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BIFURCATION AND SEGREGATION IN QUADRATIC TWO-POPULATIONS MEAN FIELD GAMES SYSTEMS

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Abstract. We search for non-constant normalized solutions to the semilinear elliptic system

$$\begin{cases} -\nu\Delta v_i + g_i(v_j^2)v_i = \lambda_i v_i, \quad v_i > 0 & \text{in } \Omega \\ \partial_n v_i = 0 & \text{on } \partial\Omega \\ \int_\Omega v_i^2 \,\mathrm{d}x = 1, & 1 \le i, j \le 2, \quad j \ne i, \end{cases}$$

where $\nu > 0$, $\Omega \subset \mathbb{R}^N$ is smooth and bounded, the functions g_i are positive and increasing, and both the functions v_i and the parameters λ_i are unknown. This system is obtained, via the Hopf−Cole transformation, from a two-populations ergodic Mean Field Games system, which describes Nash equilibria in differential games with identical players. In these models, each population consists of a very large number of indistinguishable rational agents, aiming at minimizing some long-time average criterion. Firstly, we discuss existence of nontrivial solutions, using variational methods when $g_i(s) = s$, and bifurcation ones in the general case; secondly, for selected families of nontrivial solutions, we address the appearing of segregation in the vanishing viscosity limit, *i.e.*

$$\int_\Omega v_1 v_2 \to 0 \qquad \text{as } \nu \to 0.$$

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1. Introduction

We consider the following semilinear elliptic system

$$\begin{cases} -\nu\Delta v_1 + g_1(v_2^2)v_1 = \lambda_1 v_1 & \\ -\nu\Delta v_2 + g_2(v_1^2)v_2 = \lambda_2 v_2 & \text{in } \Omega \\ \int_\Omega v_1^2 \,\mathrm{d}x = \int_\Omega v_2^2 \,\mathrm{d}x = 1, \quad v_1, v_2 > 0 & \\ \partial_n v_1 = \partial_n v_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, normalized in such a way that

$$|\Omega| = 1,$$

$\nu > 0$, and both the functions v_i and the parameters λ_i are unknown. The interaction functions $g_i \in C^2([0,\infty))$ satisfy

$$\begin{aligned} &\bullet \quad C_g^{-1} s \le g_i(s) \le C_g s \qquad \forall s \ge 0, \\ &\bullet \quad g_i \text{ is strictly increasing}, \quad g_i'(1) > 0, \end{aligned} \tag{1.2}$$

for some $C_g > 0$ $(i = 1, 2)$.

The elliptic system (1.1) arises in the context of Mean Field Games (briefly MFG) theory. MFG is a branch of Dynamic Games which has been proposed independently by Lasry and Lions [21–23] and Caines, Huang, Malhamé [18, 19] in the engineering community, with the aim of modeling and analyzing decision processes involving a very large number of indistinguishable rational agents. Here, we focus on MFG with two competing populations, where every individual of the ith population $(i = 1, 2)$ is represented by a typical agent, and whose state is driven by the controlled stochastic differential equation

$$dX_s^i = -a_s^i \mathrm{d}s + \sqrt{2\tilde{\nu}}\, \mathrm{d}B_s^i,$$

where B_s^i are independent Brownian motions. The agent chooses her own velocity a_s^i in order to minimize a cost of long-time-average form

$$\mathcal{J}^i(X_0^i, a^1, a^2) = \liminf_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\left[\frac{|a_s^i|^2}{2} + g_i((\hat{m}_j)_s)\right] \mathrm{d}s,$$

where $\hat{m}_j$ denotes the empirical density of the players belonging to the other population (*i.e.* $j = 3 - i$). It has been shown (see in particular [16]) that equilibria of the game (in the sense of Nash) are captured by the following system of non-linear elliptic equations

$$\begin{cases} -\tilde{\nu}\Delta u_i(x) + \frac{1}{2}|\nabla u_i(x)|^2 + \lambda_i = g_i(m_j(x)) & \\ -\tilde{\nu}\Delta m_i(x) - \operatorname{div}(\nabla u_i(x)\, m_i(x)) = 0 & \text{in } \Omega \\ \int_\Omega m_i \mathrm{d}x = 1,\ m_i > 0, & i = 1, 2. \end{cases} \tag{1.3}$$

The unknowns u_i, λ_i provide the value functions of typical players and the average costs respectively. On the other hand, the unknowns m_i represent the stationary distributions of players of the ith population implementing the optimal strategy, that is, the long time behavior of agents playing in an optimal way. We suppose that the state X_s^i is subject to reflection at $\partial\Omega$; this motivates the Neumann boundary conditions.

Note that the individual cost $\mathcal{J}^i$ is increasing with respect to $\hat{m}_j$, as we are supposing that g_i is increasing. In other words, every agent is lead to avoid regions of Ω where an high concentration of competitors is present. For this reason, our MFG model is expected to show phenomena of segregation between the two populations. In particular, segregation should arise distinctly in the vanishing viscosity regime, namely when the Brownian noise (whose intensity is controlled by $\tilde{\nu}$) becomes negligible with respect to interactions. We will explore this aspect in terms of qualitative properties of the two distributions m_1, m_2.

Another key feature of this model is the quadratic dependence of the cost $\mathcal{J}^i$ with respect to the velocity a^i. It has been pointed out (see [21, 24]) that the so-called Hopf−Cole transformation partially decouples the equations in (1.3), reducing the number of the unknowns. Precisely, if we let

$$v_i^2 := m_i = \mathrm{e}^{-u_i/\tilde{\nu}} \qquad \text{and} \qquad \nu = 2\tilde{\nu}^2,$$

then (1.3) becomes (1.1). We will therefore consider (1.1) and transpose the obtained results to the original system (1.3).

Before proceeding with the analysis of the reduced system (1.1), a few bibliographical remarks are in order. First of all, while the single population case has received a considerable attention, few papers deal with mathematical aspects of the multi-population setting. We mention that a preliminary study of (1.1)–(1.3) has been made in [10], while a non-stationary version of (1.3) is considered in [20]. The latter work provides also a motivation for (1.3) based on pedestrian crowd models. Our MFG system can be also seen as a simplified version of the population models presented in [1].

Since $|\Omega| = 1$,

$$v_1 \equiv v_2 \equiv 1, \qquad \lambda_1 = g_1(1), \quad \lambda_2 = g_2(1)$$

is a solution of (1.1) for every value of ν. We will refer to it (or, with some abuse, to the pair $(v_1, v_2) \equiv (1, 1)$) as the *trivial* (or *constant*) solution. The aim of our investigation is twofold: firstly, to show the existence of families, indexed by ν, of nontrivial *Nash equilibria* for (1.1); secondly, to analyze possible *segregation* phenomena for such families, as $\nu \to 0$.

Definition 1.1. The pair (v_1, v_2) is a *Nash equilibrium* for (1.1) if each v_i achieves

$$\lambda_i := \inf\left\{ \int_\Omega \left[\nu|\nabla w|^2 + g_i(v_j^2)w^2\right] \,\mathrm{d}x : w \in H^1(\Omega),\ \int_\Omega w^2 \,\mathrm{d}x = 1 \right\}.$$

It is easy to show (see Lem. 2.1 ahead) that a pair (v_1, v_2) is a Nash equilibrium if and only if (up to a change of sign of its components) it solves (1.1) with multipliers (λ_1, λ_2).

Definition 1.2. We say that a set of solutions

$$\Sigma \subset \left\{(\nu, v_1, v_2) \in \mathbb{R} \times C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}) : (\nu, v_1, v_2) \text{ satisfies (1.1) for some } (\lambda_1, \lambda_2)\right\}$$

segregates if it contains sequences $\{(\nu_n, v_{1,n}, v_{2,n})\}_n$ with $\nu_n \to 0$, and for every such sequence it holds

$$\int_\Omega v_{1,n} v_{2,n} \to 0 \quad \text{as } n \to \infty.$$

One important feature of system (1.1) is that its unknowns are both the functions v_i, which are required to be normalized (in the L^2 sense), and the parameters λ_i. Despite the large literature devoted to existence results for semilinear elliptic systems, only few papers deal with normalized solutions, mainly when searching for solitary waves associated to nonlinear Schrödinger systems [3,4,26–28]. Note that all these papers are based on variational methods, since the systems they consider are of gradient type. This is not the case for (1.1), except when the interactions g_i are linear functions.

On the other hand, segregation issues have received much attention in the last decade, and by now a large amount of literature is dedicated to this subject, see *e.g.* [6–9,11,12,15,25,29,31,33], the recent survey [32], and references therein. Mainly two types of competitions have been widely investigated, namely the Lotka−Volterra type (*e.g.* $g_i(s) = a_i\sqrt{s}$), and again the variational one. Furthermore in these papers segregation (as defined in Def. 1.2) is a first easy step, while all the effort is done to show that the convergence of $v_1 v_2$ to 0 is very much stronger than merely L^1. Conversely, in our situation, even the L^1 convergence is not clear at all, mainly due to the unknown behavior of the parameters λ_i. For instance, the set of trivial solutions does not segregate at all. Actually, this is one of the main difficulties we have to face.

Motivated by the above discussion, we first treat the variational case

$$g_i(s) = \gamma_i s, \qquad \text{for some } \gamma_i > 0.$$

In such a case, as we mentioned, (1.1) has a gradient structure, at least in dimension $N \leq 3$: Nash equilibria can be obtained as critical points of the functional

$$I_\nu(v_1, v_2) = \int_\Omega \left[\frac{1}{\gamma_1}\,|\nabla v_1|^2 + \frac{1}{\gamma_2}\,|\nabla v_2|^2 + \frac{1}{\nu}\,v_1^2 v_2^2\right] \mathrm{d}x$$

constrained to the manifold

$$M=\left\{(v_1,v_2)\in H^1(\Omega)\times H^1(\Omega):\int_\Omega v_1^2\,\mathrm{d}x=\int_\Omega v_1^2\,\mathrm{d}x=1\right\}.$$

As a consequence, existence of solutions can be obtained by direct minimization of $I_\nu|_M$. Regarding the asymptotic behaviour of such minimizers, using techniques contained in [26] we can show Γ-convergence to the following limiting problem:

$$\min\left\{\int_\Omega\left[\frac{1}{\gamma_1}|\nabla v_1|^2+\frac{1}{\gamma_2}|\nabla v_2|^2\right]\mathrm{d}x:(v_1,v_2)\in M,\ v_1\cdot v_2\equiv 0,\right\}. \tag{1.4}$$

It can be proved that such minimum is achieved, and, among other properties, that any minimizer (V_1,V_2) is such that $V_1\sqrt{\gamma_2}-V_2\sqrt{\gamma_1}\in C^{2,\alpha}(\overline{\Omega})$, for every $0<\alpha<1$ (see Prop. 3.3 ahead). As a matter of fact, we can prove the following.

Theorem 1.3 (Variational case)**.** *Let $N\le 3$, $g_i(s)=\gamma_i s$, $\gamma_i>0$, and let $\mu_1>0$ denote the first positive Neumann eigenvalue of $-\Delta$ in Ω. Then, for every*

$$0<\nu\le\frac{\gamma_1\gamma_2}{\mu_1(\gamma_1+\gamma_2)},$$

the minimum of $I_\nu|_M$ is achieved by a pair $(v_{1,\nu},v_{2,\nu})$, which is a nontrivial Nash equilibrium for (1.1).

Moreover, any family of minimizers exhibits segregation: up to subsequences,

$$v_{i,\nu}\to V_i \text{ in } H^1(\Omega)\cap C^\alpha(\overline{\Omega})\qquad \text{as } \nu\to 0,$$

for every $\alpha<1$, where (V_1,V_2) achieves (1.4).

Turning to the general case, since (1.1) has no variational structure, one is lead to search for solutions using topological methods. In particular, it is natural to use bifurcation theory to find nontrivial solutions (ν,v_1,v_2) branching off from the trivial ones

$$\mathcal{T}=\{(\nu,1,1):\nu>0\}\subset\mathbb{R}\times C^{2,\alpha}(\overline{\Omega})\times C^{2,\alpha}(\overline{\Omega}).$$

We denote by $\mathcal{S}$ the closure of the set of nontrivial solutions of (1.1), so that a bifurcation point is a point of $\mathcal{S}\cap\mathcal{T}$. The classical bifurcation theory by Rabinowitz [14, 30] can be applied to our setting to obtain the following.

Theorem 1.4. *Let g_1,g_2 satisfy assumption (1.2), let $\mu^*>0$ denote a positive Neumann eigenvalue of $-\Delta$ in Ω, and let*

$$\nu^*=\frac{2\sqrt{g_1'(1)g_2'(1)}}{\mu^*}.$$

- *If μ^* has odd multiplicity then there exists a continuum $\mathcal{C}^*\subset\mathcal{S}$ such that $(\nu^*,1,1)\in\mathcal{C}^*$ and*
  - *either $(\nu^{**},1,1)\in\mathcal{C}^*$, where $\nu^{**}=2\sqrt{g_1'(1)g_2'(1)}/\mu^{**}$ and $\mu^{**}\ne\mu^*$ is another positive Neumann eigenvalue;*
  - *or $\mathcal{C}^*$ is unbounded; furthermore, in dimension $N\le 3$, $\mathcal{C}\cap\{(\nu,v_1,v_2):\nu\ge\bar\nu\}$ is bounded for every $\bar\nu>0$, and $\mathcal{C}^*$ contains a sequence $(\nu_n,v_{1,n},v_{2,n})$ such that, as $n\to+\infty$,*

$$\nu_n\to 0,\qquad \|(v_{1,n},v_{2,n})\|_{C^{2,\alpha}}\to+\infty.$$

- *If μ^* is simple (with eigenfunction ψ^*) then the set of non-trivial solutions is, near $(\nu^*,1,1)$, a unique smooth curve with parametric representation*

$$\nu=\nu(\varepsilon),\quad v=(1,1)+\varepsilon v^*+o(\varepsilon),$$

  where $\nu(0)=\nu^*$ and $v^*=\left(-\psi^*\sqrt{g_1'(1)},\psi^*\sqrt{g_2'(1)}\right)$.

Remark 1.5. Sharper asymptotic expansions are provided in Remark 4.6 ahead, in case both g_i are more regular.

Remark 1.6. For generic domains, infinitely many eigenvalues μ_n are odd, and we have infinitely many bifurcation points $\nu_n \to 0$, with associated branches $\mathcal{C}_n$. As a consequence, picking $(\tilde{\nu}_n, v^{(n)}) \in \mathcal{C}_n$ with both $|\tilde{\nu}_n - \nu_n| \to 0$ and $\|v^{(n)} - (1,1)\| \to 0$, one can construct *families of nontrivial solutions that not only do not exhibit segregation, but even tend to the trivial solution as* $\nu \to 0$. As a matter of fact, to avoid this phenomenon and obtain segregation, it is crucial to select families belonging to a single bifurcation branch (to compare this theorem with the classical results by Rabinowitz, recall that here the natural bifurcation parameter is $1/\nu$, rather than ν itself).

The previous remark shows that one can not expect segregation for a generic family of nontrivial solutions. It is then natural to ask whether segregation occurs for the bifurcation branches above described, at least for the unbounded ones. According to Theorem 1.4, in order to find unbounded branches of nontrivial solutions we first have to find odd eigenvalues of $-\Delta$ in Ω, and then to exclude that the corresponding branch goes back to the set of trivial solutions. Usually, in the bifurcation framework, both conditions can be satisfied when working with the first eigenvalue of the linearized problem: indeed, on one hand such eigenvalue is simple; on the other hand, it is usually possible to carry over to the full branch the nodal characterization of the corresponding eigenfunction. Notice that this is not our case, since the first Neumann eigenvalue is 0 and it does not provide a bifurcation point, while the first positive eigenvalue μ_1 is actually the second one. Another way to exploit these ideas is to work in dimension $N = 1$.

