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ON NON-CONVEX ANISOTROPIC SURFACE ENERGY REGULARIZED *VIA* THE WILLMORE FUNCTIONAL: THE TWO-DIMENSIONAL GRAPH SETTING

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Abstract. We regularize non-convex anisotropic surface energy of a two-dimensional surface, given as a graph over the two-dimensional unit disk, by the Willmore functional and investigate existence of the corresponding global minimizers. Restricting to the rotationally symmetric case, we obtain a one-dimensional variational problem which permits to derive substantial qualitative information on the minimizers. We show that minimizers tend to a "cone"-like solution as the regularization parameter tends to zero. Areas where the solutions are either convex or concave are identified. It turns out that the structure of the chosen anisotropy hardly affects the qualitative shape of the minimizers.

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1. Introduction

In references [12] and [11] the authors investigated *non-convex* anisotropic mean curvature motion regularized *via* a Willmore term in the one-dimensional graph setting. There, the analysis of the stationary case is thoroughly discussed, while the evolution problem, in particular the behaviour when the regularization parameter is sent to zero, is treated *via* a numerical approach.

Motivation for our work here is the next natural step, namely the higher dimensional case. In the following we generalize the analytical results presented in [12] to the two-dimensional setting. Again we take care in presenting elementary proofs while imposing so little restrictions as possible to the choice of anisotropy function.

Our starting point is the anisotropic surface energy

$$E_0 : u \mapsto \int_{\operatorname{graph} u} \gamma(\nu)\, \mathrm{d}A \tag{1.1}$$

(which can also be thought of as a generalization of the area functional): here u belongs to $W^{1,1}(\Omega)$ for some open connected domain $\Omega \subset \mathbb{R}^2$, the function γ denotes a non-convex anisotropy map (typically a positive, postively homogeneous, $C^{0,1}(\mathbb{R}^3)$-map, *cf.* [6]), and $\nu \in \mathbb{S}^2$ is the outward unit normal to graph u. We are interested in the shape of global minimizers of E_0, since these are candidates for limit points of the corresponding gradient flow.

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It is well-known, that because of the non-convexity of γ the parabolic equation associated to steepest descent evolution is not well-defined, and henceforth a regularization of some sort is necessary in order to tackle the problem. As in [12], motivated by Angenent and Gurtin [1] and Di Carlo *et al.* [7], we consider a regularization in terms of the squared mean curvature H, *i.e.*, the Willmore energy. To this end, we define the regularized energy

$$E_\varepsilon : u \mapsto \int_{\mathrm{graph}\, u} \gamma(\nu)\, \mathrm{d}A + \varepsilon^2 \int_{\mathrm{graph}\, u} H^2\, \mathrm{d}A, \qquad \varepsilon > 0, \tag{1.2}$$

for $u \in W^{2,2}(\Omega)$. As also observed in [12], when investigating the existence of minimizers for E_ε, the regularization acts as a choice criterion among possible minimizers for E_0.

Besides its intrinsic mathematical interest and several applications related to motion by anisotropic mean curvature (see for instance [6] and [3]), the study of E_ε is significant because of its similarity to the Aviles–Giga energy. Indeed, a model problem related to (1.2) is the functional

$$F_\varepsilon : u \mapsto \int_U \left(|Du|^2 - 1\right)^2 \mathrm{d}x + \varepsilon^2 \int_U \left|D^2 u\right|^2 \mathrm{d}x, \qquad \varepsilon > 0, \tag{1.3}$$

where U denotes a domain in $\mathbb{R}^n$. The first term presents a non-convex integrand (although, when compared with (1.2), we should note that it does not have the linear growth at infinity that is typical of the anisotropy maps considered there), and the regularization is a linearized version of the one employed for (1.2). The Aviles–Giga energy F_ε was introduced by Aviles and Giga [2] in connection with the theory of smectic liquid crystal. The literature around the investigation of the Aviles–Giga functional is simply huge: for our scope we wish to highlight the work by Lorent [10], in which it is shown using methods of regularity theory and ODE that any minimizer u of $\frac{1}{\varepsilon}F_\varepsilon$ over $W_0^{2,2}(\mathbb{B}^2)$ satisfies

$$\int_{\mathbb{B}^2} \left| Du(x) + \xi \frac{x}{|x|} \right|^2 \leq c\varepsilon^{\frac{1}{6}} \left(\log(\varepsilon^{-1})\right)^{\frac{13}{6}} \tag{1.4}$$

for some $\xi \in \{\pm 1\}$. (Recall that the "cone" map $u(x) = \mathrm{dist}(x, \partial U) = 1 - |x|$ has gradient $Du(x) = -\frac{x}{|x|}$.) This theorem is somehow linked to the following discussion because in the study of (1.2) we too restrict to functions defined on the unit ball $\mathbb{B}^2 \subset \mathbb{R}^2$, and eventually we look for rotationally symmetric minimizers. Indeed, as a first step in handling (1.2), we assume that the non-convex map γ is rotationally symmetric around the z-axis. This will allow us to look for rotationally symmetric solutions and therefore to reduce by one the dimensionality of the problem.

Exploiting the dimension reduction and under some mild regularity assumptions on the anisotropy map γ we are able to show for the functional E_ε (as in (1.2))

- existence of global mimimizers u_ε for $0 < \varepsilon \ll 1$ in the class of rotational symmetric $W^{2,2}(\mathbb{B}^2)$-maps with zero boundary data, as well as
- convergence in $W^{1,p}(\mathbb{B}^2)$, $p \in [1, \infty)$, as $\varepsilon \to 0$, to a cone solution of the type described in (1.4) (the slope of the cone now being determined by the choice of anisotropy γ).

Unlike the analogous one-dimensional setting studied in [12], the global minimizers u_ε of (1.2) are not globally convex or globally concave; instead concavity/convexity can be shown to hold only in certain regions of the domain. Finally under some additional very mild assumptions on γ we are able to derive interesting qualitative information about the global minimizers: in this respect it is remarkable to note that very different choices of anisotropy maps give rise to quite similar shapes. A precise statement is formulated in Theorem 2.3 below.

The paper is organized as follows: in Section 2 we introduce notation, general assumptions, and state the main contribution of this paper, Theorem 2.3. Its proof relies on all results collected in the subsequents sections. More precisely: first of all the radial formulation and the corresponding function spaces are analysed in Section 3. Based on an alternative formulation of the problem, existence of minimizers is achieved in Section 4. Regularity properties are studied in Section 5, convergence to cones solution is described in Section 6, and, finally, the shape of the minimizers is studied in Section 7.

2. Preliminaries and notation

2.1. Anisotropy map and general assumptions

Consider an anisotropy function $\gamma : \mathbb{R}^3 \to [0, \infty)$ which is Lipschitz continuous, Positive, and positively Homogeneous of degree one, *i.e.*

(L) $\gamma \in C^{0,1}(\mathbb{R}^3)$,

(P) $\gamma(p) > 0 \quad$ for $p \neq 0$,

(H) $\gamma(\lambda p) = |\lambda|\, \gamma(p) \quad$ for $\lambda \in \mathbb{R}, p \in \mathbb{R}^3$.

We furthermore assume that γ is Rotationally invariant with respect to the p_3-axis, *i.e.*

(R) $\gamma(R_\vartheta p) = \gamma(p) \qquad$ for all $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}, p \in \mathbb{R}^3$, and

$$R_\vartheta := \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are interested in the case of non-convex anisotropy functions. A number of explicit examples of profile curves that generate the set $\{p \in \mathbb{R}^3 : \gamma(p) = 1\}$ through rotation around the p_3-axis can be found for instance in [12].

Observe that by (H) and (R), the entire information of γ is contained in

$$g(y) := \gamma(y, 0, -1), \qquad y \in \mathbb{R}. \tag{2.1}$$

The map g is even by (R). The non-convexity of γ implies that g is non-convex. Moreover from the homogeneity properties of γ we derive that that g grows linearly at $\pm\infty$, namely

$$(\min_{\partial\mathbb{B}^3} \gamma)\sqrt{1+y^2} \le g(y) \le (\max_{\partial\mathbb{B}^3} \gamma)\sqrt{1+y^2}.$$

Conditions (H) and (L) ensure the existence of a global Lipschitz constant for g.

We assume (L), (P), (H), (R) to hold throughout this paper.

In the following we denote by $z_{\min} \ge 0$ the smallest non-negative point where g attains its (positive) global minimum, that is

$$g(z_{\min}) = \min_{\mathbb{R}} g, \qquad g(\pm y) > g(z_{\min}) \quad \text{for all } y \in [0, z_{\min}). \tag{2.2}$$

Note that $z_{\min} > 0$ implies the non-convexity of g (and γ) while the converse is not true. In our case, it turns out that $z_{\min} > 0$ is the most interesting situation since we will show that

$$z_{\min} = 0 \iff (u_\varepsilon \equiv 0 \text{ is the unique global minimizer of } E_\varepsilon),$$

see Section 4 below.

Please note that, unless stated otherwise, a *minimizer* always denotes a *global* minimizer (which does not have to be unique, *cf.* Example 2.1 below).

The term 'monotonic' will generally refer to *weak* monotonicity; the same applies to 'concave' and 'convex' respectively.

By $C^k(U)$, $k \in \mathbb{N} \cup \{0\}$, we denote the set of k-times continuously differentiable functions. Unless U is compact, the respective supremum norms are not necessarily finite. By $C^{k,1}(\mathbb{R})$, $k \in \mathbb{N} \cup \{0\}$, we denote the set of $C^k(\mathbb{R})$ maps whose kth derivative is locally Lipschitz.

Finally, $C_0^\infty(0,\infty)$ denotes the subspace of compactly supported functions in $C^\infty(0,\infty)$.

2.2. Motivation

A first natural step to extend our previous results [12] to the non-scalar case is to consider the minimization of the energy

$$E_0(u) = \int_{\operatorname{graph} u} \gamma(\nu)\,\mathrm{d}A$$

in the class of functions

$$\mathcal{C}^*_\alpha := \{u \in W^{1,1}(\mathbb{B}^2) : u|_{\partial\mathbb{B}^2} = \alpha\}, \tag{2.3}$$

where $\mathbb{B}^2 \subset \mathbb{R}^2$ is the unit ball, $\nu = (u_x, u_y, -1)/\sqrt{1+|\nabla u|^2}$ is the unit normal to the graph of u and γ is a *non-convex* anisotropy function as defined above.

Since our problem is translation invariant and $\mathcal{C}^*_\alpha = \alpha + \mathcal{C}^*_0$ there is no loss of generality in assuming

$$\alpha = 0.$$

Using (H) and (R) one immediately infers

$$E_0(u) = \int_{\mathbb{B}^2} \gamma(u_x, u_y, -1)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{B}^2} \gamma(R_\vartheta(u_x, u_y, -1))\,\mathrm{d}x\,\mathrm{d}y.$$

Without loss of generality we may choose a rotation which maps the vector $(u_x, u_y, -1)$ to $(|\nabla u|, 0, -1)$ so that (recall (2.1))

$$E_0(u) = \int_{\mathbb{B}^2} \gamma(|\nabla u|, 0, -1)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{B}^2} g(|\nabla u|)\,\mathrm{d}x\,\mathrm{d}y.$$

Due to the rotational invariance of the anisotropy map and the symmetry of the domain $\mathbb{B}^2$ it is plausible to expect existence of rotationally symmetric minimizers (and it is easy to construct such examples). Hence from now on we will consider the class

$$\mathcal{C}_\alpha := \{u \in W^{1,1}(\mathbb{B}^2) : u|_{\partial\mathbb{B}^2} = \alpha, u \text{ rotationally symmetric}\}.$$

An advantage in restricting to the class $\mathcal{C}_\alpha$ is that the problem becomes essentially one-dimensional.

Example 2.1 (Double-well)**.** Let g have the shape of a symmetric double-well where the two minima are attained at $z_{\min} > 0$ and $-z_{\min}$, *i.e.*

$$0 < \min g = g(\pm z_{\min}).$$

If we consider the cone(s)

$$\Lambda : \mathbb{B}^2 \ni x \mapsto Z\,(1 - |x|)$$

with slope $Z = \pm z_{\min}$, then one can verify that $|\nabla\Lambda| = z_{\min}$ and hence Λ minimizes the energy E_0 in $\mathcal{C}_0$. In fact, from the characterization of radially symmetric $W^{2,2}$-functions given below one can also infer that $\Lambda \in W^{1,1}(\mathbb{B}^2) \setminus W^{2,2}(\mathbb{B}^2)$, see Remark 3.6.

Remark 2.2 (Eikonal equation)**.** Let g and Λ be as in Example 2.1. Since $E_0(\Lambda) = \inf_{W^{1,1}(\mathbb{B}^2)} E_0 = \pi g(z_{\min})$ we immediately deduce that any global minimizer of E_0 in $W^{1,1}(\mathbb{B}^2)$ satisfies the *Eikonal equation*

$$|\nabla u(x)| = z_{\min} \quad \text{for a.e. } x \in \mathbb{B}^2. \tag{2.4}$$

Vice versa, any solution of the Eikonal equation is a global E_0-minimizer. Note that the minimization problem of E_0 in $\mathcal{C}^*_0$ allows for non-symmetric solutions. For instance, as a consequence of Vitali's Covering Theorem (see [8], Sect. 1.5) one can cover – up to a set of measure zero – the set $\mathbb{B}^2$ with countably many disjoint closed balls of radius smaller than 1. Putting a cone with slope $z_{\min}$ on each such smaller ball gives a $W^{1,1}$ (even $W^{1,\infty}$) function that satisfies (2.4).

