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STRICT CONVEXITY AND THE REGULARITY OF SOLUTIONS TO VARIATIONAL PROBLEMS *

Arrigo Cellina[1]

Abstract. We consider the problem of minimizing

$$\int_\Omega [L(\nabla v(x)) + g(x, v(x))]dx \quad \text{on} \quad u^0 + W_0^{1,2}(\Omega)$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and L is a convex function that grows quadratically outside the unit ball, while, when $|\nabla v| < 1$, it behaves like $|\nabla v|^p$ with $1 < p < 2$. We show that, for each $\omega \subset\subset \Omega$, there exists a constant H, depending on ω but not on p, such that both

$$\|\nabla u\|_{W^{1,2}(\omega)} \leq H \text{ and } \|\frac{\nabla u}{|\nabla u|^{2-p}}\|_{W^{1,2}(\omega)} \leq \frac{H}{(p-1)^2};$$

in particular, for every $i = 1, ...N$, we have $\max\{\frac{|u_{x_i}|}{|\nabla u|^{2-p}}, |u_{x_i}|\} \in W_{loc}^{1,2}(\Omega)$.

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1. Introduction

This paper is concerned with the regularity properties of solutions to variational problems and, more precisely, with their properties of higher differentiability. We consider the problem of minimizing

$$\int_\Omega [L(\nabla v(x)) + g(x, v(x))]\mathrm{d}x \quad \text{on} \quad w^0 + W_0^{1,2}(\Omega) \tag{1.1}$$

where L is a convex function and Ω a bounded open subset of $\mathbb{R}^N$. We wish to explore the effect of an increase of the strict convexity of the Lagrangian, with respect to the variable gradient, on the regularity of the solution; more precisely, we consider a problem where L grows quadratically outside the unit ball, while, when $|\nabla v| < 1$,

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[1] Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano, Italy.
arrigo.cellina@unimib.it

it behaves like $|\nabla v|^p$ with $1 < p < 2$; hence, near the origin, the norm of the matrix of the second derivatives of L diverges, making the problem very strictly convex when $|\nabla v|$ is small. Our Theorem 2.2 below describes how this increasing in the strict convexity of L affects the higher differentiability of the solution u, when $|\nabla u|$ is small.

Regularity results in the sense of u being in $C^{1,\alpha}$ for $L(\xi) = |\xi|^p$ with $p > 1$ have been proved by Uhlenbeck [8], Lewis [6] and Tolksdorf [7] for $g = 0$ and by Di Benedetto [2], Acerbi and Fusco [1] in the general case; very recently functionals with different conditions on $\{|\xi| > 1\}$ and on $\{|\xi| < 1\}$, (with $g = fu$) have been considered by Colombo and Figalli [5] and the regularity $C^{1,\alpha}$ of the solution established; these results and techniques are different from ours.

2. Statement of the Theorem

The integrand L of (1.1) is described as follows: for some $1 < p < 2$, we shall consider the function

$$l(t) = \begin{cases} \frac{1}{2}|t|^2 + 1 & \text{for } |t| \geq 1 \\ \frac{1}{p}|t|^p + \frac{3}{2} - \frac{1}{p} & \text{for } |t| \leq 1 \end{cases} \tag{2.1}$$

and set $L(\xi) = l(|\xi|)$. We have that, calling $H_L(\xi)$ the matrix of second derivatives of L computed at ξ,

$$H_L(\xi) = \begin{cases} (p-2)|\xi|^{p-4}\xi \otimes \xi + |\xi|^{p-2} I & \text{for } |\xi| < 1 \\ I & \text{for } |\xi| > 1 \end{cases} \tag{2.2}$$

so that $z^T H_L(\xi) z \geq |z|^2$ for all ξ, while $|H_L(\xi)| \to \infty$ as $|\xi| \to 0$.

The assumptions on g are:

Assumption 2.1.

i) There exist $\tau \in L^1(\Omega)$ and a non-negative $\lambda_g \in L^2_{\text{loc}}(\Omega)$ such that for a.e. $x \in \Omega$ and every u, we have $g(x, u) \geq \tau(x) - \lambda_g |u|$.
ii) There exist non-negative $\lambda_2 \in L^2_{\text{loc}}(\Omega)$ and $\lambda_\infty \in L^\infty_{\text{loc}}(\Omega)$, such that $|g_u(x, u)| \leq \lambda_2(x) + \lambda_\infty(x)|u|$.

Functions like $g(x, u) = \lambda_2(x)u$ or $g(x, u) = (\sin(x_1)u)^2$ satisfy Assumption 2.1.

The map l, and the map l_r to be defined, are not really C^2 everywhere, but their gradients are Lipschitzian, and, by a simple modification of results that go back to [3], one proves that a solution u to the problem of minimizing (1.1), with L and g described above, is such that $\nabla u \in W^{1,2}_{\text{loc}}(\Omega)$.

