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ON THE CONDITION OF TETRAHEDRAL POLYCONVEXITY, ARISING FROM CALCULUS OF VARIATIONS *

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Abstract. We study geometric conditions for integrand f to define lower semicontinuous functional of the form $I_f(u) = \int_\Omega f(u)\mathrm{d}x$, where u satisfies certain conservation law. Of our particular interest is tetrahedral convexity condition introduced by the first author in 2003, which is the variant of maximum principle expressed on tetrahedrons, and the new condition which we call tetrahedral polyconvexity. We prove that second condition is sufficient but it is not necessary for lower semicontinuity of I_f, tetrahedral polyconvexity condition is non-local and both conditions are not equivalent. Problems we discuss are strongly connected with the rank-one conjecture of Morrey known in the multidimensional calculus of variations.

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1. Introduction

One of the most challenging problems in the modern multidimensional calculus of variations is the so-called rank-one conjecture of Morrey which reads as follows. Let us consider the classical functional of the calculus of variations:

$$I_f(u) = \int_\Omega f(Du)\mathrm{d}x,$$

where $\Omega \subseteq \mathbb{R}^n$, $u : \Omega \to \mathbb{R}^m$, $u = (u_1, \ldots, u_m), u_i \in W^{1,\infty}(\Omega)$, $i = 1, \ldots, m$, $Du = (\nabla u_1, \ldots, \nabla u_m) \in \mathbb{R}^{n\times m}$. One asks about the characterization of the space of admitted functions f such that the functional I_f is sequentially lower semicontinuous with respect to the sequential weak-$*$ convergence of its arguments (gradients) in $L^\infty(\Omega, \mathbb{R}^{n\times m})$ (to abbreviate let us call this property shortly $sw*-lsc$). In the paper [30] Morrey proved

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that I_f is $sw^* - lsc$ if and only if it satisfies the following condition called the quasiconvexity condition:

$$\frac{1}{|Q|}\int_Q f(A+D\phi)\mathrm{d}x \geq f(A), \tag{1.1}$$

whenever $\phi \in C_0^\infty(\Omega, \mathbb{R}^m)$, $A \in \mathbb{R}^{n\times m}$ is an arbitrary matrix and Q is an arbitrary cube in $\mathbb{R}^n$. The quasiconvexity condition seems to be impossible to be verified in practice. Therefore it is natural to ask if there are some geometric conditions which are equivalent to the quasiconvexity condition (1.1). It was proven by Morrey in 1952 [30] that every quasiconvex function is convex in the directions of rank-one matrices and this property is called nowadays rank-one property. He conjectured (to be more precise he had expressed his doubts) that rank-one property is equivalent to the quasiconvexity condition. Since that time this conjecture is called rank one conjecture of Morrey. It required 40 years when this conjecture was disproved by Šverák [44] in cases $m \geq 3, n \geq 2$, while up to nowadays the conjecture is open in the remaining cases, which reduce to $m = n = 2$.

Many authors contributed further to that challenging question. We refer to *e.g.* [1, 4–6, 8, 11–15, 18, 26–28, 30, 31, 33–35, 37–39, 41–44, 48, 49]. Iwaniec in [19] has pointed out the strong relation between Morrey's conjecture and some important open problems in the theory of quasiconformal mappings (see also the paper by Astala [3]). Šverák has shown in [45] that quasiconvexity is strongly related to compactness properties of approximate solutions of the system $Du \in K$.

We are interested in geometric conditions which could be helpful for better understanding the quasiconvexity condition.

For this, we consider the case $m = n = 2$ and the special subset in the space of gradients, namely

$$\begin{aligned} u(z) = \xi_1 R(\xi_1\cdot z) + \xi_2 S(\xi_2\cdot z) + \xi_3 T(\xi_3\cdot z) =: u_1(z) + u_2(z) + u_3(z),\ \text{where}\\ \xi_1 = (1,0),\ \xi_2 = (0,1),\ \xi_3 = (1,1) \end{aligned}$$

and $R, S, T : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions, $R^{'} = r, S^{'} = s, T^{'} = t$. We observe that

$$\begin{aligned} Du = \sum_{i=1}^{3} Du_i &= r(x)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + s(y)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + t(x+y)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\\ &\sim (r(x), s(y), t(x+y)). \end{aligned}$$

This way the function f defined on 2×2 symmetric matrices can be identified with the function $\tilde{f}$ defined on $\mathbb{R}^3$ and our original functional reduces to the simpler one

$$I_{\tilde{f}}(v) = \int_Q \tilde{f}\big(v_1(x), v_2(y), v_3(x+y)\big)\mathrm{d}x\mathrm{d}y \tag{1.2}$$

(here $v_1 = r, v_2 = s, v_3 = t$). Note that the function $v(x,y) = (v_1(x), v_2(y), v_3(x+y))$ belongs to the kernel of differential operator $P = (P_1, P_2, P_3)$, *i.e.* $Pv = 0$, where

$$P_1 v = \frac{\partial v_1}{\partial y},\ P_2 v = \frac{\partial v_2}{\partial x},\ P_3 v = \frac{\partial v_3}{\partial x} - \frac{\partial v_3}{\partial y}\cdot \tag{1.3}$$

In particular this reduction step links the problem of quasiconvexity with the problem in the compensated compactness theory (originated by the pioneering works [32, 46], see also [47]), where one investigates the $sw*{-}lsc$-property of functionals defined on functions which lie in the kernel of the given differential operator P. In the special case when P is the rotation operator one deals with gradients. The rather well understood case is the case when operator P satisfies the so-called constant rank condition [7, 16, 17]. For the cases when the constant rank condition might not be satisfied we refer to [20, 21] and their further extensions (involving many applications), as well to first author's works [22, 23]. For recent works in this direction we refer also to [2, 40] and to references enclosed therein. We emphasize that our operator P given by (1.3) does not satisfy constant rank condition, *i.e.* it deals with the case which is less understood.

Let us skip this general approach and concentrate on the very special functional given by (1.2) which will be called the functional of the type $(2,3)$. Integrands which define $sw*-lsc$ functional will be called $(2,3)$ quasiconvex.

As there are no constraints on the involved functions v_1, v_2, v_3 in (1.2), we thought that similarly as in the case of the classical unconstrained functional, one could expect that the $sw*-lsc$ property of this functional can be expressed by the purely geometric constraints. The candidate for such geometric condition was found by first author in the paper [24] (Thm. 3.1, see also [10,36] for the related issues). The condition has to be verified on three dimensional oriented simplex'es (oriented tetrahedrons) by the purely geometric means. To be more precise, in Theorem 3.1 in [24] it was shown that if $\tilde{f}$ defines the functional (1.2) with the $sw*-lsc$ property then it necessarily must satisfy the two conditions and one of the conditions has purely geometric interpretation. We omit the formulation of the second one, which is not that directly geometric, and focus on first one only. It says the following. Having given an arbitrary tetrahedron $D \subseteq \mathbb{R}^3$ with three edges paralel to the axis and the polynomial $P_{\tilde{f}}$ from seven dimensional space of polynomials $\mathcal{A} = \text{span}\{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3\}$ such that $\tilde{f} = P_{\tilde{f}}$ in every corner of D and its three neighbours (we omit their definition, such $P_{\tilde{f}}$ is defined uniquely), one has

$$\tilde{f} \leq P_{\tilde{f}} \text{ inside } D.$$

This property serves as the version of the maximum principle for $\tilde{f}$. We will call this condition weak tetrahedral convexity condition.

We address the following questions:

Question A. Is the weak tetrahedral convexity condition equivalent to the $(2,3)$ quasiconvexity condition, *i.e.* lower semicontinuity of the related functional?

Question B. Are there some other simple geometric conditions which guarantee $sw*-lsc$ property of the related functional (1.2), *i.e.* $(2,3)$ quasiconvexity?

In this paper we try to approach them. We did not succeed in answering Question A. However, when looking for some other simple geometric conditions, we have introduced another geometric condition called tetrahedral polyconvexity condition, similarly as one deals with polyconvexity condition in the calculus of variations [5]. Trying to approach both questions, we have shown that tetrahedral polyconvexity condition is not equivalent to $(2,3)$ quasiconvexity. For this we use the tool known in the calculus of variations, namely fourth order polynomial constructed by Alibert and Dacorogna in [1] and embedding of our special functions v in (1.2) into the space of gradients. Main statement in this direction, where we compared several convexity type conditions useful for understanding Question A, is formulated in Theorem 3.3. Moreover, we have shown that weak tetrahedral polyconvexity condition is the non-local one, *i.e.* it cannot be expressed by conditions which hold in an arbitrary small neighbourhoods of points. This is done by adapting to our setting the technique of Kristensen from [26], which is known in the calculus of variations, showing that polyconvexity is not the local condition. The adaptation required perhaps not so automatic modification of the Carathéodory Theorem (Thm. 4.4). Main statement about the non-locality is formulated as Theorem 5.2.