Theorem 1.7. *Let g_1, g_2 satisfy assumption (1.2) and $N = 1$. For any $k \in \mathbb{N}$, $k \geq 1$ there exists a continuum $\mathcal{C}_k$ of solutions, such that:*

- *if $(\nu, v_1, v_2) \in \mathcal{C}_k$ then both v_i have exactly $k-1$ critical points;*
- $\overline{\mathcal{C}_k} \cap \mathcal{T} = \left\{ \left(\frac{2\sqrt{g_1'(1) g_2'(1)}}{\pi^2 k^2}, 1, 1 \right) \right\}$*;*
- *$h \neq k$ implies $\mathcal{C}_h \cap \mathcal{C}_k = \emptyset$;*
- *each $\mathcal{C}_k$ contains sequences with $\nu \to 0$ and $\|(v_1, v_2)\|_{C^{2,\alpha}} \to +\infty$;*
- *each $\mathcal{C}_k$ segregates.*

The proof of Theorem 1.7 is split in two parts: firstly, we characterize each branch $\mathcal{C}_k$ by the number of oscillations of its components, in the spirit of the original application by Rabinowitz to nonlinear Sturm−Liouville problems [30]; secondly, we prove segregation by a blow-up analysis, exploiting some Liouville-type results for entire solutions to ODEs systems on $\mathbb{R}$ having a finite number of oscillations.

Once segregation is obtained, we have that the segregating branches converge, up to subsequences, to some limiting profiles. As a consequence, some natural questions arise, about the type of convergence as well as about the properties of the limiting profiles. Restricting the analysis to the first branch $\mathcal{C}_1$, which contains pairs with monotone components, we can deepen the analysis which leads to segregation. Indeed in this case the emerging free boundary in the segregation limit consists in exactly one point, and the rate of convergence can be estimated sharply using ideas introduced in [5], to treat the one-dimensional variational case.

Theorem 1.8. *Let g_1, g_2 satisfy assumption (1.2), $N = 1$ and let $\mathcal{C}_1$ be as in Theorem 1.7. Then, any sequence $\{(\nu_n, v_n, \lambda_n)\}_n \subset \mathcal{C}_1$ such that $\nu_n \to 0$ is uniformly bounded in Lipschitz norm, and it holds*

$$v_{i,n} \to V_i \text{ in } H^1(\Omega) \cap C^\alpha(\overline{\Omega}) \qquad \text{as } \nu_n \to 0,$$

for every $\alpha < 1$, where (V_1, V_2) is the minimizer (unique up to reflections) achieving (1.4) with $\gamma_i = g_i'(0) \geq C_g^{-1}$.

Remark 1.9. We expect that most of the results of Theorems 1.7 and 1.8 can be extended to higher dimension, in the radial setting.

It is easy to see that the convergence above is optimal: indeed, in case of Lipschitz convergence, both V_i would be C^1, a contradiction with their explicit expression provided in Proposition 3.3. Up to our knowledge, this is the first paper obtaining optimal bounds for competitions which are not of power-type, even though only in dimension $N = 1$ (or in the radial case). The only other paper dealing with generic competitions is [33], where uniform bounds in the planar case $N = 2$, not necessarily radial, are obtained.

Let us also point out that along the first branch the problem – which is not variational – inherits a variational principle in the limit. This is a remarkable fact, since it shows a deep connection between the variational problem (1.4) and the nonvariational system (1.1). This phenomenon was already observed, in a different situation, in [13].

Of course, all the results we obtained for system (1.1) can be restated for the original MFG system (1.3), recalling that

$$m_i = v_i^2, \qquad u_i = -2\tilde{\nu} \ln v_i.$$

Finally, let us also mention that the true multidimensional case $N \geq 2$, as well as the case of 3 or more populations, are of interest: they will be the object of future studies.

The present paper is structured as follows: in Section 2 we list a few preliminary results; Section 3 is devoted to the analysis of the variational case, and to the proof of Theorem 1.3, while Section 4 contains the bifurcation arguments and the proof of Theorem 1.4; the Sturm-type characterization of the nontrivial solutions in dimension $N = 1$ is developed in Section 5, and the proof of Theorem 1.7 is completed in Section 6, by showing segregation; finally, the proof of Theorem 1.8 is contained in Section 7.

Notation. Throughout the paper, i denotes an index between 1 and 2, and $j = 3 - i$. With a little abuse of terminology, we say that (v_1, v_2) solves (1.1) (or even that (ν, v_1, v_2) does) if there exist λ_1, λ_2 such that $(v_1, v_2, \lambda_1, \lambda_2)$ satisfies (1.1) (for some prescribed ν).

We will denote by $(\mu_k)_{k\geq 0}$ the non decreasing sequence of the eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions, namely μ_k is such that

$$\begin{cases} -\Delta\psi_k = \mu_k\psi_k & \text{in } \Omega, \\ \partial_n\psi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

for some eigenvector $\psi_k \in C^{2,\alpha}(\overline{\Omega})$, which constitute an orthonormal basis of $L^2(\Omega)$. The first eigenvalue $\mu_0 = 0$ is simple and its corresponding eigenfunction is $\psi \equiv 1$.

Given a function u, $u^{\pm}(x) = \max(\pm u(x), 0)$ denote its positive and negative parts. Finally, $C, C_1, C_2, \ldots$ denote (positive) constants we need not to specify.

2. Preliminaries

In this section we collect some preliminary results and some estimates of frequent use.

Lemma 2.1. *The pair (v_1, v_2) is a Nash equilibrium if and only if, up to a change of sign of each component, it is a (classical) solution of (1.1).*

Proof. Considering v_j as fixed, we have that v_i is an L^2-normalized eigenfunction of the Neumann realization of the operator

$$H^1(\Omega) \ni w \mapsto -\nu\Delta w + g_i(v_j^2)w,$$

and that λ_i is the corresponding eigenvalue. But then v_i is strictly positive (up to a change of sign) if and only if it is the first eigenfunction, *i.e.* it achieves the infimum in Definition 1.1. In particular, the proof of the strict positivity in $\overline{\Omega}$ is a routine application of the Maximum Principle and Hopf's Lemma. □

Lemma 2.2. *Let (v_1, v_2) solve (1.1). Then, either it is trivial, or*

$$\min_{\overline{\Omega}} g_i(v_j^2) < \lambda_i < \max_{\overline{\Omega}} g_i(v_j^2), \qquad i = 1, 2.$$

Proof. Integrating the equation for v_i we can write

$$\int_\Omega \left[\lambda_i - g_i(v_j^2)\right] v_i \,\mathrm{d}x = \nu \int_{\partial\Omega} \partial_n v_i \,\mathrm{d}\sigma = 0,$$

and since v_i is positive, we deduce that either $\lambda_i - g_i(v_j^2) \equiv 0$, *i.e.* v_j is constant, or $\min_{\overline{\Omega}} g_i(v_j^2) < \lambda_i < \max_{\overline{\Omega}} g_i(v_j^2)$.

Now, if both v_i and v_j are not constant, then the second alternative follows. Let v_i be constant: then its equation implies $g_i(v_j^2) \equiv \lambda_i$, so that also v_j is constant. Finally, both such constants must be 1 by the L^2-constraint (recall that $|\Omega| = 1$). □

Remark 2.3. The above lemma shows that, for Nash equilibria, having a constant component implies being the trivial solution (in this sense, the terminology "constant solution" is not ambiguous). In fact, if unique continuation for (1.1) holds, then any solution such that one component is constant in a (non-empty) open $\Omega_0 \subset \Omega$ must be the trivial one. This is always true, in particular, in dimension $N = 1$ (see Sect. 5).

Lemma 2.4. *Let (v_1, v_2) solve (1.1). The following identities hold, for every i:*

$$\nu \int_\Omega |\nabla v_i|^2 + \int_\Omega g_i(v_j^2) v_i^2 = \lambda_i;$$
$$\nu \int_\Omega \left|\frac{\nabla v_i}{v_i}\right|^2 + \lambda_i = \int_\Omega g_i(v_j^2).$$

In particular, the multipliers λ_i satisfy

$$C_g^{-1} \int_\Omega v_1^2 v_2^2 \le \lambda_i \le C_g. \tag{2.1}$$

Proof. To obtain the two identities it suffices to use integration by parts after multiplying the equation for v_i by v_i and $1/v_i$, respectively. Since $\int_\Omega v_i^2 = 1$ and $C_g^{-1} s \le g_i(s) \le C_g s$, (2.1) follows. □

Corollary 2.5. *A sufficient condition for $\{(\nu_n, v_{1,n}, v_{2,n})\}_n$ to segregate is that, for the corresponding multipliers,*

$$\text{either } \lambda_{1,n} \to 0, \quad \text{or } \lambda_{2,n} \to 0,$$

as $n \to \infty$.

3. The variational case

This section is devoted to the proof of Theorem 1.3. Such proof relies on ideas contained in [26], even though in that paper a different problem is considered (Dirichlet conditions, symmetric interaction, auto-catalytic reaction terms). For this reason we describe the main ideas here, and refer the reader to [26] for more details.

In the following we assume that $N \le 3$ and

$$g_i(s) = \gamma_i s, \qquad \gamma_i > 0.$$

As we already noticed, the corresponding system has a gradient structure. For easier notation we make a change of variables, setting

$$\beta = \frac{1}{\nu}, \quad \tilde{v}_1 = \sqrt{\gamma_2}\, v_1, \quad \tilde{v}_2 = \sqrt{\gamma_1}\, v_2. \tag{3.1}$$

With this notation system (1.1) becomes

$$\begin{cases} -\Delta\tilde v_1 + \beta\tilde v_2^2\tilde v_1 = \lambda_1\tilde v_1 & \\ -\Delta\tilde v_2 + \beta\tilde v_1^2\tilde v_2 = \lambda_2\tilde v_2 & \text{in } \Omega \\ \int_\Omega \tilde v_1^2 = \gamma_2,\ \int_\Omega \tilde v_2^2 = \gamma_1, \quad \tilde v_1,\tilde v_2 > 0 & \\ \partial_n\tilde v_1 = \partial_n\tilde v_2 = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

(of course, the multipliers λ_i here are suitable multiples of those of the original system). Also for (3.2) positive solutions are Nash equilibria, among which the trivial one is the pair $(\sqrt{\gamma_2},\sqrt{\gamma_1})$. Solutions to (3.2) are critical points of the functional

$$J_\beta(\tilde v_1,\tilde v_2) = \int_\Omega \left[|\nabla\tilde v_1|^2 + |\nabla\tilde v_2|^2 + \beta\tilde v_1^2\tilde v_2^2\right]$$

constrained to the manifold

$$\tilde M = \left\{(\tilde v_1,\tilde v_2)\in H^1(\Omega)\times H^1(\Omega) : \int_\Omega \tilde v_1^2 = \gamma_2,\ \int_\Omega \tilde v_2^2 = \gamma_1\right\}$$

(recall that, since $N\le 3$, the exponent $p=4$ is Sobolev subcritical and thus J_β is of class C^1).

Lemma 3.1. *For every $\beta>0$ the value*

$$c_\beta := \inf_{\tilde M} J_\beta \quad \textit{is achieved by} \quad (\tilde v_{1,\beta},\tilde v_{2,\beta})\in\tilde M,$$

which is a Nash equilibrium for (3.2). Furthermore, if

$$\beta \ge \frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2}\mu_1$$

(where μ_1 is the first positive Neumann eigenvalue of $-\Delta$ in Ω) then $(\tilde v_{1,\beta},\tilde v_{2,\beta})$ is nontrivial.

Proof. Since J_β is weakly l.s.c. in H^1, and $\tilde M$ is weakly closed, the minima $(\tilde v_{1,\beta},\tilde v_{2,\beta})$ exist by the direct method. Moreover, since

$$\int_\Omega \left[|\nabla\tilde v_i|^2 + \beta\tilde v_j^2\tilde v_i^2\right] = J_\beta(\tilde v_1,\tilde v_2) - \int_\Omega |\nabla\tilde v_j|^2,$$

we have that such minima correspond to Nash equilibria for the original problem (the converse, of course, is false). We are left to prove that, for β large, $(\tilde v_{1,\beta},\tilde v_{2,\beta}) \ne (\sqrt{\gamma_2},\sqrt{\gamma_1})$. To do that, we will choose a suitable competitor in the definition of c_β: let ψ_1 be an eigenfunction associated to μ_1. Then ψ_1 changes sign (indeed it is orthogonal to the eigenfunction $\psi_0=1$, associated to $\mu_0=0$) and we can find non-zero constants $a_\pm$ such that $(a_+\psi^+, a_-\psi^-)\in\tilde M$. Then

$$c_\beta < J_\beta(a_+\psi^+, a_-\psi^-) = (\gamma_1+\gamma_2)\mu_1$$

(equality can not hold since $(a_+\psi^+, a_-\psi^-)$ can not solve (3.2)) while

$$J_\beta(\sqrt{\gamma_2},\sqrt{\gamma_1}) = \gamma_1\gamma_2\beta.$$

□

Once we have solved the problem for $\beta>0$ fixed, we are ready to show Γ-convergence as $\beta\to+\infty$. Let

$$J_\infty(\tilde v_1,\tilde v_2) := \begin{cases} \int_\Omega \left[|\nabla\tilde v_1|^2 + |\nabla\tilde v_2|^2\right] & \text{when } \int_\Omega \tilde v_1^2\tilde v_2^2 = 0 \\ +\infty & \text{otherwise} \end{cases} \qquad \text{and} \qquad c_\infty := \inf_{\tilde M} J_\infty.$$

Lemma 3.2. *As* $\beta \to +\infty$,

$$c_\beta \to c_\infty \qquad \text{and (up to subs.)} \qquad \tilde{v}_{i,\beta} \to \tilde{V}_i \text{ in } H^1(\Omega) \cap C^{0,\alpha}(\overline{\Omega}),$$

where $(\tilde{V}_1, \tilde{V}_2) \in \tilde{M}$ *achieves* c_∞.

Proof. First of all, we notice that, for every $(\tilde{v}_1, \tilde{v}_2)$ fixed,

$$\beta_1 \leq \beta_2 \leq +\infty \qquad \Longrightarrow \qquad J_{\beta_1}(\tilde{v}_1, \tilde{v}_2) \leq J_{\beta_2}(\tilde{v}_1, \tilde{v}_2).$$

We deduce that c_β is increasing in β and bounded by c_∞, thus it converges. If the pair $(\tilde{v}_{1,\beta}, \tilde{v}_{2,\beta})$ achieves c_β, $\beta < +\infty$, then $c_\beta \leq c_\infty$ implies

$$\text{both } \|(\tilde{v}_{1,\beta}, \tilde{v}_{2,\beta})\|^2_{H^1} \leq c_\infty + \gamma_1 + \gamma_2, \qquad \text{and } \int_\Omega \tilde{v}^2_{1,\beta}\tilde{v}^2_{2,\beta} \leq \frac{c_\infty}{\beta}.$$

We infer the existence of $(\tilde{V}_1, \tilde{V}_2)$ such that, up to subsequences, $\tilde{v}_{i,\beta} \to \tilde{V}_i$, weakly in H^1 and strongly in L^p, $p = 2, 4$. In particular $(\tilde{V}_1, \tilde{V}_2) \in M$ and $\tilde{V}_1 \cdot \tilde{V}_2 \equiv 0$. We have

$$c_\infty \geq \lim c_\beta = \lim J_\beta(\tilde{v}_{1,\beta}, \tilde{v}_{2,\beta}) \geq \liminf \int_\Omega |\nabla \tilde{v}_{1,\beta}|^2 + |\nabla \tilde{v}_{2,\beta}|^2 \geq \int_\Omega |\tilde{V}_1|^2 + |\tilde{V}_2|^2 \geq c_\infty.$$

Thus $(\tilde{V}_1, \tilde{V}_2)$ achieves c_∞, and the inequalities above are indeed equalities, proving convergence in H^1 norm and hence strong H^1 convergence. Furthermore, by a standard Brezis−Kato argument, the H^1 bounds along the sequence imply uniform ones (see [26], Proof of Cor. 1.6, p. 1264 for further details).

