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GAMMA-CONVERGENCE OF CERTAIN MODIFIED PERONA−MALIK FUNCTIONALS *

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Abstract. In this paper, a behavior of certain modified Perona−Malik functionals is considered as the parameter, which determines the scaling and the amount of the regularization, tends to zero.

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1. Introduction

Minimizing the Mumford−Shah functional is a classical way to segment or restore an image, reference [21]. In this model, the segmentation is defined as a joint smoothing and edge detection problem. The Mumford−Shah functional is defined as

$$E(u,K) := \mu||f-u||^2_{L^2(\Omega)} + \lambda \int_{\Omega\setminus K} |\nabla u|^2 \,\mathrm{d}x + 2\int_K \mathrm{d}\sigma \tag{1.1}$$

where $\mu,\ \lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^2$, $f \in L^2(\Omega)$, $K \subset \Omega$ is a closed set modeling the edges of the image u, and $d\sigma$ denotes the line integral. In the above energy, the first term forces that the segmentation u is near the original image f, the second term forces u to be smooth outside the edge set K, and the last term forces that the image u does not contain too many edges.

The direct minimization of (1.1) is difficult both theoretically and numerically. In a weak formulation of the Mumford−Shah functional, the minimization over functions u and sets K is replaced by the minimization over functions $u \in \mathrm{SBV}(\Omega)$ where the edge set K is identified with the jump set S_u of the function u. This results in the weak form

$$E(u) := \mu||f-u||^2_{L^2(\Omega)} + \lambda \int_{\Omega} |\nabla_a u|^2 \,\mathrm{d}x + 2\mathcal{H}^1(S_u). \tag{1.2}$$

From now on we use at times the term Mumford−Shah functional to refer also to the weak form. Of course it is not directly clear how the weak form is mathematically related to the original functional. For the existence of minimizers of the energies (1.1) and (1.2), and the connections between the models, see *e.g.* the brief discussion in reference [5]. Despite the simplification, the numerical minimization of the energy (1.2) is still challenging

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due to the $\mathcal{H}^1(S_u)$ term. Thus it is natural to approximate the functional E in (1.2) (or E in (1.1)) by a family $\{F_\epsilon\}_{\epsilon>0}$ of relatively simple functionals, which are easy to minimize, and prove that in some sense these functionals converge towards the functional E as $\epsilon \to 0+$.

The literature on approximating the Mumford−Shah functional is huge. We only mention some pioneering works. In reference [3], Ambrosio and Tortorelli utilized an auxiliary function v defined in Ω to approximately represent the edge set S_u (or K) such that $v \approx 1 - \chi_{S_u}$, where χ_{S_u} denotes the characteristic function of the set S_u. Then they approximated the $\mathcal{H}^1(S_u)$ term by an energy expression depending only on v. In reference [10], Chambolle proposed an approximation based on finite differences and he proved that this family Γ-converges to an anisotropic Mumford−Shah functional. In reference [7], Braides and Dal Maso proposed a non-local approximation of (1.2) and they proved that their family of functionals Γ-converges to the Mumford−Shah functional as the index controlling the regularization and rescaling goes to zero. In these non-local models, the regularization is based on averaging over small balls. In reference [14], Gobbino proved that a family based on non-local approximation, where the gradient is replaced by finite differences, Γ-converges to the Mumford−Shah functional. In reference [9], Chambolle and Dal Maso approximated the Mumford−Shah functional in the sense of Γ-converge based on adaptive finite elements.

Another classical approach to restore or segment an image is the Perona−Malik diffusion, reference [23]. Formally, if f is the original image defined on $\Omega \subset \mathbb{R}^2$, then the Perona−Malik diffusion forms a nonlinear scale space $\{u_t\}_{t\geq 0}$ such that $u(\cdot, 0) = f(\cdot)$ and for $t > 0$, $\partial_t u = \mathrm{div}(g(|\nabla u|)\nabla u)$, with the suitable boundary conditions. In the model, the diffusivity was chosen either as $g(t) = 1/(1 + (t/\gamma)^2)$ or $g(t) = e^{-(t/\gamma)^2}$ where $\gamma > 0$ is a constant. The diffusion corresponding to the choice $g = 1/(1 + (t/\gamma)^2)$ could formally be seen to be the gradient flow of the Perona−Malik energy

$$u \mapsto \frac{\gamma^2}{2} \int_\Omega \log(1 + |\nabla u/\gamma|^2)\,\mathrm{d}x \tag{1.3}$$

with respect to the metric $L^2(\Omega)$.

The segmentation based on the minimization of the Mumford−Shah functional and the segmentation based on the Perona−Malik diffusion, although very different at first glance, are related. In reference [20], Morini and Negri proved that a suitably rescaled family of biased discrete Perona−Malik energies Γ-converges to an anisotropic Mumford−Shah functional. In reference [17], Kawohl argues that the Ambrosio-Tortorelli approximation [3] of the Mumford−Shah functional naturally leads to the diffusion model of Perona and Malik. In reference [22], Negri considers regularized Perona−Malik functionals in the non-local approximation framework and proves that the Γ-limit of these functionals is the Mumford−Shah functional. See also [13] for a discussion on the Mumford−Shah and Perona−Malik models and [19] for more recent results on Gamma-limits of convolution functionals.

In this paper, similar to [22], we also consider approximating the weak form of the Mumford−Shah functional by Perona−Malik type functionals. In contrast to [22], where the approximating functionals are defined in a Sobolev space, we consider the approximating functionals defined in the space of functions of bounded variation.

2. Preliminaries and statement of the main result

We denote by $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary. Thus it is assumed that $\partial\Omega$ can be covered by a finite number of Lipschitz graphs in some suitable coordinate systems. If $E \subset \mathbb{R}^n$ and $\delta > 0$, we denote by E_δ the open δ-neighborhood of E.

In this paper, $G \in C_c^1(\mathbb{R}^n)$ is a radially symmetric, radially decreasing function such that $\mathrm{supp}(G) = \overline{B(0,1)}$, $G \geq 0$, and $\int_{\mathbb{R}^n} G\,\mathrm{d}x = 1$. If $\sigma > 0$, let $G_\sigma(x) := \frac{1}{\sigma^n} G(x/\sigma)$. We denote

$$\ell(\sigma) := \sigma \log\left(1 + \frac{1}{\sigma}\right) \tag{2.1}$$

where $\sigma > 0$. Clearly, $\ell(\sigma) \to 0$ as $\sigma \to 0+$. If $Q \subset \mathbb{R}^n$, we use $|v|_Q$ to denote the integral average of $|v|$ over Q, *i.e.*

$$|v|_Q = \fint_Q |v| \, \mathrm{d}x.$$

Next we state some facts regarding the functions of bounded variation. For the proofs, see *e.g.* [2]. The space of functions of bounded variation, $\mathrm{BV}(\Omega)$, consists of the functions $u \in L^1(\Omega)$ whose total variation is finite. Here the total variation of u is defined as

$$\sup \left\{ \int_\Omega u \, \mathrm{div}(\varphi) \, \mathrm{d}x | \varphi \in C_c^1(\Omega; \mathbb{R}^n), \ ||\varphi||_{L^\infty(\Omega)} \leq 1 \right\}. \tag{2.2}$$

If $u \in \mathrm{BV}(\Omega)$, then it follows from the Riesz representation theorem that the distributional gradient of u is an $\mathbb{R}^n$-valued Radon measure Du, whose total variation measure $|Du|$ has the property that $|Du|(\Omega)$ equals the total variation of u defined in (2.2). If $u \in \mathrm{BV}(\Omega)$, then the measure Du can be decomposed as

$$Du = \nabla_a u \, \mathrm{d}x + D_s u = \nabla_a u \, \mathrm{d}x + D_j u + D_c u = \nabla_a u \, \mathrm{d}x + (u^+ - u^-)|\nu_u \, \mathcal{H}^{n-1}|_{S_u} + D_c u. \tag{2.3}$$

In the decomposition (2.3), $\nabla_a u \in L^1(\Omega; \mathbb{R}^n)$ is the approximate gradient and the measure $D_s u$ is the singular part of Du. The measure $D_s u$ is mutually singular to $\mathcal{L}^n = \mathrm{d}x$, *i.e.* $D_s u \perp \mathcal{L}^n$. The singular part can further be decomposed as the sum of the jump part $D_j u = (u^+ - u^-)|\nu_u \, \mathcal{H}^{n-1}|_{S_u}$ and the Cantor part $D_c u$, where S_u is the jump set of u and ν_u is a normal vector field to S_u, and $u^+(x)$, $u^-(x)$ denote the traces where $\lim_{\sigma \to 0+} \fint_{B^+(x,\sigma)} |u(y) - u^+(x)| \, \mathrm{d}y = 0$ and $\lim_{\sigma \to 0+} \fint_{B^-(x,\sigma)} |u(y) - u^-(x)| \, \mathrm{d}y = 0$. In the limits, $B^+(x, \sigma) = \{ y \mid ||x - y|| < \sigma, \ (y - x) \cdot \nu_u(x) > 0 \}$ and $B^-(x, \sigma)$ is defined analogously.

The space $\mathrm{SBV}(\Omega)$ consists of those functions $u \in \mathrm{BV}(\Omega)$ for which the Cantor part of Du vanishes, $D_c u = 0$. The space $\mathrm{SBV}^2(\Omega)$ consists of those functions $u \in \mathrm{SBV}(\Omega)$ for which $\mathcal{H}^{n-1}(S_u) < \infty$ and $\nabla_a u \in L^2(\Omega; \mathbb{R}^n)$.

If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the maximal operator is defined by

$$M(u)(x) := \sup_{r > 0} \fint_{B(x,r)} |u(y)| \, \mathrm{d}y$$

where $x \in \mathbb{R}^n$. It is well-known that $M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded. The boundedness is still true if the balls in the definition of M are replaced by cubes.

