\{\overline{v}_k\}$ converging strongly to u_0 in L^1 and weakly $*$ in L^∞ to α, respectively and such that (4.11) holds with f_p^∞ replaced by f^∞.

Next we apply Remark 2.8 ii) to $f^\infty(x_0,\cdot,\cdot,\cdot)$, obtaining (4.12) with the obvious adaptations, where $\xi_k \to u_0$ in $L^1(Q;\mathbb{R}^d)$ and $\xi_k \in \mathcal{A}(u^+(x_0), u^-(x_0), \nu_u(x_0))$, $\zeta_k \in L^\infty(Q;\mathbb{R}^m)$, converging weakly $*$ to α in $L^\infty(Q;\mathbb{R}^m)$ with $\int_Q \zeta_k \mathrm{d}y = y_0$. In particular, by (1.9) we have

$$\mu_j(x_0) \geq K_\infty(x_0, y_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) \text{ for } \mathcal{H}^{N-1} - \text{a.e. } x_0 \in J_u \cap \Omega,$$

and this, together with Proposition 3.7 concludes the proof of the lower bound inequality for the jump part.

Cantor part. We divide the proof into several steps, just emphasizing the main differences with the $BV \times L^p$ case. Recall that (4.13) – (4.17) hold.

Step 1. It suffices to observe that (4.18) holds for every $1 < p < \infty$, in particular there exist two sequences $\{\bar{u}_k\}$ and $\{\bar{v}_k\}$ such that Step 1 a) in the $BV \times L^p$ case holds.

Step 2. With easier estimates than those of the $BV \times L^p$ case, we obtain (4.20), where the sequences $\{\tilde{v}_k\}$ and $\{\tilde{u}_k\}$ are obtained through a diagonalization argument from $\{\overline{v}_k\}$ and $\{\tilde{u}_k^{r,s}\}$, where this latter sequence is defined by

$$\tilde{u}_k^{r,s} := a_k + \varphi_{r,s}\left(|\bar{u}_k - a_k)|\right)(\bar{u}_k - a_k),$$

with $a_k := \frac{1}{|Q_k|}\int_{x_0+\varepsilon Q} \bar{u}_k \,\mathrm{d}x$, and $\varphi_{r,s}$ is as in (4.22).

Step 3. Notice that, if we consider $\{\bar{w}_k\}$ and $\{\theta_k\}$ as in (4.25) and (4.26), respectively, then condition (4.20) can be written as

$$\left(1+\omega_{M,K}\left(\tfrac{1}{n}\right)\right)\,\mu^c(x_0) \geq \limsup_{k\to+\infty} \frac{1}{\theta_k}\int_{\gamma Q} f(x_0, u(x_0), \bar{v}_k(x_0+\varepsilon_k z), \theta_k \nabla \bar{w}_k(z))\,\mathrm{d}z. \tag{5.2}$$

Then, modifying $\{\bar{v}_k(x_0+\varepsilon_k\cdot)\}$ and $\{\bar{w}_k\}$ near the boundary of γQ new sequences $\{\tilde{v}_k\}$ and $\{\tilde{w}_k\}$ are obtained in order to apply the convexity-quasiconvexity of f. We consider an inner cube τQ, $\tau \in (t,\gamma)$, and we modify the sequences in a layer $\tau Q \setminus \tau(1-\delta)Q$. Indeed we construct $\tilde{v}_k \overset{*}{\rightharpoonup} \alpha$ in $L^\infty(\tau Q;\mathbb{R}^m)$, $\int_{\tau Q}\tilde{v}_k \mathrm{d}x = \int_{\tau Q}\alpha \mathrm{d}x =: y_0(\tau)$, and $\tilde{w}_k(x) = \xi_0(x) + \varphi(x)$ for some $\varphi \in W^{1,\infty}_{\text{per}}(\tau Q;\mathbb{R}^d)$ and such that

$$\left(1+\omega_{M,K}\left(\tfrac{1}{n}\right)\right)\mu^c(x_0) \geq \lim_{k\to+\infty}\frac{1}{\theta_k}\int_{\tau Q} f(x_0, u(x_0), \tilde{v}_k(z), \theta_k \nabla \tilde{w}_k(z))\,\mathrm{d}z + \Lambda(1-t), \tag{5.3}$$

for some continuous function Λ: $[0,1] \to \mathbb{R}$ with $\Lambda(0) = 0$. Observe that, as in Step 3 (lower bound) of the $BV \times L^p$ case, (4.28) – (4.30) hold. More precisely, we can apply the slicing method as in the proof of Step 3 (lower bound) for the case $BV \times L^p$, observing that (4.31) holds, with obvious adaptations. For the reader's convenience, we observe that the construction of the fields v_k is different from the L^p case, here it is identical to the proof of Lemma 2.7. We briefly recall that, for every $j \in \mathbb{N}$, we can divide $\tau Q \setminus \tau(1-\delta)Q$ into j layers thus we getting a sequence $\{k(j)\}$, layers $S_j := \{z \in \tau Q \setminus \tau(1-\delta)Q : \ \alpha_j < \mathrm{dist}(z, \partial(\tau Q)) < \beta_j\}$ and cut-off functions η_j on τQ such that estimates analogous to (2.5) hold with obvious adaptations. Then, define $\tilde w_j$ as in (4.32), and

$$\tilde v_j(z) := (1-\eta_j(z))\frac{\dfrac{1}{|\tau Q|}\displaystyle\int_{\tau Q}\alpha(x) - \eta_j(x)\bar v_{k(j)}(x_0+\varepsilon_{k(j)}x)\,\mathrm{d}x}{\dfrac{1}{|\tau Q|}\displaystyle\int_{\tau Q}(1-\eta_j(x))\,\mathrm{d}x} + \eta_j(z)\bar v_{k(j)}(x_0+\varepsilon_{k(j)}z).$$