The last thing to prove is the boundedness in $C^{0,\alpha}$ (which will imply convergence in $C^{0,\alpha}$ too, by Ascoli's Theorem). Notice that $(\tilde{v}_{1,\beta}, \tilde{v}_{2,\beta})$ satisfies (3.2), and that $0 \leq \lambda_i \leq c_\infty/\gamma_i$. As a consequence, boundedness of the Hölder seminorm can be obtained as in Theorem 1.1 of [25], which provides the same result in the case of Dirichlet boundary conditions: since the proofs in [25] use blow-up arguments, in order to cover the Neumann case one just has to replace odd extensions (from the half-space to $\mathbb{R}^N$) with even ones. More precisely, this replacement has to be performed in Lemmas 3.4–3.6 of [25]. □

End of the proof of Theorem 1.3. The proof of such theorem easily descends from Lemmas 3.1 and 3.2, when going back to the original unknowns (3.1). In particular, notice that $(\tilde{V}_1, \tilde{V}_2) \in \tilde{M}$ achieves c_∞ if and only if $(\tilde{V}_1/\sqrt{\gamma_2}, \tilde{V}_2/\sqrt{\gamma_1}) \in M$ achieves (1.4). □

To conclude this section, we collect some properties of the minimizers associated to c_∞.

Proposition 3.3. *Let* $(\tilde{V}_1, \tilde{V}_2) \in \tilde{M}$ *achieve* c_∞. *Then* $\tilde{V}_1 \cdot \tilde{V}_2 \equiv 0$ *and there exist parameters* Λ_i *such that*

$$-\Delta(\tilde{V}_1 - \tilde{V}_2) = \Lambda_1 \tilde{V}_1 - \Lambda_2 \tilde{V}_2$$

(in particular, $\tilde{V}_1 - \tilde{V}_2 \in C^{2,\alpha}(\overline{\Omega})$*).*

Furthermore, in dimension $N = 1$, *let* $\Omega = (0, 1)$. *Then, the unique minimizer is (up to the reflection* $x \leftrightarrow 1 - x$*)*

$$\tilde{V}_1(x) = \sqrt{\frac{2\gamma_2}{x_0}} \cos\left(\frac{\pi}{2x_0} x\right) \cdot \chi_{[0,x_0]}(x),$$
$$\tilde{V}_2(x) = \sqrt{\frac{2\gamma_1}{1 - x_0}} \cos\left(\frac{\pi}{2(1 - x_0)}(1 - x)\right) \cdot \chi_{[x_0,1]}(x),$$

and $x_0 = \dfrac{\sqrt[3]{\gamma_2}}{\sqrt[3]{\gamma_1} + \sqrt[3]{\gamma_2}}$.

Proof. Let

$$J^*(w) = \int_\Omega |\nabla w|^2, \qquad M^* = \left\{ w \in H^1(\Omega) : \int_\Omega (w^+)^2 = \gamma_2,\ \int_\Omega (w^-)^2 = \gamma_1 \right\}.$$

Then, for component-wise positive pairs, $J_\infty(\tilde v_1, \tilde v_2)|_{\tilde M} = J^*(\tilde v_1 - \tilde v_2)|_{M^*}$, and the first part of the proposition follows by the Lagrange multipliers rule (and by standard elliptic regularity).

Turning to the monodimensional case, we have that $(\tilde V_1, \tilde V_2) \in H^1(0,1) \times H^1(0,1)$ satisfies

$$-(\tilde V_1 - \tilde V_2)'' = \Lambda_1 \tilde V_1 - \Lambda_2 \tilde V_2, \qquad \tilde V_1 \cdot \tilde V_2 \equiv 0, \qquad \text{in } (0,1) \tag{3.3}$$

with Neumann boundary conditions. By elementary considerations we deduce the existence of (at most countable) disjoint open intervals $I_{i,n}$, with $i = 1,2$ and $n \in \mathcal{N}_i \subset \mathbb{N}$, such that

$$\tilde V_i(x) = \sum_{n \in \mathcal{N}_i} a_{i,n} \cos\left(\sqrt{\Lambda_i}(x - x_{i,n})\right) \cdot \chi_{I_{i,n}}(x),$$

where

$$I_{i,n} = \begin{cases} \left(0, \frac{\pi}{2\sqrt{\Lambda_i}}\right) & \text{if } x_{i,n} = 0 \\ \left(x_{i,n} - \frac{\pi}{2\sqrt{\Lambda_i}}, x_{i,n} - \frac{\pi}{2\sqrt{\Lambda_i}}\right) & \text{if } x_{i,n} \in \left(\frac{\pi}{2\sqrt{\Lambda_i}}, 1 - \frac{\pi}{2\sqrt{\Lambda_i}}\right), \\ \left(1 - \frac{\pi}{2\sqrt{\Lambda_i}}, 1\right) & \text{if } x_{i,n} = 1 \end{cases} \qquad \sum_n \frac{\pi}{2} |I_{i,n}| a_{i,n}^2 = \gamma_j.$$

Now, also the pair defined by

$$\tilde W_i = \frac{2\gamma_j}{\pi |I_{i,1}| a_{i,1}^2} \tilde V_i|_{I_{i,1}}$$

achieves c_∞; as a consequence, $\tilde W_1 - \tilde W_2$ solves (3.3), while $\tilde W_1 - \tilde W_2 \equiv 0$ outside $I_{1,1} \cup I_{2,1}$. We deduce that both $\mathcal{N}_i$ are singletons, and finally that

$$c_\infty = \min \left\{ \int_0^1 (w_1')^2 + (w_2')^2 : \begin{array}{l} w_1(x) = \sqrt{\frac{2\gamma_2}{x_1}} \cos\left(\frac{\pi}{2x_1} x\right) \cdot \chi_{[0,x_1]}(x) \\ w_2(x) = \sqrt{\frac{2\gamma_1}{1-x_2}} \cos\left(\frac{\pi}{2(1-x_2)}(1-x)\right) \cdot \chi_{[x_2,1]}(x) \\ 0 < x_1 \le x_2 < 1 \end{array} \right\},$$

whose unique solution can be computed by elementary tools. □

4. Bifurcation results

In this section we apply tools from global bifurcation theory in order to prove Theorem 1.4. The main references are the celebrated papers by Rabinowitz [30] and Crandall and Rabinowitz [14], which deal respectively with global bifurcation results for odd eigenvalues, and local ones for simple eigenvalues; for some details about the asymptotic expansions in the latter case, we refer also to [2], Chapter 5. For the reader's convenience, we recall here the two statements we will apply.

Theorem 4.1 ([30], Thm. 1.3)**.** *Let E be a Banach space, and let $G\colon \mathbb{R} \times E \to E$, continuous and compact, be such that*

$$G(\beta, v) = \beta L v + H(\beta, v),$$

with L linear and compact and $H(\beta, v) = o(\|v\|)$ as $v \to 0$, uniformly on bounded β intervals.

*If β^* is a characteristic value (*i.e.* $1/\beta^*$ is an eigenvalue) of L, having odd multiplicity, then*

$$\mathcal{S} := \overline{\{(\beta, v) : v = G(\beta, v),\ v \neq 0\}}$$

possesses a maximal subcontinuum $\mathcal{C}$ such that $(\beta^*, 0) \in \mathcal{C}$, and $\mathcal{C}$ either is unbounded in $\mathbb{R} \times E$, or $(\beta^{**}, 0) \in \mathcal{C}$, where $\beta^{**} \neq \beta^*$ is another characteristic value of L.

Theorem 4.2 ([14], Thms. 1.37, 1.18)**.** *Under the assumptions of Theorem 4.1, assume furthermore that G is of class C^2 and that $\partial^2_{\beta,v}G(\beta,0)=L$.*

If β^* is a simple characteristic value of L and $v^*\neq 0$ is such that

$$\mathrm{Ker}(I-\beta^* L)=\mathrm{span}\{v^*\},\qquad v^*\notin \mathrm{R}(I-\beta^* L),$$

then $\mathcal{S}$ is a continuous curve, locally near $(\beta^*,0)$, parameterized as

$$\varepsilon\mapsto(\beta,v)=(\beta^*+\varphi(\varepsilon),\varepsilon v^*+\varepsilon\psi(\varepsilon)),$$

where $\varphi(0)=0$, $\psi(0)=0$. If G is more regular, then also the above curve is, and one can write higher order expansions (see Rem. 4.6).

Among different possible choices, we will apply the above results in the ambient space

$$E:=\left\{v=(v_1,v_2)\in C^{2,\alpha}(\overline{\Omega},\mathbb{R}^2):\partial_n v_i=0\text{ on }\partial\Omega\right\}.$$

Lemma 4.3. *The map $G\colon\mathbb{R}\times E\to E$ defined as*

$$u=G(\beta,v)\qquad\Longleftrightarrow\qquad\begin{cases}-\Delta u_1+\beta g_1(v_2^2)u_1=\lambda_1 u_1 & \\ -\Delta u_2+\beta g_2(v_1^2)u_2=\lambda_2 u_2 & \text{in } \Omega\\ \int_\Omega u_1^2\,\mathrm{d}x=\int_\Omega u_2^2\,\mathrm{d}x=1,\quad u_1,u_2>0 & \\ \partial_n u_1=\partial_n u_2=0 & \text{on } \partial\Omega,\end{cases}$$

for suitable λ_i, is (well-defined and) of class C^2. Moreover it holds

$$\partial_v G(\beta,1,1)=\beta L,$$

where

$$z=Lw\qquad\Longleftrightarrow\qquad\begin{cases}-\Delta z_1=-2g_1'(1)\left[w_2-\int_\Omega w_2\right] & \\ -\Delta z_2=-2g_2'(1)\left[w_1-\int_\Omega w_1\right] & \text{in } \Omega\\ \int_\Omega z_1\,\mathrm{d}x=\int_\Omega z_2\,\mathrm{d}x=0 & \\ \partial_n z_1=\partial_n z_2=0 & \text{on } \partial\Omega.\end{cases}\tag{4.1}$$

Proof. The proof is based on standard smooth dependence of simple eigenvalues (and corresponding eigenfunctions) with respect to the potentials, see for instance the book ([17], Sect. 2.5). In turn, such smooth dependence can be shown using the Implicit Function Theorem. For the reader's convenience, we sketch some detail in the following. Let us consider the map $F\colon\mathbb{R}\times E\times E\times\mathbb{R}^2\to C^{0,\alpha}(\overline{\Omega},\mathbb{R}^2)\times\mathbb{R}^2$,

$$F(\beta,v,u,\lambda):=\begin{pmatrix}-\Delta u_1+\beta g_1(v_2^2)u_1-\lambda_1 u_1\\ -\Delta u_2+\beta g_2(v_1^2)u_2-\lambda_2 u_2\\ \int_\Omega u_1^2\,\mathrm{d}x-1\\ \int_\Omega u_2^2\,\mathrm{d}x-1\end{pmatrix}.\tag{4.2}$$

Let β,v be fixed. Then we can uniquely find positive eigenfunctions $u_i=u_i(\beta,v_j)$ and simple eigenvalues $\lambda_i=\lambda_i(\beta,v_j)$, such that $F=0$. As a consequence, it is possible to apply the Implicit Function Theorem in order to show that

$$F(\beta,v,u,\lambda)=0\qquad\Longleftrightarrow\qquad(u,\lambda)=\tilde{G}(\beta,v),$$

with $\tilde{G}\in C^2$ (recall that each g_i is of class C^2). More precisely, the invertibility of $\partial_{(u,\lambda)}F$ at any of the points above mentioned can be obtained by its injectivity (by Fredholm's Alternative).

Since G is the projection of $\tilde{G}$ on E, the first part of the lemma follows. Observing that

$$\tilde{G}(\beta,1,1)=(1,1,\beta g(1),\beta g(1)),$$

also the second part can be proved, by direct calculations. □

In order to apply the abstract results, we need to find the eigenvalues of the operator L defined in the previous lemma. In the following, for easier notation, we write

$$\alpha_i = g_i'(1) > 0 \qquad \text{(by assumption (1.2))}. \tag{4.3}$$

Lemma 4.4. *Let L be defined as in (4.1). Then*

$$\beta^* L v^* = v^*, \quad v^* \neq 0, \qquad \Longleftrightarrow \qquad \beta^* = \frac{\mu^*}{2\sqrt{\alpha_1\alpha_2}}, \quad v^* = \left(-\sqrt{\alpha_1}\psi^*, \sqrt{\alpha_2}\psi^*\right),$$

where μ^* is a positive Neumann eigenvalue of $-\Delta$ in Ω and ψ^* a corresponding eigenfunction.

Proof. Recall that $\beta^* L v^* = v^*$ if and only if, for both i,

$$\begin{cases} -\Delta v_i^* = -2\beta^*\alpha_i v_j^* & \text{in } \Omega \\ \int_\Omega v_i^* = 0 \\ \partial_n v_i^* = 0 & \text{on } \partial\Omega. \end{cases}$$

Setting

$$\psi_\pm = \sqrt{\alpha_2} v_1 \pm \sqrt{\alpha_1} v_2,$$

we obtain that the above system is equivalent to

$$\begin{cases} -\Delta \psi_\pm = \mp 2\beta^*\sqrt{\alpha_1\alpha_2}\psi_\pm & \text{in } \Omega \\ \int_\Omega \psi_\pm = 0 \\ \partial_n \psi_\pm = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, if $\beta^* \neq 0$, we infer that β^* is a characteristic value of L if and only if $\psi_+ = 0$ (by the Maximum Principle) and $2\beta^*\sqrt{\alpha_1\alpha_2}$ is an eigenvalue of $-\Delta$ with zero Neumann boundary conditions. Moreover, the characteristic vector space associated to β^* is generated by

$$\left(-\sqrt{\alpha_1}\psi^*, \sqrt{\alpha_2}\psi^*\right). \tag{4.4}$$

Finally, note that $\beta^* = 0$ is not a characteristic value of L, as $-\Delta\psi = 0$ and $\int_\Omega \psi = 0$ imply that $\psi \equiv 0$. □

The last ingredient we need is some control on the behavior of nontrivial solutions.

Lemma 4.5. *There exists a constant $C > 0$ such that*

1. $\mathcal{S} \subset \left\{(\beta, v) : \int_\Omega |\nabla v|^2 \leq C\beta\right\}$;
2. $\mathcal{S} \subset \{(\beta, v) : \beta \geq C\}$.

Proof. Recalling Lemma 2.4 we have that, in the present setting,

$$\int_\Omega |\nabla v_i|^2 \leq \lambda_i \leq \beta \int_\Omega g_i(v_j^2) \leq C_g \beta,$$

and the first inclusion follows. Concerning the second one, let by contradiction $(\beta_n, v_n)_n \subset \mathcal{S}$ be such that $\beta_n \to 0$. Then, by the first inclusion, $v_n \to (1, 1)$ in H^1 and, by elliptic regularity and a Brezis–Kato argument, also in E. We deduce that $\beta^* = 0$ corresponds to a bifurcation point, and therefore $\partial_v(\cdot - G(\beta, \cdot)) = I - \beta L$ can not be invertible at $\beta = 0$, a contradiction. □

We are ready to prove our main bifurcation results.