2.3. The regularized energy and main result

We would like now to investigate the functional E_ε from (1.2) where $u \in W^{2,2}(\mathbb{B}^2) \cap \mathcal{C}_0$ and

$$H = \varkappa_1 + \varkappa_2 = \nabla \cdot \left(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \tag{2.5}$$

denotes (twice) the mean curvature of the graph of u (see for instance [6]).

Our problem reads

$$E_\varepsilon \to \min! \qquad \text{in } \mathcal{C}_0 \cap W^{2,2}(\mathbb{B}^2). \tag{2.6}$$

In the following we will show

Theorem 2.3 (Main theorem). *Let γ be an anisotropy map satisfying (L), (P), (H), (R) and let the (even) map g be as in (2.1). Let $z_{\min} \geq 0$ be the smallest non-negative point where g attains its global minimum (cf. (2.2)).*

If $z_{\min} = 0$ then $u \equiv 0$ is the unique global minimizer for the problem (2.6) for all $\varepsilon > 0$. (It is still a global minimizer for (2.6) in case $\varepsilon = 0$, however, uniqueness depends on g, see Rem. 4.1.)

Let $z_{\min} > 0$. If $g \in C^{1,1}(\mathbb{R}) \cap C^k(\mathbb{R})$, $k \in \mathbb{N}$, then for any $0 < \varepsilon \ll 1$

(i) *there exists a global minimizer u_ε in $\mathcal{C}_0 \cap W^{2,2}(\mathbb{B}^2)$ of class $C^1(\mathbb{B}^2) \cap C^{k+2}\left(\mathbb{B}^2 \setminus \{0\}\right)$ with $|\nabla u_\varepsilon| \leq z_{\min}$ which is convex in a neighborhood of the origin; the negative $-u_\varepsilon$ is also a minimizer;*

(ii) *any sequence $(u_\varepsilon)_{\varepsilon>0}$ of global minimizers being convex in a neighborhood of the origin converges to the cone solution $-\Lambda \in W^{1,p}(\mathbb{B}^2)$, $\Lambda(x) = z_{\min}(1 - |x|)$, for any $p \in [1, \infty)$.*

If additionally g is weakly decreasing on $[z_{\min} - \delta, z_{\min}]$ for some $\delta > 0$, then the profile curves of those global minimizers that are convex in the neighborhood of the origin have following common feature: expressed in terms of the radial function $r(\varrho) = u(x)$ with $\varrho = |x| \in [0, 1]$ we have that r' is strictly monotone increasing near the origin, attains a global maximum at some point $\varrho_0 \in (0, 1)$ and then strictly decreases towards a strictly positive value on the boundary at $\varrho = 1$.

Proof. First of all notice that the entire information of $u \in \mathcal{C}_0 \cap W^{2,2}(\mathbb{B}^2)$ is captured by the radial function $r : [0, 1] \to \mathbb{R}$ *via*

$$u(x) = r(|x|) = r\left(\sqrt{x_1^2 + x_2^2}\right). \tag{2.7}$$

Therefore, our first task consists in reviewing Problem (2.6) in terms of r. It turns out (see Sect. 3.2 below) that the functional E_ε can be conveniently expressed in terms of r', namely $2\pi I_\varepsilon(\psi) = E_\varepsilon(u)$ (*cf.* (3.13)), with $\psi = r'$ as in (3.14) and I_ε as in (3.12), so that Problem (2.6) can be equivalently formulated as in (3.15). The case $z_{\min} = 0$ is dealt with at the beginning of Section 4. The statements for the case $z_{\min} > 0$ follow from Lemmas 4.4, 5.2, 3.1, Proposition 6.5, and Corollary 7.5 below. The last claim follows from Theorem 7.6. ☐

3. Radial formulation

3.1. Spaces of radially symmetric functions

The aim of this section is to determine the space X_0 consisting of the restrictions to the radial line of radially symmetric $W^{2,2}(\mathbb{B}^2)$-functions that vanish on the boundary $\partial\mathbb{B}^2$. In a second step we will show that X_0 is isomorphic to $W^{1,2}(0, \infty)$. The latter characterization will be of particular importance as it allows to write the integrand of the regularization term in (1.2) in a more convenient form as the corresponding one for X_0.

For $m \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, the space of radial symmetric functions is denoted by

$$W^{m,p}_{\text{rad}}(\mathbb{B}^2) := \left\{ u \in W^{m,p}(\mathbb{B}^2) \,|\, u \text{ is rotational symmetric with respect to the origin} \right\}.$$

It is equipped with the usual $W^{m,p}$-norm.

Furthermore, we define

$$X := \{r : (0,1) \to \mathbb{R} \,|\, r \text{ has weak derivatives up to order two and } \|r\|_X < \infty\}$$

with norm $\|\cdot\|_X := \|\cdot\|_{L^2} + [\cdot]_X$ where

$$[r]_X := \left[\int_0^1 \left(\frac{r'(\varrho)^2}{\varrho} + r''(\varrho)^2\varrho\right) \mathrm{d}\varrho\right]^{1/2}. \tag{3.1}$$

Since $\|r\|_{W^{1,2}} \le C\,\|r\|_X$ (recall that $\varrho < 1$), the space X is embedded in $W^{1,2}(0,1)$.

Lemma 3.1 ($X \hookrightarrow C^1$)**.** *The space X continuously embeds into $C^1([0,1])$. Moreover $r'(0) = 0$ for any $r \in X$.*

Thus, without further notice, we will always assume $r \in X$ to be C^1.

Proof. As $X \hookrightarrow W^{1,2}(0,1)$, the function r has an absolutely continuous representative with $\|r\|_{C^0([0,1])} \le C\,\|r\|_{W^{1,2}(0,1)} \le C\,\|r\|_X$. For any $\delta \in (0,1)$ we have $r \in W^{2,2}(\delta,1)$, so this representative is even $C^1([\delta,1])$ by Sobolev embedding theory. Consequently, r is differentiable at any point in $(0,1]$ and the derivative is continuous on $(0,1]$. We still have to show that r' exists and is continuous in 0. From

$$\left|\frac{r(\delta)-r(0)}{\delta}\right| \le \frac{1}{\delta}\int_0^\delta \left|\frac{r'(\varrho)}{\sqrt{\varrho}}\sqrt{\varrho}\right| \mathrm{d}\varrho \le \tfrac{\sqrt{2}}{2}\left(\int_0^\delta \frac{r'(\varrho)^2}{\varrho}\,\mathrm{d}\varrho\right)^{1/2} \to 0$$

as $\delta \searrow 0$ we infer $r'(0) = 0$. Using $[r]_X < \infty$ once more, we may find, for given $\varepsilon > 0$, some $\delta_0 > 0$ such that, for any $0 < \delta < \delta' < \delta_0$,

$$\varepsilon \ge \int_0^{\delta_0}\left(\frac{r'^2}{\varrho} + r''^2\varrho\right)\mathrm{d}\varrho \ge \int_\delta^{\delta'}\left(\frac{r'^2}{\varrho} + r''^2\varrho\right)\mathrm{d}\varrho \ge \left|2\int_\delta^{\delta'} r''r'\,\mathrm{d}\varrho\right| = \left|r'(\delta')^2 - r'(\delta)^2\right|. \tag{3.2}$$

Consequently $r'(\varrho)^2$ converges as $\varrho \searrow 0$. Since $\int_0^1 \frac{r'^2}{\varrho}\,\mathrm{d}\varrho < \infty$ the limit has to be zero which gives $r'(\varrho) \to r'(0) = 0$ as $\varrho \searrow 0$. Note that (3.2) also holds for $\varepsilon = [r]_X^2$, $\delta_0 = 1$, $\delta = 0$ and any $\delta' \in [0,1]$. This gives $\|r'\|_{C^0([0,1])} \le C\,\|r\|_X$. □

Proposition 3.2 ($X \cong W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)$)**.** *The linear map*

$$\Phi : X \to W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2), \qquad X \ni r \longmapsto \Big(u : x \mapsto r(|x|)\Big) \in W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2), \quad x \in \mathbb{B}^2,$$

is a homeomorphism.

Proof. For $\varrho \ge 0$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ we set

$$x_1 = \varrho\cos\varphi, \qquad x_2 = \varrho\sin\varphi.$$

From $u(x) = r(|x|) \in C^1(\mathbb{B}^2 \setminus \{0\})$ we infer for $i = 1,2$ and $x \neq 0$

$$u_{x_i}(x) = r'(|x|)\frac{x_i}{|x|} = \begin{cases} r'(\varrho)\cos\varphi & \text{if } i = 1,\\ r'(\varrho)\sin\varphi & \text{if } i = 2,\end{cases}$$

thus

$$|\nabla u(x)| = |r'(\varrho)|\,.$$

Moreover a formal computation gives, for $i \neq j$,

$$\begin{aligned} u_{x_i x_i}(x) &= r''(|x|)\frac{x_i^2}{|x|^2} + r'(|x|)\frac{x_j^2}{|x|^3}, \\ u_{x_i x_j}(x) &= r''(|x|)\frac{x_i x_j}{|x|^2} - r'(|x|)\frac{x_i x_j}{|x|^3}, \end{aligned} \tag{3.3}$$

and thus also

$$\triangle u(x) = r''(\varrho) + \frac{r'(\varrho)}{\varrho}\cdot \tag{3.4}$$

We have to discuss five points:

(i) The map Φ is well-defined, *i.e.*, $r \in X \Rightarrow u := \Phi(r) \in W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)$. Note that u and its partial derivatives as given above are measurable maps. To see that these are in fact (weak) derivatives, let $\phi \in C_0^\infty(\mathbb{B}^2)$ and $\psi(\varrho,\varphi) := \phi(x)$. Then, writing $z(\varphi) := (\cos\varphi, \sin\varphi)$, $D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$, $\tilde{D} = \left(\frac{\partial}{\partial \varrho}, \frac{\partial}{\partial \varphi}\right)$, $\nabla = D^\top$, we get

$$\tilde{D}\psi = D\phi \begin{pmatrix} \cos\varphi & -\varrho\sin\varphi \\ \sin\varphi & \varrho\cos\varphi \end{pmatrix} \qquad \text{and} \qquad D\phi = \tfrac{1}{\varrho}\tilde{D}\psi \begin{pmatrix} \varrho\cos\varphi & \varrho\sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

and we can compute

$$\begin{aligned} \int_{\mathbb{B}^2} u(x) D\phi(x)\,\mathrm{d}x &= \int_0^1 \int_0^{2\pi} (\varrho\psi_\varrho z + \psi_\varphi z_\varphi)\, r \,\mathrm{d}\varphi\,\mathrm{d}\varrho = \int_0^1 \int_0^{2\pi} (\varrho\psi_\varrho z + \psi z)\, r\,\mathrm{d}\varphi\,\mathrm{d}\varrho \\ &= \int_0^1 \int_0^{2\pi} (\varrho\psi z)_\varrho\, r\,\mathrm{d}\varphi\,\mathrm{d}\varrho = -\int_0^1\int_0^{2\pi} \varrho\psi z r_\varrho \,\mathrm{d}\varphi\,\mathrm{d}\varrho \\ &= -\int_{\mathbb{B}^2} Du(x)\phi(x)\,\mathrm{d}x. \end{aligned}$$

Similarly we get

$$\begin{aligned} \int_{\mathbb{B}^2} \nabla u(x) D\phi(x)\,\mathrm{d}x &= \int_0^1 \int_0^{2\pi} r_\varrho z^\top (\varrho\psi_\varrho z + \psi_\varphi z_\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\varrho \\ &= \int_0^1 \int_0^{2\pi} \left(\varrho r_\varrho \psi_\varrho z^\top z + r_\varrho \psi_\varphi z^\top z_\varphi\right)\mathrm{d}\varphi\,\mathrm{d}\varrho \\ &= -\int_0^1 \int_0^{2\pi} \left((\varrho r_{\varrho\varrho} + r_\varrho)\,\psi z^\top z + r_\varrho\psi\left(z_\varphi^\top z_\varphi + z^\top z_{\varphi\varphi}\right)\right)\mathrm{d}\varphi\,\mathrm{d}\varrho \\ &= -\int_0^1 \int_0^{2\pi} \left((\varrho r_{\varrho\varrho} + r_\varrho)\,\psi z^\top z + r_\varrho\psi\left(z_\varphi^\top z_\varphi - z^\top z\right)\right)\mathrm{d}\varphi\,\mathrm{d}\varrho \\ &= -\int_0^1 \int_0^{2\pi} \left(r_{\varrho\varrho}\psi z^\top z + \frac{r_\varrho}{\varrho}\psi z_\varphi^\top z_\varphi\right)\varrho\,\mathrm{d}\varphi\,\mathrm{d}\varrho \\ &= -\int_{\mathbb{B}^2} D^2 u(x)\phi(x)\,\mathrm{d}x. \end{aligned}$$

Again application of the transformation formula gives

$$\|\Phi(r)\|_{W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)} \leq C\,\|r\|_X\,. \tag{3.5}$$

(ii) The map Φ is obviously a linear map between vector spaces, and, by (3.5), it is bounded.