The purpose of this paper is to prove the following result:

Theorem 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, let l be as in (2.1) and let g satisfy Assumption 2.1. Then, there exist u, a solution to the Euler–Lagrange equation,* i.e. *such that*

$$\int_\Omega [\langle \nabla L(\nabla u(x)), \nabla \eta(x) \rangle + g_u(x, u(x))\eta(x)] \mathrm{d}x = 0$$

for every $\eta \in C^1_c(\Omega)$, and, for each $\omega \subset\subset \Omega$, a constant H, depending on ω but not on p, such that both

$$\|\nabla u\|_{W^{1,2}(\omega)} \leq H \ \text{ and } \ \|\frac{\nabla u}{|\nabla u|^{2-p}}\|_{W^{1,2}(\omega)} \leq \frac{H}{(p-1)^2};$$

in particular, for every $i = 1, \ldots N$, we have $\max\{\frac{|u_{x_i}|}{|\nabla u|^{2-p}}, |u_{x_i}|\} \in W^{1,2}_{\text{loc}}(\Omega)$.

Under some additional assumptions, mainly when g is convex in the variable v, a solution to the Euler−Lagrange equation is actually a solution to the minimization problem (1.1).

The additional regularity of the solution, provided by Theorem 1, is actually lost in the limit as $p \to 1$, as the statement of Theorem 2.2 itself suggests. In fact, the limit problem consists in minimizing (1.1) where $L(\xi) = l_*(|\xi|)$ with

$$l_*(t) = \begin{cases} \frac{1}{2}|t|^2 + 1 & \text{for } |t| \geq 1 \\ |t| + \frac{1}{2} & \text{for } |t| \leq 1; \end{cases} \tag{2.3}$$

here we have at once that $l_*''(0) = \infty$ while $l_*''(t) = 0$ for $0 < |t| < 1$. When $g(x,u) = u$, a (radial) solution to problem (1.1) is

$$u_*(x) = \begin{cases} 0 & \text{for } |x| \leq 1 \\ \frac{1}{2}(|x|^2 - 1) & \text{for } |x| \geq 1 \end{cases}$$

whose gradient is

$$\nabla u_*(x) = \begin{cases} 0 & \text{for } |x| < 1 \\ x & \text{for } |x| > 1 \end{cases}$$

so that the gradient is discontinuous along $|x| = 1$, preventing ∇u_* from being a Sobolev function.

3. Proof of Theorem 2.2

We shall use the following notations. The measure of $A \subset \mathbb{R}^N$ is $|A|$; a^T is the transpose of a; for a fixed coordinate direction e_s, we set $\delta_{he_s} u$ to be the difference quotient of the function u, defined by $\delta_{he_s} u(x) = \frac{u(x+he_s)-u(x)}{h}$. For a variation η to be defined, D_η is such that $|\nabla\eta(x)| \leq D_\eta$.

For the proof of the main result we shall need l_r, a regularization of l, defined to be

$$l_r(t) = \begin{cases} \frac{1}{2}r^{p-2}t^2 + \left(\frac{1}{p} - \frac{1}{2}\right) r^p + \frac{3}{2} - \frac{1}{p} & \text{for } |t| \leq r \\ l(t) & \text{otherwise,} \end{cases} \tag{3.1}$$

so that

$$l_r'(t) = \begin{cases} r^{p-2}t & \text{for } 0 \leq t \leq r \\ l'(t) & \text{otherwise.} \end{cases}$$

We have that l_r' is continuous and increasing, hence l_r is convex; moreover, for $t \notin \{r, 1\}$, $l_r''(t)$ exists and

$$l_r''(t) = \begin{cases} r^{p-2} & \text{for } |t| < r \\ (p-1)|t|^{p-2} & \text{for } r < |t| < 1 \\ 1 & \text{otherwise.} \end{cases}$$

In particular, l_r' is (globally) Lipschitzian with constant r^{p-2}. Set $L_r(\xi) = l_r(|\xi|)$ so that ∇L_r is Lipschitzian with Lipschitz constant r^{p-2}. In addition, we have that $\nabla L_r \to \nabla L$ uniformly as $r \to 0$.

Besides Problem 1.1, we shall also consider the problem of minimizing

$$\int_\Omega [L_r(\nabla v(x)) + g(x, v(x))]\mathrm{d}x \quad \text{on} \quad w^0 + W_0^{1,2}(\Omega) \tag{3.2}$$

and call u^r its solution. By known regularity results, the function u^r is in $W^{2,2}_{\mathrm{loc}}(\Omega)$.

Lemma 3.1. *Let Ω and g as in Theorem 2.2; let u^r be a solution to the minimization of (3.2); let $\phi \in W^{1,2}(\Omega)$ with support compactly contained in Ω. Then, for $s = 1, \ldots, N$, we have*

$$\int_\Omega \left\langle \frac{\mathrm{d}}{\mathrm{d}x_s} \nabla L_r(\nabla u^r), \nabla\phi \right\rangle = \int_\Omega g_u(\cdot, u^r)\phi_{x_s}$$