Proof of Theorem 1.4. First of all, let μ^* be a positive Neumann eigenvalue, with odd multiplicity, and

$$\beta^* = \frac{\mu^*}{2\sqrt{\alpha_1\alpha_2}}\cdot$$

Since G is compact, by Lemma 4.4 we are in a position to apply Theorem 4.1, obtaining a nontrivial branch which satisfies one of the alternatives there. Recalling that $\beta = 1/\nu$, we readily have the existence of a nontrivial branch $\mathcal{C}$ in the (ν, v)-space, satisfying all the conditions in (1.1), with the possible exception of the positivity ones. In view of Lemma 4.5 we have that

$$\mathcal{C} \subset \left\{(\nu, v) : \int_\Omega |\nabla v|^2 \leq \frac{C_1}{\nu},\ 0 < \nu \leq C_2\right\}.$$

Note that, in principle, $\mathcal{C} \cap \{\nu \geq \varepsilon > 0\}$ may be unbounded in $C^{2,\alpha}$. Recalling that, in dimension $N \leq 3$, the nolinearities in (1.1) are Sobolev subcritical, by standard elliptic regularity we have that H^1 bounds imply $C^{2,\alpha}$ ones, so that unboundedness can happen only as $\nu \to 0$. The last thing that is left to prove, to complete the first part of the theorem, is that the branch we obtained consists of componentwise positive pairs. This easily follows since, by the Maximum Principle, if the pairs $(v_{1,n}, v_{2,n})$ solve (1.1), with $\nu = \nu_n > 0$ and $\lambda_i = \lambda_{i,n}$, and

$$v_{i,n} \to \bar{v}_i, \qquad \nu_{i,n} \to \bar{\nu}_i, \qquad \lambda_{i,n} \to \bar{\lambda}_i,$$

then either $\bar{\nu} = 0$ or $\bar{\nu} > 0$ and $\bar{v}_1, \bar{v}_2$ are strictly positive in $\overline{\Omega}$.

Coming to the second part, let μ^* be a simple positive Neumann eigenvalue. By Lemma 4.4 we have that $\partial^2_{\beta,v} G(\beta, 0) = L$. In order to apply Theorem 4.2, we only have to check the compatibility condition, which in our case writes

$$(-\sqrt{\alpha_1}\psi^*, \sqrt{\alpha_2}\psi^*) \notin \mathrm{R}(I - \beta^* L)$$

(here ψ^* is an eigenfunction associated to $\mu^* = 2\sqrt{\alpha_1\alpha_2}\beta^*$). By contradiction, let us assume that $(-\sqrt{\alpha_1}\psi^*, \sqrt{\alpha_2}\psi^*) = (I - \beta^* L)w$, *i.e.*

$$\begin{cases} -\Delta w_i = -2\beta^*\alpha_i w_j + (-1)^i \mu^* \psi^* \sqrt{\alpha_i} & \text{in } \Omega \\ \int_\Omega w_i = 0 \\ \partial_n w_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Reasoning as in the proof of Lemma 4.4, it is easy to prove that $w = 0$, and hence $\psi^* = 0$, a contradiction. □

Remark 4.6. If we suppose that g_1 and g_2 are smooth, then the branch $\mathcal{S}$ bifurcating from $(\beta^*, 1, 1)$ is a smooth curve (at least in a neighborhood of that point), and its parametrization can be made more precise. In order to simplify the following computations, we set

$$(\beta, v) \in \mathcal{S} \Leftrightarrow 0 = \hat{F}(\beta, v, \lambda) := F(\beta, v, v, \beta\lambda)$$

for some $\lambda \in \mathbb{R}^2$, where F is as in (4.2). Then, $\hat{F}\colon \mathbb{R} \times E \times \mathbb{R}^2 \to C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times \mathbb{R}^2$ is smooth and satisfies

$$\hat{F}_v(\beta, v, \lambda)[w, \ell] = \left(-\Delta w_i + \beta(2g_i'(v_j^2)v_iv_jw_j + g_i(v_j^2)w_i - \ell_iv_i - \lambda_iw_i), 2\int_\Omega v_iw_i\right), \tag{4.5}$$

$$\hat{F}_{v,\beta}(\beta, v, \lambda)[w, \ell] = (2g_i'(v_j^2)v_iv_jw_j + g_i(v_j^2)w_i - \ell_iv_i - \lambda_iw_i, 0), \tag{4.6}$$

$$\begin{aligned} \hat{F}_{v,v}(\beta, v, \lambda)[w, \ell; h, p] = \big(&\beta[4g_i''(v_j^2)v_iv_j^2 + 2g_i'(v_j^2)v_i]w_jh_j \\ &+2\beta g_i'(v_j^2)v_jw_jh_i + 2\beta g_i'(v_j^2)v_jw_ih_j - \beta\ell_ih_i - \beta p_iw_i, 2\int_\Omega h_iw_i\bigg), \end{aligned} \tag{4.7}$$

$$\begin{aligned} \hat{F}_{v,v,v}(\beta, v, \lambda)[w, \ell; h, p; z, q] = \big(&\beta[8g_i'''v_iv_j^3 + 8g_i''v_iv_j + 4g_i''v_iv_j]w_jh_jz_j + \beta[4g_i''v_j^2 + 2\beta g_i']w_jh_jz_i \\ &+ \beta[4g_i''v_j^2 + 2g_i']w_jh_iz_j + \beta[4g_i''v_j^2 + 2g_i']w_ih_jz_j, 0). \end{aligned} \tag{4.8}$$

If $(\beta^*, 1, 1, g_1(1), g_2(1))$ is a simple bifurcation point, then $\mathrm{Ker}(\hat{F}_v)$ is spanned by the vector $V^* = (-\sqrt{\alpha_1}\psi^*, \sqrt{\alpha_2}\psi^*, 0, 0)$, and $R(\hat{F}_v) = \{(\Psi, \cdot) = 0\}$, where $\Psi = (-\sqrt{\alpha_2}\psi^*, \sqrt{\alpha_1}\psi^*, 0, 0)$. Therefore, arguing as in ([2] Chap. 5), if we set

$$A := (\Psi, \hat{F}_{v,\beta}[V^*]), \quad B := \frac{1}{2}(\Psi, \hat{F}_{v,v}[V^*, V^*]), \quad C := -\frac{1}{6A}(\Psi, \hat{F}_{v,v,v}, [V^*]^3)$$

where all the derivatives of $\hat{F}$ are evaluated at $(\beta^*, 1, 1, g_1(1), g_2(1))$, the following expansions hold true

$$\beta = \beta^* - \frac{B}{A}\varepsilon + o(\varepsilon), \quad (\text{if } B \neq 0),$$

and

$$\begin{aligned} v &= (1,1) + \tfrac{A}{B}(\beta - \beta^*) \cdot (\sqrt{\alpha_1}\psi^*, -\sqrt{\alpha_2}\psi^*) + o(\beta - \beta^*) \ \text{ if } B \neq 0, \\ v &= (1,1) \pm \left(\tfrac{\beta - \beta^*}{C}\right)^{1/2} \cdot (\sqrt{\alpha_1}\psi^*, -\sqrt{\alpha_2}\psi^*) + O(\beta - \beta^*) \text{ if } B = 0, C \neq 0. \end{aligned}$$

Note that in the latter case, if $C > 0$ (respectively, $C < 0$) the bifurcating branch emanates on the right (respectively, left) of β^*. In our case, the coefficients A, B, C have the explicit form

$$\begin{aligned} A &= -4g_1'g_2' \int (\psi^*)^2 < 0, \\ B &= \beta^*[2g_2''\sqrt{(g_1')^3} - 2g_1''\sqrt{(g_2')^3} + 3g_1'g_2'(\sqrt{g_1'} - \sqrt{g_2'})] \int (\psi^*)^3, \\ C &= \frac{\beta^*}{-6A}\left[12g_1'g_2'\sqrt{g_1'g_2'} + \sum_{i=1,2}(-8(g_j')^2 g_i''' + 12g_i''(g_j'\sqrt{g_i'g_j'} - (g_j')^2))\right] \int (\psi^*)^4, \end{aligned}$$

where all the derivatives of g_i are evaluated at $s = 1$.

We observe that if $N = 1$, the bifurcation is always *critical*, namely $B = 0$, as every eigenfunction ψ^* satisfies $\int (\psi^*)^3 = 0$. In the variational case, where $g_i''(1) = g_i'''(1) = 0$, the bifurcating branch emanates on the right, namely $(\beta, v) \in \mathcal{S}$ is such that $\beta \geq \beta^*$ (and therefore $\nu \leq \nu^*$), at least in a neighborhood of β^*.

5. Classification of solutions in dimension $N = 1$

In this section we restrict our attention to the case $\Omega = (0,1) \subset \mathbb{R}$. Consequently, in the following (v_1, v_2) denotes a solution of the problem $(i, j = 1, 2,\ j \neq i)$

$$\begin{cases} -\nu v_i'' + g_i(v_j^2)v_i = \lambda_i v_i & \text{in } (0,1) \\ \int_0^1 v_i^2 \,\mathrm{d}x = 1, \qquad v_i > 0 & \text{in } [0,1] \\ v_i'(0) = v_i'(1) = 0. \end{cases} \tag{5.1}$$

In particular, each v_i is $C^2([0,1])$, and it has at least an inflection point in $(0,1)$ (just apply Rolle's theorem to v_i'). Furthermore, $v_i''(x)$ has the same sign of $g_i(v_j^2(x)) - \lambda_i$, for every x.

Lemma 5.1. *v_i and v_j have opposite concavity at 0 and 1. More precisely:*

- $g_i(v_j^2(0)) > \lambda_i \iff g_j(v_i^2(0)) < \lambda_j$;
- $g_i(v_j^2(1)) > \lambda_i \iff g_j(v_i^2(1)) < \lambda_j$.

Proof. Let us assume, for instance, $g_i(v_j^2(0)) > \lambda_i$ and, by contradiction, $g_j(v_i^2(0)) \geq \lambda_j$ (the other cases are analogous).

Then $v_i''(0) > 0$, and there exists $\xi \in (0,1]$ such that

$$v_i'' > 0 \text{ in } [0,\xi), \qquad v_i''(\xi) = 0 \text{ (and hence } g_i(v_j^2(\xi)) = \lambda_i). \tag{5.2}$$

Notice that $\xi < 1$, otherwise v_i would have no inflection point in $(0,1)$. By convexity and monotonicity we deduce that

$$x \in (0,\xi] \quad \Longrightarrow \quad v_i(x) > v_i(0) \quad \Longrightarrow \quad g_j(v_i^2(x)) - \lambda_j > g_j(v_i^2(0)) - \lambda_j \geq 0.$$

But then also v_j is (convex and) increasing in $[0,\xi]$, so that

$$g_i(v_j^2(\xi)) > g_i(v_j^2(0)) > \lambda_i,$$

in contradiction with (5.2). □

Next we exploit standard uniqueness results for ODEs in order to detect a number of situations in which a considered solution is the trivial one.

Lemma 5.2. *Let one of the following conditions hold:*

1. *there exists* $\xi \in [0,1]$ *such that*

$$g_1(v_2^2(\xi)) = \lambda_1, \qquad g_2(v_1^2(\xi)) = \lambda_2, \qquad v_1'(\xi) = v_2'(\xi) = 0;$$

2. *there exist* $0 \leq x_1 < x_2 \leq 1$ *such that, for some* i*,* v_i *is constant in* $I = [x_1,x_2]$*;*
3. *for some* i*,* $g_i(v_j^2(0)) = \lambda_i$*;*
4. *for some* i*,* $g_i(v_j^2(1)) = \lambda_i$*.*

Then (v_1,v_2) *is the trivial solution.*

Proof. Under the assumptions of case 1, uniqueness for the Cauchy problem

$$\begin{cases} -\nu v_i'' + g_i(v_j^2)v_i = \lambda_i v_i & \text{in } (0,1) \\ v_i(\xi) = \sqrt{g_j^{-1}(\lambda_j)}, \quad v_i'(\xi) = 0, & i = 1,2,\ j \neq i \end{cases}$$

implies that both v_1 and v_2 are constant, and we can conclude exploiting the normalization in $L^2(0,1)$.

If 2 holds, the equation for v_i implies $g_i(v_j^2(\xi)) = \lambda_i$ on I. But then also v_j is constant in I, forcing $g_j(v_i^2(\xi)) = \lambda_j$. Since both v_i' and v_j' are identically zero in I, case 1 applies.

Recalling the Neumann boundary conditions, also cases 3 and 4 can be reduced to 1: indeed, by Lemma 5.1, $g_i(v_j^2) - \lambda_i$ vanishes at one endpoint if and only if $g_j(v_i^2) - \lambda_j$ does. □

The following key lemma asserts that between two consecutive maxima of each v_i there exists an interval of concavity of v_j.

Lemma 5.3. *Let* $0 \leq x_1 < x_2 \leq 1$ *be such that, for some* i*,*

$$v_i'(x_1) = v_i'(x_2) = 0, \qquad v_i''(x_1) \leq 0,\ v_i''(x_2) \leq 0.$$

Then either (v_1,v_2) *is the trivial solution, or there exists* $\xi \in (x_1,x_2)$ *such that*

$$g_j(v_i^2(\xi)) < \lambda_j.$$

Analogously, if v_i' *vanishes and* v_i'' *is nonnegative at* x_1,x_2 *then* $g_j(v_i^2(\xi)) > \lambda_j$ *for some* $\xi \in (x_1,x_2)$*.*

Proof. We have to show that, in case $g_j(v_i^2(x)) \geq \lambda_j$ for every $x \in [x_1, x_2]$, then (v_1, v_2) is the trivial solution. Under such assumption we have that

$$\begin{cases} v_j'' \geq 0 & \text{in } (x_1, x_2) \\ g_i(v_j^2) \leq \lambda_i & \text{at } \{x_1, x_2\}, \end{cases} \qquad \text{so that } g_i(v_j^2) \leq \lambda_i \text{ in the whole } [x_1, x_2].$$

Thus $v_i'' \leq 0$ in $[x_1, x_2]$. Since $v_i' = 0$ at x_1 and x_2, we obtain that v_i is constant in $[x_1, x_2]$, and Lemma 5.2 (case 2) applies, concluding the proof. □

The above result provides a sharp control on the critical and inflection points of each v_i, as we show in the next sequence of lemmas.

Lemma 5.4. *If (v_1, v_2) is non trivial then both components have only isolated critical points.*

Proof. Let by contradiction $\xi \in [0, 1]$ be an accumulation point for the set of critical points of v_i. Of course

$$v_i'(\xi) = 0.$$

We recall that, for any pair of critical points $x_1 < x_2$, if both $v_i''(x_1) > 0$ and $v_i''(x_2) > 0$ then there exists a third critical point $y_1 \in (x_1, x_2)$ such that $v_i''(y_1) \leq 0$ (and the same holds for opposite inequalities). Using this fact, it is not difficult to construct two sequences $x_n \to \xi$, $y_n \to \xi$ such that

$$v_i'(x_n) = v_i'(y_n) = 0, \qquad v_i''(x_n) \geq 0,\ v_i''(y_n) \leq 0.$$

Applying repeatedly Lemma 5.3 we deduce the existence of sequences $\overline{\xi}_n \to \xi$, $\underline{\xi}_n \to \xi$ such that $g_j(v_i^2(\underline{\xi}_n)) < \lambda_j < g_j(v_i^2(\overline{\xi}_n))$. This promptly yields

$$g_j(v_i^2(\xi)) = \lambda_j.$$

Now back to the sequence (x_n), applying Rolle's Theorem we first deduce the existence of a sequence $z_n \to \xi$ such that $0 = \nu v_i''(z_n) = g_i(v_j^2(z_n)) - \lambda_i$, implying

$$g_i(v_j^2(\xi)) = \lambda_i,$$

and then of a sequence $z_n' \to \xi$ with

$$0 = v_j'(z_n') \to v_j'(\xi).$$

Summing up, we are in a position to apply Lemma 5.2 (case 1), obtaining that (v_1, v_2) is trivial, a contradiction. □

Lemma 5.5. *Let $x_0 \in [0, 1]$ be a point of local minimum for v_i. Then either (v_1, v_2) is the trivial solution, or*

$$g_j(v_i^2(x_0)) < \lambda_j, \qquad g_i(v_j^2(x_0)) > \lambda_i$$

(in particular, it is non degenerate). An analogous statement (with reverse inequalities) holds for local maxima.