We hope that the presented issue will contribute to the discussion of the quasiconvexity condition in the calculus of variations, as well as will indicate on some new interesting questions in pure geometry.

2. Preliminaries and basic notation

2.1. Functions of the type (2,3)

In this section we will be dealing with the following set of functions.

Definition 2.1 (Function of the type $(2,3)$)**.** Let Ω be an arbitrary open subset of $\mathbb{R}^n$. Function $u : \Omega \to \mathbb{R}^3$ having the form

$$u(z) = \Big(r\big(z\cdot\xi_1\big), s\big(z\cdot\xi_2\big), t\big(z\cdot\xi_3\big)\Big),$$

where $(\xi_i)_{i=1}^3$ is a triple of vectors from $\mathbb{R}^n$ which is pairwise independent, but dependent as a triple, $a\cdot b$ stays for scalar product of the vectors a, b and r, s, t are scalar functions of one variable, will be called `function of the type (2,3)` defined on Ω.

Justification of this notion comes from the fact that we deal with two variables (z may be here a vector from $\mathbb{R}^n$, however a function is dependent only on its projection to a two dimensional plane: span $\{\xi_1,\xi_2,\xi_3\}$) and three functions. As an example of such function we may consider function

$$v(x,y) = \Big(r(x), s(y), t(x+y)\Big), \text{ where } (x,y)\in\Omega\subseteq\mathbb{R}^2.$$

Functions of the form $v(x,y) = (r(x), s(y), t(x+y))$ and $\Omega\subseteq\mathbb{R}^2$ will be called a `special (2,3) functions`.

However not defined so far, functions of that type appear in several papers in calculus of variations as a tool to investigate quasiconvexity condition [10,37].

In our considerations we will use the fact that functions of the type $(2,3)$ can be canonically embedded into the space of symmetric gradients. Let us explain how it is done.

For this we will use the convention:

$$Dv = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_m}{\partial x_1} & \cdots & \frac{\partial v_m}{\partial x_n} \end{pmatrix},$$

where $v = (v_1,\ldots,v_m) : \Omega\to\mathbb{R}^m$ is vector-valued function and $\Omega\subseteq\mathbb{R}^n$.

Let $r,s,t : \mathbb{R}\to\mathbb{R}$ be given scalar one-variable bounded functions and R,S,T be their absolutely continuous primitives (so Lipschitz), *i.e.* $R' = r, S' = s, T' = t$. Consider $u : \Omega\to\mathbb{R}^n$ such that for any $z\in\Omega$

$$u(z) = \xi_1 R(\xi_1\cdot z) + \xi_2 S(\xi_2\cdot z) + \xi_3 T(\xi_3\cdot z) =: u_1(z) + u_2(z) + u_3(z).$$

Then we obtain

$$Du(z) = \xi_1\otimes\xi_1 r(\xi_1\cdot z) + \xi_2\otimes\xi_2 s(\xi_2\cdot z) + \xi_3\otimes\xi_3 t(\xi_3\cdot z) = \sum_{i=1}^3 Du_i(z). \tag{2.1}$$

Taking $n = 2$ and

$$\xi_1 = (1,0),\, \xi_2 = (0,1),\, \xi_3 = (1,1),$$

we may consider the special subsets in the space of gradients:

$$Du = Du_1 + Du_2 + Du_3, \text{ where}$$

$$u_1(x,y) := (R(x),0),\;\; u_2(x,y) := (0,S(y)),\;\; u_3(x,y) := (T(x+y),T(x+y)), \tag{2.2}$$

$$Du \;=\; r(x)\begin{bmatrix}1\,0\\0\,0\end{bmatrix} + s(y)\begin{bmatrix}0\,0\\0\,1\end{bmatrix} + t(x+y)\begin{bmatrix}1\,1\\1\,1\end{bmatrix}. \tag{2.3}$$

In particular, function of the type $(2,3)$ is embedded into the space of symmetric gradients. We arrive at a following observation.

Proposition 2.2. *Any special function of the type* $(2,3)$ *may be uniquely identified with a gradient of a certain function* $u : \Omega \to \mathbb{R}^3, u = u_1 + u_2 + u_3$ *defined by* (2.2) via *expression* (2.3).

We will consider $r, s, t \in L^\infty(\mathbb{R})$, so that R, S, T are Lipschitz. In fact, for the definition of a function of a type $(2,3)$ we only need to know the values on projections of Ω into three lines along ξ_1, ξ_2, ξ_3 respectively. Note that any Lipschitz function defined on a closed subset may be extended to a Lipschitz function on whole $\mathbb{R}^N$ with no change of Lipschitz constant by the Kirszbraun theorem.

2.2. Functionals of the type (2,3)

We start with recalling the definition of special functionals from [24].

Definition 2.3 (Functional of the type $(2,3)$)**.** Let Ω be an open subset of $\mathbb{R}^n$, $\xi = (\xi_i)_{i=1}^3$ be a triple of vectors belonging to $\mathbb{R}^n$ which are linearly dependent and pairwise independent. Let

$$v(z) = \Big(r\big(z \cdot \xi_1\big), s\big(z \cdot \xi_2\big), t\big(z \cdot \xi_3\big)\Big) \text{ be given function of the type } (2,3).$$

A functional of the form

$$I_f(v,\xi) = \int_\Omega f\big(r(z\cdot\xi_1), s(z\cdot\xi_2), t(z\cdot\xi_3)\big)\mathrm{d}z$$

will be called a `functional of the type (2,3)`, while a functional of the form

$$I_f(v) = \int_\Omega f\big(r(x), s(y), t(x+y)\big)\mathrm{d}x\mathrm{d}y, \text{ where } (x,y) \in \Omega \subseteq \mathbb{R}^2$$

will be called the `special functional of the type (2,3)`.

A crucial problem for us is the investigation of lower semicontinuity property of such functionals with respect to weak-$*$ convergence of $v's$ in $L^\infty(\Omega, \mathbb{R}^3)$. For this we use the following definition.

Definition 2.4 (weak-$*$ lower semicontinuity, weak-$*$ continuity)**.**

(i) A functional of the type $(2,3)$ is lower semicontinuous with respect to the sequential weak-$*$ convergence in $L^\infty(\Omega,\mathbb{R}^3)$ whenever for any sequence $v^\nu \overset{*}{\rightharpoonup} v$ (*i.e.* v^ν weak-$*$ converges to v in $L^\infty(\Omega,\mathbb{R}^3)$) of the functions of the type $(2,3)$ we have
$$\liminf_{\nu\to+\infty} I_f(v^\nu,\xi) \geq I_f(v,\xi).$$
To abbreviate we will call this condition the `(2,3) LSC` property. Those integrands which define functionals having the $(2,3)$ LSC property will be called `(2,3) quasiconvex`.

(ii) If $\liminf\limits_{\nu\to+\infty} I_f(v^\nu,\xi) = I_f(v,\xi)$ whenever $v^\nu \overset{*}{\rightharpoonup} v$, we say that I_f is weakly* continuous with respect to the sequential weak-$*$ convergence in $L^\infty(\Omega,\mathbb{R}^3)$ of the functions of the type $(2,3)$. Integrands defining such functionals will be called `(2,3) quasiaffine`.

The weak-$*$ convergence of the v^ν's is equivalent to convergence of functions building their coordinates: r^ν, s^ν, t^ν with respect to weak-$*$ convergence in L^∞ on the respective projections of set Ω onto the lines. Moreover, the limiting function v is also of the type $(2,3)$.

Let us note that the non-abstract description of set of $(2,3)$ quasiconvex functions has not been systematically investigated.

From the following fact stated below, it follows that the lower semicontinuity of any functional of the type $(2,3)$ reduces to the case of $\Omega = [0,1]^2$ and $\xi_1 = (1,0), \xi_2 = (0,1), \xi_3 = (1,1)$.

Fact 2.5 (Lem. 2.2 in [24])**.** *Let* f *be a continuous function* $f : \mathbb{R}^3 \to \mathbb{R}$. *The following conditions are equivalent.*

(i) *For* $Q = [0,1]^2$ *the special functional*

$$I_f(v) = \int_Q f\big(r(x), s(y), t(x+y)\big)\mathrm{d}x\mathrm{d}y \tag{2.4}$$

is lower semicontinuous with respect to weak-$*$ convergence of r, s, t *in* $L^\infty(\mathbb{R})$.

(ii) *For any domain* $\Omega \subset \mathbb{R}^N$ *and arbitrary triple of vectors* $\xi = (\xi_i)_{i=1}^3$ *which are pairwise independent, but linearly dependent as a triple, the functional* $I_f(v,\xi)$ *is lower semicontinuous with respect to weak-$*$ convergence of* r, s, t *in* $L^\infty(\mathbb{R})$.

In formulation of part (i) Lemma 2.2 in [24] one dealt with certain cube, however easy translation and dilation argument shows that this is equivalent to the statement above.