Remark that $\|\tilde v_j\|_{L^\infty} \le \|\alpha\|_{L^\infty}+1+\|\bar v_{k(j)}\|_{L^\infty} \le 2M+1$, $\frac{1}{|\tau Q|}\int_{\tau Q}\tilde v_j(z)\,\mathrm{d}z = \frac{1}{|\tau Q|}\int_{\tau Q}\alpha(z)\,\mathrm{d}z := y_0(\tau)$, for all j. By (5.2), summing and subtracting $f(x_0, u(x_0), \tilde v_j(z), \theta_{k(j)}\nabla\tilde w_j(z))$ inside the integral, having in mind the definition of η_j and using $(H_1)_\infty$, we get

$$\begin{aligned}
(1+\omega(\tfrac{1}{n}))\,\mu^c(x_0) &\ge \limsup_{j\to+\infty}\frac{1}{\theta_{k(j)}}\int_{\tau Q} f(x_0,u(x_0),\bar v_{k(j)}(x_0+\varepsilon_{k(j)}z),\theta_{k(j)}\nabla\bar w_{k(j)})\,\mathrm{d}z\\
&\ge \limsup_{j\to+\infty}\frac{1}{\theta_{k(j)}}\left\{\int_{\tau Q} f(x_0,u(x_0),\tilde v_j,\theta_{k(j)}\nabla\tilde w_j)\,\mathrm{d}z - \int_{x\in\tau Q:\mathrm{dist}(x,\partial(\tau Q))\le\beta_j} f(x_0,u(x_0),\tilde v_j,\theta_{k(j)}\nabla\tilde w_j)\,\mathrm{d}z\right\}\\
&\ge \limsup_{j\to+\infty}\frac{1}{\theta_{k(j)}}\left\{\int_{\tau Q} f(x_0,u(x_0),\tilde v_j,\theta_{k(j)}\nabla\tilde w_j)\,\mathrm{d}z - \int_{S_j} c(|\nabla\bar w_{k(j)}|+|\nabla\eta_j|\,|\bar w_{k(j)}-\xi_{k(j)}|)\,\mathrm{d}z\right.\\
&\qquad\left. - \int_{\tau Q\setminus\tau(1-\delta)Q} c(1+|\nabla\xi_{k(j)}|)\,\mathrm{d}z\right\}\\
&\ge \limsup_{j\to+\infty}\frac{1}{\theta_{k(j)}}\left\{\int_{\tau Q} f(x_0,u(x_0),\tilde v_j,\theta_{k(j)}\nabla\tilde w_j)\,\mathrm{d}z - \frac{c}{j} - \int_{\tau Q\setminus\tau(1-\delta)Q} c(1+|\nabla\xi_{k(j)}|)\,\mathrm{d}z\right\}.
\end{aligned}$$

By (4.30) and (4.17), $\int_{\tau Q\setminus\tau(1-\delta)Q} c(1+|\nabla\xi_{k(j)}|)\,\mathrm{d}z \le \Lambda(1-t)$ for some continuous $\Lambda\colon [0,1]\to\mathbb{R}$ with $\Lambda(0)=0$. Therefore we have (5.3), up to a relabelling of the sequences.

Step 4. Analogously to Step 4 for the $BV\times L^p$ case, the functions $\tilde v_k$ have constant average in τQ, given by $y_0(\tau)$ and the functions ξ_k satisfies the same periodicity properties. This, together with the fact that $\tilde w_j = \xi_{k(j)}$ on $\partial(\tau Q)$, yields that $\tilde w_j \in (\frac{\zeta_{k(j)}(\frac{\tau}{2})-\zeta_{k(j)}(-\frac{\tau}{2})}{\tau}\otimes e_N)\,x + W^{1,\infty}_{\mathrm{per}}(\tau Q;\mathbb{R}^d)$. Therefore

$$(1+\omega(\tfrac{1}{n}))\,\mu^c(x_0) \ge O(1-t) + \limsup_{j\to+\infty}\tfrac{|\tau Q|}{\theta_{k(j)}} f(x_0,u(x_0),y_0(\tau),\tfrac{\zeta_k(\frac{\tau}{2})-\zeta_k(-\frac{\tau}{2})}{\tau}\otimes e_N).$$

If we add and subtract in the previous limit the quantity $\frac{|\tau Q|}{\theta_{k(j)}} f(x_0,u(x_0),y_0(\tau),\frac{\theta_{k(j)}}{|\tau Q|}A)$ we get two terms. One will raise the expected value of the f^∞ function, namely

$$\lim_{k\to+\infty}\tfrac{|\tau Q|}{\theta_{k(j)}} f(x_0,u(x_0),y_0(\tau),\tfrac{\theta_{k(j)}}{|\tau Q|}A) = f^\infty(x_0,u(x_0),y_0(\tau),A) = f^\infty(x_0,u(x_0),0,A),$$

the last identity following from Lemma 2.2 in [14] and recalling that A is a rank-one matrix. The other term can be estimated using the Lipschitz continuity of $f(x_0,u(x_0),0,\cdot)$, *i.e.* (3.2) and (4.29).