Proof. If $x_0 = 0$ or $x_0 = 1$, then the statement is a consequence of Lemma 5.1. Otherwise, since x_0 is an isolated critical point, it is a strict minimum, and the following points are well defined:

$$x_1 = \inf\{x \in [0, x_0) : v_i' < 0 \text{ in } (x, x_0)\}, \qquad x_2 = \sup\{x \in (x_0, 1] : v_i' > 0 \text{ in } (x_0, x)\}.$$

Then x_1, x_2 satisfy the assumptions of Lemma 5.3, providing the existence of $\xi \in (x_1, x_2)$ such that

$$g_j(v_i^2(x_0)) = \min_{[x_1, x_2]} g_j(v_i^2) \leq g_j(v_i^2(\xi)) < \lambda_j,$$

which is the first inequality required.

On the other hand, since x_0 is an isolated strict minimum we have that $g_i(v_j^2(x_0)) \geq \lambda_i$ in a neighborhood of x_0. Since the last inequality implies that v_j is strictly concave in a neighborhood of x_0, we deduce also the second (strict) inequality. □

Lemma 5.6. *If* (v_1, v_2) *is not the trivial solution, then any critical point of each component is non degenerate.*

Proof. Let us assume by contradiction that ξ is a degenerate critical point of v_i. By Lemmas 5.4 and 5.5 we have that $\xi \in (0,1)$ is an isolate inflection point. Therefore

$$v_i'(\xi) = 0, \qquad g_i(v_j^2(\xi)) = \lambda_i,$$

and ξ is a local extremum for v_j. But then Lemma 5.5 applies again, implying that either $g_i(v_j^2(\xi)) > \lambda_i$ or $g_i(v_j^2(\xi)) < \lambda_i$, a contradiction. □

Lemma 5.7. *Let* (v_1, v_2) *be non trivial, and* $x_1 < x_2$ *be such that, for some* i,

$$v_i'(x_1) = v_i'(x_2) = 0, \qquad v_i' > 0 \text{ in } (x_1, x_2).$$

Then both v_i *and* v_j *have exactly one inflection point in* $[x_1, x_2]$. *An analogous statement holds for the opposite monotonicity.*

Proof. By Lemma 5.5 we immediately deduce the existence of $\xi \in (x_1, x_2)$ such that

$$v_j'' < 0 \text{ in } [x_1, \xi), \qquad v_j'' > 0 \text{ in } (\xi, x_2], \tag{5.3}$$

and v_j has exactly one inflection point in $[x_1, x_2]$.

On the other hand, the inflection points of v_i are the solutions of

$$g_j(v_i^2(x)) = \lambda_j, \qquad x \in [x_1, x_2]. \tag{5.4}$$

Since $g_j(v_i^2(x_1)) > \lambda_j$ and $g_j(v_i^2(x_2)) < \lambda_j$ (and again by Lem. 5.5), equation (5.4) has an odd number of solutions. On the other hand, taking into account (5.3), equation (5.4) has at most one solution in $[x_1, \xi]$ and one in $[\xi, x_2]$, respectively. □

Collecting the previous results we have the following characterization of nontrivial solutions.

Proposition 5.8. *If* (v_1, v_2) *is not the trivial solution, then there exists* $k \in \mathbb{N}$ *such that both* v_1 *and* v_2 *have exactly* k *critical points, all non degenerate, and* $k+1$ *isolated inflection points in* $(0,1)$.

Proof. Let n_i denote the number of critical points of v_i in $(0,1)$ (they are well defined by Lem. 5.4), and m_i the number of inflection points. Recalling that also $x = 0$ and $x = 1$ are local extrema for both components, by Lemma 5.7 we have that $n_i + 1 = m_i = m_j = n_j + 1$, and the claim follows. □

We are ready to conclude the proof of the main result of this section.

Proof of Theorem 1.7 (First part). First of all let $\mathcal{C} \subset \mathcal{S}$ be a continuum of nontrivial solutions, and

$$\mathcal{C} \ni (\nu_n, v_{1,n}, v_{2,n}) \to (\bar\nu, \bar v_1, \bar v_2), \qquad \text{as } n \to +\infty.$$

Using Proposition 5.8, it is not difficult to prove that, if the number of interior critical points of each $v_{i,n}$ is constant, and equal to k, then

- either $\bar\nu = 0$;
- or $\bar\nu > 0$ and $(\bar v_1, \bar v_2)$ is the trivial solution;
- or $\bar\nu > 0$ and each $\bar v_i$ has exactly k interior critical points.

Now recall that, being $N = 1$ and $\Omega = (0,1)$, the Neumann eigenvalues and eigenfunction of $-\partial^2_{xx}$ have the form

$$\mu_k = (k\pi)^2, \quad \psi_k = A\cos(k\pi\, x)\ (A \neq 0), \qquad k \in \mathbb{N},$$

and every eigenvalue μ_k is simple. Applying Theorem 1.4 we have the existence, for every $k \geq 1$, of continua $\mathcal{C}_k \subset \mathcal{S}$ which consist, locally near $(\mu_k, 1, 1)$, of pairs having exactly $k-1$ critical points (by the local parameterization, because also ψ_k has exactly $k-1$ critical points). The initial argument tells that each $\mathcal{C}_k$ is characterized by the number of critical points of its components, so that two of them cannot meet, and each of them is unbounded in the sense of Theorem 1.4 (since we are in dimension $N = 1 \leq 3$).

We are only left to prove segregation: this is the object of the next section. □

6. Segregation in dimension $N = 1$

As we already mentioned (see Rem. 1.6), we can not expect that all arbitrary families of nontrivial solutions segregate. Nonetheless, restricting our attention to $\mathcal{C}_k$ as in Theorem 1.7, for some fixed k, we can obtain more precise results.

In the following, we focus on $(\nu_n, v_{1,n}, v_{2,n}) \subset \mathcal{C}_k$, a sequence of solutions of (1.1), with $\nu = \nu_n > 0$ and $\lambda_i = \lambda_{i,n}$, whose components have exactly $k-1$ critical points in $(0,1)$, all non-degenerate. For easier notation, we will drop the subscript n throughout the proofs, except when some confusion may arise; in particular, properties of

$$v_1, v_2, \lambda_1, \lambda_2 \qquad \text{as } \nu \to 0,$$

are those of the considered sequence, when $\nu_n \to 0$ as $n \to +\infty$.

As a first step, we want to rule out the possibility that the branch "collapses" to the trivial solution as $\nu \to 0$.

Proposition 6.1. *Suppose that*

$$v_{1,n} \to 1, \qquad v_{2,n} \to 1, \qquad \nu_n \to \bar{\nu},$$

where the convergence is uniform in $[0,1]$*. Then,* $\lambda_{i,n} \to g_i(1)$ *and* $\bar{\nu} > 0$*.*

Proof. The proof will be carried out in three steps, and considering the system solved by $u_i := v_i - 1$, which is

$$\begin{cases} -\nu u_1'' = (\lambda_1 - G_1(1+u_2))(1+u_1), \text{ in } (0,1) \\ -\nu u_2'' = (\lambda_2 - G_2(1+u_1))(1+u_2), \\ u_1' = u_2' = 0 \quad \text{at } \{0,1\}, \\ \int_0^1 (1+u_1)^2 = \int_0^1 (1+u_2)^2 = 1, \end{cases} \tag{6.1}$$

where we have set $G_i(t) := g_i(t^2)$ for all $t \geq 0$. Note that $u_i \to 0$ uniformly in $[0,1]$.

Without loss of generality, we set $(\bar{x} = \bar{x}_n)$

$$M := \max_{i=1,2,\, x\in[0,1]} |u_i(x)| = u_1(\bar{x}). \tag{6.2}$$

Step 1. $|\lambda_i - G_i(1)|/M \to 0$ as $n \to \infty$. Indeed, note first that $\int_0^1 (1+u_i)^2 = 1$ implies that

$$\int_0^1 u_i = -\frac{1}{2}\int_0^1 u_i^2. \tag{6.3}$$

Moreover, a Taylor expansion in the equations of (6.1) gives

$$-\nu u_i'' = \left(\lambda_i - G_i(1) - G_i'(1)u_j - \frac{G_i''(\xi)}{2}u_j^2\right)(1+u_i),$$

where ξ is a bounded function in $(0,1)$ (uniformly with respect to n). By integrating the equation and using the boundary conditions we obtain

$$(\lambda_i - G_i(1)) \int_0^1 (1+u_i) = G_i'(1) \int_0^1 u_j + G_i'(1) \int_0^1 u_i \, u_j + \int_0^1 \frac{G_i''(\xi)}{2} u_j^2 (1+u_i).$$

Hence, using (6.3),

$$|\lambda_i - G_i(1)| \int_0^1 (1+u_i) \leq \frac{G_i'(1)}{2} \int_0^1 u_j^2 + G_i'(1) \int_0^1 |u_i| \, |u_j| + \int_0^1 \frac{G_i''(\xi)}{2} u_j^2 (1+u_i),$$

which leads to the assertion, as $\int_0^1 (1+u_i) \to 1$, $|u_i|, |u_j| \leq M$ in $[0,1]$ and $M \to 0$. The first conclusion of the proposition also follows, as $G_i(1) = g_i(1)$.

Step 2. Assume by contradiction that $\bar{\nu} = 0$. We proceed with a blow-up analysis, setting

$$\tilde{u}_i(x) = \frac{1}{M} u_i \left(\sqrt{\nu}\, x + \bar{x}\right) \qquad \forall x \in \left(-\frac{\bar{x}}{\sqrt{\nu}}, \frac{1-\bar{x}}{\sqrt{\nu}}\right) =: \widetilde{\Omega}_n.$$

We have that $|\tilde{u}_i| \leq 1$ in $\widetilde{\Omega}_n$ and $\tilde{u}_1(0) = 1$. Moreover, $\tilde{u}_i$ solves

$$-\tilde{u}_i'' = \left(\frac{\lambda_i - G_i(1)}{M} - (G_i'(1) + o(1))\tilde{u}_j\right)(1+u_i) \qquad \text{in } \widetilde{\Omega}_n.$$

Note that (up to subsequences)

$$\widetilde{\Omega}_n \to \widetilde{\Omega}_\infty := \begin{cases} [\bar{X}, +\infty) & \text{if } -\frac{\bar{x}}{\sqrt{\nu}} \to \bar{X} < +\infty \\ (-\infty, \bar{X}] & \text{if } \frac{1-\bar{x}}{\sqrt{\nu}} \to \bar{X} < +\infty \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

Using the equation (twice) and the uniform boundedness of $\tilde{u}_i$ on $\widetilde{\Omega}_n$, we argue that $\tilde{u}_i'''$ is bounded on compact subsets of $[0,\infty)$, uniformly as $n \to \infty$. Hence, $\tilde{u}_i \to \widetilde{U}_i \in C^2(\widetilde{\Omega}_\infty)$ locally in $C^{2,\alpha}$ where $\widetilde{U}_i$ has at most k intervals of monotonicity and solve, in $\widetilde{\Omega}_\infty$,

$$\begin{cases} \widetilde{U}_1'' = G_1'(1)\widetilde{U}_2 \\ \widetilde{U}_2'' = G_2'(1)\widetilde{U}_1, \end{cases} \tag{6.4}$$

in view of the conclusion of Step 1. Note that, in case $\widetilde{\Omega}_\infty \neq \mathbb{R}$, we can use the Neumann conditions in order to extend $\widetilde{U}_i$ by even reflection around $\bar{X}$, in such a way that $\widetilde{U}_1, \widetilde{U}_2$ solve (6.4) in the whole $\mathbb{R}$.

Step 3. To reach a contradiction we are going to show that system (6.4) does not admit nontrivial bounded solutions having a finite number of oscillations (recall that $\widetilde{U}_1(0) = 1$). We can reason as in Section 4, setting

$$W_\pm = \sqrt{\alpha_2}\widetilde{U}_1 \pm \sqrt{\alpha_1}\widetilde{U}_2,$$

and obtaining a decoupled system:

$$\begin{cases} W_+'' = \sqrt{\alpha_1\alpha_2}\, W_+, \\ W_-'' = -\sqrt{\alpha_1\alpha_2}\, W_-. \end{cases}$$

Therefore, since W_+ is bounded it must be constant and identically zero. We deduce that $\widetilde{U}_1, \widetilde{U}_2$ are proportional, so that

$$\widetilde{U}_1'' = -\sqrt{\alpha_1\alpha_2}\, \widetilde{U}_1,$$

which forces $\widetilde{U}_1 \equiv 0$ (since it has at most $2k$ monotonicity intervals in $\mathbb{R}$). □

Next we turn to the case in which $\|v_i\|_{L^\infty}$ is uniformly bounded along the sequence, for both i. To treat such case we need the following Liouville-type result.

Lemma 6.2. *Let $V_i \in C^2(\mathbb{R})$, $0 \le V_i \le M$, $\Lambda_i \ge 0$ be such that*

$$-V_i'' = (\Lambda_i - g_i(V_j^2))V_i \qquad \text{in } \mathbb{R}.$$

If both V_i have at most a finite number of monotonicity intervals, then one of the following holds:

1. *either $V_1 \equiv 0$, $\Lambda_2 = 0$,*
2. *or $V_2 \equiv 0$, $\Lambda_1 = 0$,*
3. *or $V_1 \equiv V_2 \equiv 0$,*
4. *or $g_i(V_j^2) \equiv \Lambda_i$, $i = 1, 2$.*

Proof. First of all, we can reason as in Lemma 5.2 to show that, if some V_i is constant in an interval, then (V_1, V_2) is constant in $\mathbb{R}$, and as a consequence we always fall in one of the above cases. Secondly, assume that some $\Lambda_i = 0$: then V_i is constant, and again the lemma follows by elementary considerations.

We are left to deal with the case $\Lambda_1, \Lambda_2 > 0$ and V_1, V_2 non constant and strictly positive. Since both V_i have a finite number of monotonicity intervals, the equations imply that they also have a finite number of inflection points (and they have at least one, since they are bounded in $\mathbb{R}$). We deduce the existence of $a \in \mathbb{R}$ such that, say,

$$V_1', V_2', V_1'', V_2'' \text{ do not change sign in } (a, +\infty).$$

In particular, the limits $V_i(+\infty)$ exist and $V_i'(+\infty) = V_i''(+\infty) = 0$.

Assume that $V_i' \ge 0$ for $x > a$, so that $V_i'' \le 0$ in the same interval. Then $V_i(+\infty) > 0$, and

$$g_i(V_j^2(x)) \le \Lambda_i \text{ in } (a, +\infty), \qquad g_i(V_j^2(+\infty)) = \Lambda_i - \frac{V_i''(+\infty)}{V_i(+\infty)} = \Lambda_i,$$

so that also V_j' is non negative.

Now we can lower a in such a way that, say,

$$V_1''(a) = 0.$$

If $V_1' \le 0$ for $x > a$, we deduce that also $V_2' \le 0$ in the same interval. But then $\Lambda_1 - g_1(V_2^2)$ is increasing, and $V_1'' \le 0$ for $x > a$, a contradiction since V_1 is decreasing and bounded.

On the other hand, let $V_1' \ge 0$ for $x > a$. Then $V_1(+\infty) > 0$, and

$$g_1(V_2^2(a)) = \Lambda_1 = g_1(V_2^2(+\infty)) - \frac{V_1''(+\infty)}{V_1(+\infty)} = g_1(V_2^2(+\infty)),$$

forcing V_2 to be constant in $[a, +\infty)$, again a contradiction. □

Using the previous result, we can show that uniform L^∞ bounds imply segregation.

Lemma 6.3. *Assume that $\|v_{i,n}\|_\infty \le C$ for both i. Then, up to subsequences,*

$$\lambda_{i,n} \to 0, \quad i = 1, 2,$$

as $\nu_n \to 0$.