According to Proposition 2.2 an arbitrary special functional of the type $(2,3)$ can be uniquely identified with certain functional defined on the subset of gradients given by (2.3):

$$I_f(v) = \int_\Omega f(r(x), s(y), t(x+y))\mathrm{d}x\mathrm{d}y = \int_\Omega \tilde{f}(Du_1 + Du_2 + Du_3),$$

where the u_i's were defined in (2.2) and

$$f(r,s,t) = \tilde{f}\left(r\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + s\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + t\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right) \stackrel{def}{=:} \tilde{f}\left(\begin{bmatrix}r+t & t\\ t & s+t\end{bmatrix}\right),$$

$$\tilde{f} \text{ is defined on span } \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\} = M^{2\times 2}_{sym} \tag{2.5}$$

and we use the standard notation $M^{2\times 2} \cong \mathbb{R}^4$ to denote 2×2 matrices and $M^{2\times 2}_{sym}$ to denote symmetric 2×2 matrices.

2.3. Convexity-type conditions

It is not obvious whether there is a geometric interpretation of $(2,3)$ quasiconvex functions. To understand it better we discuss here several convexity-type conditions of geometric type for functions $f : \mathbb{R}^3 \to \mathbb{R}$.

For this we will start with introducing some geometric and algebraic objects we will deal with.

Definition 2.6 (Regular symplex)**.** Let p be a point in $\mathbb{R}^3$ and $\{t_i\}_{i=1}^3$ be nonzero real numbers and $(e_i)_{i=1}^3$ the standard $\mathbb{R}^3$ basis. We will call a symplex D `regular`, whenever D is the convex hull of four points: $p, \{p + t_i e_i\}_{i=1}^3$, for some $p, \{t_i\}_{i=1}^3$.

Such a D is obviously a tetrahedron with vertices $p, \{p + t_i e_i\}_{i=1}^3$, in particular having three edges parallel to the axis.

Every regular symplex D defines a cuboid, which has eight vertices, *i.e.* $p, \{p+t_ie_i\}_{i=1}^3, \{p+t_ie_i+t_je_j\}_{i\neq j}, p+\sum t_ie_i$. For any vertex q of that cuboid let us define the `neighbours` of q – that is those vertices which are linked with q by a single edge.

We introduce the subspace of polynomials

$$\mathcal{A} \stackrel{\text{def}}{=} \text{span}\,\{1, r, s, t, rs, rt, st\}.$$

Folowing [24], having given regular symplex D, we define the projection operator $P_D : C(\mathbb{R}^3) \to \mathcal{A}$ by choosing for any continuous function f such $P_D f \in \mathcal{A}$ that the equality

$$P_D f(r,s,t) = f(r,s,t),$$

holds in every vertex in D and all its neighbours.

Note that vertices p, $\{p+t_ie_i\}_{i=1}^3$ and their neighbours $\{p+t_ie_i+t_je_j\}_{i\neq j}$ form a set of seven points – vertices of the cuboid defined by D apart from $p+\sum_{i=1}^3 t_ie_i$. As $\mathcal{A}$ is seven-dimensional and those seven points are affinely independent, Kronecker–Capelli theorem shows that P_Df is uniquely defined.

We will deal with the following convexity-type conditions, which contribute to the understanding of (2,3) quasiconvexity condition.

Definition 2.7 (Convexity-type conditions)**.** The function $f:\mathbb{R}^3\to\mathbb{R}$ will be called

(i) `tetraaffine`, whenever f is a polynomial belonging to the linear space $\mathcal{A}$;
(ii) `weakly tetrahedrally convex`, whenever the inequality

$$f(r,s,t)\le P_Df(r,s,t)$$

holds for every point $(r,s,t)\in D$ and any regular symplex D.
(iii) `tetrahedrally polyconvex` if there exists convex function $g:\mathbb{R}^6\to\mathbb{R}$ such that

$$f((x_1,x_2,x_3))=g\circ e((x_1,x_2,x_3))$$

where $e:\mathbb{R}^3\to\mathbb{R}^6$ is an embedding given by

$$e((x_1,x_2,x_3))=(x_1,x_2,x_3,x_1x_2,x_1x_3,x_2x_3);$$

(iv) `reduced polyconvex` if there exists convex function $g:\mathbb{R}^4\to\mathbb{R}$ such that

$$f((x_1,x_2,x_3))=g\circ i((x_1,x_2,x_3))$$

where $i:\mathbb{R}^3\to\mathbb{R}^4$ is an embedding given by

$$i((x_1,x_2,x_3))=\left(x_1,x_2,x_3,\det\begin{bmatrix}x_1+x_3 & x_3\\ x_3 & x_2+x_3\end{bmatrix}\right).$$

For any convexity type condition $\mathcal{C}$ a class of $\mathcal{C}-$affine functions is understood as functions f such that both f and $-f$ satisfy $\mathcal{C}$. This way we will deal with weakly tetrahedrally affine, tetrahedrally polyaffine and reduced polyaffine functions, respectively.

The following remark is in order.

Remark 2.8.

(i) Tetraaffine functions have appeared in the paper [24]. The following Proposition has been obtained there.

Proposition 2.9 (Cor. 3.4)**.** *The function $f:\mathbb{R}^3\to\mathbb{R}$ is tetraaffine if and only if it is $(2,3)$ quasiaffine.*

(ii) The weak tetrahedral convexity was one of the two conditions established in [24], which together were called "tetrahedral convexity". That is the motivation of adding the word "weak" in the above definition. The following statement follows from Theorem 3.2 obtained in [24]. In the formulation we omit the second condition obtained there.

Proposition 2.10. *If f is $(2,3)$ quasiconvex then f is weakly tetrahedrally convex.*

(iii) The notions of tetrahedral polyconvexity and reduced polyconvexity are related to the classical notion of polyconvexity condition due to Ball [5]. In case of functions defined on 2×2 matrices, function F is called `polyconvex`, if there exist the convex function $G: M^{2\times2}\times\mathbb{R}\to\mathbb{R}$ such that $F=G\circ E$ and $E(X)=(X,\det X)$ for any matrix $X\in M^{2\times2}$.

Define $I : \mathbb{R}^3 \to M^{2\times 2}_{sym}$ be the isomorphism given by

$$I(x_1, x_2, x_3) = \left(x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_3 & x_3 \\ x_3 & x_2 + x_3 \end{bmatrix}.$$

Let us now define another isomorphism $J : M^{2\times 2}_{sym} \times \mathbb{R} \to \mathbb{R}^4$ by

$$J\left(\begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix}, x \right) = (r, s, t, x).$$

For any symmetric matrix M and real number x we have

$$J(M, x) = (I^{-1}(M), x) \text{ and } J \circ E \circ I(r, s, t) = i(r, s, t).$$

Therefore, when f is reduced polyconvex, we have $f(r, s, t) =$

$$g \circ i(r, s, t) = (g \circ J) \circ E \circ \big(I(r, s, t)\big) = G \circ E\big(I(r, s, t)\big) = F \circ I(r, s, t),$$

involving convex function g, where $G = g \circ J$ is convex and $F = G \circ E$ is polyconvex in the classical sense. As I was a linear isomorphism between $\mathbb{R}^3$ and $M^{2\times 2}_{sym}$, f is identified through I with the classical polyconvex function reduced to the space of symmetric matrices.

(iv) The equality

$$\det \begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix} = rs + rt + st \tag{2.6}$$

shows that $\det \circ I$ is a tetraaffine function. It is also a `reduced polyaffine` function, *i.e.* such function f that both f and $-f$ are reduced polyconvex.

The following proposition characterises tetrahedrally polyconvex functions as supremas of tetraaffine polynomials. Its rather standard proof is left to the reader.

Proposition 2.11. *Function f is tetrahedrally polyconvex if and only if it is equal to supremum of some family of tetraaffine functions.*

Our next statement compares the introduced convexity conditions.

Lemma 2.12. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the given continuous function. Then the following implications hold:*

$$f \text{ is reduced polyconvex} \overset{(1)}{\Rightarrow} f \text{ is tetrahedrally polyconvex} \overset{(2)}{\Rightarrow} f \text{ is } (2,3) \text{ quasiconvex}$$
$$\overset{(3)}{\Rightarrow} f \text{ is weakly tetrahedrally convex} \overset{(4)}{\Rightarrow} f \text{ is convex along the axis.}$$

Moreover, the inverse implications to (1) *and* (4) *do not hold.*

In the following section we will show that the inverse implication to (2) also does not hold.

Proof.

"$\overset{(1)}{\Rightarrow}$" Assume there exists a convex function $G : M^{2\times 2}_{sym} \times \mathbb{R} \to \mathbb{R}$ such that, under the notation from Remark 2.8 we have

$$G(A, \det A) = G \circ E(A) = F(A) \text{ and } f = F \circ I.$$

We construct the projection $\pi : \mathbb{R}^3 \times \mathbb{R}^3 \to M^{2\times 2}_{sym} \times \mathbb{R}$ by

$$\pi([r,s,t],[x,y,z]) \overset{\text{def}}{=} \left(\begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix}, x+y+z \right).$$

Function π is affine and $\pi \circ e(r,s,t) =$

$$\pi(r,s,t,rs,rt,st) = \left(\begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix}, \det \begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix} \right) = E \circ I(r,s,t),$$

because of (2.6). Thus $g := G \circ \pi$ is a convex function and

$$g \circ e = G \circ \pi \circ e = G \circ E \circ I = F \circ I = f,$$

as required for tetrahedral polyconvexity.