If $u \in \mathrm{BV}(\Omega)$ and $x \in \Omega$, we denote by $|Du|_\sigma$ a regularization of the measure $|Du|$ where

$$|Du|_\sigma(x) := \int_{\Omega \cap B(x,\sigma)} G_\sigma(x - y) \, d|Du|(y). \tag{2.4}$$

If $A \subset \Omega$, A open, we define

$$F_\sigma(u, A) := \int_A \frac{1}{\ell(\sigma)} \log \left(1 + \ell(\sigma) \, |Du|_\sigma(x)^2 \right) \mathrm{d}x. \tag{2.5}$$

We set $F_\sigma(u) := F_\sigma(u, \Omega)$. If $u \in L^1(\Omega) \setminus \mathrm{BV}(\Omega)$, we set $F_\sigma(u) = +\infty$.

If $u \in H^1(\Omega)$ and $x \in \Omega$, let

$$(|\nabla u|^2)_\sigma(x) := \int_{\Omega \cap B(x,\sigma)} G_\sigma(x - y) |\nabla u(y)|^2 \, \mathrm{d}y.$$

For $u \in H^1(\Omega)$ and if $A \subset \Omega$, A open, we define

$$F_\sigma^N(u, A) := \int_A \frac{1}{\ell(\sigma)} \log \left(1 + \ell(\sigma) \, (|\nabla u|^2)_\sigma(x) \right) \mathrm{d}x \tag{2.6}$$

and we set $F_\sigma^N(u) := F_\sigma^N(u, \Omega)$. If $u \in L^1(\Omega) \setminus H^1(\Omega)$, we set $F_\sigma^N(u) = +\infty$.

For $u \in \mathrm{SBV}^2(\Omega)$ and $A \subset \Omega$, A open, we define the weak Mumford−Shah functional in A by

$$\mathrm{MS}(u, A) := \int_A |\nabla_a u|^2 \,\mathrm{d}x + 2\mathcal{H}^{n-1}(S_u \cap A) \tag{2.7}$$

and we set $\mathrm{MS}(u) := \mathrm{MS}(u, \Omega)$. If $u \in L^1(\Omega) \setminus \mathrm{SBV}^2(\Omega)$, we set $\mathrm{MS}(u) = +\infty$.

Next, the Γ-lower and the Γ-upper limits of the family F_σ are defined. If $u \in L^1(\Omega)$, let

$$F'(u) := \inf \left\{ \liminf_{\sigma\to 0+} F_\sigma(u_\sigma) | u_\sigma \to u \text{ in } L^1(\Omega) \right\}$$

and

$$F''(u) := \inf \left\{ \limsup_{\sigma\to 0+} F_\sigma(u_\sigma) | u_\sigma \to u \text{ in } L^1(\Omega) \right\}.$$

It is known that both F' and F'' are lower semicontinuous in $L^1(\Omega)$. We say that the family F_σ Γ-converges to F in $L^1(\Omega)$ if $F' = F'' = F$ in $L^1(\Omega)$. See *e.g.* [6] for an introduction on these topics.

In reference [15, 16], functionals of the form

$$\int_U \frac{g(a(\sigma)|Du|_\sigma(x)^2)}{a(\sigma)} \,\mathrm{d}x \tag{2.8}$$

were studied as $\sigma \to 0+$. Here $g : [0, \infty) \to [0, \infty)$ is a potential and $a : (0, \infty) \to (0, \infty)$ is a scaling factor. In (2.8), σ determines the amount of regularization and $a(\sigma) \to 0$ as $\sigma \to 0+$. The main result was the point-wise behavior for a fixed u of these functionals as σ tends to zero. This result included *e.g.* the choices $g(t) = \frac{t}{t+1}$ and $g(t) = \log(1 + t)$. The functionals in reference (2.8) corresponding to these choices lead to variants of the Geman−McClure and Hebert−Leahy (or Perona−Malik) models. In reference [16], also the Γ-convergence was studied in the case $g(t) = \frac{t}{t+1}$ and it was shown that the Gamma-limit is the weak form of the Mumford−Shah functional.

In this paper, we also study a family of functionals which are of the form (2.8). Compared to [16], our functionals F_σ defined in (2.5) differ regarding how the functionals are defined when x is near the boundary $\partial\Omega$, namely if $\mathrm{dist}(x, \partial\Omega) < \sigma$. We consider only the choice $g(t) = \log(1 + t)$ since this leads to an interesting formal connection between the Mumford−Shah and Perona−Malik models. The main result of this paper is the Gamma-convergence of the regularized Perona−Malik energies F_σ, defined in (2.5), to the weak form of the Mumford−Shah functional, as the parameter σ, which controls the regularization and the scaling, tends to zero. The Gamma-convergence is heavily based on the results proved in reference [22].

The functionals F_σ are a modification of the functionals F_σ^N in (2.6) proposed by Negri in reference [22]. Since in general the distributional derivative Du of $u \in \mathrm{BV}(\Omega)$ may also have a non-vanishing singular part $D_s u$, it is easier to handle F_σ^N numerically than F_σ, if traditional numerical methods are used. But on the other hand, the Perona−Malik diffusion may preserve edges or even enhance them and thus in this aspect it could be natural to associate also the Perona−Malik functionals with a space which can contain images with edges. Now F_σ^N is finite only in $H^1(\Omega)$ whereas F_σ allows to consider also images which have edges along $(n-1)$-dimensional hypersurfaces, since F_σ is finite if its argument is a BV image.

We know by ([22], Thm. 3.1) that F_σ^N Gamma-converges to MS in $L^1(\Omega)$. Namely, by choosing $p = 2$, $f(t) = \log(1 + t)$ and $a_\sigma = \sigma \log(1/\sigma)$ we see that $f_\sigma(t) = f(a_\sigma t)/a_\sigma$ satisfies the required conditions regarding the rescaling of f in ([22], Lem. 4.2). It is quite straightforward to see that if a_σ is replaced by $\ell(\sigma)$ defined in (2.1), then the required conditions remain true. It then follows from ([22], Thm. 3.1) that F_σ^N Gamma-converges to MS in $L^1(\Omega)$.

The following theorem is the main result of this paper. It follows directly from Corollary 3.4 and Theorem 5.3. The theorem considers the Gamma-convergence of the regularized Perona−Malik functionals F_σ towards the Mumford−Shah functional.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then F_σ Γ-converges to* MS *in $L^1(\Omega)$.*

As in ([16], Thm. 3.8), one could also give a direct proof for the following result:

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $u \in \mathrm{SBV}^2(\Omega)$, then $\mathrm{MS}(u) \leq \liminf_{\sigma\to 0+} F_\sigma(u)$.*

3. Γ-limsup inequality

In this section we prove that the Mumford−Shah functional is an upper bound for the Gamma-upper limit of the family F_σ.

We recall that an $(n-1)$-dimensional simplex in $\mathbb{R}^n$ is the convex hull of n points $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}^n$ (called the vertices of the simplex) which are not contained in any hyperplane of dimension $n-2$. The following density theorem follows from ([11], Cor. 3.11) where it is proved under more general assumptions.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $u \in \mathrm{SBV}^2(\Omega) \cap L^2(\Omega)$ and $\epsilon > 0$. Then there exists $v \in \mathrm{SBV}^2(\Omega) \cap L^2(\Omega)$ such that $\overline{S_v}$ is an intersection of Ω with a finite number of pairwise disjoint $(n-1)$-dimensional simplexes, $\mathcal{H}^{n-1}(\overline{S_v} \setminus S_v) = 0$, $v \in W^{k,\infty}(\Omega \setminus \overline{S_v})$ for every $k \in \mathbb{N}$, $||v-u||_{L^2(\Omega)} < \epsilon$, $||\nabla_a v - \nabla_a u||_{L^2(\Omega;R^n)} < \epsilon$, and $|\mathcal{H}^{n-1}(S_v) - \mathcal{H}^{n-1}(S_u)| < \epsilon$.*

Next a technical lemma which will be used later in this section.

Lemma 3.2. *Let $c \in \mathbb{R}$, $c \geq 0$. Then there exists $\sigma_0(c) > 0$ such that if $0 < \sigma < \sigma_0(c)$ and $t \in \mathbb{R}$, $t \geq 0$, then*

$$\log\left(1 + \ell(\sigma)\left(\frac{c}{\sigma}t\right)^2\right) \Big/ \log\left(1+\frac{1}{\sigma}\right) - \frac{1}{2}t \leq 2.$$

Proof. If $c = 0$, the claim is true so we assume from now on that $c > 0$. We denote the expression on the left of the above inequality by $f(\sigma, t)$ which is defined for $\sigma > 0$ and $t \geq 0$. If σ is fixed and so small that $\ell(\sigma) < 4c^2$, then by studying the function $t \mapsto f(\sigma, t)$, $t \geq 0$, it follows that its global maximum occurs either at $t = 0$ or $t = t_2(\sigma) := \frac{2c+\sqrt{4c^2-\ell(\sigma)}}{c\log(1+1/\sigma)}$.

Now $f(\sigma, 0) = 0$. Since $\ell(\sigma)$, $t_2(\sigma) \to 0$ as $\sigma \to 0+$, then $\ell(\sigma)(ct_2(\sigma))^2 \leq 1$ when σ is small. Using this we see that $f(\sigma, t_2(\sigma)) \leq f(\sigma, t_2(\sigma)) + \frac{1}{2}t_2(\sigma) \leq 2$ when σ is small. Then the claim of the lemma follows. □

The following theorem is similar to ([16], Thm. 1.3) where actually an equality is proved. Compared to ([16], Thm. 1.3) we utilize a certain covering by cubes having mutually disjoint interiors.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $u \in \mathrm{SBV}^2(\Omega) \cap L^2(\Omega)$ be such that $\overline{S_u} \cap \Omega \subseteq \Omega \cap \left(\cup_{i=1}^N S_i\right)$, where the sets S_i are pairwise disjoint $(n-1)$-dimensional simplexes in $\mathbb{R}^n$, and $\mathcal{H}^{n-1}(\overline{S_u} \setminus S_u) = 0$. Then*

$$\limsup_{\sigma\to 0+} F_\sigma(u) \leq \mathrm{MS}(u).$$

Proof. First, we recall from Section 2 that if $E \subset \mathbb{R}^n$, then E_σ denotes the set $\{\, x \in \mathbb{R}^n \mid \mathrm{dist}(x, E) < \sigma \,\}$.