Proof. Let us assume by contradiction that, for instance, $\lambda_1 \not\to 0$. We choose a sequence $(x_{1,n})_n \subset [0,1]$ such that (omitting the subscript n)

$$v_1(x_1) := \max_{[0,1]} v_1 \geq 1, \qquad \text{and put } \tilde{v}_i(x) := v_i\left(x_1 + x\sqrt{\nu}\right).$$

Then, $\tilde{v}_i$ solves

$$-\tilde{v}_i''(x) = \left(\lambda_i - g_i(\tilde{v}_j^2(x))\right)\tilde{v}_i(x) \quad \text{in } (-x_1\nu^{-1/2}, (1-x_1)\nu^{-1/2}),$$

$\|\tilde{v}_i\|_\infty \leq C$, $\tilde{v}_i \geq 0$. The equations and (2.1) guarantee local C^3 boundedness of $\tilde{v}_i$, thus, up to subsequences, $\tilde{v}_i \to V_i$ locally in C^2. Moreover, $\lambda_i \to \Lambda_i \geq 0$. We argue that, possibly up to an even extension, each V_i has at most $2k$ intervals of monotonicity and

$$-V_i'' = (\Lambda_i - g_i(V_j^2))V_i \quad \text{in } \mathbb{R}.$$

Then, Lemma 6.2 applies, but since $V_1(0) \geq 1$ and $\Lambda_1 > 0$, we deduce that

$$g_i(v_j^2(x_1)) - \lambda_i \to 0 \text{ for both } i, \tag{6.5}$$

and also $\lambda_2 \not\to 0$ (as $g_2(v_1^2(x_1)) \geq g_2(1) > 0$). We can implement the same argument using

$$v_1(x_2) := \min_{[0,1]} v_1 \leq 1, \qquad \text{and } \tilde{w}_i(x) := w_i\left(x_2 + x\sqrt{\nu}\right).$$

Passing to the limit (we keep the same sequence $\lambda_i \to \Lambda_i > 0$ as before), and recalling Lemma 5.5, we have that $W_2(0) > 0$ and then

$$g_i(v_j^2(x_2)) - \lambda_i \to 0 \text{ for both } i. \tag{6.6}$$

Combining (6.5) and (6.6), we deduce that $v_1 \to 1$ uniformly on $[0,1]$.

Now, since also $\Lambda_2 > 0$, we can exchange the role of v_1 and v_2, obtaining that $v_2 \to 1$ too, in contradiction with Proposition 6.1. □

We are left to deal with the case of $\max_{[0,1]}(v_{1,n} + v_{2,n}) \to +\infty$, namely when v_1 or v_2 are not bounded uniformly in n. To treat this situation we need to exploit the finite number of maxima of each component along $\mathcal{C}_k$, as enlighten in the following lemma (for convenience we write explicitely the dependence on n).

Lemma 6.4. *Let* $\max_{[0,1]}(v_{1,n} + v_{2,n}) \to +\infty$. *There exist and index* i, *constants* $C, \delta > 0$ *(independent of* n*), and a sequence of intervals* $I_n \subset [0,1]$ *such that, up to subsequences:*

$$\begin{aligned} &|I_n| = \delta \\ &\max_{I_n} v_{i,n} = \max_{\partial I_n} v_{i,n} \to +\infty \\ &\max_{I_n} v_{j,n} \leq C. \end{aligned}$$

Proof. Let

$$Z_n := \left\{z \in [0,1] : z \text{ is a local maximum for } v_{i,n}, \text{ for some } i, \text{ and } v_{i,n}(z) \to +\infty\right\}.$$

Since we are considering elements of $\mathcal{C}_k$, we have that

$$Z_n = \{z_{l,n}\}_{l=1,\dots,h}, \qquad z_{1,n} < \ldots < z_{h,n}, \qquad h \leq k+1$$

(recall that, by Lemma 5.5, if z is a local maximum for v_i then $g_i(v_j^2(z)) \leq \lambda_i$). Up to subsequences, we can assume that each $z_{l,n}$ is a maximum for some $v_{i,n}$, with i independent of n; furthermore we can assume that, for each l, $z_{l,n} \to z_l \in [0,1]$. We distinguish three cases.

Case 1. For some l, $z_l < z_{l+1}$. We choose i so that $z_{l,n}$ is a local maximum for $v_{i,n}$ and

$$2\delta = z_{l+1} - z_l, \qquad I_n = [z_{l,n}, z_{l,n} + \delta].$$

By construction, neither $v_{i,n}$ nor $v_{j,n}$ can have interior maxima which go to infinity; therefore the required properties for $\max_{I_n} v_{i,n}$ follow from the fact that $v_{i,n}(z_{l,n}) \to +\infty$, while those for $\max_{I_n} v_{j,n}$ descend again by Lemma 5.5.

Case 2. $z_1 = \ldots = z_h \neq 1$. One can reason as above, by choosing i so that $z_{h,n}$ is a local maximum for $v_{i,n}$ and

$$2\delta = 1 - z_h, \qquad I_n = [z_{h,n}, z_{h,n} + \delta].$$

Case 3. $z_1 = \ldots = z_h \neq 0$. We can choose i so that $z_{1,n}$ is a local maximum for $v_{i,n}$ and

$$2\delta = z_1, \qquad I_n = [z_{1,n} - \delta, z_{1,n}].$$ □

The last tool we need is the following standard comparison lemma.

Lemma 6.5 ([12], Lem. 4.4)**.** *Suppose that* $u \in C^2(a,b) \cap C([a,b])$ *satisfies*

$$-u''(x) \leq -Mu(x), \quad 0 \leq u(x) \leq A, \qquad \text{in } (a,b)$$

for some $A, M > 0$*. Then, for every* $0 < \delta < (b-a)/2$*,*

$$u(x) \leq 2A\, \mathrm{e}^{-\delta\sqrt{M}} \qquad \text{in } [a+\delta, b-\delta].$$

Proof. By comparison with the solution of $-w'' = -Mw$ in (a,b), $w(a) = w(b) = A$. □

Remark 6.6. By even reflection, we have that if u is as in Lemma 6.5 and furthermore $u'(a) = 0$, then the estimate holds on any $[a,b'] \subset [a,b)$, choosing $\delta = b - b'$.

We are in a position to prove that segregation occurs also when some v_i is unbounded, thus completing the proof of Theorem 1.7.

Lemma 6.7. *Let* $\max_{[0,1]}(v_{1,n} + v_{2,n}) \to +\infty$*. Then (up to subs.)* $\lambda_{i,n} \to 0$*, for some* i *(and the corresponding* $v_{i,n}$ *is not uniformly bounded).*

Proof. Let i, $I_n =: [z_n, z_n + \delta]$ be as in Lemma 6.4. We can assume, w.l.o.g.,

$$\max_{I_n} v_{i,n} = v_{i,n}(z_n) \to +\infty.$$

We define the blow-up sequences

$$\tilde{v}_{i,n}(x) := \frac{1}{v_{i,n}(z_n)} v_{i,n}(z_n + x\sqrt{\nu_n})$$
$$\tilde{v}_{j,n}(x) := v_{j,n}(z_n + x\sqrt{\nu_n}).$$

Then, $\tilde{v}_{i,n} = \tilde{v}_i$ solves

$$-\tilde{v}_i'' = (\lambda_i - g_i(\tilde{v}_j^2))\tilde{v}_i$$

in $(0, \delta\nu^{-1/2})$, $0 \leq \tilde{v}_i \leq 1$ and $\tilde{v}_i(0) = 1$. Also $\lambda_i - g_i(\tilde{v}_j^2)$ is uniformly bounded in $[0, \delta\nu^{-1/2}]$, by Lemmas 2.4 and 6.4. Since both $\tilde{v}_i$ and $\tilde{v}_i''$ are uniformly bounded on compact sets, we deduce that also $\tilde{v}_i'$ is bounded, and there exists $V \in C^1([0,+\infty))$ such that $v_i \to V$ in $C^1([a,b])$, for every $[a,b] \subset [0,+\infty)$.

We claim that, if $V > 0$ in $[a', b'] \subset (0, +\infty)$, then $\tilde{v}_j \to 0$ uniformly in $[a', b']$. Indeed, let $(a, b) \supset [a', b']$ be such that $V \geq \eta > 0$ in (a, b). We deduce that, in such interval,

$$-\tilde{v}_j'' = (\lambda_j - g_j(v_i^2(z)\tilde{v}_i^2))\tilde{v}_j \leq \left(\lambda_j - C_g^{-1} v_i^2(z) \frac{1}{2} V_i^2\right) \tilde{v}_j \leq -C v_i^2(z) \tilde{v}_j,$$

where $C > 0$ depends on η and C_g. Lemma 6.5 applies, yielding

$$0 \leq \tilde{v}_j \leq C_1 \mathrm{e}^{-C_2 v_i(z)} \to 0 \qquad \text{in } [a', b'],$$

as $C_2 > 0$ and $v_i(z) \to +\infty$.

Now, let $\lambda_i \to \Lambda \geq 0$. We can pass to the limit in the equation of $\tilde{v}_i$, deducing that

$$\begin{cases} V \in C^1([0, +\infty)), \qquad 0 \leq V \leq 1, \\ V > 0 \implies -V'' = \Lambda V \\ V(0) = 1. \end{cases}$$

Let $[0, a)$, $a \leq +\infty$, be the maximal interval containing 0 in which $V > 0$. If $a < +\infty$, by convexity we obtain that $V(a) = 0$ and $V'(a) < 0$, a contradiction since $V(x)$ must be non negative also for $x > a$. Therefore $a = +\infty$ and V is a bounded, concave function on $\mathbb{R}^+$, *i.e.* $V \equiv 1$ and $\Lambda = 0$. □

End of the proof of Theorem 1.7. Taking into account Corollary 2.5, the last part of the theorem follows from Lemmas 6.3 and 6.7. □

7. Further properties of the first branch

To conclude, we complete the analysis started in Section 6 by restricting our attention to the first bifurcation branch $\mathcal{C}_1$. Since $k = 1$, such branch consists of monotone solutions, and for concreteness we assume that the sequence we are considering is such that $v_{1,n}$ is decreasing and $v_{2,n}$ is increasing (and $\nu_n \to 0$ as $n \to \infty$). As before, we will omit the subscript n, when no confusion arises. We denote by $\xi_{1,n}, \xi_{2,n} \in (0, 1)$ the unique inflection points of the considered pair:

$$-v_{1,n}'(\xi_{1,n}) = \max_{[0,1]} |v_{1,n}'(x)|, \quad v_{2,n}'(\xi_{2,n}) = \max_{[0,1]} |v_{2,n}'(x)|.$$

A number of (rather elementary) *a priori* estimates can be deduced from the monotonicity of the components. We collect them in the following three lemmas.

Lemma 7.1. *Let $v_1' < 0$, $v_2' > 0$ on $(0, 1)$. The following inequalities hold*

$$v_1^2(x) \leq \frac{1}{x} \quad \forall x > 0, \qquad v_2^2(x) \leq \frac{1}{1 - x} \quad \forall x < 1, \tag{7.1}$$

$$\xi_1[v_1^2(0) + v_1(0)v_1(\xi_1) + v_1^2(\xi_1)] \leq 3, \tag{7.2}$$

$$(1 - \xi_2)[v_2^2(1) + v_2(1)v_2(\xi_2) + v_2^2(\xi_2)] \leq 3, \tag{7.3}$$

$$|v_1'(x)|(x - x_0) \leq x_0^{-1/2} \qquad \forall x_0 \geq \xi_1, x \in [x_0, 1], \tag{7.4}$$

$$|v_2'(x)|(x_0 - x) \leq (1 - x_0)^{-1/2} \qquad \forall x_0 \leq \xi_2, x \in [0, x_0]. \tag{7.5}$$

Proof. Estimates (7.1) follow by the L^2 constraint:

$$1 \geq \int_0^x v_1^2 \geq x\, v_1^2(x), \quad 1 \geq \int_x^1 v_2^2 \geq (1 - x) v_2^2(x).$$

For the other estimates, it is crucial to observe that $\lambda_1 - g_1(v_2^2(\xi_1)) = 0$, as $v_1''(\xi_1) = 0$ (ξ_1 is a point in $(0,1)$ where v_1' achieves its minimum). The function $\lambda_1 - g_1(v_2^2)$ is decreasing, so by the equation for v_1 in (1.1) we deduce that v_1 is concave on $[0,\xi_1]$ and convex on $[\xi_1, 1]$.

Concavity implies that

$$v_1(x) \geq v_1(0) + \xi_1^{-1}(v_1(\xi_1) - v_1(0))x$$

in $[0,\xi_1]$. By invoking the L^2 constraint of v_1 and integrating,

$$1 \geq \int_0^{\xi_1} v_1^2(x)\mathrm{d}x \geq \frac{\xi_1}{3}[v_1^2(0) + v_1(0)v_1(\xi_1) + v_1^2(\xi_1)],$$

and (7.2) follows. Similarly, concavity of v_2 on $[\xi_2, 1]$ produces (7.3).

By convexity of v_1 on $[x_0, 1]$, and (7.1),

$$-v_1'(x)(x-x_0) \leq -v_1'(x)(x-x_0) + v_1(x) \leq v_1(x_0) \leq x_0^{-1/2},$$

for all $x \in [x_0, 1]$, and we have (7.4). Similarly, (7.5) follows by concavity of v_2 on $[0, x_0]$. □

Lemma 7.2. *Suppose that* $\|v_{1,n}\|_\infty \leq C_1$. *Then,*

$$v_{1,n}^2(x) \geq \frac{1}{2} \quad \textit{in } [0, a_1], \tag{7.6}$$

where $a_1 = a_1(C_1)$. *Similarly, if* $\|v_{2,n}\|_\infty \leq C_2$,

$$v_{2,n}^2(x) \geq \frac{1}{2} \quad \textit{in } [a_2, 1], \tag{7.7}$$

where $a_2 = a_2(C_2)$.

Proof. In view of the L^2 constraint on v_1 and its monotonicity we have that

$$1 = \int_0^x v_1^2 + \int_x^1 v_1^2 \leq x\, v_1^2(0) + (1-x)v_1^2(x)$$

for all $x \in [0,1]$. Therefore, if $a_1 = (2C_1^2 - 1)^{-1}$,

$$v_1^2(a_1) \geq \frac{1 - a_1 v_1^2(0)}{1 - a_1} \geq \frac{1 - a_1 C_1^2}{1 - a_1} = \frac{1}{2}.$$

The assertion for v_1 follows. The estimate (7.7) for v_2 is analogous. □

Lemma 7.3. *For both* i *it holds*

$$|v_{i,n}^2(0) - v_{i,n}^2(1)| \leq 2\frac{\lambda_{i,n}}{\nu}, \tag{7.8}$$

$$\nu\|v_{i,n}'\|_\infty^2 \leq \lambda_{i,n}\|v_{i,n}\|_\infty^2. \tag{7.9}$$

Proof. We will prove the assertion when $i=1$, the argument is analogous when $i=2$. Multiplying the equation for v_1 by v_1 and integrating on $[0,x]$ yields

$$-\nu v_1'(x)v_1(x) + \nu\int_0^x (v_1')^2 = \int_0^x (\lambda_1 - g_1)v_1^2,$$

thus

$$-\frac{\nu}{2}(v_1^2)'(x) = -\nu v_1'(x)v_1(x) \leq \lambda_1.$$

By integrating again on $[0,1]$ we obtain (7.8).

On the other hand, testing the equation for v_1 by v_1' and integrating on $[0,\xi_1]$ we obtain

$$\frac{\nu}{2}v_1'(\xi_1)^2 = \frac{\lambda_1}{2}v_1(0)^2 - \frac{\lambda_1}{2}v_1(\xi_1)^2 + \int_0^{\xi_1} g_1 v_1 v_1',$$

and (7.9) follows since $v_1' \le 0$ in $[0,1]$. □

After the above preliminary estimates, the first part of our analysis is devoted to show that $\mathcal{C}_1$ enjoys uniform L^∞ bounds as $\nu \to 0$. To this aim we need two preliminary lemmas.