"$\overset{(1)}{\not\Leftarrow}$" Let $f(r,s,t) = rs$. To verify tetrahedral polyconvexity condition for f it is sufficient to consider $g(r,s,t,x,y,z) := x$. Then g is convex and $g \circ e(r,s,t) = rs = f(r,s,t)$. However f is not reduced polyconvex. We will proceed with a contradiction. Assume there exists a convex function $G : \mathbb{R}^4 \to \mathbb{R}$ such that $G \circ E = f$. Therefore $G(r,s,t,rs+rt+st) = rs$. As G is convex, it is bounded from below by an affine function $h : \mathbb{R}^4 \to \mathbb{R}$,

$$h(r,s,t,x) = h_0 + h_1 r + h_2 s + h_3 t + h_4 x.$$

We have then for every r, s, t, that

$$h \circ E(r,s,t) = h(r,s,t,rs+rt+st) \le rs = G \circ E(r,s,t) = f(r,s,t).$$

in particular we have that $h(r,0,t,rt) = h_0 + h_1 r + h_3 t + h_4 rt \le rs$. Taking $r = 0$, from arbitrariness of t it follows that $h_0 \le 0, h_3 = 0$. Furthermore, $h(r,s,0,rs) = h_0 + h_1 r + h_2 s + h_4 rs \le rs$. Taking $s = 0$ shows that $h_1 = 0$. Analogously taking $r = 0$ shows that $h_2 = 0$. We obtain now that $h_0 + h_4 rs \le rs$ for any r, s and thus $h_4 = 1$. So, if there exists such a function h, it is $h(r,s,t,x) = h_0 + x$ for some nonpositive h_0. Thus

$$h \circ E(r,s,t) = h_0 + rs + rt + st \le rs$$

for any r, s, t. Taking now however $r = s = t$ we obtain $h_0 + 3r^2 \le r^2$ which obviously doesn't hold for any h_0.

"$\overset{(2)}{\Rightarrow}$" It's easy to check that if $\{f_j\}_{j\in J}$ is a family $(2,3)$ quasiconvex functions, then $\sup\limits_{j\in J} f_j$ is also $(2,3)$ quasiconvex. From Proposition 2.11 any tetrahedrally polyconvex is a supremum of some family $\{p_j\}_{j\in J}$ of tetraaffine functions. As any tetraaffine function p_j is $(2,3)$ quasiconvex, the proof is done.

"$\overset{(3)}{\Rightarrow}$" This implication is just Proposition 2.10.

"$\overset{(4)}{\Rightarrow}$" For simplicity let us show the convexity of weakly tetrahedrally convex function f along the axis e_1. Let $p_1 = (r_1, s, t), p_2 = (r_2, s, t)$ be two point spanning the line parallel to e_1 axis. Let us prescribe any regular symplex D on the segment connecting points p_i. Now we obtain $f \le P_D f$ in D, so also on the segment. As $P_D f$ is affine along every axis, we have shown that f is subaffine along e_1.

"$\overset{(4)}{\not\Leftarrow}$" Note that the function $f(r,s,t) = rst$ is indeed convex along every axis. It is not however weakly tetrahedrally convex. To prove that, take simplex D with vertices $(0,0,0),(1,0,0),(0,1,0)$ and $(0,0,1)$ and note that $P_D f \equiv 0$, however $p = (\frac{1}{4},\frac{1}{4},\frac{1}{4}) \in D$ and $f(p) = \frac{1}{64} > 0$. □

More precise statement holds for respective affinity conditions.

Theorem 2.13. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the given continuous function. Then the following implications hold:*

$$f \text{ is reduced polyaffine} \overset{(1)}{\Rightarrow} f \text{ is tetrahedrally polyaffine} \overset{(2)}{\Longleftrightarrow} f \text{ is } (2,3) \text{ quasiaffine}$$
$$\overset{(3)}{\Longleftrightarrow} f \text{ is weakly tetrahedrally affine} \overset{(4)}{\Rightarrow} f \text{ is affine along the axis.}$$

Moreover, the inverse implications to (1) *and* (4) *do not hold.*

Remark 2.14. Theorem 2.13 shows that the classes of tetrahedrally polyaffine, $(2,3)$ quasiaffine and weakly tetrahedrally affine functions are the same and equal to the class of tetraaffine functions.

Proof of Theorem 2.13. Implications to the right are already proven in Lemma 2.12. Also the inverse to (1) is contradicted. Let us here remind that the class of $(2,3)$ quasiaffine functions coincide with the class of tetraaffine functions (see Prop. 2.9). We will prove the following implications.

"$\overset{(2)}{\Leftarrow}$" Tetraaffine functions are tetrahedrally polyaffine because any $p \in \mathcal{A}$ satisfies $p = g \circ e$ for certain affine g.

"$\overset{(3)}{\Leftarrow}$" We will prove that a weakly tetrahedrally affine function f must be equal to certain tetraaffine p on every regular symplex D. This is the case indeed, because on such D we have $f \le P_D f$ and $-f \le -P_D f$. This however finishes the proof because two tetraaffine functions equal on any open set are equal in every point.

"$\overset{(4)}{\not\Leftarrow}$" Consider $f(r,s,t) = rst$. It is affine in the directions of the axis and not tetraaffine. □

3. Tetrahedral polyconvexity condition is not equivalent to (2,3) quasiconvexity

In our analysis we are interested in functions defined on $\mathbb{R}^3$ and the respective integrands, which define functionals of the type $(2,3)$. We are now to prove that the inverse implication to (2) in Lemma 2.12 does not hold. For that we benefit from the well-known result in calculus of variations due to Alibert and Dacorogna [1] (see also [4] for related results). This is possible due to the canonical embedding of functions of the type $(2,3)$ into the special subspace of gradients (see Prop. 2.2).

The authors of [1] have introduced the following function $\tilde{f}_\gamma : M^{2\times 2} \to \mathbb{R}$, defined by

$$\tilde{f}_\gamma(A) = |A|^2(|A|^2 - 2\gamma \det A), \tag{3.1}$$

where $|A|$ stays for the Euclidean norm of A, *i.e.* $\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|^2 = a^2 + b^2 + c^2 + d^2$.

The following theorem holds.

Theorem 3.1 (Alibert, Dacorogna [1])**.** *The following statements hold.*

(a) *Function $\tilde{f}_\gamma$ defined in* (3.1) *is convex $\iff$ $|\gamma| \le \frac{2}{3}\sqrt{2}$.*
(b) *Function $\tilde{f}_\gamma$ defined in* (3.1) *is polyconvex $\iff$ $|\gamma| \le 1$ (see Rem.* 2.8 *point* (iii)).
(c) *Function $\tilde{f}_\gamma$ defined in* (3.1) *is rank-one convex* (i.e. *the function $t \mapsto \tilde{f}_\gamma(A + tB)$ is convex for any matrix A and any rank-one matrix B) $\iff$ $|\gamma| \le \frac{2}{\sqrt{3}}$.*

(d) *There exists* $\varepsilon > 0$ *such that* $I_{\tilde{f}_\gamma}$ *is lower semicontinuous with respect to the sequential weak-$*$ convergence of gradients of Lipschitz functions in* $L^\infty(\Omega, \mathbb{R}^2)$ $\Big($i.e. *it has the property: when* $\{u^\nu\} \subseteq W^{1,\infty}(\Omega, \mathbb{R}^2)$ *is a bounded sequence and* $Du^\nu \overset{*}{\rightharpoonup} Du$ *in* $L^\infty(\Omega, M^{2\times 2})$, $u^\nu \to u$ *in* $L^1(\Omega, \mathbb{R}^2)$ *then* $\liminf_{\nu \to +\infty} I_{\tilde{f}_\gamma}(Du^\nu) \geq I_{\tilde{f}_\gamma}(Du)\Big)$ $\iff$ $|\gamma| \leq 1 + \varepsilon$.

Defining $A(r,s,t) = \begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix}$ and using identification (2.5) and (3.1) we obtain

$$f_\gamma(r,s,t) := \tilde{f}_\gamma(A(r,s,t)) = \big(r^2 + s^2 + 4t^2 + 2(r+s)t\big)\big(r^2 + s^2 + 4t^2 + 2(r+s)t - 2\gamma(rs + rt + st)\big).$$

We will investigate tetrahedral polyconvexity condition for function $f_\gamma : \mathbb{R}^3 \to \mathbb{R}$.

Theorem 3.2. *Function* $f_\gamma(r,s,t)$, *is tetrahedrally polyconvex if and only if* $|\gamma| \leq 1$.

Proof.