If $x \in \Omega \setminus (\overline{S_u})_\sigma$, then $|Du|_\sigma(x) = \int_{B(x,\sigma)\cap\Omega} G_\sigma(x-y)|\nabla_a u(y)|\,\mathrm{d}y = |\nabla_a u|_\sigma(x)$ where we assume that $|\nabla_a u| = 0$ outside Ω. Using the inequality $\log(1+t) \leq t$, for $t \geq 0$, we get

$$F_\sigma(u, \Omega \setminus (\overline{S_u})_\sigma) \leq \int_{\Omega\setminus(\overline{S_u})_\sigma} |\nabla_a u|_\sigma(x)^2\,\mathrm{d}x.$$

Since $u \in \mathrm{SBV}^2(\Omega)$, we have $|\nabla_a u| \in L^2(\Omega)$. Now $|\nabla_a u|_\sigma(x)^2 \leq (w_n||G||_{L^\infty}M(|\nabla_a u|)(x))^2$ where w_n is the volume of the unit ball in $\mathbb{R}^n$ and M is the maximal operator defined in Section 2. Since M is bounded in

$L^2(\mathbb{R}^n)$, it follows that $(w_n||G||_{L^\infty} M(|\nabla_a u|))^2$ is an integrable majorant of $|\nabla_a u|^2_\sigma$ on Ω and thus we can apply the dominated convergence theorem and we get

$$\limsup_{\sigma\to 0+} F_\sigma(u, \Omega \setminus (\overline{S_u})_\sigma) \leq \int_\Omega |\nabla_a u|^2 \,\mathrm{d}x.$$

Since the sets S_i are pairwise disjoint and if we apply ([2], p. 32 and [2], Thm. 3.16) to a change of coordinates transformation which is a rotation followed by a translation (so the transformation is bi$-$Lipschitz with the Lipschitz constant 1), we see that for the rest of the proof it is sufficient to assume that $S_k \cap \Omega \neq \emptyset$, $S_k \subset \mathbb{R}^{n-1}$ and to show that

$$\limsup_{\sigma\to 0+} F_\sigma(u, \Omega \cap (\overline{S_u} \cap S_k)_\sigma) \leq 2\mathcal{H}^{n-1}(S_u \cap S_k).$$

Let $\epsilon > 0$ and $x \in \Omega \cap (\overline{S_u} \cap S_k)_\sigma$. Since $Du = \nabla_a u \,\mathrm{d}x + D_s u$ and the measures $\nabla_a u \,\mathrm{d}x$ and $D_s u$ are mutually singular, then

$$|Du|_\sigma(x) = \int_{B(x,\sigma)\cap\Omega} G_\sigma(x-y)\, d|D_s u|(y) + \int_{B(x,\sigma)\cap\Omega} G_\sigma(x-y)|\nabla_a u(y)|\,\mathrm{d}y =: |D_s u|_\sigma(x) + |\nabla_a u|_\sigma(x).$$

Using this and the inequality $|a+b|^2 \leq (1+\epsilon)|a|^2 + (1+1/\epsilon)|b|^2$, the inequality $\log(1+x+y) \leq y + \log(1+x)$, for x, $y \geq 0$, and Bernoulli's inequality we see that

$$\begin{aligned} F_\sigma(u, \Omega \cap (\overline{S_u} \cap S_k)_\sigma) \leq & \frac{1+\epsilon}{\epsilon} \int_{\Omega\cap(\overline{S_u}\cap S_k)_\sigma} |\nabla_a u|^2_\sigma \,\mathrm{d}x \\ & + (1+\epsilon) \int_{\Omega\cap(\overline{S_u}\cap S_k)_\sigma} \frac{1}{\ell(\sigma)} \log(1 + \ell(\sigma)|D_s u|^2_\sigma)\,\mathrm{d}x =: (1) + (2). \end{aligned}$$

Since $|\nabla_a u|_\sigma(x) \leq w_n||G||_{L^\infty} M(|\nabla_a u|)(x)$, $u \in \mathrm{SBV}^2(\Omega)$ and $\lim_{\sigma\to 0+} \mathcal{L}^n((\overline{S_u})_\sigma) = 0$, it follows from the absolute continuity of the integral that (1) tends to zero as $\sigma \to 0$. Using this and since ϵ is arbitrary, it is sufficient to show that

$$\limsup_{\sigma\to 0+} \int_{\Omega\cap(\overline{S_u}\cap S_k)_\sigma} \frac{1}{\ell(\sigma)} \log(1+\ell(\sigma)|D_s u|^2_\sigma)\,\mathrm{d}x \leq 2\mathcal{H}^{n-1}(S_u \cap S_k). \tag{3.1}$$

If $x \in \Omega$, then since $\log(1+\ell(\sigma)[c/\sigma^n]^2) \leq \log(1+1/\sigma^{2n}) \leq 2n\log(1+1/\sigma)$ provided σ is small, we see that

$$\frac{1}{\ell(\sigma)} \log(1+\ell(\sigma)|Du|_\sigma(x)^2) \leq \frac{1}{\ell(\sigma)} \log\left(1+\ell(\sigma)\left[\frac{1}{\sigma^n}||G||_{L^\infty}|Du|(\Omega)\right]^2\right) \leq \frac{2n}{\sigma} \tag{3.2}$$

provided $\sigma > 0$ is small enough. If $\sigma_0 > 0$, then $\Omega \cap (\overline{S_u} \cap S_k)_\sigma \subset \Omega \cap (S_k)_\sigma \subset (\Omega \cap [S_k \cap (\partial\Omega)_{\sigma_0}]_\sigma) \cup (\Omega \cap [S_k \setminus (\partial\Omega)_{\sigma_0}]_\sigma)$.

Since $S_k \subset \mathbb{R}^{n-1}$, $\Omega \cap [S_k \cap (\partial\Omega)_{\sigma_0}]_\sigma \subset \Omega \cap [(S_k \cap (\partial\Omega)_{\sigma_0})_{\sigma,n-1} \times (-\sigma,\sigma)]$ where $(\cdot)_{\sigma,n-1}$ denotes the open σ-neighbourhood of a set in $\mathbb{R}^{n-1}$. If $\sigma < \sigma_0$, then

$$\mathcal{L}^n(\Omega \cap [S_k \cap (\partial\Omega)_{\sigma_0}]_\sigma) \leq 2\sigma\mathcal{L}^{n-1}(\Omega \cap (S_k \cap (\partial\Omega)_{\sigma_0})_{\sigma,n-1}) \leq 2\sigma\mathcal{L}^{n-1}(\Omega \cap (S_k \cap (\partial\Omega)_{\sigma_0})_{\sigma_0,n-1})$$

which combined with the inequality (3.2) gives

$$F_\sigma(u, \Omega \cap [S_k \cap (\partial\Omega)_{\sigma_0}]_\sigma) \leq 4n\mathcal{L}^{n-1}(\Omega \cap (S_k \cap (\partial\Omega)_{\sigma_0})_{\sigma_0,n-1}) < \epsilon$$

provided σ_0 is small enough.

By the preceding analysis, to prove (3.1) it is sufficient to show that

$$\limsup_{\sigma\to 0+} \int_{\Omega\cap[S_k\setminus(\partial\Omega)_{\sigma_0}]_\sigma} \frac{1}{\ell(\sigma)} \log(1+\ell(\sigma)|D_s u|_\sigma^2)\,\mathrm{d}x \le 2\mathcal{H}^{n-1}(S_u\cap S_k). \tag{3.3}$$

To prove the inequality (3.3) we utilize a covering of the set $\Omega\cap S_k\setminus(\partial\Omega)_{\sigma_0}\subset\mathbb{R}^{n-1}$ by $(n-1)$-dimensional cubes where the cubes have mutually disjoint interiors and the area where some of the cubes overlap can be made arbitrarily small.

Let us make the preceding discussion precise. Let $\epsilon>0$ again be arbitrary. By assumption, $S_k\subset\mathbb{R}^{n-1}$. We extend the set $S_k\setminus(\partial\Omega)_{\sigma_0}$ slightly in $\mathbb{R}^{n-1}$. Let

$$S_k' := \{x\in\Omega\cap\mathbb{R}^{n-1} \mid d(x, S_k\setminus(\partial\Omega)_{\sigma_0})<\delta_0\}$$

and

$$S_k'' := \left\{x\in\Omega\cap\mathbb{R}^{n-1} \mid d(x, S_k\setminus(\partial\Omega)_{\sigma_0})\le\frac{\delta_0}{2}\right\}$$

where, since $S_k\setminus(\partial\Omega)_{\sigma_0}$ is closed, $\delta_0>0$ can be selected to be so small that $\delta_0<\sigma_0$ and

$$\mathcal{H}^{n-1}(S_k'\setminus[S_k\setminus(\partial\Omega)_{\sigma_0}])<\epsilon. \tag{3.4}$$

Now $\Omega\cap S_k\setminus(\partial\Omega)_{\sigma_0}\subset S_k''\subset S_k'$. We can also assume that δ_0 is so small that S_k' is at positive distance from other sets S_l, $l\ne k$, since it is assumed that the simplexes S_j are pairwise disjoint.