Then, passing to the limit on k, and using (4.28), (4.17) and (4.19), we get

$$(1+\omega(\tfrac{1}{n}))\,\mu^c(x_0) \ge O(1-t) + f^\infty(x_0,u(x_0),0,A) + O(1-t).$$

Finally the desired estimate is obtained letting $\varepsilon\to 0^+$ and $t\to 1^-$. □

5.2. Upper bound in $BV \times L^\infty$

Let

$$\mathcal{J}_\infty(u,v;A) := \inf\left\{\liminf_{n\to+\infty} J(u_n,v_n;A):\ u_n \in BV(\Omega;\mathbb{R}^d),\ v_n \in L^\infty(\Omega;\mathbb{R}^m),\ u_n \to u \text{ in } L^1, v_n \overset{*}{\rightharpoonup} v \text{ in } L^\infty\right\}$$

for open sets $A \subset \Omega$ and for any $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^\infty(\Omega;\mathbb{R}^m)$, where, $J(u,v;A)$ is as in (4.33) with L^p replaced by L^∞. We observe that $(H_1)_\infty$ implies that for every $u \in BV(\Omega;\mathbb{R}^d)$ and for every $v \in L^\infty(\Omega;\mathbb{R}^m)$, with $\|v\|_{L^\infty} \le M$ there exists $C_M > 0$ such that $\mathcal{J}_\infty(u,v;A) \le C_M\left(|A| + |Du|(A)\right)$. Moreover, $\mathcal{J}_\infty$ is a variational functional.

Theorem 5.2. *Let $f : \Omega\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{d\times N} \to [0,+\infty)$ be a continuous function satisfying (H_0), $(H_1)_\infty$–$(H_3)_\infty$, and $\overline{J}_\infty$ be defined in (1.2). Then for every $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^\infty(\Omega;\mathbb{R}^m)$:*

$$\overline{J}_\infty(u,v;\Omega) \le \int_\Omega f(x,u,v,\nabla u)\mathrm{d}x + \int_{J_u\cap\Omega} K_\infty(x,0,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega f^\infty(x,u,0,\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|})\mathrm{d}|D^c u|. \quad (5.4)$$

Proof. The representation (5.4) is achieved first for $u \in BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$, then, via an approximation argument as in [6], the result will be obtained in the whole space.

Part 1. Let $u \in BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$. As in Part 1 of Theorem 4.2 it suffices to prove the equivalent of (4.35)–(4.37) with K_p and f_p^∞ replaced by K_∞ and f^∞, respectively.

Bulk part. It follows from ([23], Thm. 12).

Cantor part. We consider $u \in BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ and $v \in L^\infty(\Omega;\mathbb{R}^m)$. Again we follow [14, 18]. As usual we identify u with its approximate limit defined in $\Omega \setminus J_u$. Considering $u_n := u * \varrho_n$, where $\{\varrho_n\}$ is a sequence of mollifiers, one has (4.38). Recalling that u is $|D^c u|$-measurable, $|Du| = |D^c u| + \eta$, where η and $|D^c u|$ are mutually singular Radon measures, we consider $x_0 \in \Omega$ such that $\frac{\mathrm{d}\mathcal{J}_\infty(u,v;\cdot)}{\mathrm{d}|D^c u|}(x_0)$ exists and is finite, (4.39)–(4.41) hold, for every $1 < p < \infty$,

$$\lim_{\varepsilon\to 0}\frac{1}{|D^c u|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} f^\infty(x_0,u(x_0),0,A(x))\mathrm{d}|D^c u| = f^\infty(x_0,u(x_0),0,A(x_0)),$$

and (4.42) hold. Fixing $\delta > 0$ and arguing as in the Cantor part of Theorem 4.2 we obtain

$$\mathcal{J}_\infty(u,v;B(x_0,\varepsilon)) \le \liminf_{\varepsilon\to 0^+}\liminf_{n\to+\infty}\frac{1}{|D^c u|(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} f(x_0,u(x_0),v,Du*\varrho_n)\mathrm{d}x + O(\delta).$$

Then the Cantor upper bound inequality is achieved as in the proof of ([14], (6.6)).

Jump part. We claim that

$$\mathcal{J}_\infty(u,v;J_u\cap\Omega) \le \int_{J_u\cap\Omega} K_\infty(x,0,u^-,u^+,\nu_u)\mathrm{d}\mathcal{H}^{N-1},$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d) \times L^\infty(\Omega;\mathbb{R}^m)$. The proof develops exploiting the arguments in ([18], Step 3 of Sect. 5), being divided into Cases 1, 2 and 3 as in the $BV \times L^p$ case. Here we just present Case 1, since the others are entirely similar to the ones in Theorem 4.2.