(iii) The map Φ is injective as $u = 0$ a.e. implies for a radial symmetric function u that the restriction to the radius also vanishes a.e.

(iv) The map Φ is surjective. Indeed, let $u \in W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)$. By embedding theory the map u is continuous. We will show that the restriction of u to the radius, $r(\varrho) := u(x)$ with $\varrho = |x|$ belongs to X; the relation $\Phi(r) = u$ immediately follows. The fact that r admits weak derivatives of first and second order and that these are given by

$$r'(|x|) = u_{x_1}(x)\frac{x_1}{|x|} + u_{x_2}(x)\frac{x_2}{|x|}, \tag{3.6}$$

$$r''(|x|) = u_{x_1x_1}(x)\cos^2\varphi + 2u_{x_1x_2}(x)\cos\varphi\sin\varphi + u_{x_2x_2}(x)\sin^2\varphi \tag{3.7}$$

for a.e. $|x| \in (0,1)$ is shown in ([5], Thm. 2.2). The idea is to take radially symmetric test functions $\phi(x) = \phi(|x|) = \psi(\varrho) \in C_0^\infty(0,1)$, perform similar integral transformation as above and use the fact that $\mathrm{div}(\frac{x}{|x|^2}) = 0$.

By the Sobolev embedding we obtain

$$\|r\|_{L^2} \le C\,\|r\|_{C^0([0,1])} \le C\,\|u\|_{W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)}\,.$$

Furthermore (3.7) gives

$$\int_0^1 r''(\varrho)^2\varrho\,\mathrm{d}\varrho = \tfrac{1}{2\pi}\int_{\mathbb{B}^2} r''(|x|)^2\,\mathrm{d}x \le C\int_{\mathbb{B}^2}\left|D^2u\right|^2\,\mathrm{d}x \le \|u\|^2_{W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)}\,.$$

From (3.4) we infer that $\triangle u$ is rotationally symmetric and

$$\int_0^1 \frac{r'(\varrho)^2}{\varrho}\,\mathrm{d}\varrho \le C\,\|u\|^2_{W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)}\,,$$

which gives

$$\|r\|_X \le C\,\|\Phi(r)\|_{W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)}\,. \tag{3.8}$$

(v) Φ^{-1} is continuous. This follows from the bijectivity of Φ and (3.8).

□

In order to fit our setting we restrict to elements in X with fixed boundary data. Let

$$X_\alpha := \{r \in X \,|\, r(1) = \alpha\}\,.$$

Without loss of generality we may choose $\alpha = 0$ which makes X_0 a linear subspace of X. Moreover observe that $\|\cdot\|_X$ and $[\cdot]_X$ are equivalent norms on X_0 due to Poincaré's inequality.

Proposition 3.3 ($X_0 \cong W^{1,2}(0,\infty)$)**.** *The linear map*

$$\Psi : X_0 \to W^{1,2}(0,\infty), \qquad X_0 \ni r \longmapsto \Big(\sigma \mapsto r'(\mathrm{e}^{-\sigma})\Big) \in W^{1,2}(0,\infty), \quad \sigma \in (0,\infty),$$

is a homeomorphism.

Proof. As before, we have to comment on the following items:

(i) The map Ψ is well-defined, *i.e.*, $r \in X_0 \Rightarrow \psi := \Psi(r) \in W^{1,2}(0,\infty)$. Both the firstly formally defined maps $\psi : \sigma \mapsto r'(\mathrm{e}^{-\sigma})$ and $\psi' : \sigma \mapsto -\mathrm{e}^{-\sigma}r''(\mathrm{e}^{-\sigma})$ are measurable. Next, we show that ψ' is in fact the weak derivative of ψ. For $\phi \in C_0^\infty(0,\infty)$ we compute

$$\int_0^\infty \psi(\sigma)\phi'(\sigma)\,\mathrm{d}\sigma = \int_0^\infty r'(\mathrm{e}^{-\sigma})\phi'(\sigma)\,\mathrm{d}\sigma = \int_0^1 r'(\tau)\phi'(-\log\tau)\frac{\mathrm{d}\tau}{\tau}$$
$$= \int_0^1 r''(\tau)\phi(-\log\tau)\,\mathrm{d}\tau = -\int_0^\infty \psi'(\sigma)\phi(\sigma)\,\mathrm{d}\sigma.$$

Finally, by

$$\int_0^\infty r'(\mathrm{e}^{-\sigma})^2\,\mathrm{d}\sigma = \int_0^1 r'(\tau)^2\frac{\mathrm{d}\tau}{\tau}, \qquad \int_0^\infty \mathrm{e}^{-2\sigma} r''(\mathrm{e}^{-\sigma})^2\,\mathrm{d}\sigma = \int_0^1 \tau r''(\tau)^2\,\mathrm{d}\tau,$$

we have

$$\|\Psi(r)\|_{W^{1,2}(0,\infty)} \le C\,[r]_X\,. \tag{3.9}$$

(ii) The map Ψ is obviously a linear map between vector spaces, and, by (3.9), it is bounded.
(iii) The map Ψ is injective as $\psi = 0$ a.e. implies $r' \equiv 0$ from which $r \equiv 0$ follows by the boundary condition.
(iv) The map Ψ is surjective. Indeed, let $\psi \in W^{1,2}(0,\infty)$. We will show that the function $r : \varrho \mapsto -\int_0^{-\log\varrho} \psi(\sigma)\mathrm{e}^{-\sigma}\,\mathrm{d}\sigma$ belongs to X_0; the relation $\Psi(r) = \psi$ follows immediately. Of course, $r(\varrho)$, $r'(\varrho) = \psi(-\log\varrho)$, and $r''(\varrho) = -\frac{1}{\varrho}\psi'(-\log\varrho)$ are measurable. Since ψ is continuous by embedding theory, it follows that r is continuously differentiable and r' is both classical and weak derivative. Analogously to (i) we obtain $\int_0^1 r'\phi' = -\int_0^1 r''\phi$ for any $\phi \in C_0^\infty(0,1)$. Finally, $\|r\|_{L^2} \le \|r'\|_{L^2} \le C\,\|\psi\|_{L^2}$ and $[r]_X \le \|\psi\|_{W^{1,2}(0,\infty)}$, *i.e.*,

$$\|r\|_X \le C\,\|\Psi(r)\|_{W^{1,2}(0,\infty)}\,. \tag{3.10}$$

(v) The map Ψ^{-1} is continuous. This follows from the bijectivity of Ψ and (3.10).

□

Corollary 3.4 ($W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)\big|_{\partial\mathbb{B}^2\mapsto 0} \cong W^{1,2}(0,\infty)$)**.** *The map $\Phi\circ\Psi^{-1}$ defines a linear homeomorphism from $W^{1,2}(0,\infty)$ to the $W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)$-functions with vanishing boundary data $\alpha = 0$ via*

$$W^{1,2}(0,\infty) \ni \psi \longmapsto \left(x \mapsto -\int_0^{-\log|x|} \psi(\sigma)\mathrm{e}^{-\sigma}\,\mathrm{d}\sigma\right) \in W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2), \tag{3.11}$$

and the respective norms are equivalent due to (3.5)*,* (3.8)*,* (3.9)*,* (3.10)*.*

With the aid of the characterization of functions $u \in W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)$ by elements ψ in the Sobolev space $W^{1,2}$ on the positive real axis, we are able to pass to an equivalent formulation of our problem (see Sect. 3.2 below). The key relation is the formula $\psi(\sigma) = r'(\mathrm{e}^{-\sigma})$.

Remark 3.5 ($\psi(\infty) = 0$)**.** Note that $\psi \in W^{1,2}(0,\infty)$ implies $\psi(\sigma) \to 0$ as $\sigma \nearrow \infty$ since, for $0 \le \sigma \le \sigma' < \infty$,

$$\left|\psi(\sigma')^2 - \psi(\sigma)^2\right| \le 2\int_\sigma^{\sigma'} |\psi\psi'| \le \int_\sigma^{\sigma'} \left(\psi^2 + \psi'^2\right).$$

The right hand side tends to zero as $\sigma \nearrow \infty$ for $\psi, \psi' \in L^2(0,\infty)$, therefore $\psi(\sigma)^2$ converges as $\sigma \nearrow \infty$. Again $\psi \in L^2(0,\infty)$ implies $\psi(\sigma)^2 \to 0$. This fact corresponds to $r'(0) = 0$.

Remark 3.6 (Higher dimensions)**.** Note that the characterization of $W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)$ crucially depends on the fact that $\mathbb{B}^2$ is two-dimensional. Let $\mathbb{B}^N$ denote the unit ball $\mathbb{B}^N := \left\{ x \in \mathbb{R}^N \,\middle|\, |x| < 1 \right\}$. For general dimension $N \geq 3$, Figueiredo *et al.* ([5], Thm. 2.3(3)) have shown that $W^{2,2}_{\mathrm{rad}}(\mathbb{B}^N)$ can be identified with the space X_N consisting of functions $r : (0,1) \to \mathbb{R}$ with weak derivatives up to order two and with finite norm

$$\|r\|_{X_N} := \left(\int_0^1 \left(r(\varrho)^2 + r'(\varrho)^2 + r''(\varrho)^2 \right) \varrho^{N-1}\, \mathrm{d}\varrho \right)^{1/2}.$$

Moreover they have shown that $W^{1,1}_{\mathrm{rad}}(\mathbb{B}^2)$ is characterized by radial functions r that are once weakly differentiable and with finite norm $\left(\int_0^1 \left(r^2 + r'^2 \right) \varrho \, \mathrm{d}\varrho \right)^{1/2}$ (see [5], Thm. 2.3(2)). Consequently, for the cone function of Example 2.1 we infer that $\Lambda \in W^{1,1}_{\mathrm{rad}}(\mathbb{B}^2) \setminus W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2)$ (recall (3.1)) while its N-dimensional equivalent ($N \geq 3$) belongs to $W^{2,2}_{\mathrm{rad}}(\mathbb{B}^N)$.

3.2. A radially symmetric formulation for the problem

In this section we will derive the equivalent formulation of our problem (2.6) under the transformation $\Phi \circ \Psi^{-1}$. To this end, we define for $\psi \in W^{1,2}(0,\infty)$

$$\begin{aligned} I_\varepsilon(\psi) :&= \int_0^\infty \mathrm{e}^{-2\sigma} g(\psi)\, \mathrm{d}\sigma + \varepsilon^2 \int_0^\infty \frac{(\psi' - \psi(1+\psi^2))^2}{(1+\psi^2)^{5/2}}\, \mathrm{d}\sigma \\ &= \int_0^\infty \mathrm{e}^{-2\sigma} g(\psi)\, \mathrm{d}\sigma + \varepsilon^2 \int_0^\infty \left(\frac{\psi'^2}{(1+\psi^2)^{5/2}} + \frac{\psi^2}{(1+\psi^2)^{1/2}} \right) \mathrm{d}\sigma \\ &\quad + 2\varepsilon^2 \left(1 - \frac{1}{\sqrt{1+\psi(0)^2}} \right) \end{aligned} \tag{3.12}$$

in order to derive

$$2\pi I_\varepsilon(\psi) = E_\varepsilon(u) \tag{3.13}$$

where u and ψ are related through $\psi = (\Psi \circ \Phi^{-1})u$ (recall Cor. 3.4).

Note that, in contrast to the respective radial symmetric version for E_ε, the integrand of the regularization term in I_ε only depends on ψ and its derivatives and does not explicitly contain the integration variable σ.