Next we write S_k' as $S_k'=\cup_{j=1}^\infty Q_j$, where the cubes Q_j are closed in $\mathbb{R}^{n-1}$ and have mutually disjoint interiors. For instance, the Whitney decomposition can be used to accomplish this. If $\ell(Q_j)$ denotes the side length of Q_j, let $Q_{j,a}\subset\mathbb{R}^{n-1}$ denote the open cube co-centric with Q_j and whose side length is $\ell(Q_j)+a$.

We select $\epsilon_j>0$ for each j such that $Q_{j,2\epsilon_j}$ is at positive distance from other sets S_l, $l\ne k$, and

$$\sum_j \mathcal{H}^{n-1}(Q_{j,\epsilon_j}\setminus Q_j)<\epsilon. \tag{3.5}$$

Then $\mathcal{H}^{n-1}(\cup_j Q_{j,\epsilon_j}\setminus(\cup_j Q_j))<\epsilon$. Now S_k'' can be covered by the open cubes Q_{j,ϵ_j} and since S_k'' is compact in $\mathbb{R}^{n-1}$, then S_k'' can be covered by a finite number of cubes Q_{j,ϵ_j}, $j=1,\ldots,M$. Provided $\sigma>0$ is small, we also have

$$\Omega\cap[S_k\setminus(\partial\Omega)_{\sigma_0}]_\sigma\subset S_k''\times(-\sigma,\sigma) \tag{3.6}$$

where S_k'' is viewed as a subset of $\mathbb{R}^{n-1}$.

By ([2], Thm. 3.77), $D_s u\big|_{\Omega\cap\mathbb{R}^{n-1}}=(u^+-u^-)\overline{e_n}\mathcal{H}^{n-1}\big|_{\Omega\cap\mathbb{R}^{n-1}}$ where $\overline{e_n}$ is the nth Cartesian unit coordinate vector. If $x=(x',x^n)\in Q_{i,\epsilon_i}\times(-\sigma,\sigma)$, then due to the choice of ϵ_i, $Q^n(x,2\sigma)$, the n-dimensional cube of the side length 2σ centered at x, is at positive distance from other sets S_l, $l\ne k$, provided σ is small. Then

$$|D_s u|_\sigma(x',x^n)\le\frac{c}{\sigma}|u^+-u^-|_{Q^{n-1}(x',2\sigma)}$$

where $c=2^{n-1}||G||_{L^\infty}$ and $|u^+-u^-|_{Q^{n-1}(x',2\sigma)}$ is the integral average of $|u^+-u^-|$ in $Q^{n-1}(x',2\sigma)\subset\mathbb{R}^{n-1}$. Using (3.6) and since S_k'' is covered by the cubes Q_{i,ϵ_i}, $i=1,\ldots,M$, then

$$\int_{\Omega\cap[S_k\setminus(\partial\Omega)_{\sigma_0}]_\sigma} \frac{1}{\ell(\sigma)}\log(1+\ell(\sigma)|D_s u|_\sigma^2)\,\mathrm{d}x \le \sum_{i=1}^M \int_{Q_{i,\epsilon_i}} \frac{2}{\log(1+\frac{1}{\sigma})}\log\left(1+\ell(\sigma)\left[\frac{c}{\sigma}|u^+-u^-|_{Q^{n-1}(x',2\sigma)}\right]^2\right)\mathrm{d}x'. \tag{3.7}$$

Since $u\in \mathrm{SBV}^2(\Omega)\cap L^2(\Omega)$, $S_k\subset\mathbb{R}^{n-1}$ and $Q_{i,2\epsilon_i}$ is at positive distance from other sets S_l, $l\ne k$, it follows that provided $a>0$ is small enough, then $g:=u\big|_{Q_{i,2\epsilon_i}\times(0,a)}\in H^1(Q_{i,2\epsilon_i}\times(0,a))$ and $h:=u\big|_{Q_{i,2\epsilon_i}\times(-a,0)}\in$

$H^1(Q_{i,2\epsilon_i}\times(-a,0))$. It follows from the Sobolev trace theorem that $\mathrm{tr}(g)\in H^{1/2}(\partial(Q_{i,2\epsilon_i}\times(0,a)))$ and $\mathrm{tr}(h)\in H^{1/2}(\partial(Q_{i,2\epsilon_i}\times(-a,0)))$. Actually, $\mathrm{tr}(g)=u^+$ and $\mathrm{tr}(h)=u^-$ in $Q_{i,2\epsilon_i}$ (see *e.g.* [2], Thm. 3.77 and [12], Rem., Sect. 4.3). Let $v:=\mathrm{tr}(g)-\mathrm{tr}(h)$ in $Q_{i,2\epsilon_i}$ and $v\equiv 0$ otherwise in $L^2(\mathbb{R}^{n-1})$. Then $v\in L^2(\mathbb{R}^{n-1})$ and $v\big|_{Q_{i,2\epsilon_i}}\in H^{1/2}(Q_{i,2\epsilon_i})$.

It follows from Lemma 3.2 that the integrand in (3.7) has a majorant $|v|_{Q^{n-1}(x',2\sigma)}+4$ in Q_{i,ϵ_i} provided $\sigma>0$ is small. Since $|v|_{Q^{n-1}(x',2\sigma)}\le M(|v|)(x')$ and since the maximal operator is bounded in $L^2(\mathbb{R}^{n-1})$, we see that we can apply the dominated convergence theorem in (3.7) as $\sigma\to 0+$.

Next we examine how the integrand in (3.7) behaves for a fixed $x'\in Q_{i,\epsilon_i}$ as $\sigma\to 0+$. If $x'\in Q_{i,\epsilon_i}\cap S_u$, then $|v|_{Q^{n-1}(x',2\sigma)}\to|v(x')|\ne 0$ $\mathcal{H}^{n-1}$ a.e. Thus for $\mathcal{H}^{n-1}$ a.e. x' there exists $\gamma>0$ such that $0<\gamma\le|v|_{Q^{n-1}(x',2\sigma)}\le|v(x')|+1$ provided $\sigma>0$ is small. If $C>0$ is a constant, we have

$$\lim_{\sigma\to 0}\frac{2}{\log(1+\frac{1}{\sigma})}\log\left(1+\ell(\sigma)\frac{c^2}{\sigma^2}C\right)=2$$

and thus for $\mathcal{H}^{n-1}$ a.e. $x'\in Q_{i,\epsilon_i}\cap S_u$, the integrand in (3.7) tends to 2 as $\sigma\to 0$.

Next we examine the case $x'\in Q_{i,\epsilon_i}\setminus S_u$. If $x'\in Q_{i,\epsilon_i}\setminus S_u$, then $v(x')=0$ and since $v\in H^{\frac12}(Q_{i,2\epsilon_i})$, a differentiation theorem in ([16], Lem. 4.1) implies that $\lim_{\sigma\to 0}\frac{1}{\sigma^{\frac12}}|v|_{Q^{n-1}(x',2\sigma)}=0$. Thus $\frac{1}{\sigma^{\frac12}}|v|_{Q^{n-1}(x',2\sigma)}<1$ provided $\sigma>0$ is small enough. We get for the integrand in (3.7) that

$$\lim_{\sigma\to 0}\frac{\log\left(1+\ell(\sigma)\frac{c^2}{\sigma}\left(\frac{|v|_{Q^{n-1}(x',2\sigma)}}{\sqrt{\sigma}}\right)^2\right)}{\log(1+\frac{1}{\sigma})}\le\lim_{\sigma\to 0}\frac{\log(1+c^2\log(1+\frac{1}{\sigma}))}{\log(1+\frac{1}{\sigma})}=0.$$

Applying the dominated convergence theorem in (3.7), we then see that

$$\limsup_{\sigma\to 0+}\int_{Q_{i,\epsilon_i}}\frac{2}{\log\left(1+\frac{1}{\sigma}\right)}\log\left(1+\ell(\sigma)\Big(\frac{c}{\sigma}|v|_{Q^{n-1}(x',2\sigma)}\Big)^2\right)\mathrm{d}x'=2\mathcal{H}^{n-1}(S_u\cap Q_{i,\epsilon_i}).$$

Since the cubes Q_i have mutually disjoint interiors, since $\sum_i\mathcal{H}^{n-1}(Q_{i,\epsilon_i}\setminus Q_i)<\epsilon$ by (3.5) and since $\mathcal{H}^{n-1}(S_k'\setminus[S_k\setminus(\partial\Omega)_{\sigma_0}])<\epsilon$ by (3.4), we see that

$$\begin{aligned}\sum_{i=1}^{M}\mathcal{H}^{n-1}(S_u\cap Q_{i,\epsilon_i})&\le\sum_{i=1}^{M}\mathcal{H}^{n-1}(S_u\cap Q_i)+\mathcal{H}^{n-1}(Q_{i,\epsilon_i}\setminus Q_i)\le\mathcal{H}^{n-1}(S_u\cap(\cup_{i=1}^{M}Q_i))+\epsilon\\&\le\mathcal{H}^{n-1}(S_u\cap S_k')+\epsilon=\mathcal{H}^{n-1}(S_u\cap S_k\setminus(\partial\Omega)_{\sigma_0})+\mathcal{H}^{n-1}(S_u\cap S_k'\setminus[S_k\setminus(\partial\Omega)_{\sigma_0}])\\&\quad+\epsilon\le\mathcal{H}^{n-1}(S_u\cap S_k)+2\epsilon,\end{aligned}$$

and since $\epsilon>0$ is arbitrary, we see that (3.3) is true. □

Corollary 3.4. $F''(u)\le\mathrm{MS}(u)$ *for all* $u\in L^1(\Omega)$.

Proof. The proof is based on a density argument ([6], Rem. 1.29). If $u\in L^1(\Omega)\setminus\mathrm{SBV}^2(\Omega)$, the claim is trivially true. If $u\in\mathrm{SBV}^2(\Omega)$, let u_k denote the truncation of u with the levels $-k$ and k. Then $u_k\in\mathrm{SBV}^2(\Omega)\cap L^\infty(\Omega)$, $u_k\to u$ in $L^1(\Omega)$ and $\lim_{k\to+\infty}\mathrm{MS}(u_k)=\mathrm{MS}(u)$, see *e.g.* the proof of ([8], Thm. A.1).