Proof.
a) First, we claim that the map $\nabla L_r(\nabla u^r)$ is in $W^{1,2}(\Omega)$; we have that $\nabla L_r(\xi) = l'_r(|\xi|)\frac{\xi}{|\xi|}$ and that $|\nabla u^r|$ is in $W^{1,2}(\Omega)$, with $\frac{\mathrm{d}}{\mathrm{d}x_i}|\nabla u^r| = \left\langle \frac{\nabla u^r}{|\nabla u^r|}, \nabla u^r_{x_i} \right\rangle$. The map

$$\frac{l'_r(t)}{t} = \begin{cases} r^{p-2} & \text{for } 0 \le |t| \le r \\ |t|^{p-2} & \text{for } r \le |t| \le 1 \\ 1 & \text{for } |t| \ge 1 \end{cases}$$

is (uniformly) Lipschitzian and it is not differentiable only at at $|t| = r$ and $|t| = 1$; then, as it is known, $x \to \frac{l'_r(|\nabla u^r(x)|)}{|\nabla u^r(x)|}$ is a Sobolev function with

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l'_r(|\nabla u^r(x)|)}{|\nabla u^r(x)|} &= \left[\left(\frac{l'_r(t)}{t} \right)' \circ |\nabla u^r(x)| \right] \left\langle \frac{\nabla u^r}{|\nabla u^r|}, \nabla u^r_{x_i} \right\rangle \\ &= \begin{cases} 0 & \text{for } |\nabla u^r(x)| \le r \text{ or } |\nabla u^r(x)| \ge 1 \\ (p-2)|\nabla u^r(x)|^{p-3} \left\langle \frac{\nabla u^r}{|\nabla u^r|}, \nabla u^r_{x_i} \right\rangle & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x_i} \left[\frac{l'_r(|\nabla u^r(x)|)}{|\nabla u^r(x)|} \cdot \nabla u^r(x) \right] = \frac{l'_r(|\nabla u^r(x)|)}{|\nabla u^r(x)|} \nabla u^r_{x_i}(x) + \left(\frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l'_r(|\nabla u^r(x)|)}{|\nabla u^r(x)|} \right) \nabla u^r(x).$$

Both terms above are in $L^2_{\mathrm{loc}}(\Omega)$: in fact, $\frac{l'_r(t)}{t}$ is bounded and, from (3.3), the absolute value of the second term is at most $|\nabla u_{x_i}|$. Hence, $\nabla L^r(\nabla u^r)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$.

b) Under the assumptions of the Lemma, the Euler–Lagrange equation holds for u^r in the sense that for $\psi \in W^{1,2}_0(\Omega)$ we have

$$\int_\Omega [\langle \nabla L_r(\nabla u^r), \nabla \psi \rangle + g_u(x,u)\psi] \mathrm{d}x = 0.$$

For h sufficiently small, consider the variation $\psi = \delta_{-he_s}\phi$ to obtain

$$\int_\Omega \left\langle \frac{\nabla L^r(\nabla u^r(x+he_s)) - \nabla L^r(\nabla u^r(x))}{h}, \nabla \phi(x) \right\rangle \mathrm{d}x = \int_\Omega g_u(x, u^r(x)) \frac{\phi(x-he_s) - \phi(x)}{-h} \mathrm{d}x. \tag{3.4}$$

Since $\nabla L^r(\nabla u^r)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$, the family $\left(\frac{\nabla L^r(\nabla u^r(x+he_i)) - \nabla L^r(\nabla u^r(x))}{h} \right)_h$ is bounded in $L^2_{\mathrm{loc}}(\Omega)$ and we can assume the existence of a sequence (h_n) such that

$$\frac{\nabla L^r(\nabla u^r(x+h_ne_i)) - \nabla L^r(\nabla u^r(x))}{h_n} \rightharpoonup \frac{\mathrm{d}}{\mathrm{d}x_i} \nabla L^r(\nabla u^r)$$

so that the left hand side of (3.4) converges to $\int_\Omega \langle \frac{\mathrm{d}}{\mathrm{d}x_s} \nabla L^r(\nabla u^r), \nabla \phi \rangle$.

We also have

$$\begin{aligned} \int_\Omega g_u(x, u^r(x)) \frac{\phi(x-he_s) - \phi(x)}{-h} \mathrm{d}x &= \int_0^1 \int_{\mathrm{supp}(\phi)+the_s} g_u(x, u^r(x)) \phi_{x_s}(x - the_s) \mathrm{d}x \mathrm{d}t \\ &= \int_0^1 \int_{\mathrm{supp}(\phi)} g_u(x+the_s, u^r(x+the_s)) \phi_{x_s}(x) \mathrm{d}x \mathrm{d}t \\ &= \int_\Omega g_u(x, u^r(x)) \phi_{x_s}(x) \mathrm{d}x + \int_0^1 \int_\Omega [g_u(x+he_s, u^r(x+he_s)) - g_u(x, u^r(x)] \phi_{x_s}(x) \mathrm{d}x \mathrm{d}t. \end{aligned}$$

By Assumption 2.1, ii), we obtain that the map $x \to g_u(x, u^r(x))$ is in $L^2_{\mathrm{loc}}(\Omega)$, so that $\|g_u(\cdot + he_s, u^r(\cdot + he_s)) - g_u(\cdot, u^r(\cdot))\|_{L^2(\mathrm{supp}(\phi))} \to 0$, thus proving the lemma. □

Lemma 3.2. *There exists K, depending neither on r nor on p, such that $\|\nabla u^r\|_{L^2(\Omega)} \le K$ and $\|u^r\|_{L^2(\Omega)} \le K$.*

Proof. Set $L^0(\xi) = \frac{1}{2}|\xi|^2 + 1$, so that, for any $1 < p < 2$ and any $r \le 1$, we have $L^r(\xi) \le L^0(\xi) + 1$.