Using the fact that g is even and $|\nabla u| = |r'|$ we can write

$$\int_{\mathrm{graph}\, u} \gamma(\nu)\, \mathrm{d}A = 2\pi \int_0^1 \varrho g(r'(\varrho))\, \mathrm{d}\varrho.$$

Next, we consider the Willmore term. By (2.5) and

$$\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}} \right)_{x_i} = \frac{\left(1 + u_{x_1}^2 + u_{x_2}^2\right) u_{x_i x_i} - u_{x_1 x_i} u_{x_1} u_{x_i} - u_{x_2 x_i} u_{x_2} u_{x_i}}{\left(1+|\nabla u|^2\right)^{3/2}}$$

we infer

$$H = \frac{u_{x_1x_1} + u_{x_2x_2} + u_{x_1x_1} u_{x_2}^2 + u_{x_2x_2} u_{x_1}^2 - 2u_{x_1x_2} u_{x_1} u_{x_2}}{\left(1+|\nabla u|^2\right)^{3/2}}.$$

Using (3.3) we compute

$$\begin{aligned} H\left(1+|\nabla u|^2\right)^{3/2} &= \left(1+r'^2\frac{x_2^2}{|x|^2}\right)\left(r''\frac{x_1^2}{|x|^2}+r'\frac{x_2^2}{|x|^3}\right)+\left(1+r'^2\frac{x_1^2}{|x|^2}\right)\left(r''\frac{x_2^2}{|x|^2}+r'\frac{x_1^2}{|x|^3}\right)\\ &\quad -2x_1x_2\left(\frac{r''}{|x|^2}-\frac{r'}{|x|^3}\right)r'^2\frac{x_1x_2}{|x|^2}\\ &= \left(1+r'^2\sin^2\right)\left(r''\cos^2+r'\frac{\sin^2}{\varrho}\right)+\left(1+r'^2\cos^2\right)\left(r''\sin^2+r'\frac{\cos^2}{\varrho}\right)\\ &\quad -2\varrho^2\cos^2\sin^2\left(\frac{r''}{\varrho^2}-\frac{r'}{\varrho^3}\right)r'^2\\ &= r''+\frac{r'}{\varrho}+\frac{r'^3}{\varrho}\cdot \end{aligned}$$

Since $|\nabla u|^2 = r'^2$ we can write

$$\begin{aligned} \int_{\operatorname{graph} u} H^2\,\mathrm{d}A &= \int_0^1\int_0^{2\pi}\frac{\left(\varrho r''+r'+r'^3\right)^2}{\varrho^2\left(1+r'^2\right)^3}\sqrt{1+r'^2}\;\varrho\,\mathrm{d}\varphi\,\mathrm{d}\varrho\\ &= 2\pi\int_0^1\frac{\left(\varrho r''+r'+r'^3\right)^2}{\varrho\left(1+r'^2\right)^{5/2}}\,\mathrm{d}\varrho. \end{aligned}$$

Summing up we obtain

$$\frac{E_\varepsilon(u)}{2\pi} = \int_0^1 \varrho g(r'(\varrho))\,\mathrm{d}\varrho + \varepsilon^2\int_0^1\frac{\left(\varrho r''+r'(1+r'^2)\right)^2}{\varrho\left(1+r'^2\right)^{5/2}}\,\mathrm{d}\varrho.$$

Next we perform another change of variables, namely

$$(0,1]\ni\quad \varrho=\mathrm{e}^{-\sigma},\qquad \sigma\in[0,\infty),$$

and set (recall Prop. 3.3)

$$\psi(\sigma) = r'(\mathrm{e}^{-\sigma}). \tag{3.14}$$

This gives (3.13). Finally observe that, in view of (3.13) and Corollary 3.4, our problem (2.6) turns into

$$I_\varepsilon \to \min! \qquad \text{in } W^{1,2}(0,\infty). \tag{3.15}$$

Minimizers of E_ε correspond to minimizers of I_ε. The same holds true for stationary points: this is a consequence of the following remark.

Remark 3.7. Consider functionals $\mathcal{J}: A\to\mathbb{R}$, $\mathcal{K}: B\to\mathbb{R}$ defined on Banach spaces A, B which are related by some isomorphism $\omega: B\to A$ through $\mathcal{K}=\mathcal{J}\circ\omega$. Assuming that the first variation of $\mathcal{K}$ at $b\in B$ in direction $q\in B$ exists, we have

$$\begin{aligned} \delta\mathcal{K}(b;q) &= \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\,\mathcal{K}(b+\varepsilon q) = \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\,\mathcal{J}(\omega(b+\varepsilon q))\\ &= \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\,\mathcal{J}(\omega(b)+\varepsilon\omega(q)) = \delta\mathcal{J}(\omega(b);\omega(q)), \end{aligned}$$

hence $\delta\mathcal{J}(\omega(b);\omega(q))$ also exists. Moreover, $b\in B$ is a critical point of $\mathcal{K}$, i.e.,

$$\delta\mathcal{K}(b;q)=0 \qquad \text{for all } q\in B,$$

if and only if $\omega(b)$ is a critical point of $\mathcal{J}$.

4. Existence of minimizers for I_ε

In this section we prove existence of minimizers for I_ε in $W^{1,2}(0,\infty)$. Because of the lack of an estimate for $|\psi'|$ we cannot immediately apply direct methods. Instead, we have to employ a refined coercivity argument.

Remark 4.1 ($z_{\min} = 0$)**.** Notice that if

$$g(0) = \min_{\mathbb{R}} g \qquad (\iff z_{\min} = 0)$$

(recall (2.2)) then the map $\psi \equiv 0$ is the unique global minimizer of I_ε for all $\varepsilon > 0$. If $\varepsilon = 0$, it is still a global minimizer which fails to be unique if and only if g vanishes in some neighborhood of zero.

Because of the above remark, it is interesting to look at the case where

$$z_{\min} > 0,$$

a situation that we shall assume henceforth (although many of the results shown below hold also in the limit case where $z_{\min} = 0$).

Proposition 4.2 (Minimizers remain in $[-z_{\min}, z_{\min}]$)**.**
Assume $\psi \in W^{1,2}(0,\infty)$ *with* $\operatorname{image}\psi \not\subset [-z_{\min}, z_{\min}]$.
Then $\hat\psi := \min(\max(\psi, -z_{\min}), z_{\min})$ *satisfies* $I_\varepsilon(\hat\psi) < I_\varepsilon(\psi)$.

Proof. Note that $\operatorname{image}\psi \cap [-z_{\min}, z_{\min}] \neq \emptyset$ by $\psi \in W^{1,2}(0,\infty)$ since $\psi(\infty) = 0$. By construction $\hat\psi \in W^{1,2}(0,\infty)$ (*cf.* Gilbarg and Trudinger [9], Lem. 7.6). For those points $\sigma \in \mathbb{R}$ where $\psi(\sigma) \neq \hat\psi(\sigma)$ we have $\hat\psi(\sigma) = \pm z_{\min}$, so $g(\hat\psi(\sigma)) \le g(\psi(\sigma))$. This gives $I_0(\hat\psi) \le I_0(\psi)$. Furthermore, we obtain (recall (3.12))

$$\begin{aligned}&\int_0^\infty \left(\frac{\psi'^2}{(1+\psi^2)^{5/2}} + \frac{\psi^2}{(1+\psi^2)^{1/2}} \right) \mathrm{d}\sigma + 2\left(1 - \frac{1}{\sqrt{1+\psi(0)^2}}\right)\\ &\ge \int_0^\infty \left(\frac{\hat\psi'^2}{(1+\hat\psi^2)^{5/2}} + \frac{\hat\psi^2}{(1+\hat\psi^2)^{1/2}} \right) \mathrm{d}\sigma + 2\left(1 - \frac{1}{\sqrt{1+\hat\psi(0)^2}}\right)\end{aligned}$$

where we used $\left|\hat\psi\right| \le |\psi|$ and the fact that $x \mapsto \dfrac{x^2}{\sqrt{1+x^2}}$ is monotone increasing on $[0,\infty)$. By continuity the above inequality is in fact a strict inequality on some positive-measure set where $\psi \neq \hat\psi$ and the claim follows. $\square$

Lemma 4.3 (Weak lower semi-continuity)**.** *For each $\varepsilon > 0$ the functional I_ε is sequentially weakly lower semi-continuous on $W^{1,2}(0,\infty)$.*

Proof. Consider an arbitrary sequence $(\psi_k)_{k\in\mathbb{N}} \subset W^{1,2}(0,\infty)$ with $\psi_k \rightharpoonup \psi$ in $W^{1,2}(0,\infty)$. Letting $L := \liminf_{k\to\infty} I_\varepsilon(\psi_k)$ we may (after relabeling) pass to a subsequence $(I_\varepsilon(\psi_k))_{k\in\mathbb{N}}$ with $I_\varepsilon(\psi_k) \to L$ as $k \to \infty$. We have $\|\psi_k\|_{W^{1,2}(0,\infty)} \le C$. Hence, for $K \in (0,\infty)$ and $\sigma, \sigma' \in [0, K]$,

$$|\psi_k(\sigma) - \psi_k(\sigma')| \le \left| \int_\sigma^{\sigma'} \psi_k'(s)\,\mathrm{d}s \right| \le \|\psi_k'\|_{L^2(0,\infty)} |\sigma - \sigma'|^{1/2}$$

and

$$|\psi_k(\sigma)| \le |\psi_k(\sigma')| + |\psi_k(\sigma) - \psi_k(\sigma')| \le |\psi_k(\sigma')| + C\sqrt{K},$$

so that integration in σ' gives

$$|\psi_k(\sigma)| \le \frac{1}{K}\int_0^K |\psi_k(\sigma')|\, \mathrm{d}\sigma' + C\sqrt{K} \le C\left(\frac{1}{\sqrt{K}} + \sqrt{K}\right).$$

We infer that $(\psi_k)_{k\in\mathbb{N}}$ is uniformly bounded and equicontinuous on $[0,K]$. Applying the Arzelà–Ascoli theorem, we may pass to a subsequence which uniformly converges to a continuous function $\tilde{\psi}$. Since $\|\psi_k\|_{W^{1,2}(0,K)} \le C$ implies that (for a subsequence) $\psi_k \to \psi$ in $L^2(0,K)$, then $\psi = \tilde{\psi}$ and we deduce that $\psi_k(0) \to \psi(0)$ as $k \to \infty$. Thus we may omit the boundary term of I_ε in the arguments that follow. For $K \in [0,\infty]$ let

$$I_{\varepsilon,K}(\psi) := \int_0^K \mathrm{e}^{-2\sigma} g(\psi)\, \mathrm{d}\sigma + \varepsilon^2 \int_0^K \left(\frac{\psi'^2}{(1+\psi^2)^{5/2}} + \frac{\psi^2}{(1+\psi^2)^{1/2}}\right) \mathrm{d}\sigma.$$

As any sequence $(\psi_k)_{k\in\mathbb{N}} \subset W^{1,2}(0,\infty)$ with $\psi_k \rightharpoonup \psi \in W^{1,2}(0,\infty)$ also satisfies $\psi_k|_{(0,K)} \rightharpoonup \psi|_{(0,K)}$ in $W^{1,2}(0,K)$, we obtain using Tonelli's Theorem ([4], Thm. 3.5) and the non-negativity of the integrands of I_ε

$$I_{\varepsilon,K}(\psi) \le \liminf_{k\to\infty} I_{\varepsilon,K}(\psi_k) \le \liminf_{k\to\infty} I_{\varepsilon,\infty}(\psi_k) = L.$$

Finally, for any $\delta > 0$ there is some $K > 0$ with $I_{\varepsilon,\infty}(\psi) \le I_{\varepsilon,K}(\psi) + \delta$, thus

$$I_{\varepsilon,\infty}(\psi) \le \delta + \liminf_{k\to\infty} I_{\varepsilon,\infty}(\psi_k) = \delta + L.$$

□

Lemma 4.4 (Existence of minimizers)**.** *For any $\varepsilon > 0$ there exists a minimizer of I_ε in $W^{1,2}(0,\infty)$.*

Proof. Let $(\psi_k)_{k\in\mathbb{N}} \subset W^{1,2}(0,\infty)$ be a minimizing sequence for I_ε converging to $\inf_{W^{1,2}(0,\infty)} I_\varepsilon \in [0, \frac{1}{2}g(0)] = [0, I_\varepsilon(0)]$. By Proposition 4.2 the sequence $\left(\hat{\psi}_k\right)_{k\in\mathbb{N}}$ is another minimizing sequence with

$$C \ge I_\varepsilon(\hat{\psi}_k) \ge \varepsilon^2 \int_0^\infty \frac{\hat{\psi}_k'^2 + \hat{\psi}_k^2}{(1+\hat{\psi}_k^2)^{5/2}} \ge \frac{\varepsilon^2}{(1+z_{\min}^2)^{5/2}} \left\|\hat{\psi}_k\right\|_{W^{1,2}}^2.$$

Passing to a subsequence, this gives the existence of a limit function $\psi_0 \in W^{1,2}(0,\infty)$ with $\psi_k \rightharpoonup \psi_0$ weakly in $W^{1,2}(0,\infty)$. As I_ε is weakly lower semicontinuous with respect to $W^{1,2}(0,\infty)$ we infer $I_\varepsilon(\psi_0) \le \inf_{W^{1,2}(0,\infty)} I_\varepsilon$. □

5. Regularity of stationary points

Our next task is to compute the first variation of I_ε and derive the Euler–Lagrange equation. We will infer regularity not only for minimizers but for all stationary points.

Lemma 5.1 (First variation)**.** *For any $\psi, \phi \in W^{1,2}(0,\infty)$ and $g \in C^1(\mathbb{R})$ the first variation $\delta I_\varepsilon(\psi,\phi) := \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} I_\varepsilon(\psi + \tau\phi)$ exists and amounts to*

$$\delta I_\varepsilon(\psi,\phi) = \int_0^\infty \mathrm{e}^{-2\sigma} g'(\psi)\phi\, \mathrm{d}\sigma + \varepsilon^2 \int_0^\infty \left(2\frac{\psi'\phi'}{(1+\psi^2)^{5/2}} - 5\frac{\psi'^2\psi\phi}{(1+\psi^2)^{7/2}} + \right.$$
$$\left. + 2\frac{\psi\phi}{(1+\psi^2)^{1/2}} - \frac{\psi^3\phi}{(1+\psi^2)^{3/2}}\right) \mathrm{d}\sigma + 2\varepsilon^2 \frac{\psi(0)\phi(0)}{(1+\psi(0)^2)^{3/2}}.$$

Proof. The result follows by standard computations using the continuity of g', the fact that $\psi, \phi \in C^0([0,\infty))$ by embedding theory and that they are bounded due to Remark 3.5. □

Note that we do not obtain the above result for $g \in C^{0,1}$ as $g' \circ \psi$ might be undefined on a positive measure set.