By Theorem 3.1 there exists $v_k\in\mathrm{SBV}^2(\Omega)\cap L^2(\Omega)$ such that $\mathrm{MS}(v_k)\le\mathrm{MS}(u_k)+\frac{1}{k}$ and $v_k\to u$ in $L^1(\Omega)$. Applying Theorem 3.3 to v_k we get $\limsup_{\sigma\to 0+}F_\sigma(v_k)\le\mathrm{MS}(v_k)$ and thus $F''(v_k)\le\mathrm{MS}(v_k)$. Using the lower semicontinuity of F'' and the analysis we just made we see that $F''(u)\le\liminf_{k\to+\infty}F''(v_k)\le\mathrm{MS}(u)$. □

4. BV extension theorem

In this section we consider an extension of a BV function where the extension does not cause a jump along the boundary. The extension will be used as a technical tool later in this paper. The extension is different from the one in ([2], Prop. 3.21).

The following reflection result follows *e.g.* from ([1], Rem. 8.2 or [18], Appendix A).

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exist an open neighborhood W of $\partial\Omega$ and a bi–Lipschitz map $\phi : W \to W$ such that if $W^+ := W \cap \Omega$ and $W^- := W \setminus \overline{\Omega}$, then $\phi(W^\pm) = W^\mp$ and $\phi(x) = x$ for $x \in \partial\Omega$.*

Next the actual result. Similar extension was considered in the proof of ([8], Thm. 1.2).

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $u \in \mathrm{BV}(\Omega)$. Let W, $W^\pm$ and ϕ be as in Theorem 4.1. We set*

$$u_{\mathrm{ext}}(x) = \begin{cases} u(x), & x \in \Omega \\ u(\phi(x)), & x \in W^-. \end{cases}$$

On the boundary $\partial\Omega$, u_{ext} can be set to be e.g. a constant. Then $u_{\mathrm{ext}} \in \mathrm{BV}(\overline{\Omega} \cup W^-)$ and $|Du_{\mathrm{ext}}|(\partial\Omega) = 0$.

Proof. Applying ([2], Thm. 3.16) and since $\partial\Omega$ is a Lipschitz boundary we see that $u_{\mathrm{ext}} \in \mathrm{BV}(\overline{\Omega} \cup W^-)$. We can suppose that $\partial\Omega \subset \cup_{i=1}^N R_i$ where each R_i is an open rectangle and $R_i \subset \overline{\Omega} \cup W^-$. So, $R_i = L_i \times (-t, t)$ in some local coordinate system where L_i is an $(n-1)$-dimensional rectangle and $t > 0$. Let $R_i' := L_i' \times (-t, t)$ where L_i' is a rectangle such that $L_i' \subset\subset L_i$. We can suppose that also the rectangles R_i' cover $\partial\Omega$. Next we select i arbitrary and set $R := L \times (-t, t)$ and $R' := L' \times (-t, t)$. To prove the theorem, it is sufficient to show that $|Du_{\mathrm{ext}}|(R' \cap \partial\Omega) = 0$.

We can assume that in some local coordinate system, $R \cap \Omega = \{x \in R \mid x_n > f(x_1, \ldots, x_{n-1}),\ (x_1, \ldots, x_{n-1}) \in L\}$ where f is a Lipschitz function. We can assume that in L, $\inf f > -t$ and $\sup f < t$. Let $S : R \to R$ be the vertical deformation from the proof of ([2], Prop. 3.21) which straightens out $\partial\Omega \cap R$. The function S is bi–Lipschitz.

Next we consider how smooth functions behave under certain extension. These smooth functions are then used to approximate a general BV function. Let $\beta > 0$, $c_1 := 1 + \mathrm{Lip}(S \circ \phi \circ S^{-1})$, $a \in C^\infty(\overline{L \times (0, c_1\beta)})$ and define a_{ext} in $L' \times (-\beta, \beta)$ by

$$a_{\mathrm{ext}}(x) = \begin{cases} a(x), & x \in L' \times (0, \beta) \\ a \circ S \circ \phi \circ S^{-1}(x), & x \in L' \times (-\beta, 0). \end{cases} \tag{4.1}$$

When β is small, then $S \circ \phi \circ S^{-1}(L' \times (-\beta, 0)) \subset L \times (0, c_1\beta)$ and a_{ext} is then well defined. We prove that $|Da_{\mathrm{ext}}|(L' \times (-\beta, \beta)) \leq (1 + c_2)|Da|(L \times (0, c_1\beta))$ where c_2 depends only on S, ϕ and n. Namely, let $\varphi \in C_c^1(L' \times (-\beta, \beta))$, $||\varphi||_{L^\infty} \leq 1$. By the integration by parts, since the outward normals for the boundaries of $L' \times (-\beta, 0)$ and $L' \times (0, \beta)$ are opposite on $L' \times \{0\}$, since $\phi\big|_{\partial\Omega} = id$, and since by the chain rule, $J_x(a \circ S \circ \phi \circ S^{-1}) = J_{S\circ\phi\circ S^{-1}(x)} a \cdot J_x(S \circ \phi \circ S^{-1})$ for $\mathcal{L}^n$ a.e. x where $J_a(g)$ denotes the Jacobian matrix of a function g at a, we see after some calculations that

$$\begin{aligned} \int_{L' \times (-\beta, \beta)} a_{\mathrm{ext}}(x) \frac{\partial \varphi}{\partial x_i}(x)\,\mathrm{d}x &= -\int_{L' \times (0, \beta)} \frac{\partial}{\partial x_i} a(x)\varphi(x)\,\mathrm{d}x \\ &\quad - \int_{L' \times (-\beta, 0)} \sum_{j=1}^n \frac{\partial a}{\partial_j}(S \circ \phi \circ S^{-1}(x)) \cdot \frac{\partial (S \circ \phi \circ S^{-1})_j}{\partial x_i}(x) \cdot \varphi(x)\,\mathrm{d}x = (1) + (2). \end{aligned}$$

Using the substitution $y := S \circ \phi \circ S^{-1}(x)$ for (2), since $||\varphi||_{L^\infty} \leq 1$, and since $||J_x(S \circ \phi \circ S^{-1})|| \leq \mathrm{Lip}(S \circ \phi \circ S^{-1})$, we see that

$$|Da_{\mathrm{ext}}|(L' \times (-\beta, \beta)) \leq |Da|(L' \times (0, \beta)) + C(S, \phi)|Da|(S \circ \phi \circ S^{-1}(L' \times (-\beta, 0))) \leq (1 + c_2)|Da|(L \times (0, c_1\beta)).$$

We return to the general case. Let $c_3 := \mathrm{Lip}(S)$. Now $u \circ S^{-1} \in \mathrm{BV}(L \times (0,a))$. Let $r > 0$ be so small that $c_1 c_3 r < t$. *E.g.* by ([4], Thm. 10.1.2) it is possible to find a sequence $\{a_k\} \subset C^\infty(\overline{L \times (0, c_1 c_3 r)})$ such that $a_k \to u \circ S^{-1}$ in $L^1(L \times (0, c_1 c_3 r))$ and $|Da_k|(L \times (0, c_1 c_3 r)) \to |D[u \circ S^{-1}]|(L \times (0, c_1 c_3 r))$.

Consider the extension (4.1) for each a_k where $\beta = c_3 r$. This implies that $(a_k)_{\mathrm{ext}} \to u_{\mathrm{ext}} \circ S^{-1}$ in $L^1(L' \times (-c_3 r, c_3 r))$ provided $r > 0$ is small enough. Let $c_4 := [\mathrm{Lip}(S^{-1})]^{n-1}$ and $c_5 := [\mathrm{Lip}(S)]^{n-1}$. Using the preceding analysis we see that

$$\begin{aligned}|Du_{\mathrm{ext}}|(R' \cap (\partial\Omega)_r) &\le c_4 |D[u_{\mathrm{ext}} \circ S^{-1}]|(L' \times (-c_3 r, c_3 r)) \le \liminf_{k\to\infty} c_4 |D(a_k)_{\mathrm{ext}}|(L' \times (-c_3 r, c_3 r)) \\ &\le \liminf_{k\to\infty} c_4(1+c_2)|Da_k|(L \times (0, c_1 c_3 r)) = c_4(1+c_2)|D[u \circ S^{-1}]|(L \times (0, c_1 c_3 r)) \\ &\le c_4(1+c_2)c_5 |Du|(S^{-1}(L \times (0, c_1 c_3 r))) \le c_4(1+c_2)c_5|Du|(\Omega \cap (\partial\Omega)_{c_6 r})\end{aligned}$$

where also c_6 depends only on S, ϕ and n. Then $|Du_{\mathrm{ext}}|(R' \cap \partial\Omega) = 0$ follows by letting r tend to zero. □

The next theorem shows that $F_\sigma(u_{\mathrm{ext}}, \Omega)$ can be majorized by $F_\sigma(u)$ up to a multiplicative constant which depends neither on u nor σ.

Theorem 4.3. *Let Ω be a bounded Lipschitz domain. Let $u \in \mathrm{BV}(\Omega)$. Let u_{ext} be as in Theorem 4.2. If $\sigma > 0$ is small enough, then*

$$F_\sigma(u_{\mathrm{ext}}, \Omega) \le c\, F_\sigma(u)$$

where the constant $c \ge 0$ depends neither on u nor σ.