Lemma 7.4. *Suppose that, for some i, $\|v_{i,n}\|_\infty \le C$ and $\lambda_{j,n} \to 0$. Then, there exists $C' > 0$ that does not depend on n such that*

$$\lambda_{i,n} \le C'\nu_n.$$

Proof. We will detail the proof in the case $i = 1$. Note that

$$g_2(v_1^2(x)) - \lambda_2 \ge g_2(1/2) - \lambda_2 \ge C_g^{-1}/2 - \lambda_2 \ge C_g^{-1}/4 \quad \text{in } [0,a_1]$$

by the monotonicity of g_2, (7.6), (1.2) and $\lambda_2 \to 0$. Hence,

$$-v_2'' = -\frac{g_2(v_1^2) - \lambda_2}{\nu}\, v_2 \le \frac{C_g^{-1}}{4\nu}\, v_2$$

in $(0,a_1)$, and Lemma 6.5 (or better Rem. 6.6) allows to conclude that

$$v_2(x) \le 2v_2(a_1)\mathrm{e}^{-C/\sqrt{\nu}} \quad \text{in } [0,a_1/2], \tag{7.10}$$

for some $C = C(a_1, C_g^{-1}) > 0$.

Recalling Definition 1.1, we choose $w(x) := \sqrt{\frac{4}{a_1}}\cos\left(\frac{\pi}{a_1}x\right)$ for $x \in [0,a_1/2]$ and $w \equiv 0$ in $[a_1/2,1]$ to conclude that, for some $C' > 0$,

$$\lambda_1 \le \int_0^{a_1/2} \nu(w')^2 + g_1(v_2^2)w^2 \le \frac{\nu\pi}{a_1} + g_1\left(4v_2^2(a_1)\mathrm{e}^{-2C/\sqrt{\nu}}\right) \le \frac{\nu\pi}{a_1} + \frac{4C_g\mathrm{e}^{-2C/\sqrt{\nu}}}{1-a_1} \le C'\nu,$$

by (1.2), (7.10) and (7.1). □

Lemma 7.5. *Suppose that, for some i, $\|v_{i,n}\|_\infty \to +\infty$. Then,*

$$\nu\|v_{i,n}'\|_\infty^2 \le C(\lambda_{j,n} + \nu) \tag{7.11}$$

for some $C > 0$ that does not depend on n.

Proof. We will detail the proof in the case $i = 1$, thus assuming

$$v_1(0) \to +\infty.$$

Note that $|v_2'| \le c_2$ in $[0,1/2]$ for some $c_2 > 0$. Indeed, if v_2 is bounded then Lemmas 6.7 and 7.4 imply that $\lambda_2 \le C_2'\nu$ for some $C_2' > 0$, and by (7.9) it follows that $\|v_2'\|_\infty^2 \le C_2'\|v_2\|_\infty^2$. On the other hand, if $v_2(1)$ is unbounded, then $\xi_2 \to 1$ (see (7.3)), and then we have the required bound by (7.5) (choose, for example, $x_0 = 3/4$).

We now integrate the equation for v_2 on $[\xi_1, 1/2]$, use (1.2) and $\int v_2^2 = 1$ to obtain

$$C_g^{-1}\int_{\xi_1}^{1/2} v_1^2 v_2 \le \int_{\xi_1}^{1/2} g_2(v_1^2)v_2 = \lambda_2 \int_{\xi_1}^{1/2} v_2 + \nu(v_2'(1/2) - v_2'(\xi_1)) \le \lambda_2 + 2c_2\nu. \tag{7.12}$$

Let $\overline{T}_1$ be the function

$$\overline{T}_1(x) = \nu(v_1'(x))^2 + [\lambda_1 - g_1(v_2^2(x))]v_1^2(x) + 2\int_{1/2}^{x} g_1'(v_2^2(\sigma))v_2'(\sigma)v_2(\sigma)v_1^2(\sigma)\mathrm{d}\sigma.$$

$\overline{T}_1$ is easily verified to be constant in $[0,1]$. Since $\lambda_1 - g_1(v_2^2(x))$ is decreasing and $\lambda_1 - g_1(v_2^2(\xi_1)) = 0$, $\lambda_1 - g_1(v_2^2(1/2)) \le 0$, as $\xi_1 \le 1/2$ ($\xi_1 \to 0$ because $v_1(0) \to +\infty$). Hence,

$$\nu(v_1'(\xi_1))^2 + 2\int_{1/2}^{\xi_1} g_1'(v_2^2)v_2'v_2v_1^2\mathrm{d}\sigma = \overline{T}_1(\xi_1) = \overline{T}_1(1/2) = \nu(v_1'(1/2))^2 + [\lambda_1 - g_1(v_2^2(1/2))]v_1^2(1/2)$$

and

$$\nu\|v_1'\|_\infty^2 = \nu(v_1'(\xi_1))^2 \le \nu(v_1'(1/2))^2 + 2\int_{\xi_1}^{1/2} g_1'(v_2^2)v_2'v_1^2v_2\mathrm{d}\sigma \le C(\nu+\lambda_2).$$

The last bound comes from $|v_2'| \le c_2$, $|v_2| \le 1 + c_2/2$, (7.12) and $|v_1'| \le c_1$ in $[1/2,1]$ (use (7.4): v_1 is unbounded and $\xi_1 \to 0$). □

As already mentioned, the previous results allow to obtain uniform bounds for the sequence we are considering.

Lemma 7.6. *There exists $C_\infty > 0$, that does not depend on n, such that*

$$\|v_{i,n}\|_\infty \le C_\infty, \qquad i = 1,2.$$

Proof. Without loss of generality, we can assume by contradiction that

$$v_1(0) \to \infty, \quad \text{and } \lambda_2 \le \lambda_1.$$

Indeed, if both $v_1(0)$ and $v_2(1)$ are unbounded, such condition can be guaranteed by interchanging the role of v_1 and v_2. Otherwise, suppose that, say, $v_1(0) \to \infty$ and v_2 is bounded: by Lemmas 6.7 and 7.4 there exists $C > 0$ such that $\lambda_2 \le C\nu$, while $\lambda_1/\nu \to \infty$ (otherwise v_1 would be bounded in view of (7.8)). Therefore, $\lambda_2 \le \lambda_1$ whenever ν is sufficiently small and we infer, by Lemma 7.5, the existence of $C > 0$ such that

$$\|v_1'\|_\infty \le C\sqrt{\frac{\lambda_1}{\nu} + 1}. \tag{7.13}$$

We proceed as in the proof of Lemma 6.7, by defining the blow-up sequences

$$\tilde v_1(x) := \frac{1}{v_1(0)} v_1\left(x\sqrt{\frac{\nu}{\lambda_1}}\right), \qquad \tilde v_2(x) := v_2\left(x\sqrt{\frac{\nu}{\lambda_1}}\right),$$

Note that $0 \le \tilde v_1 \le 1$ in $[0, \lambda_1^{1/2}\nu^{-1/2}]$, and that $\tilde v_1(0) = 1$. Since, in such interval,

$$|\tilde v_1'(x)| = \frac{1}{v_1(0)}\sqrt{\frac{\nu}{\lambda_1}}\left|v_1'\left(x\sqrt{\frac{\nu}{\lambda_1}}\right)\right| \le \frac{C}{v_1(0)} \to 0$$

(we used (7.13)), we deduce that $\tilde v_1 \to V \equiv 1$, uniformly in every $[a,b] \subset [0,+\infty)$. As a consequence, in any such interval,

$$-\tilde v_2'' = \left(\frac{\lambda_2}{\lambda_1} - \frac{g_2(v_1(0)\tilde v_1^2)}{\lambda_1}\right)\tilde v_2 \le \left(1 - \frac{C_g^{-1}}{2\lambda_1}v_1^2(0)\right)\tilde v_2 \le -C^2\frac{v_1^2(0)}{\lambda_1}\tilde v_2,$$

with $C > 0$, and Remark 6.6 applies, yielding

$$\tilde v_2(x) \le \tilde v_2(b+1)\mathrm{e}^{-Cv_1(0)/\sqrt{\lambda_1}} \le 2\mathrm{e}^{-Cv_1(0)/\sqrt{\lambda_1}} \qquad \text{for } x \in [0,b],$$

for ν sufficiently small (recall (7.1)). Then

$$\frac{g_1(\tilde{v}_2^2)}{\lambda_1} \le \frac{C_1 \mathrm{e}^{-C_2 v_1(0)/\sqrt{\lambda_1}}}{\lambda_1} \le \frac{C_3}{v_1^2(0)} \to 0.$$

We can plug such estimate in the equation for $\tilde{v}_1$

$$-\tilde{v}_1'' = \left(1 - \frac{g_1(\tilde{v}_2^2)}{\lambda_1}\right)\tilde{v}_1,$$

in order to pass to the limit and obtain

$$-V'' = V \qquad \text{in } (0, +\infty),$$

in contradiction with the fact that $V \equiv 1$. ☐

Uniform L^∞ bounds readily provide Lipschitz ones, thus yielding convergence to some limiting profiles.

Proposition 7.7. *There exists $C'_\infty > 0$, not depending on n, such that*

$$\|v'_{i,n}\|_\infty \le C'_\infty \quad i = 1, 2.$$

As a consequence, up to subsequences,

$$v_{i,n} \to V_i \text{ in } C^{0,\alpha}([0,1]), \qquad \text{with } \int_0^1 V_1^2 = \int_0^1 V_2^2 = 1 \text{ and } V_1 \cdot V_2 \equiv 0 \text{ in } [0,1], \tag{7.14}$$

and

$$\frac{\lambda_{i,n}}{\nu_n} \to \ell_i > 0, \tag{7.15}$$

as $n \to +\infty$.

Proof. Lemma 7.6 guarantees the uniform L^∞ bound for v_1, v_2, hence $\lambda_1, \lambda_2 \to 0$ by Lemma 6.3. As a consequence we can apply Lemma 7.4, for both i, obtaining that there exists $C'_i > 0$ that does not depend on ν such that

$$\lambda_i \le C'_i \nu.$$

This implies that both $\nu \|v_i'\|_\infty^2 \le \nu C'_i \|v_1\|_\infty^2$, by (7.9), and, up to subsequences, both $v_i \to V_i$ in $C^{0.\alpha}$ and $\lambda_i/\nu \to \ell_i \ge 0$. Since uniform convergence implies L^2-one, the required properties for the limiting profiles V_i follow (recall Cor. 2.5), and the only thing that remains to be proved is that both $\ell_i > 0$.

Assume by contradiction that, for instance, $\ell_1 = 0$. Then we can use equation (7.8) to infer that $V_1 \equiv 1$, in contradiction with (7.14). ☐

Remark 7.8. Once we know that $v_{i,n} \to V_i$ uniformly, the strong H^1 convergence follows by standard arguments. Indeed, by integrating the equations we have

$$0 \le \frac{1}{\nu} \int_0^1 g_i(v_{j,n}^2) v_{i,n} \,\mathrm{d}x = \frac{\lambda_{i,n}}{\nu} \int_0^1 v_{i,n} \,\mathrm{d}x \le C;$$

therefore, testing with $v_{i,n} - V_i$ we infer

$$\int_0^1 v'_{i,n}(v'_{i,n} - V'_i) \,\mathrm{d}x \le \max_{[0,1]} |v_{i,n} - V_i| \cdot \frac{1}{\nu} \int_0^1 (\lambda_{i,n} + g_i(v_{j,n}^2)) v_{i,n} \,\mathrm{d}x \to 0.$$

As a consequence, weak H^1 convergence implies convergence in norm, and finally strong H^1 one.

The remaining part of the section will be devoted to fully characterize the limits V_i, ℓ_i. To this aim, we need a sharper analysis of the convergence of $v_{i,n}$.

Lemma 7.9. *Suppose that, as $n \to +\infty$, $v_{1,n}(y_n) \geq c\nu_n^{1/2-\epsilon}$ for some $y_n \in [0,1)$, $c > 0$, $0 < \epsilon \leq 1/2$. Then there exists $c_1 > 0$ such that*

$$v_{2,n}(x) \leq 2v_{2,n}(y_n)\mathrm{e}^{-c_1(y_n-x)\nu_n^{-\epsilon}} \quad \text{in } [0, y_n]. \tag{7.16}$$

Proof. By the monotonicity of v_1, (1.2) and (7.15),

$$g_2(v_1^2(x)) - \lambda_2 \geq C_g^{-1}v_1^2(x) - \lambda_2 \geq \frac{C_g^{-1}c^2}{2}\nu^{1-2\epsilon} \quad \text{in } [0, y]$$

as $\nu \to 0$. Hence,

$$-v_2'' = -\frac{g_2(v_1^2) - \lambda_2}{\nu}v_2 \leq -\frac{C_g^{-1}c^2}{2}\nu^{-2\epsilon}\, v_2$$

in $(0, y)$, and we can conclude using Remark 6.6. □

Remark 7.10. A direct consequence of the previous lemma, which will be used thoroughly in the sequel, is that if $\liminf_{\nu\to 0} v_1(y) > 0$ for some $y \in [0,1)$, then there exists $c_2 > 0$, $y < b < 1$ (that does not depend on ν) such that

$$v_2(x) \leq C_\infty \mathrm{e}^{-\frac{c_2}{\sqrt{\nu}}} \quad \text{in } [0, b]. \tag{7.17}$$

Indeed, the assumption guarantees that $v_1(y) \geq 2c > 0$ for some $c > 0$, so, by Proposition 7.7, $v_1(y') \geq c$ for some $y' > y$. Hence,

$$v_2(x) \leq C_\infty \mathrm{e}^{-c_1(y'-x)\nu^{-1/2}} \quad \text{in } [0, y'],$$

that implies (7.17) if we choose $y < b < y'$, and $c_2 = c_2(c_1, b, y, y') > 0$.

Note that $v_1(0) \geq 1$ for all ν (otherwise the mass constraint $\int_0^1 v_1^2 \mathrm{d}x = 1$ would be violated), thus

$$v_2(0) \leq C_\infty \mathrm{e}^{-c_2\nu^{-1/2}} = o(\nu^a) \quad \text{for all } a > 0. \tag{7.18}$$

for some $c_2 > 0$.

Analogous conclusions hold if v_1 and v_2 are interchanged.

Lemma 7.11. *The limits V_i, ℓ_i satisfy, in $[0,1]$,*

$$V_1(x) = \frac{2}{\sqrt{\pi}}\sqrt[4]{\ell_1}\cos\left(\sqrt{\ell_1}x\right)\cdot \chi_{\left[0,\frac{\pi}{2\sqrt{\ell_1}}\right]}(x), \tag{7.19}$$

$$V_2(x) = \frac{2}{\sqrt{\pi}}\sqrt[4]{\ell_2}\cos\left(\sqrt{\ell_2}\,(x-1)\right)\cdot \chi_{\left[1-\frac{\pi}{2\sqrt{\ell_2}},1\right]}(x). \tag{7.20}$$

Moreover, as $n \to +\infty$,

$$\xi_{1,n} \to \frac{\pi}{2\sqrt{\ell_1}}, \quad \xi_{2,n} \to 1 - \frac{\pi}{2\sqrt{\ell_2}}. \tag{7.21}$$

Proof. Let $x_1 > 0$ be such that $[0, x_1) = \{x : V_1(x) > 0\}$ (V_1 is identically zero in $[x_1, 1]$). If $y < x_1$, $v_1(y)$ is bounded away from zero, uniformly with respect to ν, hence $v_2(x) \leq C_\infty \mathrm{e}^{-\frac{c_2}{\sqrt{\nu}}}$ in $[0, y]$ by (7.17). Therefore, $g_1(v_2^2) = o(\nu)$ uniformly in $[0, y]$, that is

$$\frac{\lambda_1 - g_1(v_2^2)}{\nu} = \ell_1 + o(1)$$

uniformly on compact subsets of $[0, x_1)$. Hence, we might pass to the limit (weakly) into the equation for v_1: let φ be a smooth test function, with support laying in $[0, x_1)$. The equation reads

$$\int_0^{x_1} v_1'\varphi'\mathrm{d}x = \int_0^{x_1} \frac{\lambda_1 - g_1(v_2^2(x))}{\nu} v_1\varphi\mathrm{d}x,$$

and passing to the limit (Prop. 7.7 ensures weak convergence in $H^1((0,1))$ of v_i to V_i),

$$-V_1'' = \ell_1 V_1 \quad \text{in } (0, x_1),$$

$V_1'(0) = 0$ and $V_1(x_1) = 0$. Thus, being V_1 positive, it has to be of the form $A\cos\left(\sqrt{\ell_1}x\right)$ in $(0, x_1)$, for some $A > 0$. This forces $x_1 = \pi/(2\sqrt{\ell_1})$. Moreover, $\int_0^1 V_1^2 = 1$, since by uniform convergence the L^2-constraint passes to the limit, and A must satisfy $A = \frac{2}{\sqrt{\pi}}\sqrt[4]{\ell_1}$. The characterization of V_2 is analogous.