Case 1. We start to consider $u := c\chi_E + d(1-\chi_E)$, with $\mathrm{Per}(E;\Omega) < +\infty$, and $v \in L^\infty(\Omega;\mathbb{R}^m)$ and we aim to prove that

$$\mathcal{J}_\infty(u,v;A) \le \int_A f(x,u,v,0)\mathrm{d}x + \int_{J_u\cap A} K_\infty(x,0,c,d,\nu)\mathrm{d}\mathcal{H}^{N-1},\ \text{for every } A \in \mathcal{A}(\Omega). \quad (5.5)$$

This inequality is achieved in several steps, and we present just the main differences with the case $BV \times L^p$.

Step 1. First we assume that u has a planar interface and we keep the same notations as in Step 1 of (4.37). Suppose that f does not depend on x, and we claim that there exist $\{u_n\}$ as in the proof of (4.37) and a sequence $\{v_n\} \subset L^\infty(a_0 + \lambda Q_\nu; \mathbb{R}^m)$, such that $v_n(x) = v(x)$ if $|(x - a_0)\cdot\nu| > \frac{\lambda}{2(2n+1)}$, with $u_n \to u$ in $L^1(a_0 + \lambda Q_\nu; \mathbb{R}^d)$, $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(a_0 + \lambda Q_\nu; \mathbb{R}^m)$ and

$$\lim_{n\to+\infty} \int_{a_0+\lambda Q_\nu} f(u_n, v_n, \nabla u_n)\mathrm{d}x = \int_{a_0+\lambda Q_\nu} f(u, v, 0)\mathrm{d}x + \lambda^{N-1} K_\infty(0, c, d, \nu). \tag{5.6}$$

Step 1 a). As in the proof of Theorem 4.2 we claim that for all $\xi \in \mathcal{A}(c, d, e_N)$ and for all $\varphi \in L^\infty(Q; \mathbb{R}^m)$, with $\int_Q \varphi \mathrm{d}x = 0$, there exists $\xi_n \in \mathcal{A}(c, d, e_N)$ and $v_n \in L^\infty(Q; \mathbb{R}^m)$ such that $v_n(x) = v(x)$ if $|x_N| > \frac{1}{2(2n+1)}$ and (4.48) hold replacing $\rightharpoonup$ in L^p by $\overset{*}{\rightharpoonup}$ in L^∞ and (4.49) hold with f_p^∞ replaced by f^∞.

Let Σ be as in the proof of (4.49). For $k \in \mathbb{N}$, we label the elements of $(\mathbb{Z} \cap [-k, k]^N) \times \{0\}$ by $\{a_i\}_{i=1}^{(2k+1)^{N-1}}$ and we recall that $(2k+1)\overline{\Sigma} = \bigcup_{i=1}^{(2k+1)^{N-1}} (a_i + \overline{\Sigma})$, with $(a_i + \Sigma) \cap (a_j + \Sigma) =$ if $i \neq j$. We extend $\xi(\cdot, x_N)$ to $\mathbb{R}^{N-1}$ by periodicity and define $\{\xi_{2k+1}\}$ as in (4.50). Clearly $\xi_{2k+1} \in \mathcal{A}(c, d, e_N)$ and $\|\xi_{2k+1} - u\|_{L^1(Q;\mathbb{R}^d)} \to 0$ as $k \to +\infty$. Extending $\varphi(\cdot, x_N)$ to $\mathbb{R}^{N-1}$ by periodicity define

$$v_{2k+1}(x) := \begin{cases} v(x) & \text{if } |x_N| > \frac{1}{2(2k+1)}, \\ \varphi((2k+1)x) & \text{if } |x_N| \leq \frac{1}{2(2k+1)}. \end{cases}$$

As in Step 1 a) of Theorem 4.2 we observe that $v_{2k+1} \overset{*}{\rightharpoonup} v$ in $L^\infty(Q; \mathbb{R}^m)$.

We can argue as in the $BV \times L^p$ case, exploiting the periodicity of φ.

$$\int_Q f(\xi_{2k+1}, v_{2k+1}, \nabla\xi_{2k+1})\mathrm{d}x = \int_\Sigma \int_{-\frac{1}{2}}^{-\frac{1}{2(2k+1)}} f(c, v(x), 0)\mathrm{d}x + \int_\Sigma \int_{\frac{1}{2(2k+1)}}^{\frac{1}{2}} f(d, v(x), 0)\mathrm{d}x$$
$$+ \int_\Sigma \int_{|x_N| < \frac{1}{2(2k+1)}} f(\xi((2k+1)x), \varphi((2k+1)x), (2k+1)\nabla\xi((2k+1)x)\mathrm{d}x.$$

The first two integrals in the right hand side, converge as $k \to +\infty$, to $\int_Q f(u(x), v(x), 0)\mathrm{d}x$. The latter integral, after a change of variables becomes

$$\int_\Sigma \int_{|x_N| < \frac{1}{2(2k+1)}} f(\xi((2k+1)x), \varphi((2k+1)x), (2k+1)\nabla\xi((2k+1)x))\mathrm{d}x$$
$$= \frac{1}{2k+1} \int_Q f(\xi(y), \varphi(y), (2k+1)\nabla\xi(y))\mathrm{d}y \underset{k\to+\infty}{\to} \int_Q f^\infty(\xi(y), \varphi(y), \nabla\xi(y))\mathrm{d}y.$$

From the last two convergence we obtain the desired limit.