Proof. By Theorem 4.2, $|Du_{\mathrm{ext}}|(\partial\Omega) = 0$. Let $x \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) < \sigma$. Using ([2], Thm. 3.16 and [2], p. 32) we see that

$$\begin{aligned}|Du_{\mathrm{ext}}|_\sigma(x) &= \int_{B(x,\sigma)\cap\Omega} G_\sigma(x-y)\, d|Du|(y) + \int_{B(x,\sigma)\setminus\overline{\Omega}} G_\sigma(x-y)\, d|Du_{\mathrm{ext}}|(y) \\ &\le |Du|_\sigma(x) + [\mathrm{Lip}(\phi^{-1})]^{n-1} \int_{\phi(B(x,\sigma)\setminus\overline{\Omega})} G_\sigma(x - \phi^{-1}(y))\, d|Du|(y).\end{aligned} \tag{4.2}$$

If σ is small, then ϕ maps $B(x,\sigma) \setminus \overline{\Omega}$ into Ω. We will show that there exists $R > 0$ which depends only on Ω and ϕ such that $\phi(B(x,\sigma) \setminus \overline{\Omega})$ is contained in the ball $B(x, R\sigma)$. Then we will cover a ball co-centric with $B(0,R)$ and whose radius is larger than R by small cubes whose centers a_j lie on a regular grid and where 0 is one of the points a_j. Then we will show that $|Du_{\mathrm{ext}}|_\sigma(x) \le c \sum |Du|_\sigma(x + \sigma a_j)$ where $c > 0$ depends neither on σ nor x and where the summation is over those j for which $x + \sigma a_j \in \Omega$.

Let us make the preceding discussion precise. Since $\mathrm{dist}(x, \partial\Omega) < \sigma$ and $\phi\big|_{\partial\Omega} = id$, we see after some calculations that $\phi(B(x,\sigma) \setminus \overline{\Omega}) \subseteq B(x, [2\,\mathrm{Lip}(\phi) + 1]\sigma)$. Set $R := 2\,\mathrm{Lip}(\phi) + 1$. Then

$$|Du_{\mathrm{ext}}|_\sigma(x) \le |Du|_\sigma(x) + \beta \int_{B(x,R\sigma)\cap\Omega\cap\phi(B(x,\sigma)\setminus\overline{\Omega})} G_\sigma(x - \phi^{-1}(y))\, d|Du|(y)$$

where $\beta = [\mathrm{Lip}(\phi^{-1})]^{n-1}$. Since G is radially symmetric, there exists a function h such that $G(x) = h(|x|)$. Let $c > 0$ whose exact value will be fixed later. Select $T_{1/2} > 0$ such that $h(T_{1/2}) = \frac{1}{2}||G||_{L^\infty}$. Since $\partial\Omega$ is a Lipschitz boundary, it follows that if $z \in \Omega$ is near $\partial\Omega$, then there exists $\theta(z) \in \mathbb{R}^n$, $|\theta(z)| = 1$, such that if $z^\sigma := z + \frac{\sigma}{c}\alpha\theta(z)$ where $\alpha > 0$ is a constant depending only on $\partial\Omega$, then $B(z^\sigma, \frac{\sigma}{c}) \subset \Omega$ (this is argued locally *e.g.* in the proof of ([12], Thm. 3).

We fix c such that $c > \frac{1+\alpha}{T_{1/2}}$. Then $B(z^\sigma, \frac{\sigma}{c}) \subset B(z, \sigma T_{1/2})$. We cover $B(0, R + \frac{1}{c} + \frac{\alpha}{c})$ by small cubes. Let $Q(a_j, a)$ denote the cube centered at a_j and of side length a. We assume that a is so small that $a < \frac{1}{3\sqrt{n}c}$ and $G(a, \ldots, a) \ge \frac{1}{2}||G||_{L^\infty}$. We choose a regular grid $\{a_1, \ldots, a_N\}$ such that $B(0, R + \frac{1}{c} + \frac{\alpha}{c}) \subset \cup_{j=1}^N Q(a_j, a)$. We can assume that 0 is one of the points a_j.

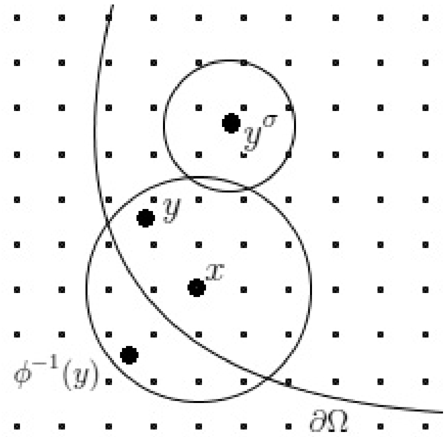

FIGURE 1. In the figure, the dots denote the grid points $x+\sigma a_k$. The large ball is $B(x,R\sigma)$ and the small ball is $B(y^\sigma,\dfrac{\sigma}{c})$. We prove that if x is near $\partial\Omega$, then for all $y\in B(x,R\sigma)\cap\Omega\cap\phi(B(x,\sigma)\setminus\overline{\Omega})$ there exists $x+\sigma a_j$ for some j such that y and $x+\sigma a_j\in\Omega$ are so near that $G_\sigma(x+\sigma a_j-y)\geq\dfrac{1}{2}||G||_{L^\infty}\dfrac{1}{\sigma^n}$. To prove this, we will show that the ball $B(y^\sigma,\dfrac{\sigma}{c})$ contains such a point.

For each $x\in\Omega$ and $\sigma>0$, let $N(x,\sigma):=\{a_j\mid j\in\{1,\ldots,N\},\ x+\sigma a_j\in\Omega\}$. Since 0 belongs to the grid, $N(x,\sigma)\neq\emptyset$.

Let $x\in\Omega$, $\operatorname{dist}(x,\partial\Omega)<\sigma$, and $y\in B(x,R\sigma)\cap\Omega\cap\phi(B(x,\sigma)\setminus\overline{\Omega})$. Then $B(y^\sigma,\frac{\sigma}{c})\subset B(x,(R+\frac{1}{c}+\frac{\alpha}{c})\sigma)\cap\Omega$ where $y^\sigma=y+\frac{\sigma}{c}\alpha\theta(y)$. Since $a<\frac{1}{3\sqrt{n}c}$, it is possible to place a cube whose side length is $3a\sigma$ inside the ball $B(y^\sigma,\frac{\sigma}{c})$. Since also $B(x,(R+\frac{1}{c}+\frac{\alpha}{c})\sigma)\subseteq x+\cup_{j=1}^N Q(a_j\sigma,a\sigma)$, we see that $B(y^\sigma,\frac{\sigma}{c})$ contains the cube $x+Q(a_j\sigma,a\sigma)$ for some j. Thus there exists $a_j\in N(x,\sigma)$ such that $x+\sigma a_j\in B(y^\sigma,\frac{\sigma}{c})$. See also Figure 1.

Since $B(y^\sigma,\frac{\sigma}{c})\subset B(y,\sigma T_{1/2})$, then $G_\sigma(x+\sigma a_j-y)\geq\frac{1}{2}||G||_{L^\infty}\frac{1}{\sigma^n}\geq\frac{1}{2}G_\sigma(x-\phi^{-1}(y))$. Using this and since $\operatorname{supp}(G_\sigma)=\overline{B(0,\sigma)}$, we get

$$\int_{B(x,R\sigma)\cap\Omega\cap\phi(B(x,\sigma)\setminus\overline{\Omega})}G_\sigma(x-\phi^{-1}(y))\,d|Du|(y)\leq 2\sum_{a_j\in N(x,\sigma)}|Du|_\sigma(x+\sigma a_j)$$

and since 0 belongs to the grid, we get recalling (4.2) that

$$|Du_{\text{ext}}|_\sigma(x)\leq 2[1+2[\operatorname{Lip}(\phi^{-1})]^{n-1}]\sum_{a_j\in N(x,\sigma)}|Du|_\sigma(x+\sigma a_j).$$

Let $c(n,\phi):=2[1+2[\operatorname{Lip}(\phi^{-1})]^{n-1}]$. We get using the above inequality and Bernoulli's inequality that

$$\begin{aligned}F_\sigma(u_{\text{ext}},\Omega)&\leq(c(n,\phi)N)^2\int_\Omega\frac{1}{\ell(\sigma)}\log\left(1+\ell(\sigma)\max_{a_j\in N(x,\sigma)}|Du|_\sigma(x+\sigma a_j)^2\right)\mathrm{d}x\\&\leq c(n,\phi)^2N^2(N+1)\int_\Omega\frac{1}{\ell(\sigma)}\log\left(1+\ell(\sigma)|Du|_\sigma(x)^2\right)\mathrm{d}x\end{aligned}$$

and so the claim of the theorem is true with the choice $c=c(n,\phi)^2N^2(N+1)$ which depends neither on u nor σ. □

5. Γ-LIMINF INEQUALITY

The following lemma is kind of a change-of-scale inequality.

Lemma 5.1. *Let $0 < \alpha,\ \beta < 1$. There exists $A_{\alpha,\beta} > 0$ such that if $\sigma > 0$, $x \in \mathbb{R}^n$ and $y \in \overline{B}(x, \alpha(1-\beta)\sigma)$, then*

$$G_{(1-\alpha)(1-\beta)\sigma}(z-y) \le A_{\alpha,\beta} G_\sigma(x-z) \tag{5.1}$$

for all $z \in B(y,(1-\alpha)(1-\beta)\sigma)$. Furthermore, if $d_{\alpha,\beta} := \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta}$, and if $d_{\alpha,\beta} \to 0$ as $\alpha,\ \beta \to 0$, then $A_{\alpha,\beta} \to 1$.

Proof. It is sufficient to prove (5.1) in the case x, y and z are collinear. Namely, let us assume that (5.1) is true in this case. Then, if x, y and z satisfy the assumptions of the lemma but are not necessarily collinear, let $y' := x + |x-y|\frac{z-x}{|z-x|}$. Then x, y' and z are collinear, $y' \in \overline{B}(x, \alpha(1-\beta)\sigma)$ and $z \in B(y',(1-\alpha)(1-\beta)\sigma)$. It follows that $G_{(1-\alpha)(1-\beta)\sigma}(z-y') \le A_{\alpha,\beta}G_\sigma(x-z)$. Combining this with $|y'-z| \le |y-z|$ we see that $G_{(1-\alpha)(1-\beta)\sigma}(z-y) \le A_{\alpha,\beta}G_\sigma(x-z)$.