As for the second assertion, we argue that $v_1(\xi_1) \to 0$. If not, $v_2(\xi_1) \le C_\infty \mathrm{e}^{-\frac{c_2}{\sqrt{\nu}}} = o(\nu^{1/2})$ by (7.17), that is not compatible with $g_1(v_2^2(\xi_1)) = \lambda_1 \ge c_1\nu$. Hence, $\lim \xi_1 \ge x_1$. Suppose that $\lim \xi_1 > x_1$; note that v_1 is concave on $(0, \xi_1)$, so $v_1(x) \ge v_1(0) + (v_1(\xi_1) - v_1(0))x/\xi_1$ in $[0, \xi_1]$. We infer

$$\lim v_1(x_1) \ge v_1(0)\left(1 - \lim \frac{x_1}{\xi_1}\right) > 0,$$

which contradicts $v_1(x_1) \to V_1(x_1) = 0$. Then, $\xi_1 \to x_1 = \pi/(2\sqrt{\ell_1})$. □

The last part of our analysis focuses on the "interface" between v_1 and v_2, namely we are going to consider the point $x_m = x_{m,n} \in (0,1)$ such that

$$m_n = v_1(x_{m,n}) = v_2(x_{m,n}).$$

We follow ideas introduced in [5] to treat the one-dimensional variational case. Note that by strict monotonicity of $v_{i,n}$, $x_{m,n} \in (0,1)$ is well-defined, and

$$m_n \to 0, \quad x_{m,n} \to x_0 \in (0,1),$$

in view of (7.14) and the fact that $v_{1,n}$ and $v_{2,n}$ are bounded away from zero in neighborhoods of $x = 0$ and $x = 1$ respectively (see Lem. 7.2).

In what follows, we will write

$$g_i(s) = \gamma_i s + h_i(s), \quad \text{for all } s \ge 0,$$

where $\gamma_i = g_i'(0) > 0$, $h_i(0) = 0$, $h_i'(0) = 0$.

Remark 7.12. The functions h_i have to be considered as "lower order terms" in the vanishing viscosity limit, and we will use their Taylor expansions around $s = 0$, namely

$$h_i(v_j^2(x)) = a_i(x)v_j^4(x), \quad h_i'(v_j^2) = b_i(x)v_j^2(x),$$

where $|a_i(x)|, |b_i(x)| \le C$ for some universal constant $C > 0$ (depending on g'').

The "joint energy" is going to be crucial in our analysis:

$$\begin{aligned} T(x) := \frac{1}{\gamma_1}\left[\nu(v_1'(x))^2 + [\lambda_1 - h_1(v_2^2(x))]v_1^2(x) + 2\int_{x_m}^x h_1'(v_2^2(\sigma))v_2'(\sigma)v_2(\sigma)v_1^2(\sigma)\mathrm{d}\sigma\right] \\ + \frac{1}{\gamma_2}\left[\nu(v_2'(x))^2 + [\lambda_2 - h_2(v_1^2(x))]v_2^2(x) - 2\int_x^{x_m} h_2'(v_1^2(\sigma))v_1'(\sigma)v_1(\sigma)v_2^2(\sigma)d\sigma\right] \\ - v_1^2(x)v_2^2(x). \end{aligned} \tag{7.22}$$

Of course, along any pair $(v_{1,n}, v_{2,n})$, $T_n(x) = T(x)$ is constant (indeed $T_n'(x) \equiv 0$).

Lemma 7.13. *It holds*

$$\frac{\lambda_{1,n}}{\gamma_1} v_{1,n}^2(0) + o(\nu_n) = T_n = \frac{\lambda_{2,n}}{\gamma_2} v_{2,n}^2(1) + o(\nu_n), \tag{7.23}$$

as $n \to +\infty$.

Proof. Note firstly that

$$\left| \int_{x_m}^{x} h_i'(v_j^2) v_j' v_j v_i^2 \mathrm{d}\sigma \right| \le \int_0^1 |b_i v_j'| v_j^3 v_i^2 \mathrm{d}\sigma \le C \int_0^1 v_j^3 v_i^2 \le C \|v_i v_j\|_\infty \int_0^1 v_j^2 v_i$$
$$= C C_g \|v_i v_j\|_\infty \int_0^1 g_i(v_j^2) v_i = C' \|v_i v_j\|_\infty \int_0^1 (\nu v_i'' + \lambda_i v_i) = o(\nu), \tag{7.24}$$

by Remark 7.12 and uniform convergence of $v_i v_j$ to 0. Note also that $v_2^2(0) = o(\nu)$ by (7.18).

Therefore, being v_1 bounded by C_∞,

$$T(0) = \frac{\lambda_1}{\gamma_1} v_1^2(0) - \frac{v_1^2(0)}{\gamma_1} h_1(v_2^2(0)) + \frac{2}{\gamma_1} \int_{x_m}^0 h_1'(v_2^2) v_2' v_2 v_1^2 \mathrm{d}\sigma$$
$$+ \frac{\lambda_2 - h_2(v_1^2(0))}{\gamma_2} v_2^2(0) - \frac{2}{\gamma_2} \int_0^{x_m} h_2'(v_1^2) v_1' v_1 v_2^2 \mathrm{d}\sigma - v_1^2(0) v_2^2(0) = \frac{\lambda_1}{\gamma_1} v_1^2(0) + o(\nu).$$

Similarly,

$$T(1) = \frac{\lambda_2}{\gamma_2} v_2^2(1) + o(\nu). \qquad \square$$

Lemma 7.14. *It holds true that*

$$\limsup_{n \to +\infty} \frac{m_n^4}{\nu_n} < +\infty.$$

Proof. Arguing by contradiction,

$$\frac{m^4}{\nu} \to \infty,$$

possibly along a subsequence. Let $\tilde{v}_i(x) := \frac{1}{m} v_i \left(x_m + x \frac{\sqrt{\nu}}{m} \right)$. Then, $\tilde{v}_i$ solves

$$-\tilde{v}_i'' = \left(\frac{\lambda_i}{m^2} - \gamma_i \tilde{v}_j^2 - \frac{h_i(m^2 \tilde{v}_j^2)}{m^2} \right) \tilde{v}_i, \quad \text{in } I_\nu = \left(-\frac{m^2}{\nu} x_m, (1 - x_m) \frac{m^2}{\nu} \right).$$

Note that I_ν tends to the whole real line as $\nu \to 0$ (x_m is bounded away from $x = 0$ and $x = 1$, and $m^2 \nu^{-1} \to \infty$), $\tilde{v}_i(0) = 1$ and

$$|\tilde{v}_i(y) - \tilde{v}_i(0)| \le |y| \|v_i'\|_\infty \frac{\sqrt{\nu}}{m^2} \to 0, \quad \text{for all } y \in [a, b] \subset I_\nu.$$

Thus, $\tilde{v}_i$ converges uniformly on compact subsets of I_ν, as $\|v_i'\|_\infty$ is bounded by C_∞'. Moreover, $\lambda_i m^{-2} \to 0$ (by (7.15)) and $h_i(m^2 \tilde{v}_j^2) m^{-2} \to 0$ uniformly on compact subsets of I_ν (see Rem. (7.12)). Hence,

$$\tilde{v}_i \to 1, \quad \tilde{v}_i' \to 0 \quad \text{locally uniformly.} \tag{7.25}$$

We then have

$$\sum_{i=1,2} \frac{1}{\gamma_i} \left[(\tilde{v}_i'(0))^2 + \left(\frac{\lambda_i}{m^2} - \frac{h_i(m^2 \tilde{v}_j^2(0))}{m^2} \right) \tilde{v}_i^2(0) \right] - \tilde{v}_1^2(0) \tilde{v}_2^2(0) =$$
$$\frac{T(x_m)}{m^4} = \frac{\lambda_1}{\gamma_1 m^4} v_1^2(0) + o(\nu/m^4) \ge \frac{c_1 \nu}{\gamma_1 m^4} v_1^2(0) + o(\nu/m^4) \ge 0 \tag{7.26}$$

by (7.23) and (7.15) when ν is close enough to zero. On the other hand, the left hand side of (7.26) goes to -1 as $\nu \to 0$ by (7.25), a contradiction. $\square$

Lemma 7.15. *As $n \to +\infty$, there exists $L > 0$ such that, up to a subsequence,*

$$\frac{m_n^4}{\nu_n} \to L. \tag{7.27}$$

Moreover,

$$\liminf |v'_{i,n}(x_{m,n})| > 0. \tag{7.28}$$

Proof. Let us assume by contradiction that

$$\frac{m^4}{\nu} \to 0.$$

Let $\tilde{v}_i(x) := \frac{1}{m} v_i\,(x_m + m\,x)$. Then, $\tilde{v}_i$ solves

$$-\tilde{v}_i'' = \frac{m^4}{\nu}\left(\frac{\lambda_i}{m^2} - \gamma_i \tilde{v}_j^2 - \frac{h_i(m^2\tilde{v}_j^2)}{m^2}\right)\tilde{v}_i, \quad \text{in } I_\nu = \left(-\frac{x_m}{m}, \frac{1-x_m}{m}\right).$$

Note that I_ν tends to the whole real line as $\nu \to 0$, $\tilde{v}_i(0) = 1$ and $\|\tilde{v}_i'\|_\infty = \|v_i'\|_\infty \le C'_\infty$, so $\tilde{v}_i$ converges uniformly on compact subsets of I_ν. Moreover, $\lambda_i m^{-2} \to 0$ and $h_i(m^2\tilde{v}_j^2)m^{-2} \to 0$ uniformly on compact subsets of I_ν (as in the proof of the Lem. 7.14). Hence, $\tilde{v}_i \to W_i$ locally in $C^2(\mathbb{R})$, and $W_i'' = 0$. Since W_i is positive we have

$$W_i \equiv 1 \quad \text{in } \mathbb{R},\ i = 1, 2. \tag{7.29}$$

Therefore,

$$\sum_{i=1,2} \frac{1}{\gamma_i}\left[(\tilde{v}_i'(0))^2 + \frac{m^4}{\nu}\left(\frac{\lambda_i}{m^2} - \frac{h_i(m^2\tilde{v}_j^2(0))}{m^2}\right)\tilde{v}_i^2(0)\right] - \frac{m^4}{\nu}\tilde{v}_1^2(0)\tilde{v}_2^2(0) =$$
$$\frac{T(x_m)}{\nu} = \frac{\lambda_1}{\gamma_1 \nu} v_1^2(0) + o(1) \ge \frac{c_1}{\gamma_1} v_1^2(0) + o(1) > 0 \tag{7.30}$$

by (7.23) and (7.15) when $\nu \to 0$. However, the left hand side of (7.26) goes to zero as $\nu \to 0$ by (7.29), that is not possible. Hence, $L > 0$.

To prove (7.28) we proceed as before, setting $\tilde{v}_i(x) := \frac{1}{m} v_i\,(x_m + m\,x)$. We have that $\tilde{v}_i \to W_i$ locally in $C^2(\mathbb{R})$, and (W_1, W_2) solves

$$\begin{cases} W_1'' = L\gamma_1 W_2^2 W_1, \\ W_2'' = L\gamma_2 W_1^2 W_2. \end{cases}$$

in $\mathbb{R}$. W_1 and W_2 are also positive and monotone, so $W_i'(0) \ne 0$. We conclude by observing that $|v_i'(x_m)| = |\tilde{v}_i'(0)| \to |W_i'(0)| > 0$. $\square$

Lemma 7.16. *As $n \to +\infty$, it holds true that*

$$\xi_{1,n} \le x_{m,n} \le \xi_{2,n},$$

and

$$\xi_{1,n} \to x_0, \quad \xi_{2,n} \to x_0. \tag{7.31}$$

Proof. Since ξ_1 is the inflection point of v_1 we have

$$C_g^{-1} v_2^2(\xi_1) \le g_1(v_2^2(\xi_1)) = \lambda_1 \le C_1' \nu,$$

also by invoking (1.2) and (7.15). Hence, $v_2(\xi_1) \le \sqrt{(C_g C_1')\nu}$, but $v_2(x_m) = m \sim \sqrt[4]{L\nu}$ by (7.27), so $v_2(\xi_1) \le v_2(x_m)$ for ν sufficiently small. Monotonicity of v_2 implies that $\xi_1 \le x_m$, while $x_m \le \xi_2$ is obtained by an analogous argument at the inflection point ξ_2 of v_2.

Suppose now that $\xi_2 - \xi_1 = 4\eta$ and η is uniformly bounded away from zero as $\nu \to 0$. Assume, without loss of generality, that $x_m \in [\xi_1, \xi_1 + 2\eta]$ (on the other hand, if $x_m \in [\xi_1 + 2\eta, \xi_2]$ we interchange the roles of v_1 and v_2). Note that ξ_2 is the inflection point of v_2, so v_2 is convex on $(0, \xi_2)$, which provides $v_2(x) \ge v_2(x_m) + v_2'(x_m)(x - x_m)$ for all $x \in (0, \xi_2)$. Therefore,

$$v_2(\xi_1 + 3\eta) \ge v_2(x_m) + v_2'(x_m)(\xi_1 + 3\eta - x_m) \ge c(\xi_1 + 3\eta - x_m) \ge c\eta > 0,$$

for some positive c in view of (7.28). Now we reason as in Remark 7.10 (in particular we apply (7.17) with v_1 and v_2 interchanged) to get

$$v_1(\xi_1 + 3\eta) \le C_\infty \mathrm{e}^{-\frac{c_2}{\sqrt{\nu}}} = o(\nu^{1/2}),$$

but $C_g^{-1} v_1^2(\xi_2) \ge g_2(v_1^2(\xi_2)) = \lambda_2 \ge c_2 \nu$ (again by (1.2) and (7.15)), so $v_1(\xi_2) \ge v_1(\xi_1 + 3\eta)$ as $\nu \to 0$. Being v_1 decreasing, $\xi_2 \le \xi_1 + 3\eta = \xi_2 - \eta$, which is impossible. Then, $0 \le \xi_2 - \xi_1 \to 0$ follows, and the second assertion is proved as $x_m \to x_0$. □

Proof of Theorem 1.8. In view of Propositions 3.3, 7.7 and Remark 7.8, the theorem will follow once we show that the following equalities hold:

$$\frac{\ell_2}{\ell_1} = \left(\frac{\gamma_2}{\gamma_1}\right)^{2/3} \qquad \text{and} \qquad x_0 = \frac{\sqrt[3]{\gamma_2}}{\sqrt[3]{\gamma_1} + \sqrt[3]{\gamma_2}}.$$

To this aim, we put together all the asymptotic information (as $\nu \to 0$) we obtained so far. Firstly, $\pi v_1^2(0) \sim 4\sqrt{\ell_1}$ and $\pi v_2^2(1) \sim 4\sqrt{\ell_2}$ by (7.19) and (7.20). Hence, if we divide (7.23) by ν we obtain

$$\frac{\ell_1 \sqrt{\ell_1}}{\gamma_1} = \frac{\ell_2 \sqrt{\ell_2}}{\gamma_2}, \tag{7.32}$$

which is the first stated equality. Then, $x_0 = \frac{\pi}{2\sqrt{\ell_1}}$ by (7.21) and (7.31). Moreover,

$$\frac{\pi}{2\sqrt{\ell_1}} + \frac{\pi}{2\sqrt{\ell_2}} = 1.$$

By plugging (7.32) in the last equality we conclude. □

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