"$\Leftarrow$" At first we note that f_γ is reduced polyconvex for $|\gamma| \leq 1$ as we have $\tilde{f}_\gamma = G \circ E$ under the notation of Remark 2.8 point (iii), where G is convex. We have then that

$$f_\gamma = G \circ E \circ I = G \circ J^{-1} \circ J \circ E \circ I = \big(G \circ J^{-1}\big) \circ i =: \tilde{G} \circ i,$$

where $\tilde{G}$ is convex. As reduced polyconvexity implies tetrahedral polyconvexity (see Lem. 2.12) the proof is done.

"$\Rightarrow$" Assume on the contrary that there exists γ such that $|\gamma| > 1$ and f_γ is tetrahedrally polyconvex. Then there exists an affine function $p : \mathbb{R}^6 \to \mathbb{R}$,

$$p(r,s,t,x,y,z) := p_0 + v \cdot (r,s,t,x,y,z),$$

where $(v_1, v_2, v_3, v_4, v_5, v_6) \in \mathbb{R}^6$ is a constant vector and p is such that $f_\gamma(r,s,t) \geq p \circ e(r,s,t)$ for any r,s,t. We compute that

$$\begin{aligned} f_\gamma(r, cr, 0) &= r^4(1 + 2c^2 + c^4 - 2\gamma(c + c^3)), \\ p(r, cr, 0, cr^2, 0, 0) &= p_0 + v_1 r + c v_2 r + c v_4 r^2. \end{aligned}$$

This implies the inequality

$$r^4(1 + 2c^2 + c^4 - 2\gamma(c + c^3)) \geq p_0 + (v_1 + c v_2) r + c v_4 r^2,$$

holding for every $r, c \in \mathbb{R}$. We obviously need a coefficient $\kappa_\gamma(c) := (1 + 2c^2 + c^4 - 2\gamma(c + c^3))$ to be nonegative for every c. For $\gamma > 1$ we obtain that $\kappa_\gamma(1) = 4 - 4\gamma < 0$, for $\gamma < -1$ we get $\kappa_\gamma(-1) = 4 + 4\gamma < 0$. Therefore f_γ cannot be tetrahedrally polyconvex. □

We end up with the following statement, which is one of our main results.

Theorem 3.3. *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *be the given continuous function. Then the following implications hold:*

$$\begin{aligned} &f \text{ is reduced polyconvex} \overset{(1)}{\Rightarrow} f \text{ is tetrahedrally polyconvex} \overset{(2)}{\Rightarrow} f \text{ is } (2,3) \text{ quasiconvex} \\ &\overset{(3)}{\Rightarrow} f \text{ is weakly tetrahedrally convex} \overset{(4)}{\Rightarrow} f \text{ is convex along the axis.} \end{aligned}$$

Moreover, the inverse implications to (1), (2), (4) *do not hold.*

Proof. Implications $\overset{(1)}{\Rightarrow}, \overset{(2)}{\Rightarrow}, \overset{(3)}{\Rightarrow}, \overset{(4)}{\Rightarrow}$, as well as $\overset{(1)}{\not\Leftarrow}$ and $\overset{(4)}{\not\Leftarrow}$ have been already established in Lemma 2.12. For the proof of the property $\overset{(2)}{\not\Leftarrow}$ we have to show that there exists a $(2,3)$ quasiconvex function that is not tetrahedrally polyconvex. Let $\gamma = 1+\varepsilon$ as in point d) of Theorem 3.1. Now f_γ is $(2,3)$ quasiconvex due to embedding (2.3). Theorem 3.2 shows however, that f_γ is not tetrahedrally polyconvex. □

We address the following problem.

Open problem 3.4. *We do not know whether the inverse implication to* (3) *holds.*

However the problem is open, let us note that the case is much simpler with bilinear forms.

Fact 3.5. *A bilinear form P is convex along each axis if and only if P is tetrahedrally polyconvex.*

Proof. Of course we only need to prove to prove the implication "$\Rightarrow$". Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $P(x) = \sum_{i=1}^3 a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j =: Q + T$ be convex along the axis. As T is affine along the axis, it follows that Q must be convex along the axis. It shows that $a_i \geq 0$ for any $1 \leq i \leq 3$. Thus P is equal to the sum of tetrahedrally polyaffine form T and convex form Q. It is obvious that any convex function is tetrahedrally polyconvex and thus P is tetrahedrally polyconvex. □

4. Carathéodory type theorem for the class of tetrahedrally polyconvex functions

Our goal now is to obtain a variant of Carathéodory theorem for tetrahedrally polyconvex functions. This statement will be needed later to discus the locality properties of tetrahedral polyconvexity condition. For our analysis we have to define and investigate the tetrahedral polyconvex envelope of the given function f.

Let us start by recalling the definitions of convex hulls of sets, as well as the classical Carathéodory theorem.

Definition 4.1 (Convex hull, convex envelope)**.** For any subset X of a linear space $\mathcal{V}$ we define `convex hull` of X as

$$CH\ X \overset{\text{def}}{=} \bigcap \{C \mid C \text{ is convex and } X \subseteq C\}.$$

For any function $f : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ we define `convex envelope` of f as

$$Cf \overset{\text{def}}{=} \sup\{g(x) \mid g \text{ is convex and } g \leq f\}.$$

We use the convention that $\sup \emptyset = -\infty$.

Theorem 4.2 (Carathéodory theorem, 1911, [9])**.** *Let X be a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then*

(i) $CH\ X = \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{n+1} \lambda_i x_i, x_i \in X, \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \in [0,1]\}$,

(ii) $Cf(x) = \inf \left\{ \sum_{i=1}^{n+1} \lambda_i f(x_i) \mid \sum_{i=1}^{n+1} \lambda_i x_i = x, \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \in [0,1] \right\}$.

To proceed further we require the following definition.

Definition 4.3 (Tetrahedral polyconvex envelope)**.** For function $f : \mathbb{R}^3 \to \mathbb{R}$ the `tetrahedral polyconvex envelope` is defined by

$$TPEf(r,s,t) \overset{\text{def}}{=} \sup\{g(r,s,t) \mid g \text{ is tetrahedrally polyconvex and } g \leq f\}.$$

It is clear that if $TPEf \neq -\infty$, it is then a tetrahedrally polyconvex function. This is because at the same time $TPEf(\cdot,\cdot,\cdot)$ is a supremum of tetraaffine functions.

We are now to prove the variant of part (ii) in Carathéodory theorem dealing with tetrahedral polyconvex envelopes of functions. Our arguments are based on variants of Carathéodory Theorem similar as presented in [12], Chapter 5. For readers convenience we present the proof in detail, as it contains not so trivial arguments.

Theorem 4.4 (Carathéodory theorem with tetrahedrally polyconvex envelope)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a given function. Then the following statements hold.*

(i) *The function $g^f : \mathbb{R}^6 \to \mathbb{R} \cup \{-\infty\}$ given by*

$$g^f(r,s,t,x,y,z) \stackrel{def}{=} \inf \left\{ \sum_{i=1}^{7} \lambda_i f(r_i,s_i,t_i) \mid \sum_{i=1}^{7} \lambda_i e(r_i,s_i,t_i) = (r,s,t,x,y,z) \right.$$
$$\left. \textit{where } \{\lambda_i\}_{i=1}^{7} : \ \textit{such that } \sum_{i=1}^{7} \lambda_i = 1, \lambda_i \in [0,1] \right\},$$

is well defined and convex.

(ii) *If f is tetrahedrally polyconvex, then*

$$f(r,s,t) = g^f(e(r,s,t)) \textit{ for any } (r,s,t) \in \mathbb{R}^3.$$

Moreover, for any f we have

$$g^f \circ e(r,s,t) = g^f(r,s,t,rs,rt,st) = TPEf.$$

Proof. Part (i):
We begin with constructing desired convex function g^f. For integers $N \geq 7$ we define

$$g^f_N(r,s,t,x,y,z) \stackrel{\text{def}}{=} \inf S_N(r,s,t,x,y,z), \text{ where we set}$$

$$S_N = \left\{ \sum_{i=1}^{N} \lambda_i f(r_i,s_i,t_i) \mid \sum_{i=1}^{N} \lambda_i e(r_i,s_i,t_i) = (r,s,t,x,y,z), \lambda_i \in [0,1], \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

We divide the proof into steps.

Step 1. We show that

$$CH\ e(\mathbb{R}^3) = \mathbb{R}^6.$$

Step 2. We prove that g^f_7 is well defined (and thus also g^f_N whenever $N \geq 7$) and for any $N \geq 7$ we have $g^f_N = g^f_7 \stackrel{\text{def}}{=} g^f$.

Step 3. We prove that g^f is convex.