Lemma 5.2 (Regularity of stationary points)**.** *For $\varepsilon > 0$ and $g \in C^k(\mathbb{R})$, $k \in \mathbb{N}$, any stationary point ψ of I_ε in $W^{1,2}(0,\infty)$ belongs to $C^{k+1}([0,\infty))$ and satisfies the Euler–Lagrange equation*

$$\psi'' = \frac{(1+\psi^2)^{5/2}}{2\varepsilon^2}\mathrm{e}^{-2\sigma} g'(\psi) + \frac{5\psi'^2\psi}{2(1+\psi^2)} + \tfrac{1}{2}\psi(1+\psi^2)(2+\psi^2). \tag{5.1}$$

Note that Equation (5.1) is non-autonomous as it contains the factor $\mathrm{e}^{-2\sigma}$.

Since the L^p-spaces are not nested in the case of infinite domains, equation (5.1) does not yield much information as to which L^p-space ψ'' may belong. In fact, since $g'(\psi)$ is bounded (due to the continuity of g and the boundedness of ψ by Rem. 3.5), the first summand on the right-hand side of (5.1) belongs to L^p for $p \in [1,\infty]$, the second one to L^1, and the third one to L^2.

Proof. For $\phi \in C_0^\infty(0,\infty)$, the weak Euler–Lagrange equation reads

$$\begin{aligned} 0 &= \int_0^\infty \mathrm{e}^{-2\sigma} g'(\psi)\phi \,\mathrm{d}\sigma + \varepsilon^2 \int_0^\infty \Bigg(2\frac{\psi'\phi'}{(1+\psi^2)^{5/2}} - 5\frac{\psi'^2\psi\phi}{(1+\psi^2)^{7/2}} + \\ &\quad + 2\frac{\psi\phi}{(1+\psi^2)^{1/2}} - \frac{\psi^3\phi}{(1+\psi^2)^{3/2}} \Bigg) \mathrm{d}\sigma \\ &= \int_0^\infty \phi' \Bigg[-\int_0^\sigma \mathrm{e}^{-2\sigma'} g'(\psi)\,\mathrm{d}\sigma' + \varepsilon^2 \Bigg(2\frac{\psi'}{(1+\psi^2)^{5/2}} + 5\int_0^\sigma \frac{\psi'^2\psi}{(1+\psi^2)^{7/2}}\,\mathrm{d}\sigma' - \\ &\quad - 2\int_0^\sigma \frac{\psi}{(1+\psi^2)^{1/2}}\,\mathrm{d}\sigma' + \int_0^\sigma \frac{\psi^3}{(1+\psi^2)^{3/2}}\,\mathrm{d}\sigma' \Bigg) \Bigg] \mathrm{d}\sigma. \end{aligned} \tag{5.2}$$

Since the terms in the bracket belong to $L^1_{loc}(0,\infty)$ we may apply DuBois−Reymond's Lemma, which gives

$$\begin{aligned} 2\psi' = (1+\psi^2)^{5/2} \Bigg[\frac{1}{\varepsilon^2}\int_0^\sigma \mathrm{e}^{-2\sigma'} g'(\psi)\,\mathrm{d}\sigma' - 5\int_0^\sigma \frac{\psi'^2\psi}{(1+\psi^2)^{7/2}}\,\mathrm{d}\sigma' \\ + 2\int_0^\sigma \frac{\psi}{(1+\psi^2)^{1/2}}\,\mathrm{d}\sigma' - \int_0^\sigma \frac{\psi^3}{(1+\psi^2)^{3/2}}\,\mathrm{d}\sigma' + c \Bigg] \end{aligned} \tag{5.3}$$

for some constant $c \in \mathbb{R}$ and any $\sigma \in (0,\infty)$. Due to the continuity of ψ and g', and the boundedness of ψ we infer that the terms in the bracket on the right-hand side of (5.3) belong to $W^{1,1}(0,K)$ for any positive K and more generally they belong to $W^{1,1}_{\mathrm{loc}}(0,\infty)$. By embedding theory we infer that they are continuous on $[0,\infty)$. The fact that Sobolev spaces in one dimension are Banach algebras yields $\psi' \in W^{1,1}_{\mathrm{loc}}(0,\infty)$. Integrating by parts in (5.2) and applying the Fundamental Lemma we deduce (5.1) for any $\sigma \in (0,\infty)$. As the right-hand side of (5.1) is continuous, the function ψ is twice continuously differentiable. Bootstrapping we infer higher regularity for $k > 1$. □

Integrating by parts in the expression for the first variation given in Lemma 5.1 and taking $\phi \in C^\infty[0,\infty)$, with $\phi(0) \neq 0$ and $\phi(\sigma) = 0$ for $\sigma \geq K$, $K \in (0,\infty)$, yields

Corollary 5.3 (Natural boundary conditions)**.** *For $\varepsilon > 0$ and $g \in C^1(\mathbb{R})$, a stationary point ψ of I_ε in $W^{1,2}(0,\infty)$ satisfies*

$$\psi'(0) = \psi(0)(1+\psi(0)^2). \tag{5.4}$$

6. Convergence

In this section, for purely technical reasons we consider

$$\tilde{I}_\varepsilon := I_\varepsilon - \int_0^\infty \mathrm{e}^{-2\sigma}(\min_{\mathbb{R}} g)\,\mathrm{d}\sigma = I_\varepsilon - \tfrac{1}{2}\min_{\mathbb{R}} g \tag{6.1}$$

and we write

$$\tilde{g} := g - \min_{\mathbb{R}} g \tag{6.2}$$

which results in $\tilde{g} \geq 0$ and $\min_{\mathbb{R}} \tilde{g} = \tilde{g}(\pm z_{\min}) = 0$.

Remark 6.1 (Uniqueness)**.** If $g \in C^{1,1}(\mathbb{R})$, the Euler–Lagrange equation (5.1) reads

$$\psi''(\sigma) = F(\sigma, \psi(\sigma), \psi'(\sigma))$$

where F is locally Lipschitz-continuous. By the Picard–Lindelöf theorem, any (global) solution ψ is uniquely determined by its values $\psi(\sigma)$ and $\psi'(\sigma)$ at an arbitrary position $\sigma \in [0,\infty)$.

Lemma 6.2 (Trichotomy)**.** *If $g \in C^{1,1}(\mathbb{R})$, any* local *$I_\varepsilon$-minimizer having at least one zero identically vanishes. Moreover, the image of any global I_ε-minimizer is contained in either $(-z_{\min}, 0)$, $\{0\}$, or $(0, z_{\min})$.*

Proof. Let $\psi \in W^{1,2}(0,\infty)$ be an I_ε-minimizer with $\psi(\sigma_0) = 0$ for some $\sigma_0 \in [0,\infty)$. As ψ satisfies the Euler–Lagrange equation (5.1) and the null function is also a solution of (5.1) (recall $g'(0) = 0$ since g is even) we infer $\psi \equiv 0$ from Remark 6.1 provided $\psi'(\sigma_0) = 0$. In case $\sigma_0 = 0$ the latter directly follows from the natural boundary condition (5.4). Otherwise, if $\sigma_0 > 0$, note that the absolute value of ψ is another I_ε-minimizer since $|\psi| \in W^{1,2}(0,\infty)$ by Gilbarg and Trudinger ([9], Lem. 7.6) and

$$I_\varepsilon(|\psi|) = I_\varepsilon(\psi) \tag{6.3}$$

since g is even by (R). By Lemma 5.2 both ψ and $|\psi|$ are C^2. From $\psi(\sigma_0) = 0$ we infer $|\psi|'(\sigma_0) = 0$, so $|\psi| \equiv 0$ which gives $\psi \equiv 0$. The same arguments apply if ψ is a local minimizer. This gives the first statement.

Next, observe that $\psi(0) = z_{\min}$ contradicts Proposition 4.2 due to equation (5.4). If $\psi(\sigma) = z_{\min}$ for some $\sigma \in (0,\infty)$, we deduce $\psi'(\sigma) = 0$ and $\psi''(\sigma) \leq 0$ again by Proposition 4.2 while (5.1) implies $\psi''(\sigma) > 0$. The same arguments apply to the case $\psi(\sigma) = -z_{\min}$. Consequently, the second claim of the statement now follows by continuity. □

Lemma 6.3 (Lower bound for $\tilde{I}_\varepsilon$)**.** *Let $\tilde{I}_\varepsilon$ be as in (6.1). We obtain*

$$\inf_{W^{1,2}(0,\infty)} \tilde{I}_\varepsilon = \mathcal{O}(\varepsilon^2\,|\log\varepsilon|) \qquad \text{as } \varepsilon \searrow 0. \tag{6.4}$$

An immediate consequence is that if $z_{\min} > 0$ then $\psi \equiv 0$ is not a minimizer for sufficiently small $\varepsilon > 0$.

Proof. We introduce some comparison function

$$\tilde{\psi}_S : \sigma \longmapsto \begin{cases} z_{\min} & \text{if } \sigma \in [0,S], \\ z_{\min}\mathrm{e}^{S-\sigma} & \text{if } \sigma \in [S,\infty), \end{cases}$$

where $S > 0$ will be chosen later. Of course we have $\tilde{\psi}_S \in W^{1,2}(0,\infty)$ and

$$\begin{aligned}\tilde{I}_\varepsilon(\tilde{\psi}_S) &\leq (\max_{[0,z_{\min}]}\tilde{g}) \int_S^\infty \mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma + \varepsilon^2\left(z_{\min}^2\int_S^\infty \mathrm{e}^{2(S-\sigma)}\,\mathrm{d}\sigma + z_{\min}^2 S + z_{\min}^2\int_S^\infty \mathrm{e}^{2(S-\sigma)}\,\mathrm{d}\sigma + 2\right) \\ &\leq \frac{1}{2}(\max_{[0,z_{\min}]}\tilde{g})\,\mathrm{e}^{-2S} + \varepsilon^2\left(z_{\min}^2 + z_{\min}^2 S + 2\right).\end{aligned}$$

Letting $S := -\log\varepsilon$ we arrive at

$$\begin{aligned}\tilde{I}_\varepsilon(\tilde{\psi}_S) &\le \tfrac{1}{2}(\max_{[0,z_{\min}]}\tilde{g})\varepsilon^2 + \varepsilon^2(z_{\min}^2+2) + \varepsilon^2\left|\log\varepsilon\right| z_{\min}^2 \\ &\le C\left(\varepsilon^2 + \varepsilon^2\left|\log\varepsilon\right|\right). \end{aligned}$$ □

Note that equation (6.3) reflects the fact that it is not relevant to E_ε whether we consider u or $-u$.

According to Lemma 6.3 the null function is not a minimizer for sufficiently small $\varepsilon > 0$. Together with Lemma 6.2 and (3.14) this gives

Corollary 6.4 (Strong monotonicity of radial functions)**.** *Let $g \in C^{1,1}(\mathbb{R})$ with $z_{\min} > 0$ and $\varepsilon \ll 1$. Then the radial function of an E_ε-minimizer is strongly monotonic,* i.e., *either $r' > 0$ or $r' < 0$ on $(0,1)$.*

Proposition 6.5 (Convergence of minimizers)**.** *Assume $g \in C^{1,1}(\mathbb{R})$ and let $(\psi_\varepsilon)_{\varepsilon>0} \subset W^{1,2}(0,\infty)$ be a sequence of minimizers for I_ε. Then there is a subsequence converging to the constant function $z_{\min}$ or $-z_{\min}$ in $L^p_{\mathrm{e}^{-2\cdot}}(0,\infty)$ for any $p \in [1,\infty)$ as $\varepsilon \searrow 0$, more precisely*

$$\int_0^\infty \left|\psi_{\varepsilon_k} \pm z_{\min}\right|^p \mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma \xrightarrow{k\to\infty} 0.$$

Consequently,

$$\left\|u_{\varepsilon_k} \mp \Lambda\right\|_{W^{1,p}_{\mathrm{rad}}(\mathbb{B}^2)} \xrightarrow{k\to\infty} 0$$

where Λ denotes the cone $\Lambda(x) = z_{\min}\,(1-|x|)$.