Let u^0 be a solution to the problem of minimizing

$$\int_\Omega [L^0(\nabla v(x)) + 1 + g(x, v(x))]\mathrm{d}x \quad \text{on} \quad w^0 + W_0^{1,2}(\Omega) \tag{3.5}$$

and set $V = \int_\Omega [L^0(\nabla u^0(x)) + 1 + g(x, u^0(x))]\mathrm{d}x$. Then

$$V \ge \int_\Omega [L^r(\nabla u^0) + g(x,u^0)] \ge \int_\Omega [L^r(\nabla u^r) + g(x,u^r)] \ge \int_\Omega \left[\frac{1}{2}|\nabla u^r|^2 + g(x,u^r)\right];$$

on the other hand, recalling Assumption 2.1, for a constant α to be fixed, from $\int \lambda_g |u| \le \frac{1}{2}\alpha^2 \int (\lambda_g)^2 + \frac{1}{2}\frac{1}{\alpha^2}\int |u|^2$ we obtain

$$\int_\Omega g(x, u^r(x))\mathrm{d}x \ge \int \tau - \frac{1}{2}\alpha^2 \int (\lambda_g)^2 - \frac{1}{2}\frac{1}{\alpha^2}\int |u^r|^2.$$

Call P the Poincaré constant in $W^{1,2}(\Omega)$; from $\int |u^r|^2 = \int |w^0 - (u^r - w^0)|^2 \le 2\int |w^0|^2 + 2P \int |\nabla(u^r - w^0)|^2 \le 2\int |w^0|^2 + 4P\int |\nabla u^r|^2 + 4P \int |\nabla w^0|^2$, we obtain

$$\int_\Omega g(x,u^r) \ge \int \tau - \frac{1}{2}\alpha^2 \int (\lambda_g)^2 - \frac{1}{2}\frac{1}{\alpha^2}\left[4P\int |\nabla u^r|^2 + \int |\nabla w^0|^2 (2 + 4P)\right].$$

Hence,

$$\int_\Omega \frac{1}{2}|\nabla u^r|^2 \le V - \int_\Omega g(x,u^r) \le V - \int \tau + \frac{1}{2}\alpha^2 \int (\lambda_g)^2 + \frac{2P}{\alpha^2}\int |\nabla u^r|^2 + \frac{1}{2\alpha^2}\int |\nabla w^0|^2(2+4P).$$

Choose α such that $\frac{2P}{\alpha^2} = \frac{1}{4}$ to obtain $\int |\nabla u^r|^2 \le 4[V + \int (-\tau + \frac{1}{2}\alpha^2(\lambda_g)^2 + \frac{2+4P}{2\alpha^2}|\nabla w^0|^2)] = k_1$.

From this, making use of $w^0 \in W^{1,2}$ and of Poincaré's inequality, we infer that for some k_2, we also have $\int_\Omega |u^r|^2 \le k_2$ for all $r \le 1$. □

A similar estimate was proved in [4].

Proof of Theorem 2.2.

a) Consider the function

$$\gamma_r(t) = \frac{l_r'(t)}{t} = \begin{cases} r^{p-2} & \text{for } |t| \le r \\ |t|^{p-2} & \text{for } r < |t| < 1 \\ 1 & \text{otherwise;} \end{cases}$$

then, as in the Proof of Lemma 3.1, the map $x \to \gamma_r(|\nabla u^r(x)|)$ is in $W^{1,2}_{loc}$ and

$$\frac{\mathrm{d}}{\mathrm{d}x_s}\gamma_r(|\nabla u^r(x)|) = \begin{cases} 0 & \text{for } |\nabla u^r| \le r \text{ or } |\nabla u^r| \ge 1 \\ (p-2)|\nabla u^r|^{p-3}\left\langle \dfrac{\nabla u^r}{|\nabla u^r|}, \nabla u^r_{x_s}\right\rangle & \text{for } r < |\nabla u^r| < 1. \end{cases}$$

Moreover, $1 \le \gamma_r \le r^{p-2}$ and $|\frac{\mathrm{d}}{\mathrm{d}x_s}\gamma_r(|\nabla u^r|)| \le (2-p)r^{p-3}|\nabla u^r_{x_s}|$. Then, the map $x \to \gamma_r(|\nabla u^r(x)|)u_{x_i}(x)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and, setting H_{u^r} to be the Hessian matrix of u^r, we obtain

$$\nabla(\gamma_r(|\nabla u^r|)u^r_{x_i}) = \begin{cases} \gamma_r(|\nabla u^r|)\nabla u^r_{x_i} & \text{for } |\nabla u^r| \le r \text{ or } |\nabla u^r| \ge 1 \\ (p-2)|\nabla u^r|^{p-2} H_{u^r}\dfrac{\nabla u^r}{|\nabla u^r|}\dfrac{u^r_{x_i}}{|\nabla u^r|} + \gamma_r(|\nabla u^r|)\nabla u^r_{x_i} & \text{for } r \le |\nabla u^r| \le 1. \end{cases} \tag{3.6}$$