Proof of Step 1*:* Assume than $CH\ e(\mathbb{R}^3) \neq \mathbb{R}^6$. Thus $CH\ e(\mathbb{R}^3)$, as a convex set, lies in some halfspace of the form

$$\mathcal{H} = \{v \in \mathbb{R}^6 \mid \alpha \cdot v < \beta\}$$

for some nonzero $\alpha \in (\mathbb{R}^6)^* \cong \mathbb{R}^6$ and real β. Now $e(\mathbb{R}^3) \subseteq CH\ e(\mathbb{R}^3) \subseteq \mathcal{H}$. To show a contradiction we will find a triple (r,s,t) such that $\alpha \cdot (r,s,t,rs,rt,st)$ is not less then β. Let $\alpha = (\alpha_i)_{i=1}^6$ and i_0 be the smallest index i such that $\alpha_i \neq 0$. Assume first that $i_0 \leq 3$. Let then $(\bar{r},\bar{s},\bar{t})$ be equal to the i_0^{th} vector of the standard basis of $\mathbb{R}^3$. Now $e(\bar{r},\bar{s},\bar{t}) = (e_{i_0},0,0,0)$ (which is the i_0th vector of the standard basis of $\mathbb{R}^6$) and for $(r,s,t) = (\frac{\beta}{\alpha_{i_0}}(\bar{r},\bar{s},\bar{t}))$ we arrive at

$$\alpha \cdot e(r,s,t) = \beta,$$

which contradicts the inclusion $e(\mathbb{R}^3) \subseteq \mathcal{H}$. For the case where $i_0 = 4$ take $(r,s,t) = \frac{\beta}{\alpha_4}(1,1,0)$ (so that α and $e(r,s,t)$ meet only in fourth place). Similar reasoning holds for $i_0 = 5, 6$, which finishes the proof of Step 1.

Proof of Step 2*:* To begin let us note that from Carathéodory theorem (Thm. 4.2) and Step 1 we see that

$$CH\ e(\mathbb{R}^3) = \{\sum_{i=1}^{7} \lambda_i e(r_i, s_i, t_i), \sum_{i=1}^{7} \lambda_i = 1, \lambda_i \in [0,1]\} = \mathbb{R}^6$$

and thus g_7^f is well defined, that is $S_7 \neq \emptyset$.

Now let us introduce two substeps.

Substep 2A: We prove that for $N \geq 8$ we have $S_N = S_8$.

Let us recall the definition of the epigraph of a function f:

$$epif \stackrel{\text{def}}{=} \{(r,s,t,x) \in \mathbb{R}^4 \mid f(r,s,t) \leq x\}$$

and define

$$\hat{e}(epif) \stackrel{\text{def}}{=} \{(e(r,s,t),x) \mid f(r,s,t) \leq x\} \subseteq \mathbb{R}^7.$$

Note that $(e(r,s,t), f(r,s,t)) \in \hat{e}(epif)$, therefore any convex combination of such points belongs to $CH\ \hat{e}(epif)$.

As $\hat{e}(epif) \subseteq \mathbb{R}^7$, from Carathéodory theorem (Thm. 4.2) it follows that

$$CH\ \hat{e}(epif) = \left\{ \sum_{i=1}^{8} \lambda_i (e(r_i,s_i,t_i), f(r_i,s_i,t_i) \mid \lambda_i \in [0,1], \sum_{i=1}^{8} \lambda_i = 1 \right\}.$$

It implies that $S_N \subseteq S_8$. Indeed, let $\bar{f} = \sum_{i=1}^{N} \lambda_i f(r_i,s_i,t_i) \in S_N(r,s,t,x,y,z)$ *i.e.* $\sum_{i=1}^{N} \lambda_i e(r_i,s_i,t_i) = (r,s,t,x,y,z), \lambda_i \in [0,1], \sum_{i=1}^{N} \lambda_i = 1$. Hence

$$\sum_{i=1}^{N} \lambda_i (e(r_i,s_i,t_i), f(r_i,s_i,t_i)) \in CH\ \hat{e}(epif).$$

Therefore there exist $\{\bar{\lambda_i}\}_{i=1}^{8}$ and $\{(\bar{r_i}, \bar{s_i}, \bar{t_i})\}_{i=1}^{8}$ such that

$$\sum_{i=1}^{8} \bar{\lambda_i} (e(\bar{r_i}, \bar{s_i}, \bar{t_i}), f(\bar{r_i}, \bar{s_i}, \bar{t_i})) = \sum_{i=1}^{N} \lambda_i (e(r_i,s_i,t_i), f(r_i,s_i,t_i)).$$

We obtain $\bar{f} \in S_8$. As sequence of sets S_N is nondecreasing, we see that for $N \geq 8$ we have $S_N = S_8$. As $g_N^f(r,s,t,x,y,z) = \inf S_N(r,s,t,x,y,z)$, we establish $g_N^f = g_8^f$ for any $N \geq 8$.

Substep 2B: We show that $g^f = g_7^f = g_8^f$. It suffices to prove $g_7^f \leq g_8^f$.

Take any $v \in \mathbb{R}^6 = CH\ e(\mathbb{R}^3)$, a sequence $\{\alpha_i\}_{i=1}^{8}$ satisfying $\alpha_i \in [0,1], \sum_{i=1}^{8} \alpha_i = 1$ and points $\{(r_i,s_i,t_i)\}_{i=1}^{8}$ such that

$$\sum_{i=1}^{8} \alpha_i e(r_i,s_i,t_i) = v.$$

From Carathéodory theorem (Thm. 4.2) applied to the set $\{(r_i,s_i,t_i)\}_{i=1}^{8}$, there exists a sequence $\{\beta_i\}_{i=1}^{8}$ satisfying $\beta_i \in [0,1], \sum_{i=1}^{8} \beta_i = 1$ such that at least one of β_i vanishes and

$$\sum_{i=1}^{8} \beta_i e(r_i,s_i,t_i) = v.$$

To finish the proof of this substep it suffices to show that

$$\sum_{i=1}^{8} \alpha_i f(r_i,s_i,t_i) \geq \sum_{i=1}^{8} \beta_i f(r_i,s_i,t_i). \tag{4.1}$$

We may assume that all $\alpha_i s$ are positive, as otherwise we could take $\{\beta_i\}_{i=1}^8 = \{\alpha_i\}_{i=1}^8$. Assume that this inequality does not hold, *i.e.*

$$\sum_{i=1}^{8} \alpha_i f(r_i, s_i, t_i) < \sum_{i=1}^{8} \beta_i f(r_i, s_i, t_i). \tag{4.2}$$

In particular the set $C := \{i \in \{1, 2, \ldots, 8\} \mid \beta_i > \alpha_i\}$ is nonempty, because $\{\alpha_i\}$ and $\{\beta_i\}$ are two different sequences with equal sum of coefficients. Set now

$$\gamma \overset{\text{def}}{=} \min_{i \in C} \left\{ \frac{\alpha_i}{\beta_i - \alpha_i} \right\}.$$

Note that γ is positive from the definition of C. Moreover, let

$$\lambda_i \overset{\text{def}}{=} \alpha_i + \gamma(\alpha_i - \beta_i).$$

We have now

$$\sum_{i=1}^{8} \lambda_i = \sum_{i=1}^{8} \alpha_i + \gamma \sum_{i=1}^{8} (\alpha_i - \beta_i) = 1$$

and from the definition of γ we have $\lambda_i \geq 0$ for every i. It follows that every $\lambda_i \leq 1$ for every $i \in \{1, \ldots, 8\}$. From the definition of γ it also follows that there exists i such that $\lambda_i = 0$ – this is exactly index i on which we obtain a minimum in the definition of γ. Furthermore,

$$\sum_{i=1}^{8} \lambda_i e(r_i, s_i, t_i) = (1 + \gamma)v - \gamma v = v,$$

and

$$\sum_{i=1}^{8} \lambda_i f(r_i, s_i, t_i) = \sum_{i=1}^{8} \alpha_i f(r_i, s_i, t_i) + \gamma\Big(\sum_{i=1}^{8} (\alpha_i - \beta_i) f(r_i, s_i, t_i)\Big) \overset{(4.2)}{<} \sum_{i=1}^{8} \alpha_i f(r_i, s_i, t_i),$$

because we assumed $\sum(\alpha_i - \beta_i) f(r_i, s_i, t_i) < 0$ and dealt positive γ. We have shown that when the inequality (4.1) did not hold with coefficients $\{\beta_i\}$, it holds with coefficient $\{\lambda_i\}$. This finishes Step 2.