Proof. Without loss of generality we may assume $z_{\min} > 0$. Of course, $(\psi_\varepsilon)_{\varepsilon>0} \subset W^{1,2}(0,\infty)$ is by (6.1) also a minimizing sequence for $\tilde{I}_\varepsilon$. For $\varepsilon > 0$, $\eta \in (0, z_{\min})$ let

$$B_{\varepsilon,\eta} := \left\{\sigma \in [0,\infty) \,|\, \psi_\varepsilon(\sigma) \in [-z_{\min}+\eta, z_{\min}-\eta]\right\}.$$

We obtain using Lemma 6.3

$$\min_{[-z_{\min}+\eta, z_{\min}-\eta]}\tilde{g}\int_{B_{\varepsilon,\eta}}\mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma \le \int_{B_{\varepsilon,\eta}}\mathrm{e}^{-2\sigma}\tilde{g}(\psi_\varepsilon)\,\mathrm{d}\sigma \le \tilde{I}_\varepsilon(\psi_\varepsilon) = \mathcal{O}(\varepsilon).$$

Thus by (2.2), for any $\eta \in (0, z_{\min})$ we have $\int_{B_{\varepsilon,\eta}}\mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma \to 0$ as $\varepsilon \searrow 0$. By Proposition 4.2, Lemma 6.2, and $\varepsilon \ll 1$ minimizers are contained either in $(0, z_{\min})$ or $(-z_{\min}, 0)$. Thus we can find a subsequence whose values are either strictly positive or strictly negative. For simplicity of exposition let us assume that $\psi_\varepsilon \in (0, z_{\min})$. Then

$$\begin{aligned}\int_0^\infty |\psi_\varepsilon - z_{\min}|^p \mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma &\le \eta^p\int_{(0,\infty)\setminus B_{\varepsilon,\eta}}\mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma + z_{\min}^p\int_{B_{\varepsilon,\eta}}\mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma \\ &\le \tfrac{1}{2}\eta^p + z_{\min}^p\int_{B_{\varepsilon,\eta}}\mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma\end{aligned}$$

which gives

$$\limsup_{\varepsilon\searrow 0}\int_0^\infty |\psi_\varepsilon - z_{\min}|^p \mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma \le \tfrac{1}{2}\eta^p.$$

Now let $\eta \searrow 0$. Substituting we arrive at

$$\int_{\mathbb{B}^2}\left|\nabla\left(u_\varepsilon + \Lambda\right)\right|^p\,\mathrm{d}x = 2\pi\int_0^1\left|r_\varepsilon' - z_{\min}\right|^p\varrho\,\mathrm{d}\varrho = 2\pi\int_0^\infty\left|\psi_\varepsilon - z_{\min}\right|^p\mathrm{e}^{-2\sigma}\,\mathrm{d}\sigma$$

and, by $r_\varepsilon(1) = \Lambda(1) = 0$, using Poincaré's inequality,

$$\int_{\mathbb{B}^2}\left|u_\varepsilon + \Lambda\right|^p\,\mathrm{d}x = 2\pi\int_0^1\left|r_\varepsilon + z_{\min}(1-\varrho)\right|^p\varrho\,\mathrm{d}\varrho \le 2\pi\int_0^1\left|r_\varepsilon' - z_{\min}\right|^p\varrho\,\mathrm{d}\varrho.$$ □

Remark 6.6 (Optimality of convergence rate)**.** Observe that we cannot replace the right-hand side of (6.4) by $\mathcal{O}(\varepsilon^2)$. Otherwise this would imply a uniform $W^{1,2}(0,\infty)$-bound on a sequence of I_ε-minimizers ψ_ε (recall (3.12) and Prop. 4.2), thus (after passing to a subsequence) $\psi_\varepsilon \rightharpoonup \psi_0$ for some $\psi_0 \in W^{1,2}(0,\infty)$. Proposition 6.5 (together with Prop. 4.2) would imply $\psi_0 = \pm z_{\min}$ (at least for sufficiently smooth g), but a constant function does not belong to $W^{1,2}(0,\infty)$ unless it is the null function.

Corollary 6.7 (Convergence of boundary data)**.** *Let* $(\psi_\varepsilon)_{\varepsilon>0} \subset W^{1,2}(0,\infty)$ *be a sequence of minimizers for* I_ε *and* $g \in C^1(\mathbb{R})$*. Let* $\tilde{g}$ *be as in* (6.2)*. Then*

$$\tilde{g}(\psi_\varepsilon(0)) = \mathcal{O}(\varepsilon^2 \left|\log \varepsilon\right|)$$

and $|\psi_\varepsilon(0)| \to z_{\min}$.

Proof. The idea is to consider the first variation of $\tilde{I}_\varepsilon$ and use ψ' as a test function. However, as pointed out in the context of (5.1), we do not know whether $\psi'' \in L^2$ (which would imply $\psi' \in W^{1,2}(0,\infty)$ and give rise to a straightforward argument), so we first have to construct an admissible test function. We fix $\varepsilon > 0$ and write ψ instead of ψ_ε for simplicity of notation. For any $S > 0$ we define

$$\phi_S(\sigma) := \begin{cases} \psi'(\sigma), & \sigma \in [0,S], \\ (S+1-\sigma)\psi'(S), & \sigma \in [S,S+1], \\ 0, & \sigma \in [S+1,\infty), \end{cases}$$

which obviously belongs to $W^{1,2}(0,\infty)$. Since ψ is a minimizer and $|\psi(\cdot)| \le z_{\min}$ by Proposition 4.2 and, according to (3.12),

$$\tilde{I}_\varepsilon(\psi) = \int_0^\infty \mathrm{e}^{-2\sigma}\tilde{g}(\psi)\,\mathrm{d}\sigma + \varepsilon^2 \int_0^\infty \frac{(\psi' - \psi(1+\psi^2))^2}{(1+\psi^2)^{5/2}}\,\mathrm{d}\sigma,$$

we obtain

$$\begin{aligned}
0 &= \delta\tilde{I}_\varepsilon(\psi,\phi_S) \\
&= \int_0^\infty \mathrm{e}^{-2\sigma}\tilde{g}'(\psi)\phi_S \\
&\quad + \varepsilon^2 \int_0^\infty \left(2\frac{\psi' - \psi(1+\psi^2)}{(1+\psi^2)^{5/2}} \left(\phi_S' - \phi_S - 3\psi^2\phi_S\right) - 5\frac{\left(\psi' - \psi(1+\psi^2)\right)^2}{(1+\psi^2)^{7/2}}\psi\phi_S \right) \\
&= \int_0^\infty \mathrm{e}^{-2\sigma}\tilde{g}'(\psi)\phi_S + \varepsilon^2 \left.\frac{\left(\psi' - \psi(1+\psi^2)\right)^2}{(1+\psi^2)^{5/2}}\right|_0^S \\
&\quad + \varepsilon^2 \int_S^{S+1} \left(2\frac{\psi' - \psi(1+\psi^2)}{(1+\psi^2)^{5/2}} \left(\phi_S' - \phi_S - 3\psi^2\phi_S\right) - 5\frac{\left(\psi' - \psi(1+\psi^2)\right)^2}{(1+\psi^2)^{7/2}}\psi\phi_S \right) \\
&\overset{(5.4)}{\le} \int_0^S \mathrm{e}^{-2\sigma}\left(\tilde{g}(\psi)\right)'\,\mathrm{d}\sigma + \int_S^{S+1} \mathrm{e}^{-2\sigma}\tilde{g}'(\psi)\psi'(S)(S+1-\sigma)\,\mathrm{d}\sigma \\
&\quad + \varepsilon^2 \left(\psi' - \psi(1+\psi^2)\right)^2(S) \\
&\quad + C\varepsilon^2 \int_S^{S+1} \left|\psi' - \psi(1+\psi^2)\right|\,\mathrm{d}\sigma\,|\psi'(S)| \\
&\quad + C\varepsilon^2 \int_S^{S+1} \left|\psi' - \psi(1+\psi^2)\right|^2\,\mathrm{d}\sigma\,|\psi'(S)|
\end{aligned}$$

$$
\begin{aligned}
&\le \mathrm{e}^{-2\sigma}\tilde g(\psi)\big|_0^S + 2\int_0^S \mathrm{e}^{-2\sigma}\tilde g(\psi)\,\mathrm{d}\sigma \\
&\quad + |\psi'(S)| \int_S^{S+1} \mathrm{e}^{-2\sigma} \max_{[0,z_{\min}]} |\tilde g'|\,\mathrm{d}\sigma \\
&\quad + \varepsilon^2 \left(\psi' - \psi(1+\psi^2)\right)^2(S) \\
&\quad + C\varepsilon^2 \int_S^{S+1} \left(|\psi'| + |\psi| + |\psi'|^2 + |\psi|^2\right)\mathrm{d}\sigma\,|\psi'(S)| \\
&\le -\tilde g(\psi(0)) + 2\tilde I_\varepsilon(\psi) \\
&\quad + \mathrm{e}^{-2S} \underbrace{\tilde g(\psi(S))}_{\le \max_{[0,z_{\min}]}\tilde g} + c\varepsilon^2 \left(\psi'(S)^2 + \underbrace{\psi(S)^2}_{\to 0}\right) \\
&\quad + |\psi'(S)| \left[\tfrac12 \max_{[0,z_{\min}]} |\tilde g'|\,\mathrm{e}^{-2S} + C\varepsilon^2 \underbrace{\left(\|\psi\|_{W^{1,2}(S,S+1)} + \|\psi\|^2_{W^{1,2}(S,S+1)}\right)}_{\to 0}\right].
\end{aligned}
$$

As $\psi' \in L^2(0,\infty)$ is continuous(ly differentiable) by Lemma 5.2, we may choose $S_k \in \operatorname{argmin}_{[k-1,k]} |\psi'|$, so $\sum_{k\in\mathbb{N}} |\psi'(S_k)|^2 \le \|\psi'\|^2_{L^2}$ by Riemann integration theory. Thus we obtain a monotone sequence $(S_k)_{k\in\mathbb{N}} \subset (0,\infty)$, $S_k \nearrow \infty$, satisfying $\psi'(S_k) \to 0$ as $k \to \infty$. Finally, we arrive at

$$
0 \le -\tilde g(\psi(0)) + 2\tilde I_\varepsilon(\psi) \overset{(6.4)}{\le} -\tilde g(\psi(0)) + C\varepsilon^2 |\log\varepsilon|\,.
$$

To see the second statement, recall $|\psi_\varepsilon(0)| \in [0, z_{\min}]$ by Proposition 4.2, $\tilde g(z_{\min}) = 0$, and $\tilde g > 0$ on $[0, z_{\min})$. From $0 \le \tilde g(\psi_\varepsilon(0)) \le C\varepsilon^2 |\log\varepsilon|$ and the continuity of $\tilde g$ it follows that $\tilde g(\xi) = 0$ holds for any accumulation point ξ of $\psi_\varepsilon(0)$. In other words $\xi = z_{\min}$ for any accumulation point ξ of $|\psi_\varepsilon(0)|$. Since $|\psi_\varepsilon(0)| \in [0, z_{\min}]$ it follows that $|\psi_\varepsilon(0)| \to z_{\min}$. □

7. Monotonicity and convexity of minimizers

In order to investigate convexity of (radially symmetric) minimizers of E_ε we first show that minimizers of I_ε are monotonic on certain regions.

In contrast to [12], where the authors were able to infer global convexity/concavity properties of local and global minimizers, our following results here can deal only with global minimizers. Moreover we show that convexity/concavity can be expected only in certain regions. Therefore we notice that, in spite of the dimension reduction that we obtained by working in the set of rotationally symmetric maps belonging to $\mathcal{C}_0 \cap W^{2,2}(\mathbb{B}^2)$, the higher dimensionality of the original problem plays a significant role. In ([12], Prop. 4.10) one could exploit the fact that the Euler−Lagrange equation did not depend explicitly on the space variable and study the related autonomous system; in our situation this no longer possible since equation (5.1) is non-autonomous. Hence new ideas must be employed.

Recall our convention: unless otherwise stated, by the term '(monotonic) de/increasing' we always refer to *weak* monotonicity; the same applies to convexity and concavity. Moreover let us underline, that due to (3.14),

$$
\begin{aligned}
r \text{ monotonic increasing} &\iff \psi \ge 0, \\
r \text{ (weakly) convex} &\iff \psi \text{ monotonic decreasing.}
\end{aligned}
$$

Proposition 7.1 (The case of decreasing g)**.** *Let $g \in C^{1,1}(\mathbb{R})$ be (weakly) decreasing on $[0, z_{\min}]$, $z_{\min} > 0$, and $\psi \in W^{1,2}(0,\infty)$ an I_ε-minimizer.*

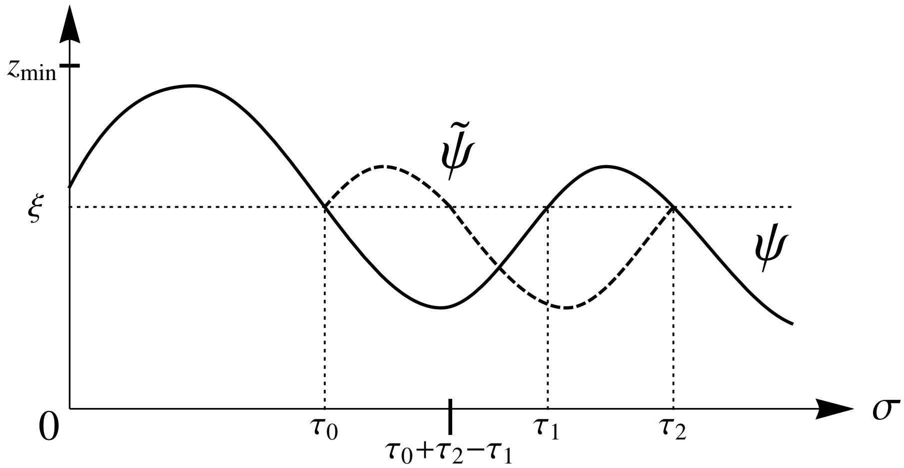

FIGURE 1. Situation in Lemma 7.2.