b) Let x^0 and δ^0 be such that $B(x^0, 4\delta^0) \subset\subset \Omega$. Let $\eta \in C_0^\infty(B(x^0, 2\delta^0))$ be such that $0 \le \eta \le 1$ and that $\eta(x) = 1$ for $x \in B(x^0, \delta^0)$; we recall that $D_\eta = \sup\{|\nabla\eta(x)|\}$. Then, the function $\phi = [\eta^2\gamma_r(|\nabla u^r|)u^r_{x_i}]$ is in $W_0^{1,2}(B(x^0, 3\delta^0))$ and from Lemma 3.1 we have

$$\int_\Omega \left\langle \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r, \nabla\phi \right\rangle = \int_\Omega g_u(\cdot, u^r)\phi_{x_i},$$

i.e.,

$$\int_{B(x^0,3\delta^0)} \left\langle \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r_{x_i} + \left(\frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|}\right) \nabla u^r, \qquad 2\eta\nabla\eta\gamma_r(|\nabla u^r|)u^r_{x_i} + \eta^2\nabla(\gamma_r(|\nabla u^r|)u_{x_i}) \right\rangle \mathrm{d}x$$
$$= \int_{B(x^0,3\delta^0)} g_u(\cdot, u^r) \left[2\eta\eta_{x_i}\gamma_r(|\nabla u^r|)u^r_{x_i} + \eta^2 \frac{\mathrm{d}}{\mathrm{d}x_i}(\gamma_r(|\nabla u^r|)u^r_{x_i})\right] \mathrm{d}x \tag{3.7}$$

We shall call G_i the term at the right hand side.

Since the above equality holds for every i, we obtain

$$\sum_i \int_{B(x^0,3\delta^0)} \left\langle \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r, \eta^2\nabla(\gamma_r(|\nabla u^r|)u^r_{x_i}) \right\rangle \mathrm{d}x$$
$$\le \sum_i \Big| \int_{B(x^0,3\delta^0)} \left\langle \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r, 2\eta\nabla\eta\gamma_r(|\nabla u^r|)u^r_{x_i} \right\rangle \Big| \mathrm{d}x + \sum_i G_i \tag{3.8}$$

c) For $j = 1, \ldots, N$, we have

$$\left(\nabla_x \left(\frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r\right)\right)_{i,j} = \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} u^r_{x_j x_i} + \left(\frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|}\right) u^r_{x_j}$$
$$= \frac{l_r'}{|\nabla u^r|} u^r_{x_j x_i} + \left\langle \frac{\nabla u^r}{|\nabla u^r|}, \nabla u^r_{x_i} \right\rangle \left(l_r'' - \frac{l_r'}{|\nabla u^r|}\right) \frac{u^r_{x_j}}{|\nabla u^r|}$$

i.e.,

$$\nabla_x \left(\frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r\right) = \frac{l_r'}{|\nabla u^r|} H_{u^r} + \left(l_r'' - \frac{l_r'}{|\nabla u^r|}\right) \left(\frac{\nabla u^r}{|\nabla u^r|} H_{u^r}\right) \otimes \frac{\nabla u^r}{|\nabla u^r|}$$

and we obtain

$$\left|\nabla_x \left(\frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r\right)\right|^2$$
$$= \left(\frac{l_r'}{|\nabla u^r|}\right)^2 |H_{u^r}|^2 + \left(l_r'' - \frac{l_r'}{|\nabla u^r|}\right)^2 \left|\frac{\nabla u^r}{|\nabla u^r|} H_{u^r}\right|^2 + 2\left(l_r'' - \frac{l_r'}{|\nabla u^r|}\right) \frac{l_r'}{|\nabla u^r|} \left|\frac{\nabla u^r}{|\nabla u^r|} H_{u^r}\right|^2$$
$$= \left(\frac{l_r'}{|\nabla u^r|}\right)^2 |H_{u^r}|^2 + \left((l_r'')^2 - \left(\frac{l_r'}{|\nabla u^r|}\right)^2\right) \left|\frac{\nabla u^r}{|\nabla u^r|} H_{u^r}\right|^2$$

so that

$$\inf\left\{(l_r'')^2, \left(\frac{l_r'}{|\nabla u^r|}\right)^2\right\} |H_{u^r}|^2 \le |\nabla_x \left(\frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r\right)|^2 \le \sup\left\{(l_r'')^2, \left(\frac{l_r'}{|\nabla u^r|}\right)^2\right\} |H_{u^r}|^2. \tag{3.9}$$