Proof of Step 3: From Step 2 we already know $g^f = g_7^f$. We are now to show that for $\lambda \in [0, 1]$ and any vectors $v, w \in \mathbb{R}^6$ we have inequality

$$\lambda g^f(v) + (1 - \lambda) g^f(w) \geq g^f(\lambda v + (1 - \lambda) w).$$

From the definition of g^f we have that for any $\varepsilon > 0$ there exist $(\mu_i)_{i=1}^7, (\nu_i)_{i=1}^7$ satisfying $\mu_i, \nu_i \in [0, 1]$, $\sum_{i=1}^7 \mu_i = \sum_{i=1}^7 \nu_i = 1$ and $(r_i, s_i, t_i)_{i=1}^7, (\bar{r_i}, \bar{s_i}, \bar{t_i})_{i=1}^7$ such that

$$\sum_{i=1}^{7} \mu_i e(r_i, s_i, t_i) = v, \sum_{i=1}^{7} \nu_i e(\bar{r_i}, \bar{s_i}, \bar{t_i}) = w,$$

and

$$\lambda g^f(v) + (1 - \lambda) g^f(w) + \varepsilon \geq \lambda \sum_{i=1}^{7} \mu_i f(r_i, s_i, t_i) + (1 - \lambda) \sum_{i=1}^{7} \nu_i f(\bar{r_i}, \bar{s_i}, \bar{t_i}).$$

Defining new sequence as

$$\lambda_i \overset{\text{def}}{=} \lambda \mu_i, \;\; \lambda_{7+i} \overset{\text{def}}{=} (1 - \lambda) \nu_i,$$

and new points as

$$(r_i, s_i, t_i) \stackrel{\text{def}}{=} (r_i, s_i, t_i),\ (r_{7+i}, s_{7+i}, t_{7+i}) \stackrel{\text{def}}{=} (\bar{r_i}, \bar{s_i}, \bar{t_i}),\ \text{where } i = 1, \ldots, 7,$$

we arrive at

$$\lambda g^f(v) + (1-\lambda) g^f(w) + \varepsilon \geq \sum_{i=1}^{14} \lambda_i f(r_i, s_i, t_i), \text{ where}$$

$$\sum_{i=1}^{14} \lambda_i e(r_i, s_i, t_i) = \lambda v + (1-\lambda) w.$$

Using the definition of g^f and the fact that $g^f = g^f_{14}$ we get

$$\lambda g^f(v) + (1-\lambda) g^f(w) + \varepsilon \geq g^f\left(\lambda v + (1-\lambda) w\right)$$

and arbitrariness of ε finishes the proof of Step 3 and of Part (i).

Part (ii):
At first we will show that if f is tetrahedrally polyconvex, then $f = g^f \circ e$.

For this let us consider a convex function $g:\ \mathbb{R}^6 \to \mathbb{R}$ such that $g \circ e = f$. As g is convex we have that for any choice of points $v_i \in \mathbb{R}^6$

$$\sum_{i=1}^{7} \lambda_i g(v_i) \geq g\left(\sum_{i=1}^{7} \lambda_i v_i\right),$$

where $\lambda_i \in [0,1]$ and $\sum_{i=1}^{7} \lambda_i = 1$. Taking $\{\lambda_i\}_{i=1}^{7}$ and $\{(r_i, s_i, t_i)\}_{i=1}^{7}$ such that

$$\sum_{i=1}^{7} \lambda_i e(r_i, s_i, t_i) = e\left(\sum_{i=1}^{7} \lambda_i (r_i, s_i, t_i)\right) = e(r, s, t) \tag{4.3}$$

shows that

$$\sum_{i=1}^{7} \lambda_i f(r_i, s_i, t_i) \geq f\left(\sum_{i=1}^{7} \lambda_i (r_i, s_i, t_i)\right) = f(r, s, t). \tag{4.4}$$

Taking infimum over all possible coefficients $\{\lambda_i\}_{i=1}^{7}$ and points $\{(r_i, s_i, t_i)\}_{i=1}^{7}$ satisfying (4.3), (4.4) and using the definition of g^f we obtain $g^f(e(r,s,t)) \geq f(r,s,t)$. As we obviously have $g^f(e(r,s,t)) \leq f(r,s,t)$, we obtain $g^f \circ e = f$.

To prove the second statement note that $g^f \circ e$ is tetrahedrally polyconvex, because g^f is convex. What is left is to establish that $g^f = TPEf$. From the definition of $TPEf$ and tetrahedral polyconvexity of $g^f \circ e$, we have that $g^f \circ e \leq TPEf$, because $g^f \circ e \leq f$. Observe that the following monotonicity property holds: when $h \leq f$ we have $g^h \leq g^f$. Moreover, from the already established first statement in this part, $h \mapsto g^h \circ e$ is a projection onto tetrahedrally polyconvex functions. Therefore we have $h = g^h \circ e \leq g^f \circ e$, whenever $h \leq f$ and h is tetrahedrally polyconvex. Taking $h = TPEf$ finishes the proof. □

We end this section with the following characterisation of tetrahedrally polyconvex functions, which is the consequence of the Theorem 4.4.

Corollary 4.5. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the given function. The following conditions are equivalent:*

(a) *f is tetrahedrally polyconvex;*

(b) *for any* $(r,s,t) \in \mathbb{R}^3$*, any coefficients* $\{\lambda_i\}_{i=1}^7$ *such that* $\sum_{i=1}^7 \lambda_i = 1$*,* $\lambda_i \in [0,1]$ *and any triples of real numbers* $\{(r_i,s_i,t_i)\}_{i=1}^7$ *such that* $\sum_{i=1}^7 \lambda_i e(r_i,s_i,t_i) = e(r,s,t)$ *the following Jensen-type inequality holds:*

$$f(r,s,t) \le \sum_{i=1}^{7} \lambda_i f(r_i,s_i,t_i).$$

Proof.

"(a) $\Rightarrow$ (b)" We proceed exactly like in the proof of Part (ii) of Theorem 4.4.

"(a) $\Leftarrow$ (b)" Having a function f satisfying (b), from the definition of the g^f in Theorem 4.4, Part (i), we see that $g^f \circ e = f$ (while we always have $g^f \circ e \le f$ and the converse inequality follows from (b)). As a function g^f is convex (see Thm. 4.4, Part (i), Step 3) the proof is done. □

5. Non-locality of tetrahedral polyconvexity

In this section we are going to prove, that there exist no local condition for tetrahedral polyconvexity. We proceed in a similar way to Kristensen in [26]. We begin with some definitions, useful in the reasoning.

For f – a function of class $C^2(\mathbb{R}^3;\mathbb{R})$ recall the Taylor formula

$$f(z+w) = f(z) + Df(z)w + \frac{1}{2}D^2 f(z)(w;w) + \rho(z,w),$$

where $\rho(z,w)$ is given by

$$\rho(z,w) = \int_0^1 (1-t)\left(D^2 f(z+tw)(w;w) - D^2 f(z)(w;w)\right)\mathrm{d}t.$$

We also define function

$$\varLambda(r,s) = \sup\{|D^2 f(z+w) - D^2 f(z)| \ : \ |z| \le r, |w| \le s\}. \tag{5.1}$$

The function $\varLambda$ is defined in such a way that we obtain an obvious estimate, for $|z| \le r$ and any w we have

$$|\rho(z,w)| \le \frac{1}{2}\varLambda(r,|w|)|w|^2. \tag{5.2}$$

We start with the following result.

Lemma 5.1. *Let* f *be any function of class* $C^2(\mathbb{R}^3;\mathbb{R})$ *such that* $D^2 f(z)(w;w) \ge 0$ *for any* z, w *such that* $|z| \le r$ *and* w *is parallel to one of the axis. Take any* $\epsilon > 0$ *and define* $\delta \overset{def}{=} \frac{1}{2}\sup\{t \in (0,r) \ : \ \epsilon \ge \varLambda(r,t)\}$*. Then there exists a tetrahedrally polyconvex function* g *such that*

$$g(z) = f(z) + \epsilon|z|^2 \text{ for } |z| < \delta.$$

Proof. Define

$$f_\epsilon(z) = f(z) + \epsilon|z|^2,$$

$$G(z) := \begin{cases} f_\epsilon(z) \text{ for } |z| \le \delta. \\ \sup_{|w|<\delta}\left(f_\epsilon(w) + Df_\epsilon(w)(z-w) + \frac{1}{2}D^2 f_\epsilon(w)(z-w;z-w)\right) \text{ for } |z| > \delta \end{cases}$$

and $g = TPE(G)$. It's now obvious that g is tetrahedrally polyconvex and that $g \le f_\epsilon$ for $|z| < \delta$. To check that $g = f_\epsilon$ for $|z| < \delta$ take any z such that $|z| \le \delta$. By Theorem 4.4 for any σ we may choose a convex combination $\{\lambda_j z_j\}_{i=1}^7$ of z such that

$$e(z) = \sum \lambda_j e(z_j) \tag{5.3}$$

and

$$g(z)+\sigma>\sum_{j=1}^{7}\lambda_j G(z_j).$$

From the definition of G it follows that

$$\begin{aligned} g(z)+\sigma &> \sum_{|z_j|\le\delta}\lambda_j f_\epsilon(z_j)+\sum_{|z_j|>\delta}\lambda_j\left(f_\epsilon(z)+Df_\epsilon(z)(z_j-z)+\frac{1}{2}D^2 f_\epsilon(z)(z_j-z;z_j-z)\right)\\ &=\sum_{|z_j|\le\delta}\ldots+\sum_{|z_j|>\delta}\ldots=: A+B. \end{aligned}$$