Step 1 b). To prove (5.6), let $\{(\eta_n, \varphi_n)\} \subset \mathcal{A}(c, d, e_N) \times L^\infty(Q; \mathbb{R}^m)$ with $\int_Q \varphi_n \mathrm{d}y = 0$ be a minimizing sequence for $K_\infty(c, d, 0, \varepsilon_N)$.

By Step 1 a), for every $n \in \mathbb{N}$ we can find $k_n \in N$ and choose $u_n \in \mathcal{A}(c, d, e_N)$ and $v_n \in L^\infty(Q; \mathbb{R}^m)$ such that $\|u_n - u\|_{L^1(Q;\mathbb{R}^d)} \leq \frac{1}{n}$, $\left|\int_Q (v_n - v)\psi_l \mathrm{d}x\right| < \frac{1}{n}$, (for $l = 1, \dots, n$ with $\{\psi_l\}$ a dense sequence of functions in $L^1(Q; \mathbb{R}^m)$), with v_n defined as

$$v_n(x) := \begin{cases} v(x) & \text{if } |x_N| > \frac{1}{2(2k_n+1)}, \\ \varphi_n((2k_n+1)x) & \text{if } |x_N| \leq \frac{1}{2(2k_n+1)} \end{cases}$$

and

$$\left|\int_Q f(u_n, v_n, \nabla u_n)\mathrm{d}x - \int_Q f(u, v, 0)\mathrm{d}x - \int_Q f^\infty(\eta_n, \varphi_n, \nabla\eta_n)\mathrm{d}x\right| < \frac{1}{n}\cdot$$

By the lower bound inequality we have that

$$\int_Q f(u,v,0)\mathrm{d}x + K_\infty(0,c,d,e_N) \leq \liminf_{n\to+\infty}\int_Q f(u_n,v_n,\nabla u_n)\mathrm{d}x \leq \limsup_{n\to+\infty}\int_Q f(u_n,v_n,\nabla u_n)\mathrm{d}x$$
$$\leq \lim_{n\to+\infty}\left\{\int_Q f(u,v,0)\mathrm{d}x + \int_Q f^\infty(\eta_n,\varphi_n,\nabla\eta_n)\mathrm{d}x + \frac{1}{n}\right\} = \int_Q f(u,v,0)\mathrm{d}x + K_\infty(0,c,d,e_N), \tag{5.7}$$

which proves (5.6), up to relabeling $\{u_n\}$ and $\{v_n\}$ with the same indices k_n, when $\lambda = 1$ and $a_0 = 0$.

Considering the case of $A := \lambda Q$, for $\lambda > 0$, we define f_λ, u_0 and v_0 as in (4.51). By (5.7) there exists $(u_n, v_n) \in \mathcal{A}(c,d,e_N) \times L^\infty(Q;\mathbb{R}^m)$ such that $u_n \to u_0$ in $L^1(Q;\mathbb{R}^d)$, $v_n \overset{*}{\rightharpoonup} v_0$ in $L^\infty(Q;\mathbb{R}^m)$ and (4.52) with $(K_p)_\lambda$ replaced by $(K_\infty)_\lambda$, where $(K_\infty)_\lambda$ is the function defined in (1.9), with f replaced by f_λ. Consider any $a_0 \in \mathbb{R}^N$ and set $\bar{u}_n$ and $\bar{v}_n$ as in (4.53).

Clearly $\{\bar{u}_n\}$ satisfies all the properties stated in the proof of Theorem 4.2, Step 1 b), $\bar{v}_n \overset{*}{\rightharpoonup} v$ in $L^\infty(a_0+\lambda Q)$ and (4.54) holds with $(K_p)_\lambda$ replaced by $(K_\infty)_\lambda$. Moroever, (4.55) holds with the obvious adaptation $(K_\infty)_\lambda = \frac{1}{\lambda}K_\infty$. Hence we conclude that (5.10) holds.

Step 1 c). We allow f to have explicit x-dependence and, given $r \geq 0$, we prove (5.5) with K_∞ replaced by K_r as in Proposition 3.10.

Let A, A^*, Q_ν, A' and Q' as in Theorem 4.2, Step 1 c). Since A^* is a compactly included in Ω, fixing $\varepsilon > 0$, it is possible to find a $\delta > 0$ such that $(H_2)_\infty$ and (3.10) hold uniformly in A^*, *i.e.*

$$x, y \in A^*, |x-y| < \delta \Rightarrow |f(x,u,b,\xi) - f(y,u,b,\xi)| \leq \varepsilon C_M(1+|\xi|), \tag{5.8}$$

for any $b \in \mathbb{R}^m$, and

$$x, y \in A^*, |x-y| < \delta \Rightarrow |K_r(x,b,c,d,\nu) - K_r(y,b,c,d,\nu)| \leq \varepsilon C'_{|b|+r}(1+|d-c|). \tag{5.9}$$

Let $h \in \mathbb{N}$ be as in (4.58), partition Ω' according to (4.59) and denote $Q'_i := a_i + \eta Q'$ and $Q_i := a_i + \eta Q$.