We can also assume that $|x-z| \ge |z-y|$. Namely, let us assume that (5.1) is true in this case. Then, if x, y and z satisfy the assumptions of the lemma but $|x-z| < |z-y|$, let $z' := y-(-y+z)$. Then $z' \in B(y,(1-\alpha)(1-\beta)\sigma)$. We also have $|z'-x| \ge |y-z+y-z+z-x| + |z-x| - |z-y| \ge |y-z| = |z'-y|$. Using the above analysis we see that $G_{(1-\alpha)(1-\beta)\sigma}(z-y) = G_{(1-\alpha)(1-\beta)\sigma}(z'-y) \le A_{\alpha,\beta}G_\sigma(z'-x) \le A_{\alpha,\beta}G_\sigma(z-x)$.

We can also assume that $|x-z| \ge |x-y|$. Namely, let us assume that (5.1) is true in this case. Then, if x, y and z satisfy the assumptions of the lemma but $|x-z| < |x-y|$, let $z' := 2y-z$. Then $z' \in B(y,(1-\alpha)(1-\beta)\sigma)$. We get $|x-z'| \ge |x-z'| + |x-z| - |x-y| = |2x-2y+z-x| + |x-z| - |x-y| \ge |x-y|$. We get $G_{(1-\alpha)(1-\beta)\sigma}(z-y) = G_{(1-\alpha)(1-\beta)\sigma}(z'-y) \le A_{\alpha,\beta}G_\sigma(x-z') \le A_{\alpha,\beta}G_\sigma(x-z)$.

The assumptions that x, y and z are collinear such that $|x-z| \ge |z-y|$ and $|x-z| \ge |x-y|$ imply that $|z-y| + |-x+y| = |z-x|$. Considering when there is an equality in the triangle inequality we see that $z-y = r(-x+y)$ for some $r \ge 0$.

In addition to the preceding paragraph, we can also assume that $|x-y| = \alpha(1-\beta)\sigma$. Namely, let us assume that (5.1) is true in this case. Then, suppose that $|x-y| < \alpha(1-\beta)\sigma$. If $y \ne x$, let $y' := x + \alpha(1-\beta)\sigma\frac{y-x}{|y-x|}$ and $z' := y'+z-y$. Then $z' \in B(y',(1-\alpha)(1-\beta)\sigma)$. Since $z-y = r(-x+y)$ for some $r \ge 0$, we see that $|z'-x| = |z-y| + \alpha(1-\beta)\sigma = |z'-y'| + |y'-x|$ and still using the previous paragraph, $|x-z| = |y-x| + |y-z| \le |x-y'| + |y'-z'| = |x-z'|$. We get $G_{(1-\alpha)(1-\beta)\sigma}(z-y) = G_{(1-\alpha)(1-\beta)\sigma}(z'-y') \le A_{\alpha,\beta}G_\sigma(x-z') \le A_{\alpha,\beta}G_\sigma(x-z)$ when $y \ne x$. If $y = x$, let $y' := x + \alpha(1-\beta)\sigma\frac{z-x}{|z-x|}$ and $z' := y'+z-y$. Then $|z'-x| = [\alpha(1-\beta)\sigma\frac{1}{|z-x|}+1]|z-x| \ge |z-x|$. Thus $G_{(1-\alpha)(1-\beta)\sigma}(z-y) = G_{(1-\alpha)(1-\beta)\sigma}(z'-y') \le A_{\alpha,\beta}G_\sigma(x-z') \le A_{\alpha,\beta}G_\sigma(x-z)$.

Next we show that (5.1) is true when x, y and z satisfy the assumptions made in the preceding two paragraphs. So, $|x-z| = |x-y| + |y-z|$ and $|x-y| = \alpha(1-\beta)\sigma$. Let $h(|x|) := G(x)$ where $x \in \mathbb{R}^n$. By using the substitution $d := |z-y|/((1-\alpha)(1-\beta)\sigma)$ we see that to show (5.1) to be true it is sufficient to prove that

$$A_{\alpha,\beta}(d) := \frac{h(d)}{(1-\alpha)^n(1-\beta)^n h(\alpha(1-\beta)+(1-\alpha)(1-\beta)d)}, \tag{5.2}$$

where $d \in [0,1]$, is bounded. First, $A_{\alpha,\beta}(0) \ge 1$. Now $d = \alpha(1-\beta)+(1-\alpha)(1-\beta)d$ if and only if $d = \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta}$. Set $d_{\alpha,\beta} := \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta}$. Clearly $d_{\alpha,\beta} \in]0,1[$. If $d \in [d_{\alpha,\beta},1]$, then $d \ge (1-\beta)[\alpha+(1-\alpha)d]$ which implies that $A_{\alpha,\beta}(d) \le 1/((1-\alpha)^n(1-\beta)^n)$. It follows from the continuity of h that there exists $y_{\alpha,\beta} \in [0,d_{\alpha,\beta}]$ such that $1 \le A_{\alpha,\beta}(y_{\alpha,\beta}) < \infty$ and

$$y_{\alpha,\beta} = \operatorname*{arg\,max}_{d\in[0,d_{\alpha,\beta}]} A_{\alpha,\beta}(d) = \operatorname*{arg\,max}_{d\in[0,1]} A_{\alpha,\beta}(d).$$

Set $A_{\alpha,\beta} := A_{\alpha,\beta}(y_{\alpha,\beta})$. We have shown that (5.1) is true and clearly $A_{\alpha,\beta} \to 1$ if $d_{\alpha,\beta} \to 0$ as $\alpha,\ \beta \to 0$. $\square$

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $u_i \to u$ in $L^1(\Omega)$, $\sigma_i > 0$, $\sigma_i \to 0$ as $i \to +\infty$, and $\liminf_{i\to\infty} F_{\sigma_i}(u_i) < \infty$, then $u \in \mathrm{SBV}^2(\Omega)$.*

Proof. Let $u_j \to u$ in $L^1(\Omega)$. If necessary, by passing to a subsequence and renaming it we can assume that $u_j \in \mathrm{BV}(\Omega)$ for all j. Consider the extension of Theorem 4.2 where ϕ is the reflection map. We denote the extension of u_j by v_j. Let $0 < \alpha, \beta < 1$ such that $(1-\beta)(\alpha + [2\,\mathrm{Lip}(\phi)+1](1-\alpha)) < 1$. Let

$$w_j := (G_{(1-\alpha)(1-\beta)\sigma_j} * v_j)\,|_{\Omega}.$$

Then $w_j \in H^1(\Omega)$.

Let $x \in \Omega$, $y \in B(x, \alpha(1-\beta)\sigma_j) \cap \Omega$.

If $\mathrm{dist}(y, \partial\Omega) < (1-\alpha)(1-\beta)\sigma_j$, then by ([2], Thm. 3.16 and [2], p. 32)

$$|\nabla w_j(y)| \le |Du_j|_{(1-\alpha)(1-\beta)\sigma_j}(y) + [\mathrm{Lip}(\phi^{-1})]^{n-1} \int_{\phi(B(y,(1-\alpha)(1-\beta)\sigma_j)\setminus\overline{\Omega})} G_{(1-\alpha)(1-\beta)\sigma_j}(y - \phi^{-1}(z))\, d|Du_j|(z).$$

Since $\phi\big|_{\partial\Omega} = id$ and $\mathrm{dist}(y, \partial\Omega) < (1-\alpha)(1-\beta)\sigma_j$, we see after some calculations that $\phi(B(y,(1-\alpha)(1-\beta)\sigma_j) \setminus \overline{\Omega}) \subseteq B(y, (1-\alpha)(1-\beta)\sigma_j[1 + 2\,\mathrm{Lip}(\phi)])$.

Let h be a function such that $G(x) = h(|x|)$. We select $C_{\alpha,\beta} > 0$ such that

$$C_{\alpha,\beta} \ge \frac{1}{((1-\alpha)(1-\beta))^n} \max\left\{ \frac{h(0)}{h((1-\beta)(\alpha + [2\,\mathrm{Lip}(\phi)+1](1-\alpha)))}, \frac{h(0)}{h(1-\beta)} \right\}.$$

If $z \in \phi(B(y,(1-\alpha)(1-\beta)\sigma_j) \setminus \overline{\Omega})$, then the inequality $|z - x| < (1-\beta)\sigma_j(\alpha + (1-\alpha)[1 + 2\,\mathrm{Lip}(\phi)])$ and the choice of $C_{\alpha,\beta}$ imply that if $h_\sigma(r) := \frac{1}{\sigma^n} h(r/\sigma)$, then

$$G_{(1-\alpha)(1-\beta)\sigma_j}(y - \phi^{-1}(z)) \le h_{(1-\alpha)(1-\beta)\sigma_j}(0) \le C_{\alpha,\beta} G_{\sigma_j}(z - x).$$

If $z \in B(y, (1-\alpha)(1-\beta)\sigma_j) \cap \Omega$, then the inequality $|z - x| < (1-\beta)\sigma_j$ and the choice of $C_{\alpha,\beta}$ imply that

$$G_{(1-\alpha)(1-\beta)\sigma_j}(y - z) \le h_{(1-\alpha)(1-\beta)\sigma_j}(0) \le C_{\alpha,\beta} G_{\sigma_j}(z - x). \tag{5.3}$$

By the previous analysis we see that for $y \in \Omega \cap B(x, \alpha(1-\beta)\sigma_j)$, $\mathrm{dist}(y, \partial\Omega) < (1-\alpha)(1-\beta)\sigma_j$, we have

$$|\nabla w_j(y)| \le [1 + [\mathrm{Lip}(\phi^{-1})]^{n-1}] C_{\alpha,\beta} |Du_j|_{\sigma_j}(x). \tag{5.4}$$

