

## AN OPTIMAL IRRIGATION NETWORK WITH INFINITELY MANY BRANCHING POINTS

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**Abstract.** The Gilbert–Steiner problem is a mass transportation problem, where the cost of the transportation depends on the network used to move the mass and it is proportional to a certain power of the “flow”. In this paper, we introduce a new formulation of the problem, which turns it into the minimization of a convex functional in a class of currents with coefficients in a group. This framework allows us to define calibrations. We apply this technique to prove the optimality of a certain irrigation network in the separable Hilbert space  $\ell^2$ , having countably many branching points and a continuous amount of endpoints.

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### 1. INTRODUCTION

The Gilbert–Steiner problem is a variant of the optimal transport problem with a cost depending not only on the position of the masses at the beginning and at the end of the process, but also on the path that has been chosen to move them. As in the optimal transport problem, the Gilbert–Steiner’s datum consists of a positive measure  $\mu_-$  (source) and a positive measure  $\mu_+$  (well), with the same total mass  $\|\mu_-\| = \|\mu_+\|$ . One wants to move  $\mu_-$  onto  $\mu_+$ , minimizing the global cost. The cost of the transport is concave with respect to the density of the transported mass, as it is in many natural phenomena and engineering problems<sup>3</sup>.

In particular, if  $\mu_-$  and  $\mu_+$  are sums of Dirac masses on a set of sources  $X$  and a set of wells  $Y$ , respectively, then a transporting network is described by a 1-dimensional rectifiable set  $\Sigma$  and a multiplicity function  $\theta \in L^1(\Sigma; \mathbb{Z})$ . The set  $\Sigma$  is a union of paths joining points of  $X$  to points of  $Y$  and the multiplicity represents the “flow” of the transported mass through each point of the set  $\Sigma$ . The cost associated to each transporting network is the integral on the set  $\Sigma$  of the quantity  $|\theta|^\alpha$  for some  $\alpha \in (0, 1)$ . It is interesting to observe the limit cases: if  $\alpha = 0$ , the problem corresponds to the minimization of the 1-dimensional measure of the set  $\Sigma$ ,

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<sup>3</sup>See [7, 21, 29, 33], for instance, for the models arising from nature and [6, 13, 15] for the models arising from engineering and optimal design. Actually, the Gilbert–Steiner problem is motivated not only by concrete applications. Indeed, it originated from operational research and graph theory, as one can see in [34], and other very interesting mathematical problems can be related with it, such as the study of topological singularities for Ginzburg–Landau models (see [5, 10]).

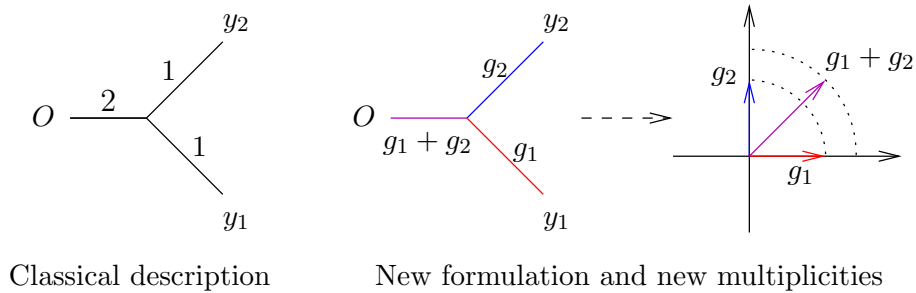


FIGURE 1. Irrigation network with two wells.

if  $\alpha = 1$ , we recover the Monge–Kantorovich energy and the Monge problem of optimal transport. When  $\mu_-$  is supported on a single point, we talk about irrigation problem. One immediately understands that, by its own nature, the problem tends to favor large flows and therefore it easily leads to branched structures.

Several descriptions of this problem have been given by many authors: we refer to [4] for a detailed overview of the subject, though different models have been proposed in [2, 19, 23, 28, 31]. Those models are summarized and proved to be equivalent in [17, 18]. Another model is available in [8], with further developments and a comparison with the model proposed in [2, 9].

Our starting point is Xia’s approach in [31]: the problem is described in terms of minimization of a certain energy, depending on  $\alpha$ , in a family of transport paths, that is, normal 1-dimensional currents with boundary  $\mu_+ - \mu_-$ . An important feature which is evident in this approach is the non-convex nature of the problem: the energy of a convex combination of two competitors may be strictly larger than the energy of each single competitor.