We also have, computing l_r', l_r'', γ_r and γ_r' at $|\nabla u^r|$,

$$\sum_i \left\langle \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r, \nabla(\gamma_r u^r_{x_i}) \right\rangle$$

$$= \sum_i \sum_j \left(\frac{l_r'}{|\nabla u^r|} u^r_{x_j x_i} + \left(\sum_s \frac{u^r_{x_s}}{|\nabla u^r|} u^r_{x_i x_s} \right) \left(l_r'' - \frac{l_r'}{|\nabla u^r|} \right) \frac{u^r_{x_j}}{|\nabla u^r|} \right)$$

$$\cdot \left(\gamma_r u^r_{x_j x_i} + (\gamma_r)' \left(\sum_l u^r_{x_j x_l} \frac{u^r_{x_l}}{|\nabla u^r|} \right) u^r_{x_i} \right)$$

$$= \frac{l_r'}{|\nabla u^r|} \gamma_r |H_{u^r}|^2 + \left(l_r'' - \frac{l_r'}{|\nabla u^r|} \right) \gamma_r' |\nabla u^r| \left(\sum_{i,s} \frac{u^r_{x_i}}{|\nabla u^r|} \frac{u^r_{x_s}}{|\nabla u^r|} u^r_{x_i x_s} \right)$$

$$\cdot \left(\sum_{j,l} \frac{u^r_{x_j}}{|\nabla u^r|} \frac{u^r_{x_l}}{|\nabla u^r|} u^r_{x_j x_l} \right)$$

$$+ \gamma_r \left(l_r'' - \frac{l_r'}{|\nabla u^r|} \right) \sum_i \left(\sum_j \frac{u^r_{x_j}}{|\nabla u^r|} u^r_{x_j x_i} \right) \left(\sum_s \frac{u^r_{x_s}}{|\nabla u^r|} u^r_{x_i x_s} \right)$$

$$+ \frac{l_r'}{|\nabla u^r|} \gamma_r' |\nabla u^r| \left(H_{u^r} \frac{\nabla u^r}{|\nabla u^r|} \right)^2$$

$$= \frac{l_r'}{|\nabla u^r|} \gamma_r |H_{u^r}|^2 + \left(l_r'' - \frac{l_r'}{|\nabla u^r|} \right) \gamma_r' |\nabla u^r| \left(\frac{\nabla u^r}{|\nabla u^r|}^T H_{u^r} \frac{\nabla u^r}{|\nabla u^r|} \right)^2$$

$$+ \gamma_r \left(l_r'' - \frac{l_r'}{|\nabla u^r|} \right) \left| H_{u^r} \frac{\nabla u^r}{|\nabla u^r|} \right|^2 + \frac{l_r'}{|\nabla u^r|} \gamma_r' |\nabla u^r| |H_{u^r} \frac{\nabla u^r}{|\nabla u^r|}|^2$$

$$= \frac{l_r'}{|\nabla u^r|} \gamma_r |H_{u^r}|^2 + \left(l_r'' - \frac{l_r'}{|\nabla u^r|} \right) \gamma_r \left| H_{u^r} \frac{\nabla u^r}{|\nabla u^r|} \right|^2$$

$$+ \frac{l_r'}{|\nabla u^r|} \gamma_r' |\nabla u^r| \left[|H_{u^r} \frac{\nabla u^r}{|\nabla u^r|}|^2 - \left(\frac{\nabla u^r}{|\nabla u^r|}^T H_{u^r} \frac{\nabla u^r}{|\nabla u^r|} \right)^2 \right]$$

$$+ l_r'' \gamma_r' |\nabla u^r| \left(\frac{\nabla u^r}{|\nabla u^r|}^T H_{u^r} \frac{\nabla u^r}{|\nabla u^r|} \right)^2$$

$$\geq \gamma_r [\frac{l_r'}{|\nabla u^r|} |H_{u^r}|^2 + \left(l_r'' - \frac{l_r'}{|\nabla u^r|} \right) \left| H_{u^r} \frac{\nabla u^r}{|\nabla u^r|} \right|^2]$$

$$\geq \gamma_r [\inf \left\{ l_r'', \frac{l_r'}{|\nabla u^r|} \right\}] |H_{u^r}|^2$$

where

$$\gamma_r(t) \inf \left\{ l_r''(t), \frac{l_r'(t)}{t} \right\} = \begin{cases} r^{2(p-2)} & \text{for } |t| \leq r \\ (p-1)|t|^{2(p-2)} & \text{for } r \leq |t| \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Hence we have obtained

$$\int_{B(x^0, 3\delta^0)} \eta^2 \gamma_r \left[\inf \left\{ l_r'', \frac{l_r'}{|\nabla u^r|} \right\} \right] |H_{u^r}|^2 \mathrm{d}x \leq \sum_i \int_{B(x^0, 3\delta^0)} \eta^2 \left\langle \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|^r} \nabla u^r, \nabla(\gamma_r(|\nabla u^r|) u_{x_i}) \right\rangle \mathrm{d}x$$

so that, from (3.8),

$$\int_{B(x^0,3\delta^0)} \eta^2\gamma_r \left[\inf\left\{l_r'', \frac{l_r'}{|\nabla u^r|}\right\}\right] |H_{u^r}|^2 \mathrm{d}x \leq \sum_i \left| \int_{B(x^0,3\delta^0)} \left\langle \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{l_r'(|\nabla u^r|)}{|\nabla u^r|} \nabla u^r, 2\eta\nabla\eta\gamma_r(|\nabla u^r|)u^r_{x_i} \right\rangle \right| \mathrm{d}x + \sum_i G_i \quad (3.10)$$