(i) *If* $\psi(0) > 0$ *then* ψ *is strictly increasing on* $(0, \sigma_0)$ *for some* $\sigma_0 \in (0, \infty)$ *and strictly decreasing on* (σ_0, ∞).
(ii) *If* $\psi(0) < 0$ *the situation is reversed.*
(iii) *If* $\psi(0) = 0$ *then* ψ *vanishes on* $[0, \infty)$.

The third statement is covered by Lemma 6.2, the second one is obtained from the first by changing sign. We will establish the proof of the first one with the aid of the following two statements.

Lemma 7.2 (Cardinality of points mapping to regular values for ψ)**.** *Under the hypotheses of Proposition* 7.1, *let* $\psi(\tilde{\sigma}) > 0$ *for some* $\tilde{\sigma} \in (0, \infty)$. *Then for almost every* $\xi \in (0, \psi(\tilde{\sigma}))$ *we have* $\#(\psi^{-1}(\xi)) \in \{1, 2\}$.

Proof. From Lemma 6.2 we infer that $\operatorname{image}\psi \subset (0, z_{\min})$. Moreover $\psi \in C^2[0, \infty)$ by Lemma 5.2. Let $\xi \in (0, \psi(\tilde{\sigma}))$ be a regular value of ψ, *i.e.*,

$$\psi'(\tau) \neq 0 \qquad \text{for any } \tau \in [0, \infty) \text{ with } \psi(\tau) = \xi. \tag{7.1}$$

By Sard's theorem [13] this holds for a.e. $\xi \in (0, \psi(\tilde{\sigma}))$. Note that the points τ satisfying (7.1) can not accumulate (otherwise we would have a sequence with $\tau_n \to \tau \in [0, \infty)$, with $\psi(\tau_n) = \psi(\tau) = \xi$ and hence $\psi'(\tau) = 0$ contradicting the fact that ξ is a regular value), therefore they are isolated and there are countably (in fact finitely) many of them.

Using $\psi(\tilde{\sigma}) > 0$ and $\psi(\infty) = 0$ there is (by continuity) at least one element in $\psi^{-1}(\xi)$ and, if there is more than one, they can be ordered in a sequence

$$0 \leq \tau_0 < \tau_1 < \tau_2 < \ldots.$$

We distinguish two cases. Firstly, suppose that $\psi'(\tau_0) < 0$. If there are further points satisfying (7.1), then there are at least three (due to $\psi(\infty) = 0$) and

$$\operatorname{sign}\psi'(\tau_k) = (-1)^{k+1} \qquad \text{for any } 0 \leq k < \#(\psi^{-1}(\xi)). \tag{7.2}$$

We construct $\tilde{\psi} \in W^{1,2}(0, \infty)$ from ψ by interchanging (τ_0, τ_1) and (τ_1, τ_2) (see Fig. 1), more precisely

$$\tilde{\psi} : \sigma \mapsto \begin{cases} \psi(\sigma) & \text{if } \sigma \in [0, \tau_0], \\ \psi(\sigma + (\tau_1 - \tau_0)) & \text{if } \sigma \in [\tau_0, \tau_0 + (\tau_2 - \tau_1)], \\ \psi(\sigma - (\tau_2 - \tau_1)) & \text{if } \sigma \in [\tau_2 - (\tau_1 - \tau_0), \tau_2], \\ \psi(\sigma) & \text{if } \sigma \in [\tau_2, \infty). \end{cases} \tag{7.3}$$

Of course, the regularization term in I_ε remains unchanged, that is $(I_\varepsilon - I_0)(\tilde{\psi}) = (I_\varepsilon - I_0)(\psi)$. On the other hand,

$$\begin{aligned} I_0(\tilde{\psi}) - I_0(\psi) &= \int_{\tau_0}^{\tau_0+(\tau_2-\tau_1)} \mathrm{e}^{-2\sigma} g(\psi(\sigma + (\tau_1 - \tau_0))) - \int_{\tau_1}^{\tau_2} \mathrm{e}^{-2\sigma} g(\psi(\sigma)) \\ &\quad + \int_{\tau_2-(\tau_1-\tau_0)}^{\tau_2} \mathrm{e}^{-2\sigma} g(\psi(\sigma - (\tau_2 - \tau_1))) - \int_{\tau_0}^{\tau_1} \mathrm{e}^{-2\sigma} g(\psi(\sigma)) \\ &= \int_{\tau_1}^{\tau_2} \mathrm{e}^{-2\sigma+2(\tau_1-\tau_0)} g(\psi(\sigma)) - \int_{\tau_1}^{\tau_2} \mathrm{e}^{-2\sigma} g(\psi(\sigma)) \\ &\quad + \int_{\tau_0}^{\tau_1} \mathrm{e}^{-2\sigma-2(\tau_2-\tau_1)} g(\psi(\sigma)) - \int_{\tau_0}^{\tau_1} \mathrm{e}^{-2\sigma} g(\psi(\sigma)). \end{aligned}$$

By $\psi(\cdot) < \xi$ on (τ_0, τ_1) and $\psi(\cdot) > \xi$ on (τ_1, τ_2) we infer from the fact that g is decreasing

$$g(\psi(\cdot)) \geq g(\xi) \quad \text{on } (\tau_0, \tau_1), \qquad g(\psi(\cdot)) \leq g(\xi) \quad \text{on } (\tau_1, \tau_2).$$

Let $\hat{\tau} \in (\tau_1, \tau_2)$ be a global maximizer of $\psi|_{[\tau_1,\tau_2]}$ (recall (7.2)). Then $g(\psi(\hat{\tau})) < g(\xi)$, for otherwise g would be constant on $[\xi, \psi(\hat{\tau})]$ and by defining

$$\hat{\psi} := \begin{cases} \psi & \text{on } [0, \tau_1] \cup [\tau_2, \infty), \\ \xi & \text{on } [\tau_1, \tau_2], \end{cases}$$

we would get $I_\varepsilon(\hat{\psi}) < I_\varepsilon(\psi)$ (due to the regularization term), a fact that contradicts the minimality of ψ. Therefore

$$\begin{aligned} g(\psi(\cdot)) \geq g(\xi) \quad \text{on } (\tau_0, \tau_1), \qquad & g(\psi(\cdot)) \leq g(\xi) \quad \text{on } (\tau_1, \tau_2), \\ & g(\psi(\cdot)) < g(\xi) \quad \text{on some neighborhood of } \hat{\tau} \in (\tau_1, \tau_2), \end{aligned} \tag{7.4}$$

and

$$\int_{\tau_0}^{\tau_1} \mathrm{e}^{-2\sigma} g(\psi(\sigma))\,\mathrm{d}\sigma \geq \int_{\tau_0}^{\tau_1} \mathrm{e}^{-2\sigma} g(\xi)\,\mathrm{d}\sigma, \qquad \int_{\tau_1}^{\tau_2} \mathrm{e}^{-2\sigma} g(\psi(\sigma))\,\mathrm{d}\sigma < \int_{\tau_1}^{\tau_2} \mathrm{e}^{-2\sigma} g(\xi)\,\mathrm{d}\sigma. \tag{7.5}$$

It follows

$$I_0(\tilde{\psi}) - I_0(\psi) < \left(\mathrm{e}^{2(\tau_1-\tau_0)} - 1\right) \int_{\tau_1}^{\tau_2} \mathrm{e}^{-2\sigma} g(\xi) + \left(\mathrm{e}^{-2(\tau_2-\tau_1)} - 1\right) \int_{\tau_0}^{\tau_1} \mathrm{e}^{-2\sigma} g(\xi) = 0.$$

This contradicts the fact that ψ is a global minimizer. Therefore $(\psi)^{-1}(\xi) = \{\tau_0\}$. On the other hand, if $\psi'(\tau_0) > 0$ then, since $\psi(\infty) = 0$, there is at least one further point $\tau_1 > \tau_0$ in $\psi^{-1}(\xi)$ and $\psi'(\tau_1) < 0$. Repeating the above arguments (for τ_1, τ_2, τ_3) we infer that necessarily $\psi^{-1}(\xi) = \{\tau_0, \tau_1\}$. □

Corollary 7.3. *Under the hypothesis of Proposition 7.1, assume $\psi(0) > 0$. Then ψ is (weakly) increasing on $(0, \sigma_1)$ where σ_1 denotes any global maximizer of ψ.*

Proof. Assuming the contrary there are points $0 \leq \sigma_+ < \sigma_- < \sigma_1$ with $\psi(\sigma_-) < \psi(\sigma_+) \leq \psi(\sigma_1)$ (see Fig. 2 for a possible configuration). But since $\psi(\infty) = 0$ we have $\#\psi^{-1}(\xi) \geq 3$ for all $\xi \in (\psi(\sigma_-), \psi(\sigma_+))$ (by continuity) contradicting Lemma 7.2. □

Proof of Proposition 7.1 (i). Recall that image $\psi \subset (0, z_{\min})$ by Lemma 6.2. Since $\psi \in C^2[0,\infty)$, $\psi'(0) > 0$ by (5.4) and $\psi(\infty) = 0$ by $\psi \in W^{1,2}(0,\infty)$, the function ψ must have at least one global maximum. Let $\sigma_0 > 0$ denote the smallest point in $(0,\infty)$ where the global maximum is attained. By Corollary 7.3, the function ψ is monotonic increasing on $(0, \sigma_0)$. We infer from Lemma 7.2 that ψ is monotonic decreasing on (σ_0, ∞). On the

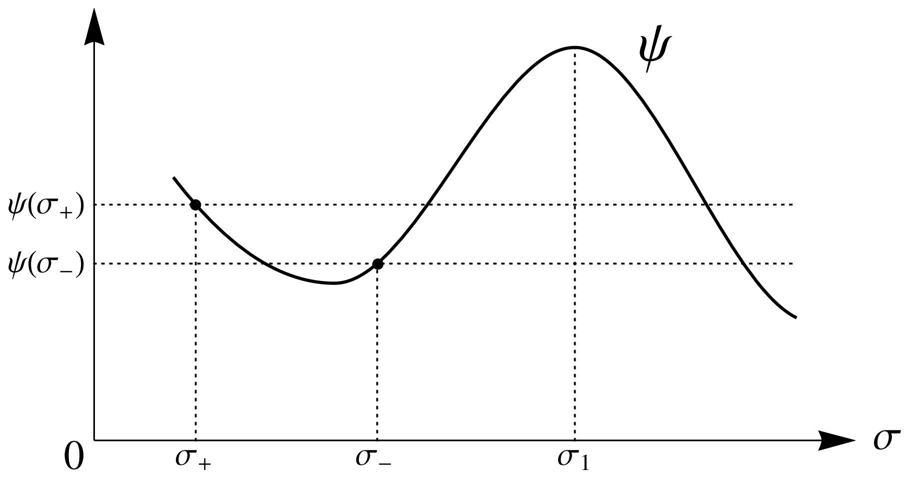

FIGURE 2. Situation in Corollary 7.3.

other hand, ψ can not be locally constant on some interval of positive measure, otherwise we would get a contradiction by using (5.1): precisely, we would arrive at

$$0 = \frac{(1+\psi^2)^{5/2}}{2\varepsilon^2} \mathrm{e}^{-2\sigma} g'(\psi) + \tfrac{1}{2}\psi(1+\psi^2)(2+\psi^2).$$

If $g'(\psi)$ vanishes, the right-hand side is positive; otherwise the first term on the right-hand side varies due to the factor $\mathrm{e}^{-2\sigma}$ while the second one is constant.

Hence ψ must be strictly monotone increasing on $(0, \sigma_0)$ and strictly decreasing on (σ_0, ∞). □

Having made transparent some important lines of reasoning, we are now in the position to relax the conditions imposed for Proposition 7.1.

Proposition 7.4 (The case of general g)**.** *Let $g \in C^{1,1}(\mathbb{R})$ with $z_{\min} > 0$ and $\psi \in W^{1,2}(0, \infty)$ be an I_ε-minimizer with $\psi(0) > 0$ attaining a global maximum at $\sigma_0 \in (0, \infty)$. Then ψ is strictly decreasing on $[\sigma_0, \infty)$. Analogously, ψ is strictly increasing on $[\sigma_0, \infty)$ provided $\psi(0) < 0$ and σ_0 denotes a point where a global minimum is attained.*

Proof. First of all note that ψ cannot be locally constant, otherwise we get a contradiction by (5.1). Proceeding as in Lemma 7.2, we infer $\operatorname{image} \psi \subset (0, z_{\min})$ from Lemma 6.2 as well as $\psi \in C^2[0, \infty)$ by Lemma 5.2. Again, by Sard's theorem (7.1) holds for a.e. $\xi \in (0, \psi(\sigma_0))$, and the elements of $\psi^{-1}(\xi)$ are isolated and can be ordered in an ascending sequence.