We claim that there exists $\{u_k\} \subset W^{1,1}(A^*;\mathbb{R}^d)$ and a sequence $\{v_k\} \subset L^\infty(A^*;\mathbb{R}^m)$ such that $u_k \to u$ in $L^1(A;\mathbb{R}^d)$, $v_k \overset{*}{\rightharpoonup} v$ in $L^\infty(A;\mathbb{R}^m)$ and

$$\lim_{k\to+\infty}\int_{A^*} f(x,u_k,v_k,\nabla u_k)\mathrm{d}x \leq \int_{J_u\cap A^*} K_r(x,0,c,d,e_N)\mathrm{d}\mathcal{H}^{N-1} + \int_{A^*} f(x,u,v,0)\mathrm{d}x.$$

By Step 1 b), there exist sequences $\{u_k^{(1)}\} \subset \mathcal{A}(c,d,e_N)$, related to the cube Q_1 and $\{v_k^{(1)}\} \in L^\infty(Q_1;\mathbb{R}^m)$, such that (4.60) holds with K_p replaced by K_∞, thus, by (3.9)

$$\lim_{k\to+\infty}\int_{Q_1} f(a_1,u_k^{(1)},v_k^{(1)},\nabla u_k^{(1)})\mathrm{d}x \leq \eta^{N-1}K_r(a_1,0,c,d,e_N) + \int_{Q_1} f(a_1,u,v,0)\mathrm{d}x.$$

By iii) in Remark 2.8 there exists $\{\xi_k^{(1)}\} \subset W^{1,1}(Q_1;\mathbb{R}^d)$ and a sequence $\{\overline{v}_k^{(1)}\} \subset L^\infty(Q_1;\mathbb{R}^m)$ such that $\xi_k^{(1)} \to u$ in $L^1(Q_1;\mathbb{R}^d)$, with $\xi_k^{(1)}(x) = U_k^{(1)}((x-a_1)/\eta)$ on ∂Q_1, ($U_k^{(1)}$ is a mollification of u) and $\overline{v}_k^{(1)} \overset{*}{\rightharpoonup} v$ in $L^\infty(Q_1;\mathbb{R}^m)$, and

$$\limsup_{k\to+\infty}\int_{Q_1} f(a_1,\xi_k^{(1)},\overline{v}_k^{(1)},\nabla\xi_k^{(1)})\mathrm{d}x \leq \liminf_{k\to+\infty}\int_{Q_1} f(a_1,u_k^{(1)},v_k^{(1)},\nabla u_k^{(1)})\mathrm{d}x$$
$$\leq \eta^{N-1}K_r(a_1,0,c,d,e_N) + \int_{Q_1} f(a_1,u,v,0)\mathrm{d}x.$$

We can repeat the same induction argument as in Step 1 c) in Theorem 4.2, obtaining h^{N-1} sequences $\{\xi_k^{(j)}\} \subset \mathcal{A}(c,d,e_N)$ related to the cube Q_j, with $\xi_k^{(j)} \to u$ in $L^1(Q_j;\mathbb{R}^d)$, $\xi_k^{(j)} = U_k^{(j)}$ on ∂Q_j and $\{\overline{v}_k^{(j)}\} \subset L^\infty(Q_j;\mathbb{R}^m)$, with $\overline{v}_k^{(j)} \overset{*}{\rightharpoonup} v$ in $L^\infty(Q_j;\mathbb{R}^m)$ and

$$\eta^{N-1}K_r(a_j,0,c,d,e_N) + \int_{Q_j} f(a_j,u,v,0)\mathrm{d}x \geq \lim_{k\to+\infty}\int_{Q_j} f(a_j,\xi_k^{(j)},\overline{v}_k^{(j)},\nabla\xi_k^{(j)})\mathrm{d}x$$

for every $j = 1,\ldots,h^{N-1}$.

Next for all $j = 1, \dots, h^{N-1}$ we consider the subsequences $\{\zeta_k\}$, with $\zeta_k := \xi_k^{(j)}$ and $\{\tilde{v}_k\}$, with $\tilde{v}_k := \overline{v}_k^{(j)}$, denoted by the same index k such that

$$\lim_{k\to+\infty} \int_{Q_j} f(a_j, \zeta_k, \tilde{v}_k, \nabla\zeta_k)\mathrm{d}x \le \eta^{N-1} K_r(a_j, 0, c, d, e_N) + \int_{Q_j} f(a_j, u, v, 0)\mathrm{d}x. \tag{5.10}$$

Define the sequences $\{u_{k,\varepsilon}\}$ and $\{v_{k,\varepsilon}\}$ almost everywhere on A^*, as in (4.62), thus (4.63) holds.

Since $\{\tilde{v}_k\} \subset L^\infty(A^*; \mathbb{R}^m)$ and coincides with $v(x)$ if $|x_N| > \eta/2$, $v_{k,\varepsilon} \overset{*}{\rightharpoonup} v$ in $L^\infty(A^*; \mathbb{R}^m)$ as $k \to +\infty$ and as $\varepsilon \to 0^+$. Moreover, we can write, as in (4.64)

$$\int_\Omega f(x, u_{k,\varepsilon}, v_{k,\varepsilon}, \nabla u_{k,\varepsilon})\mathrm{d}x =: I_1 + I_2 + I_3 + I_4.$$

Then it is easily seen that by (5.10), we have

$$\limsup_{k\to+\infty} I_1 \le \sum_{i=1}^{h^{N-1}} \left(\eta^{N-1} K_r(a_i, 0, c, d, e_N) + \int_{Q_i} f(a_i, u, v, 0)\mathrm{d}x \right).$$