d) From (3.6) we have that

$$\begin{aligned}\sum_i G_i &= \int_{B(x^0,3\delta^0)} [g_u(x,u^r)2\eta\gamma_r(|\nabla u^r|)\langle\nabla\eta,\nabla u^r\rangle + \eta^2 g_u(x,u^r)\Delta u^r]\mathrm{d}x \\ &+ \int_{B(x^0,3\delta^0)\cap\{r\leq|\nabla u^r(x)|\leq 1\}} g_u(x,u^r)\eta^2(p-2)|\nabla u^r|^{p-2}\left(\frac{\nabla u^r}{|\nabla u^r|}\right)^T H_{u^r}\frac{\nabla u^r}{|\nabla u^r|}\mathrm{d}x \\ &\leq \int_{B(x^0,3\delta^0)} [\eta^2 g_u^2 + |\nabla\eta|^2 + \frac{4}{p-1}\eta^2 g_u^2 + \frac{p-1}{4}\eta^2|H_{u^r}|^2]\mathrm{d}x \\ &+ \int_{B(x^0,3\delta^0)\cap\{r\leq|\nabla u^r(x)|\leq 1\}} \left[\frac{4}{p-1}\eta^2 g_u^2 + \eta^2\frac{p-1}{4}|\nabla u^r|^{2(p-2)}|H_{u^r}|^2\right]\mathrm{d}x \end{aligned} \quad (3.11)$$

By Assumption 2.1, $g_u(x,u^r)^2 \leq 2[(\lambda_2)^2 + (\lambda_\infty|u^r|)^2]$; hence, applying Lemma 3.2 we infer that there exists a constant K^0, independent of r and p, such that the right hand side of (3.11) is bounded above by

$$\frac{K^0}{p-1} + \frac{1}{2}\int_{B(x^0,3\delta^0)} \eta^2(p-1)|\nabla u^r|^{2(p-2)}|H_{u^r}|^2\mathrm{d}x.$$

Then, from $\inf\{l_r'', \frac{l_r'}{|\nabla u^r|}\} \geq p-1$, (3.10) gives

$$\begin{aligned}&\frac{1}{2}\int_{B(x^0,3\delta^0)} \eta^2\gamma_r\left[\inf\left\{l_r'', \frac{l_r'}{|\nabla u^r|}\right\}\right]|H_{u^r}|^2\mathrm{d}x \\ &\leq \frac{1}{p-1}K^0 + \sum_i\left|\int_{B(x^0,3\delta^0)}\left\langle\frac{\mathrm{d}}{\mathrm{d}x_i}\frac{l_r'(|\nabla u^r|)}{|\nabla u^r|}\nabla u^r, 2\eta\nabla\eta\gamma_r(|\nabla u^r|)u^r_{x_i}\right\rangle\right|\mathrm{d}x \\ &\leq \frac{1}{p-1}K^0 + \int_{B(x^0,3\delta^0)}\left[\frac{p-1}{4}|\nabla_x(\nabla L(\nabla u^r))|^2\eta^2 + \frac{4}{p-1}N|\nabla\eta|^2\right]\mathrm{d}x \\ &\leq \frac{1}{p-1}K^0 + \int_{B(x^0,3\delta^0)}\left[\frac{1}{4}\gamma_r\inf\left\{l_r'', \frac{l_r'}{|t|}\right\}|\nabla_x(\nabla L(\nabla u^r))|^2\eta^2 + \frac{4}{p-1}N|\nabla\eta|^2\right]\mathrm{d}x \end{aligned} \quad (3.12)$$

and we obtain

$$\frac{p-1}{4}\int_{B(x^0,3\delta^0)}\eta^2|\nabla_x(\nabla L_r(\nabla u^r)|^2\mathrm{d}x \leq \frac{1}{4}\int_{B(x^0,3\delta^0)}\eta^2\gamma_r\left[\inf\left\{l_r'', \frac{l_r'}{|\nabla u^r|}\right\}\right]|H_{u^r}|^2\mathrm{d}x \leq \frac{K^1}{p-1}.$$

f) In particular,

$$\int_{B(x^0,\delta^0)}|\nabla_x\nabla L_r(\nabla u^r)|^2 \leq \frac{4}{(p-1)^2}K^1; \quad (3.13)$$

hence, the family $(\nabla_x\nabla L_r(\nabla u^r))_r$ is bounded in $L^2(B(x^0,\delta^0))$. The arbitrariness of x^0 and of δ^0 then shows that, for every $\omega \subset\subset \Omega$, there exists H, independent of r and p, such that $(\|\nabla_x\nabla L_r(\nabla u^r)\|_{L^2(\omega)})_r \leq \frac{H}{(p-1)^2}$. Then, from (3.9) and since $\inf\{(l_r'')^2, (\frac{l_r'}{|\nabla u^r|})^2\} \geq (p-1)^2$, we infer that

$$\int_\omega |H_{u^r}|^2 \leq \frac{1}{(p-1)^2}\int_\omega|\nabla_x\nabla L_r(\nabla u^r)|^2 \leq \frac{1}{(p-1)^4}H^2. \quad (3.14)$$