We aim at showing that there is only one element in $\psi^{-1}(\xi)$ which is larger than σ_0, in other words we want to show that $(\psi|_{(\sigma_0,\infty)})^{-1}(\xi)$ contains just one element. To this end we assume the contrary and denote by τ_0 the point in $(\psi|_{(\sigma_0,\infty)})^{-1}(\xi)$ that is closest to σ_0. Let $\sigma_{\max}$ denote the point closest to τ_0 where the global maximum of $\psi|_{[\tau_0,\infty)}$ is attained and let $\sigma_{\min}$ be the point closest to τ_0 where the global minimum of $\psi|_{[\tau_0,\sigma_{\max}]}$ is achieved. Thus $\sigma_0 < \tau_0 < \sigma_{\min} < \sigma_{\max}$. A possible configuration is depicted in Figure 3. Since ξ is by assumption a regular value (recall (7.1)) we infer

$$\xi_{\min} := \psi(\sigma_{\min}) < \xi < \psi(\sigma_{\max}) =: \xi_{\max}$$

and

$$\psi(\cdot) \le \xi_{\max} \quad \text{on } [\tau_0, \infty), \qquad \psi(\cdot) \ge \xi_{\min} \quad \text{on } [\tau_0, \sigma_{\max}].$$

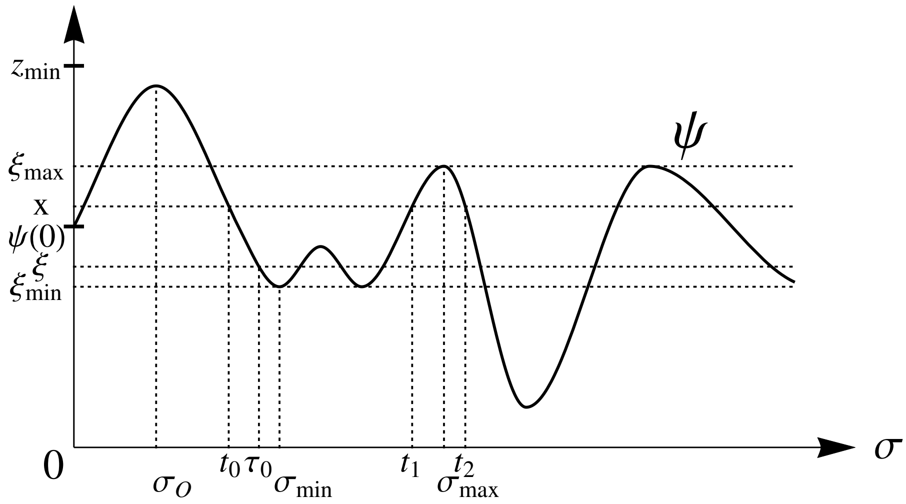

FIGURE 3. Situation in Proposition 7.4.

From the minimality of ψ we now derive some important information on the shape of g on the interval $[\xi_{\min}, \xi_{\max}]$. We first claim that

$$g(\xi_{\max}) < g(y) \qquad \text{for any } y \in [\xi_{\min}, \xi_{\max}). \tag{7.6}$$

Otherwise, if $g(\xi_{\max}) \geq g(\hat{y})$ for some $\hat{y} \in [\xi_{\min}, \xi_{\max})$, then we infer that the global minimum of $g|_{[\xi_{\min}, \xi_{\max}]}$ is attained at some $\tilde{y} \in [\xi_{\min}, \xi_{\max})$, thus $I_\varepsilon(\hat{\psi}) < I_\varepsilon(\psi)$ where

$$\hat{\psi} := \begin{cases} \psi & \text{on } [0, \sigma_{\min}], \\ \min(\psi, \tilde{y}) & \text{on } [\sigma_{\min}, \infty), \end{cases}$$

due to the regularization terms (contradicting the minimality of ψ).

Next we claim that $g'(\xi_{\max}) < 0$: indeed, if this were not the case, then, by (5.1),

$$\psi''(\sigma_{\max}) \geq \tfrac{1}{2}\xi_{\max}\left(1 + \xi_{\max}^2\right)\left(2 + \xi_{\max}^2\right) > 0,$$

which contradicts the fact that $\sigma_{\max}$ is a maximizer. Thus, by continuity there exists some $\delta > 0$, such that $\xi_{\min} < \xi_{\max} - \delta < \xi_{\max}$ and g is strictly monotone decreasing on $[\xi_{\max} - \delta, \xi_{\max}]$. On the other hand $g|_{[\xi_{\min}, \xi_{\max}-\delta]}$ attains a minimum, that is strictly larger than $g(\xi_{\max})$ due to (7.6). This implies that there is some regular value $x \in (\xi, \xi_{\max})$ close to $\xi_{\max}$ such that

$$g(\eta) > g(x) > g(\eta') \qquad \text{for all } \eta \in [\xi_{\min}, x), \eta' \in (x, \xi_{\max}]. \tag{7.7}$$

We may choose consecutive (!) elements $t_0, t_1, t_2 \in \psi^{-1}(x)$ with $\sigma_0 < t_0 < t_1 < \sigma_{\max} < t_2$, $\operatorname{sign} \psi'(t_k) = (-1)^{k+1}$, $k = 0, 1, 2$, and

$$\psi(\cdot) \in [\xi_{\min}, x) \text{ on } (t_0, t_1), \qquad \psi(\cdot) \in (x, \xi_{\max}] \text{ on } (t_1, t_2). \tag{7.8}$$

Equations (7.7), (7.8) give that

$$g(\psi(\cdot)) > g(x) \text{ on } (t_0, t_1), \qquad g(\psi(\cdot)) < g(x) \text{ on } (t_1, t_2). \tag{7.9}$$

So we are in the situation analogous to (7.4), (7.5). This permits to employ the argument from Lemma 7.2 (construct $\tilde{\psi}$ as in (7.3) by replacing τ_i with t_i, $i = 0, 1, 2$), which leads to a contradiction.

So far we have shown that for almost every $\xi \in (0, \psi(\sigma_0))$ (the regular values of ψ) the cardinality of the set $(\psi|_{(\sigma_0,\infty)})^{-1}(\xi)$ is equal to one. Since $\psi(\infty) = 0$ and since ψ can not be locally constant we infer that ψ must be strictly decreasing on (σ_0, ∞). □

Corollary 7.5 (Minimizers are concave or convex near the origin)**.** *Let γ be so that $g \in C^{1,1}(\mathbb{R})$. Then any minimizer of E_ε in $W^{2,2}_{\mathrm{rad}}(\mathbb{B}^2) \cap \mathcal{C}_0$ is concave or convex in a neighborhood of the origin (whose radius depends on ε).*

Proof. If $z_{\min} = 0$ we have $u \equiv 0$ which is both concave and convex, thus we may assume $z_{\min} > 0$. We only have to show that (recall (3.3) and (3.14))

$$\det D^2 u(x) = \frac{r''(\varrho) r'(\varrho)}{\varrho} = -\frac{\psi'(\sigma)\psi(\sigma)}{\mathrm{e}^{-2\sigma}}$$

is non-negative and the sign of

$$u_{x_1x_1}(x) = r''(\varrho)\cos^2\varphi + \frac{r'(\varrho)}{\varrho}\sin^2\varphi = \frac{-\psi'(\sigma)\cos^2\varphi + \psi(\sigma)\sin^2\varphi}{\mathrm{e}^{-\sigma}}$$

does not change for $\varrho \ll 1\,(\iff \sigma \gg 1)$. But this is immediate since either $\psi \geq 0$ and $\psi' \leq 0$ or $\psi \leq 0$ and $\psi' \geq 0$ in a neighborhood of infinity by Proposition 7.4. The claim now follows from the fact that the determinants of the leading principal minors are all positive or have alternating sign. □

With a minor extra assumption on g we are now able to infer even more information on the shape of ψ and basically extend Proposition 7.1 to the case of (almost) arbitrary g.

Theorem 7.6 (Minimizers are strictly monotonic)**.** *Let $g \in C^{1,1}(\mathbb{R})$ be (weakly) decreasing on $[z_{\min} - \delta, z_{\min}]$ for $z_{\min} > 0$ and some $\delta > 0$, and $\psi \in W^{1,2}(0, \infty)$ be an I_ε-minimizer for $0 < \varepsilon \ll 1$ with $\psi(0) > 0$. Then ψ is strictly increasing on $(0, \sigma_0)$ for some $\sigma_0 \in (0, \infty)$ and strictly decreasing on (σ_0, ∞). The situation is reversed in case $\psi(0) < 0$.*

Note that the case $\psi(0) = 0$ is excluded by Lemma 6.3.

Proof. Let $\psi(0) > 0$. By Proposition 7.4 we merely have to show that ψ is weakly increasing on $[0, \sigma_0]$ where $\sigma_0 > 0$ denotes the point where the global maximum of ψ is attained and which is unique due to Proposition 7.4. Strict monotonicity will follow again by employing (5.1) in order to show that ψ can not be locally constant.

By taking a smaller $\delta > 0$ if necessary, we may additionally assume

$$g(y) \geq g(z_{\min} - \delta) \qquad \text{for all } y \in [0, z_{\min} - \delta]. \tag{7.10}$$

(Indeed g attains a minimum on $[0, z_{\min} - \delta]$ and g is weakly decreasing on $[z_{\min} - \delta, z_{\min}]$ by the monotonicity assumption.) From Corollary 6.7 we infer $\psi(0) \to z_{\min}$ as $\varepsilon \searrow 0$, so we may assume $\psi(0) \geq z_{\min} - \delta$. Arguing as in Proposition 7.4 (recall Sard's theorem), we may choose a regular value $\xi \in (\psi(0), \psi(\sigma_0))$: aiming at showing that $(\psi|_{[0,\sigma_0)})^{-1}(\xi)$ contains just one element, we first assume that the opposite is true and obtain a contradiction.

Let $\tau_0 < \tau_1$ denote the two largest elements of $\psi^{-1}(\xi)$ being smaller than σ_0 and τ_2 the smallest one being larger than σ_0, see Figure 4 for a possible configuration. We obtain $\operatorname{sign}\psi'(\tau_k) = (-1)^{k+1}$, $k = 0, 1, 2$, and $\psi(\cdot) \in (0, \xi)$ on (τ_0, τ_1), $\psi(\cdot) \in (\xi, \psi(\sigma_0)]$ on (τ_1, τ_2).

By (7.10) and the monotonicity of g on $[z_{\min} - \delta, z_{\min}]$ we obtain $g(\psi(\cdot)) \geq g(\xi)$ on (τ_0, τ_1) and $g(\psi(\cdot)) \leq g(\xi)$ on (τ_1, τ_2). Next we would like to infer that we are in a situation analogous to (7.4).

First we claim that

$$g(\psi(\sigma_0)) < g(y) \qquad \text{for all } y \in [\xi, \psi(\sigma_0)).$$

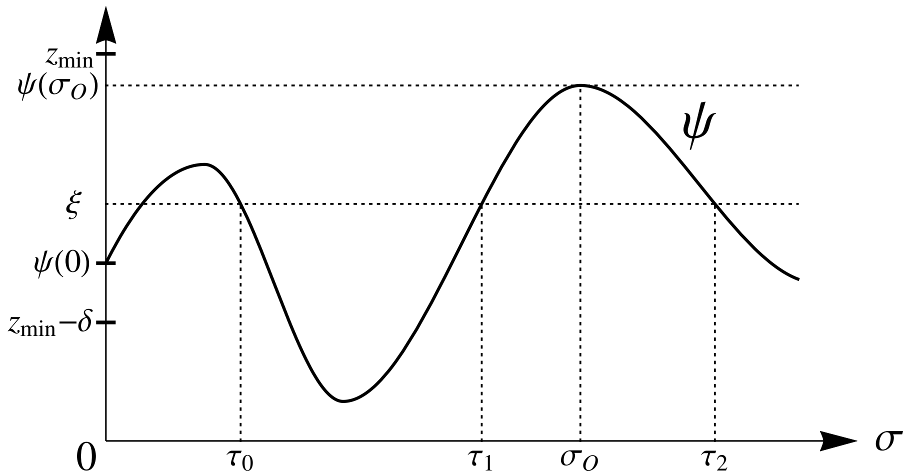

FIGURE 4. Situation in Theorem 7.6.

If this were not true, then, due to monotonicity, g would be constant on $[\psi(\sigma_0) - \delta', \psi(\sigma_0)]$ for some $\delta' > 0$. Choosing $\hat{\psi} := \min(\psi, \psi(\sigma_0) - \delta')$ we would arrive at $I_\varepsilon(\hat{\psi}) < I_\varepsilon(\psi)$ due to the regularization term. So there is some subinterval of (τ_1, τ_2) where $g(\psi(\cdot)) < g(\xi)$, and we arrive at (7.5). Constructing $\tilde{\psi}$ as in (7.3) we obtain a contradiction to the minimality of ψ, and the claim follows. □

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