On the other hand, the asymptotic behaviour of $I_3 + I_4$ is given by (4.65). Regarding I_2 we can observe that, by Remark 3.11 and (5.8),

$$\limsup_{k\to\infty} I_2 \le \limsup_{k\to+\infty} \sum_{i=1}^{h^{N-1}} \int_{Q_i} \varepsilon C_{|b|+r}(1 + |\nabla\zeta_k|)\mathrm{d}x = O(\varepsilon),$$

since, by construction and $(H_1)_\infty$, $\{\tilde{v}_k^{(i)}\}$ and $\{\zeta_k^{(i)}\}$ are bounded in $L^\infty(Q_i; \mathbb{R}^m)$ and $W^{1,1}(Q_i; \mathbb{R}^d)$, respectively. By Remark 3.11 and (5.9),

$$\left| \int_{A^*\cap J_u} K_r(x, 0, c, d, e_N)\mathrm{d}\mathcal{H}^{N-1} - \eta^{N-1} \sum_{i=1}^{h^{N-1}} K_r(a_i, 0, c, d, e_N) \right|$$
$$= \sum_{i=1}^{h^{N-1}} \int_{Q_i} |K_r(x, 0, c, d, e_N) - K_r(a_i, 0, c, d, e_N)| \,\mathrm{d}\mathcal{H}^{N-1} = O(\varepsilon).$$

Finally, putting together this estimate, the limits of I_2, I_3, I_4 and estimating $\sum_{i=1}^{h^{N-1}} \int_{Q_i} f(a_i, u, v, 0)\mathrm{d}x$ in I_1 *via* (5.8), we obtain the desired approximating sequence, just letting $\varepsilon \to 0^+$ and using a diagonalization procedure. In fact, we can say that there exist $\{\overline{\zeta}_k\} \subset L^1(A^*; \mathbb{R}^d)$ and $\{\overline{v}_k\} \subset L^\infty(A^*; \mathbb{R}^m)$, converging to u in L^1 and weakly $*$ in L^∞ to v, respectively, such that

$$\lim_{k\to+\infty} \int_{A^*} f(x, \overline{\zeta}_k, \overline{v}_k, \nabla\overline{\zeta}_k)\mathrm{d}x \le \int_{J_u\cap A^*} K_r(x, 0, c, d, \nu)\mathrm{d}\mathcal{H}^{N-1} + \int_{A^*} f(x, u(x), v(x), 0)\mathrm{d}x.$$

Hence (5.5) follows by (3.9) sending $r \to +\infty$. From the lower bound, the equality is achieved.

Step 1 d). In order to consider A^* any open subset of Ω such that (4.66) holds, it suffices to argue exactly as in Theorem 4.2, Step 1 d), replacing K_p by K_∞, weak convergence in L^p by weak $*$ convergence in L^∞ and invoking iii) in Remark 2.8.

Step 2. In order to obtain the representation

$$\mathcal{J}_\infty(u, v; E) = \int_E K_\infty(x, 0, c, d, \nu)\mathrm{d}x,$$

when $u := c\chi_E + d(1-\chi_E)$, E being a set of finite perimeter with unit normal ν, we can argue as in Steps 2, 3 and 4 of Theorem 4.2, replacing the densities f_p^∞ and K_p by f^∞ and K_∞, respectively, and hypotheses $(H_1)_p - (H_3)_p$ by $(H_1)_\infty - (H_3)_\infty$.

Part 2. If $(u,v) \in BV \times L^\infty$ the proof is identical to the one of Part 2 in Theorem 4.2. It is enough to replace K_p and f_p^∞ therein by K_∞ and f^∞, respectively, and invoking the correspondent properties. □

Proof of Theorem 1.2. The proof follows by Theorems 5.1 and 5.2. □

Appendix.

The following theorem is devoted to remove assumption (H_0) in Theorem 1.1.

Theorem A.1. *Let J be given by (1.1), with f verifying $(H_1)_p - (H_3)_p$, $1 < p < \infty$, and let $\overline{J}_p$ be given by (1.2) then*

$$\overline{J}_p(u,v) = \int_\Omega \mathcal{CQ}f(x,u,v,\nabla u)\mathrm{d}x + \int_{J_u} K_p(x,0,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega \mathcal{CQ}f_p^\infty\left(x,u,0,\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\right)\mathrm{d}|D^cu|, \quad \text{(A.1)}$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$, where $\mathcal{CQ}f$ denotes the convex-quasiconvex envelope of f, given by

$$\mathcal{CQ}f(x,u,b,\xi) := \inf\left\{\frac{1}{|D|}\int_D f(x,u,b+\eta,\xi+\nabla\varphi)\,\mathrm{d}y : \eta \in L^\infty(D;\mathbb{R}^m), \int_D \eta \mathrm{d}y = 0, \varphi \in W_0^{1,\infty}(D;\mathbb{R}^d)\right\},$$

and K_p is given by (1.6) with f_p^∞ replaced by $\mathcal{CQ}f_p^\infty$, where

$$\mathcal{CQ}f_p^\infty(x,u,b,\xi) := \limsup_{t\to+\infty} \frac{\mathcal{CQ}f(x,u,t^{\frac{1}{p}}b,t\xi)}{t}.$$

Proof. First we recall that the convex-quasiconvex envelope $\mathcal{CQ}f(x,u,\cdot,\cdot)$ of a function $f(x,u,\cdot,\cdot)$ is the largest convex-quasiconvex function below f.