Then, we can assume that, for $s = 1, \dots, N$, there exists a sequence $(r^n)_n$ such that $\frac{\mathrm{d}}{\mathrm{d}x_s}\nabla L_{r_n}(\nabla u^{r_n})$ converges weakly in $L^2(\omega)$ to some d_λ, that $\nabla L_{r_n}(\nabla u^{r_n})$ converges in $L^2(\omega)$ to a function λ, that $u^{r_n} \to u$ and, finally, that $\nabla u^{r_n} \to \nabla u$ in $L^2(\omega)$.

g) We claim that:

i) $\lambda = \nabla L(\nabla u)$; $d_\lambda = \frac{\mathrm{d}}{\mathrm{d}x_s}\nabla L(\nabla u)$ and

$$\left\| \frac{\mathrm{d}}{\mathrm{d}x_s}\nabla L(\nabla u)\right\|_{L^2(\omega)} \le \frac{1}{(p-1^2}H.$$

ii) u is a solution to the Euler Lagrange equation, *i.e.*, that, for every $\eta \in C_c^1(\Omega)$,

$$\int_\Omega [\langle \nabla L(\nabla u(x)), \nabla\eta(x)\rangle + g_u(x, u(x))\eta(x)]\mathrm{d}x = 0.$$

To prove the claim, notice that, possibly passing to a subsequence, we can assume that both $\nabla u^{r_n} \to \nabla u$ and $\nabla L_{r_n}(\nabla u^{r_n}) \to \lambda$ pointwise a.e.. Fix x such that the above holds and fix ε. By the continuity of ∇L, let δ be such that $|\nabla u(x) - \xi| \le \delta$ implies $|\nabla L(\nabla u(x)) - \nabla L(\xi)| < \frac{\varepsilon}{2}$; let n be so large that both $|\nabla u^{r_n}(x) - \nabla u(x)| < \delta$, and $\|\nabla L_{r_n} - \nabla L\|_C < \frac{\varepsilon}{2}$. Hence, for n large,

$$|\nabla L_{r_n}(\nabla u^{r_n}(x)) - \nabla L(\nabla u(x))|$$

$$\le |\nabla L_{r_n}(\nabla u^{r_n}(x)) - \nabla L(\nabla u^{r_n}(x))| + |\nabla L(\nabla u^{r_n}(x)) - \nabla L(\nabla u^{r_n}(x))| < \varepsilon,$$

so that $|\lambda(x) - \nabla L(\nabla u(x))| \le \varepsilon$, and by the arbitrariness of ε, we obtain

$$\lambda(x) = \nabla L(\nabla u(x)).$$

Moreover, $\nabla L_{r_n}(\nabla u^{r_n}) \to \lambda$ in $L^2(\omega)$ and $\frac{\mathrm{d}}{\mathrm{d}x_s}\nabla L_{r_n}(\nabla u^{r_n}) \rightharpoonup d_\lambda$ weakly, imply $d_\lambda = \frac{\mathrm{d}}{\mathrm{d}x_s}\nabla L(\nabla u)$, so that from $\|\frac{\mathrm{d}}{\mathrm{d}x_s}\nabla L_{r_n}(\nabla u^r)\|_{L^2(\omega)} \le \frac{1}{(p-1)^2}H$ we obtain that $\|\frac{\mathrm{d}}{\mathrm{d}x_s}\nabla L(\nabla u)\|_{L^2(\omega)} \le \frac{1}{(p-1)^2}H$, thus proving i).

To prove ii), fix $\eta \in C_c^1(\Omega)$. We have that

$$\int_\Omega [\langle \nabla L_{r_n}(\nabla u^{r_n}(x)), \nabla\eta(x)\rangle + g_u(x, u^{r_n}(x))\eta(x)]\,\mathrm{d}x = 0;$$

since $u^{r_n} \to u$ in $L^2(\omega)$ and $\nabla L_{r_n}(\nabla u^{r_n}) \rightharpoonup \nabla L(\nabla u)$, we obtain

$$\int_\Omega [\langle \nabla L(\nabla u(x)), \nabla\eta(x)\rangle + g_u(x, u(x))\eta(x)]\mathrm{d}x = 0.$$

Hence, we have obtained the existence of a solution u to the Euler−Lagrange equation such that, for every $s = 1, \dots, N$,

$$\frac{\mathrm{d}}{\mathrm{d}x_s}\nabla L(\nabla u) = \frac{\mathrm{d}}{\mathrm{d}x_s}\begin{cases} \dfrac{\nabla u}{|\nabla u|^{2-p}} & \text{for } |\nabla u| \le 1 \\ \nabla u & \text{for } |\nabla u| \ge 1 \end{cases}$$

belongs to $L^2(\omega)$; in particular, for every $i = 1, \dots, N$, both u_{x_i} and $\frac{u_{x_i}}{|\nabla u|^{2-p}}$ belong to $W^{1,2}_{\mathrm{loc}}$. □

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