If $y \in \Omega \cap B(x, \alpha(1-\beta)\sigma_j)$ such that $\mathrm{dist}(y, \partial\Omega) \ge (1-\alpha)(1-\beta)\sigma_j$, then $B(y, (1-\alpha)(1-\beta)\sigma_j) \subset \Omega$ so $w_j(y) = G_{(1-\alpha)(1-\beta)\sigma_j} * u_j(y)$ and using (5.3) we get

$$|\nabla w_j(y)| \le C_{\alpha,\beta} |Du_j|_{\sigma_j}(x). \tag{5.5}$$

Let $E := [1 + [\mathrm{Lip}(\phi^{-1})]^{n-1}] C_{\alpha,\beta}$. Using (5.4) and (5.5) we see that

$$\int_{B(x,\alpha(1-\beta)\sigma_j)\cap\Omega} G_{\alpha(1-\beta)\sigma_j}(x - y) |\nabla w_j(y)|^2 \,\mathrm{d}y \le E^2 |Du_j|_{\sigma_j}(x)^2$$

for all $x \in \Omega$. We denote the expression on the left of the above inequality by $(|\nabla w_j|^2)_{\alpha(1-\beta)\sigma_j}(x)$. Then

$$\left(\left| \nabla \left(\frac{w_j}{E} \right) \right|^2 \right)_{\alpha(1-\beta)\sigma_j} (x) \le |Du_j|_{\sigma_j}(x)^2.$$

Let $\epsilon > 0$. Now $\lim_{\sigma\to 0+} \frac{\ell((1-\beta)\alpha\sigma)}{(1-\beta)\alpha\ell(\sigma)} = 1$ and thus there exists $\sigma(\alpha, \beta, \epsilon) > 0$ such that if $\sigma < \sigma(\alpha, \beta, \epsilon)$, then $\frac{\ell(\sigma)}{1+\epsilon} \le \frac{\ell((1-\beta)\alpha\sigma)}{(1-\beta)\alpha} \le (1+\epsilon)\ell(\sigma)$. Thus if j is so large that $\sigma_j < \sigma(\alpha, \beta, \epsilon)$, we get

$$\frac{(1-\beta)\alpha}{1+\epsilon} F^N_{(1-\beta)\alpha\sigma_j} \left(\frac{w_j}{[(1+\epsilon)(1-\beta)\alpha]^{1/2} E} \right) \le F_{\sigma_j}(u_j).$$

Since $u_j \to u$ in $L^1(\Omega)$ implies that $w_j \to u$ in $L^1(\Omega)$ as $j \to +\infty$, since the family $\{F^N_\epsilon\}_\epsilon$ Gamma-converges to MS and since $\liminf_{i\to\infty} F_{\sigma_i}(u_i) < \infty$, it follows that

$$\frac{(1-\beta)\alpha}{1+\epsilon} \mathrm{MS} \left(\frac{u}{[(1+\epsilon)(1-\beta)\alpha]^{1/2} E} \right) \le \liminf_{j\to\infty} F_{\sigma_j}(u_j) < \infty$$

so $u \in \mathrm{SBV}^2(\Omega)$. □

The following theorem is similar to the lower bound condition of the Gamma-convergence in ([16], Thm. 1.2).

Theorem 5.3. $\mathrm{MS}(u) \le F'(u)$ *for all* $u \in L^1(\Omega)$.

Proof. Let $u_i \to u$ in $L^1(\Omega)$, $\sigma_i > 0$, $\sigma_i \to 0$ as $i \to +\infty$, and suppose that $\liminf_{i\to\infty} F_{\sigma_i}(u_i) < \infty$. It is sufficient to show that $\mathrm{MS}(u) \le \liminf_{i\to\infty} F_{\sigma_i}(u_i)$. We can assume that $u_i \in \mathrm{BV}(\Omega)$ for all i by passing to a subsequence if necessary. By Proposition 5.2, $u \in \mathrm{SBV}^2(\Omega)$.

Let $A \subset \Omega$ be open such that $\overline{A} \subset \Omega$. Let $0 < \alpha, \beta < 1$. Let $w_i \in H^1(\Omega)$ be as in the proof of Proposition 5.2. Then $w_i \to u$ in $L^1(\Omega)$ and $w_i = G_{(1-\alpha)(1-\beta)\sigma_i} * u_i$ in A provided i is large enough. Let $x \in A$. Using Lemma 5.1 we see that $|\nabla w_i(y)| \le A_{\alpha,\beta}|Du_i|_{\sigma_i}(x)$ for all $y \in B(x, \alpha(1-\beta)\sigma_i)$. Then

$$G_{\alpha(1-\beta)\sigma_i} * |\nabla w_i|^2(x) \le A^2_{\alpha,\beta}|Du_i|_{\sigma_i}(x)^2$$

for all $x \in A$.

Let $\epsilon > 0$. As in the proof of Proposition 5.2, we see that provided i is large enough, we have

$$\frac{(1-\beta)\alpha}{1+\epsilon} F^N_{(1-\beta)\alpha\sigma_i}\left(\frac{w_i}{[(1+\epsilon)(1-\beta)\alpha]^{1/2}A_{\alpha,\beta}}, A\right) \le F_{\sigma_i}(u_i, A) \le F_{\sigma_i}(u_i).$$

Using the above inequality and ([22], Sect. 6.3) we see that

$$\frac{(1-\beta)\alpha}{1+\epsilon}\,\mathrm{MS}\left(\frac{u}{[(1+\epsilon)(1-\beta)\alpha]^{1/2}A_{\alpha,\beta}}, A\right) \le \liminf_{i\to\infty} F_{\sigma_i}(u_i, A).$$

Letting ϵ tend to zero we obtain

$$\frac{1}{A^2_{\alpha,\beta}}\int_A |\nabla_a u|^2\,\mathrm{d}x + (1-\beta)\alpha 2\mathcal{H}^{n-1}(S_u \cap A) \le \liminf_{i\to\infty} F_{\sigma_i}(u_i, A).$$

Letting $\alpha \to 1$ and $\beta \to 0$ we get

$$2\mathcal{H}^{n-1}(S_u \cap A) \le \liminf_{i\to\infty} F_{\sigma_i}(u_i, A). \tag{5.6}$$

Let $\alpha_k := 2^{-k}$ and $\beta_k := \frac{1}{k}$. By Lemma 5.1, $A_{\alpha_k,\beta_k} \to 1$ as $k \to +\infty$. Thus

$$\int_A |\nabla_a u|^2\,\mathrm{d}x \le \liminf_{i\to\infty} F_{\sigma_i}(u_i, A). \tag{5.7}$$

We want to show that $\int_A |\nabla_a u|^2\,\mathrm{d}x + 2\mathcal{H}^{n-1}(S_u \cap A) \le \liminf_{i\to\infty} F_{\sigma_i}(u_i)$ for all $A \subset\subset \Omega$. Since $S_u \cap A$ is countably rectifiable, there exist sets $K_i \subset A$, K_i compact in $\mathbb{R}^n$, K_i a subset of a C^1 surface C_i, and $N \subset A$, $\mathcal{H}^{n-1}(N) = 0$, such that $S_u \cap A = N \cup \cup_{i=1}^{\infty} K_i$. It is sufficient to show that

$$\int_A |\nabla_a u|^2\,\mathrm{d}x + 2\mathcal{H}^{n-1}(\cup_{i=1}^m K_i) \le \liminf_{i\to\infty} F_{\sigma_i}(u_i)$$

for all m.

Let $\epsilon > 0$ and $K := \cup_{i=1}^m K_i$. Let $K_1' := K_1$. Let $V_1 := (K_1')_{a_1}$ where $a_1 > 0$ is so small that $\mathcal{H}^{n-1}(V_1 \cap K) \le \mathcal{H}^{n-1}(K_1') + \epsilon/m$. Let $K_2' := K_2 \setminus V_1$ and let $V_2 := (K_2')_{a_2}$ where $a_2 > 0$ such that $\mathcal{H}^{n-1}(V_2 \cap K) \le \mathcal{H}^{n-1}(K_2') + \epsilon/m$. In general, let

$$K_j' := K_j \setminus (V_1 \cup \ldots \cup V_{j-1}) \quad \text{and} \quad V_j := (K_j')_{a_j}$$

where $a_j > 0$ is so small that $\mathcal{H}^{n-1}(V_j \cap K) \le \mathcal{H}^{n-1}(K_j') + \epsilon/m$. It follows that the sets K_i' are mutually disjoint and the sets V_i cover K. Now $\mathcal{H}^{n-1}(\cup_{j=1}^m K_j) \le \sum_{j=1}^m \mathcal{H}^{n-1}(K_j') + \epsilon$.

Select $a_0 > 0$ such that $\int_{\cup_{i=1}^m \overline{(K_i')_{3a_0}}} |\nabla_a u|^2 \, \mathrm{d}x < \epsilon$ and for the mutually disjoint sets K_i', the sets $(K_i')_{a_0}$ are also mutually disjoint. Using the preceding analysis, the fact that $K_j' \subseteq S_u \cap (K_j')_{a_0}$, applying (5.7) in the set $A \setminus \cup_{j=1}^m \overline{(K_j')_{3a_0}}$ and (5.6) in the set $(K_j')_{a_0}$, we get

$$\int_A |\nabla_a u|^2 \, \mathrm{d}x + 2\mathcal{H}^{n-1}(\cup_{j=1}^m K_j) \leq \liminf_{i \to \infty} F_{\sigma_i}(u_i) + 3\epsilon.$$

Letting $\epsilon \to 0$, $m \to \infty$ and $A \uparrow \Omega$ it follows that $\mathrm{MS}(u) \leq \liminf_{i\to\infty} F_{\sigma_i}(u_i)$. □

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