In analogy with (1.1) and (1.2) we define, for every $(u,v) \in W^{1,1}(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$, the functional

$$J_{\mathcal{CQ}f}(u,v) := \int_\Omega \mathcal{CQ}f(x,u(x),v(x),\nabla u(x)\mathrm{d}x,$$

and for every $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$, the functional

$$\overline{J_{\mathcal{CQ}f_p}}(u,v) := \inf\{\liminf_{n\to+\infty} J_{\mathcal{CQ}f}(u_n,v_n) : u_n \in W^{1,1}(\Omega;\mathbb{R}^d),\ v_n \in L^p(\Omega;\mathbb{R}^m),\ u_n \to u \text{ in } L^1,\ v_n \rightharpoonup v \text{ in } L^p\}.$$

Clearly, it results that for every $(u,v) \in BV(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$, and for every $1 < p < \infty$,

$$\overline{J_{\mathcal{CQ}f}}_p(u,v) \le \overline{J}_p(u,v).$$

It remains to prove the opposite inequality. First we observe that $\mathcal{CQ}f$ satisfies $(H_1)_p - (H_3)_p$. Regarding $(H_1)_p$ and $(H_2)_p$ it is enough to argue as in ([22], Prop. 2.2). For what concerns $(H_3)_p$ we observe that it is equivalent to say that there exist $0 < \tau < 1$ such that

$$|f_p^\infty(x,u,b,\xi) - f(x,u,b,\xi)| \le C(1+|b|^{(1-\tau)p} + |\xi|^{1-\tau})$$

for every $(x,u,b,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$, then, arguing as in ([22], Prop. 2.3) one can prove that $\mathcal{CQ}f_p^\infty$ inherits the same property. Thus, applying the same arguments as in ([23], Lem. 8 and Rem. 9), the proof is concluded. □

We also observe that an argument entirely similar to ([7], Prop. 3.4), guarantees that there is no ambiguity in (A.1) when omitting the parentheses in $CQ(f_p^\infty)$, since $(CQf)_p^\infty = CQ(f_p^\infty)$. When $p = \infty$ removing (H_0), arguing as above, one can prove the following result.

Theorem A.2. *Let J be given by (1.1), with f and $\mathcal{CQ}f$ satisfying $(H_1)_\infty - (H_3)_\infty$, and let $\overline{J}_\infty$ be given by* (1.3) *then*

$$\overline{J}_\infty(u,v) = \int_\Omega \mathcal{CQ}f(x,u,v,\nabla u)\mathrm{d}x + \int_{J_u} K_\infty(x,0,u^+,u^-,\nu_u)\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega (\mathcal{CQ}f)^\infty \left(x,u,0,\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\right)\mathrm{d}|D^c u|,$$

where K_∞ is given by (1.9) *with f^∞ replaced by $(\mathcal{CQ}f)^\infty$.*

Remark A.3. Let $\alpha : [0,+\infty) \to [0,+\infty)$ be a convex and increasing function, with $\alpha(0) = 0$ such that the following assumptions hold:
$(H_1)_\alpha$ There exists a constant $C > 0$ such that

$$\frac{1}{C}(\alpha(|b|) + |\xi|) - C \leq f(x,u,b,\xi) \leq C(1 + \alpha(|b|) + |\xi|)$$

for a.e. $(x,u) \in \Omega \times \mathbb{R}^d$ and for every $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$.
$(H_2)_\alpha$ For every compact set $K \subset \Omega \times \mathbb{R}^d$ there exists a continuous function ω'_K: $\mathbb{R} \to [0,+\infty)$ such that $\omega'_K(0) = 0$ and

$$|f(x,u,b,\xi) - f(x',u',b,\xi)| \leq \omega'_K(|x-x'| + |u-u'|)(1 + \alpha(|b|) + |\xi|),$$

for every $(x,u),(x',u') \in K,\ \forall\ (b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$.
For each $x_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| \leq \delta \ \Rightarrow\ f(x,u,b,\xi) - f(x_0,u,b,\xi) \geq -\varepsilon(1 + \alpha(|b|) + |\xi|),$$

for every $(u,b,\xi) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$.
$(H_3)_\alpha$ There exist $C > 0$ and $0 < \tau < 1$ such that

$$|f(x,u,b,\xi) - f^\infty(x,u,b,\xi)| \leq C(1 + \alpha^{1-\tau}(|b|) + |\xi|^{1-\tau}),$$

for every $(x,u,b,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$.

We observe that if one replaces $(H_1)_\infty - (H_3)_\infty$ by $(H_1)_\alpha - (H_3)_\alpha$ then, arguing as in ([22], Props. 2.1–2.3)

$$\mathcal{CQ}(f^\infty)(x,u,b,\xi) = (\mathcal{CQ}f)^\infty(x,u,b,\xi),$$

for every $(x,u,b,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$, and $\mathcal{CQ}f$ satisfies $(H_1)_\alpha - (H_3)_\alpha$.

Thus an analogous of Theorem A.2 holds with obvious modifications just imposing $(H_1)_\alpha - (H_3)_\alpha$ on f.

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