Applying Taylor's formula to $f_\epsilon(z_j)$ in A yields

$$A=\sum_{|z_j|\le\delta}\lambda_j f_\epsilon(z)+\sum_{|z_j|\le\delta}\lambda_j Df_\epsilon(z)(z_j-z)+\frac{1}{2}\sum_{|z_j|\le\delta}\lambda_j D^2 f_\epsilon(z)(z_j-z;z_j-z)+\sum_{|z_j|\le\delta}\lambda_j\rho(z,z_j-z),$$

hence

$$g(z)+\sigma>\sum_{|z_j|\le\delta}\lambda_j\rho(z,z_j-z)+\sum_j\lambda_j\left(f_\epsilon(z)+Df_\epsilon(z)(z_j-z)+\frac{1}{2}D^2 f_\epsilon(z)(z_j-z;z_j-z)\right).$$

From linearity of $Df(z)$ and the fact that $z=\sum\lambda_j z_j$ we obtain

$$\begin{aligned} g(z)+\sigma &> \sum_{|z_j|\le\delta}\lambda_j\rho(z,z_j-z)+f_\epsilon(z)+\sum_j\lambda_j\left(\frac{1}{2}D^2 f_\epsilon(z)(z_j-z;z_j-z)\right)\\ &=\sum_{|z_j|\le\delta}\ldots+f_\epsilon(z)+\sum_j\ldots=: C+f_\epsilon(z)+D. \end{aligned} \tag{5.4}$$

Having in mind (5.2) we have

$$|\rho(z;z_j-z)|\le\frac{1}{2}\Lambda(r,|z_j-z|)|z_j-z|^2.$$

Note that for $|z_j|\le\delta$

$$|z_j-z|\le|z|+|z_j|\overset{|z|<\delta}{<}2\delta=\sup\{t\in(0,r):\epsilon\ge\Lambda(r,t)\}$$

and therefore $|z_j-z|\in\{t\in(0,r):\epsilon\ge\Lambda(r,t)\}$. Consequently $\Lambda(r,|z_j-z|)\le\epsilon$ and $\rho(z;|z_j-z|)\ge-\frac{1}{2}\epsilon|z_j-z|^2$. It follows that

$$C\ge-\frac{\epsilon}{2}\sum_{|z_j|\le\delta}\lambda_j|z_j-z|^2\ge-\frac{\epsilon}{2}\sum_j\lambda_j|z_j-z|^2.$$

We also notice that $D^2 f_\epsilon(z)=D^2 f(z)+2\epsilon Id$ and so

$$D=\sum_j\lambda_j\left(\frac{1}{2}D^2 f(z)(z_j-z;z_j-z)\right)+\epsilon\sum_j\lambda_j|z_j-z|^2.$$

Therefore

$$C+D\ge\sum_j\lambda_j\left(\frac{1}{2}D^2 f(z)(z_j-z;z_j-z)\right)+\frac{\epsilon}{2}\sum_j\lambda_j|z_j-z|^2.$$

What we need is to show that $C+D\ge 0$, so that from (5.4) we get

$$g(z)+\sigma>f_\epsilon(z).$$

Take now any bilinear symmetric form P and note that $P(x-y;x-y) = P(x;x) + P(y;y) - 2P(x;y)$. This shows however that

$$\begin{aligned}\sum_j \lambda_j \left(\frac{1}{2}D^2 f(z)(z_j - z; z_j - z)\right) &= \sum_j \lambda_j \left(\frac{1}{2}D^2 f(z)(z_j;z_j)\right)\\ &\quad + \sum_j \lambda_j \left(\frac{1}{2}D^2 f(z)(z;z)\right) - 2\sum_j \lambda_j \left(\frac{1}{2}D^2 f(z)(z_j;z)\right)\\ &= \sum_j \lambda_j \left(\frac{1}{2}D^2 f(z)(z_j;z_j)\right) + \frac{1}{2}D^2 f(z)(z;z) - D^2 f(z)(z;z)\\ &= \sum_j \lambda_j \left(\frac{1}{2}D^2 f(z)(z_j;z_j)\right) - \frac{1}{2}D^2 f(z)(z;z).\end{aligned}$$

From our assumptions $P(v) = D^2 f(z)(v;v)$ is the bilinear form convex along each axis. Therefore, from the Fact 3.5, it is tetrahedrally polyconvex. According to Corollary 4.5 and (5.3) we get the following Jensen-type inequality

$$\sum_j \lambda_j \left(D^2 f(z)(z_j;z_j)\right) = \sum_j \lambda_j P(z_j) \geq P(\sum_j \lambda_j z_j) = P(z) = D^2 f(z)(z;z),$$

which concludes the proof. □

We end this section with the following result.

Theorem 5.2. *There exists a function that is not tetrahedrally polyconvex such that its restriction to any ball of radius one may be extended to a tetrahedrally polyconvex function.*

Proof. Let $h(r,s,t) = -rst : \mathbb{R}^3 \to \mathbb{R}$. We note that h is not tetrahedrally polyconvex, but convex in the direction of each axis (see Lem. 2.12). Take now two functions $\alpha, \beta : [0,\infty) \to \mathbb{R}$, $\alpha, \beta \in C^1(0,\infty)$ such that

$$\alpha(t) = \begin{cases} 1 & \text{for } t < 4,\\ \cos^2\left((t-4)\frac{\pi}{2}\right) & \text{for } t \in [4,5]\\ 0 & \text{for } t > 5,\end{cases}$$

$$\beta(t) = \begin{cases} 0 & \text{for } t < 3,5\\ (t-\frac{7}{2})^2 & \text{for } t \geq 3,5\end{cases}$$

We consider trunk function $\varphi_\delta(t) = \delta^{-1}\varphi(\frac{t}{\delta})$, where $\varphi \in C_0^\infty(\mathbb{R}), 0 \leq \varphi \leq 1, \int \varphi = 1, \varphi \equiv 1$ in some neighbourhood of 0 and supp $\varphi \subseteq [-1,1]$. Then we set $\alpha_\delta := \alpha * \varphi_\delta, \beta_\delta := \beta * \varphi_\delta$.

It is easy to check that there exist $k > 0, \delta \in (0, \frac{1}{2})$ such that the function g given by

$$g(z) \stackrel{\text{def}}{=} h(z)\alpha_\delta(|z|) + k\beta_\delta(|z|)$$

is smooth and convex in the direction of each axis. It is not tetrahedrally polyconvex. To confirm that, we use the argument from paper by Šverák [44] and substitute the sequence $u^\nu(x,y) = (\cos(2\pi x\nu), \cos(2\pi y\nu), \cos(2\pi(x+y)\nu))$, $\Omega = [0,1]^2$. Applying the Riemann–Lebesgue Lemma (see [12], Thm. 1.5) we see immediately that $u^\nu \rightharpoonup u = 0$ weakly-$*$ in $L^\infty(\Omega, \mathbb{R}^3)$. However, a direct computation shows that

$$\liminf_{\nu\to\infty} I_g(u^\nu) = -\frac{1}{4} < I_g(u) = 0, \tag{5.5}$$

which shows that h is not $(2,3)$ quasiconvex and consequently it is not tetrahedrally polyconvex as well. Furthermore, we may find $\epsilon > 0$ such that

$$g_\epsilon(z) = g(z) + \epsilon|z|^2$$

is not tetrahedrally polyconvex because of minor modification of (5.5). Take now Λ defined in (5.1) for function $f = g_\epsilon$. As g_ϵ is smooth and its third derivative has a compact support, we get that

$$\left|\frac{\partial^2}{\partial z_i \partial z_j} g_\epsilon(z+w) - \frac{\partial^2}{\partial z_i \partial z_j} g_\epsilon(z)\right| = \left|\int_0^1 \nabla \frac{\partial^2}{\partial z_i \partial z_j} g_\epsilon(z+\theta w)\cdot w\mathrm{d}\theta\right| \le ||\nabla^3 g_\epsilon||_\infty |w|$$

and it follows that there exists a constant C such that $\Lambda(r,t) \le Ct$, where C is independent on r. In particular, $\epsilon \ge \Lambda(r, \frac{\epsilon}{C})$ and so for any r

$$\frac{\epsilon}{2C} \le \frac{1}{2}\sup\{t \in (0,r) \ : \ \epsilon \ge \Lambda(r,t)\}.$$

We claim that for fixed z_0 there exists a tetrahedrally polyconvex extension of g_ϵ from a ball with center in z_0 and of radius $\frac{\epsilon}{2C}$. Note that the radius does not depend on z_0. The existence of such extension follows from Lemma 5.1, when we substitute g_ϵ by a shifted function

$$g_\epsilon^{z_0}(z) \stackrel{\text{def}}{=} g_\epsilon(z_0 + z),$$

so that we extend the function $g_\epsilon^{z_0}$ from the ball centred at 0. Defining now

$$f(z) \stackrel{\text{def}}{=} g_\epsilon^{z_0}\left(\frac{2C}{\epsilon} z\right)$$

provides the radius 1 in the extension property and finishes the proof of existence of the function which is not tetrahedrally polyconvex, having however a tetrahedrally polyconvex extension from any ball of radius 1. $\square$

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