One of the achievements of the present paper is a new formulation of the Gilbert–Steiner problem which turns it into the minimization of a convex functional (see Sect. 4.5). By associating to the measures  $\mu_-$  and  $\mu_+$  a normed group  $G$  and a 0-dimensional current  $B$  with coefficients in  $G$ , we manage to prove that the Gilbert–Steiner problem is equivalent to the problem of finding a mass minimizer, among all 1-dimensional currents  $Z$  with coefficients in  $G$  having boundary  $\partial Z = B$ . The group  $G$  is chosen depending only on the number of the transported unit Dirac masses. In this new framework we can introduce the notion of calibration, that is, a functional analytic tool to prove that a given candidate is a minimizer.

Such an approach is not completely new: in [20], we applied the same procedure to the study of the Steiner tree problem, which corresponds to the limit case  $\alpha = 0$  of the Gilbert–Steiner problem, with an additional connectedness constraint.

Let us explain with an easy example the rough idea of the new formulation, with the help of Figure 1. Assume we want to irrigate the two wells  $y_1 = (2, -1)$  and  $y_2 = (2, 1)$  from a (unique) source at the origin, which generates a flow of intensity 2. Fix  $\alpha = 1/2$ . The classical description of a competitor for this problem consists of a superposition of two paths  $\Gamma_1$  and  $\Gamma_2$ , both with multiplicity 1: the first path goes from the origin to  $y_1$ , the second from the origin to  $y_2$ . Therefore the irrigation network can be identified with the set  $\Gamma_1 \cup \Gamma_2$  with a multiplicity, which is equal to 2 on  $\Gamma_1 \cap \Gamma_2$  and 1 elsewhere on  $\Gamma_1 \cup \Gamma_2$ . One computes the energy of such competitor as:

$$\mathcal{H}^1((\Gamma_1 \cup \Gamma_2) \setminus (\Gamma_1 \cap \Gamma_2)) + \sqrt{2}\mathcal{H}^1(\Gamma_1 \cap \Gamma_2),$$

where  $\mathcal{H}^1$  denotes the 1-dimensional Hausdorff measure. In our formulation we associate to  $\Gamma_1$  and  $\Gamma_2$  vectors in  $\mathbb{R}^2$  as multiplicities:  $g_1$  and  $g_2$ , respectively, where  $(g_1, g_2)$  is an orthonormal basis of  $\mathbb{R}^2$ . In this way the stretch  $\Gamma_1 \cap \Gamma_2$  is associated to the vector  $g_1 + g_2$ . In order to compute the mass of the corresponding 1-current with coefficients in  $\mathbb{R}^2$  one has to integrate on  $\Gamma_1 \cup \Gamma_2$  the Euclidean norm of the corresponding multiplicity. Clearly this mass coincides with the energy computed as above, because the Euclidean norm of  $g_1 + g_2$  is exactly  $\sqrt{2}$ .

In the last part of the paper we apply the equivalence result and the calibration method to prove the optimality of an irrigation tree having countably many branching points and a continuous amount of endpoints. The occurrence of such a topological behaviour for energy-minimizers has been the aim of various attempts: in [3] some characterizing principles for infinite irrigation trees are discussed, while in [25] the authors prove the optimality of an analogous tree for the generalized Steiner tree problem (see also [24, 26]). Their proof relies on a fine and rather delicate study of the geometry of the candidate minimizer. On the contrary, our proof is based on a calibration argument and we believe its simplicity to be an encouraging sign for further developments. Notice that the existence of an optimal irrigation network with infinitely many branching points has already been proved in [22], in order to exhibit a counterexample to the regularity of paths (the regularity of the optimal network, instead, is established when the irrigated measure is the Lebesgue measure on a smooth open set). In such example, anyway, the amount of endpoints is countable.

Let us describe now, in more detail, the content of each section.

In Section 2, we collect those definitions and basic results in geometric measure theory that are necessary to formulate both the discrete version of the Gilbert–Steiner problem, as introduced in [31] and our new formulation in terms of 1-dimensional currents with coefficients in the group  $\mathbb{Z}^n$ , being  $n$  the number of transported unit Dirac masses. Moreover we define a suitable notion of calibration for these objects and we prove that a calibrated current minimizes the mass in its homology class.

In Section 3, we recall and rework the discrete version of the Gilbert–Steiner problem given by [31], in terms of an energy minimization problem in the family of classical integral currents, having a fixed boundary, which is supported on the given set of sources and wells (this is what we call problem (P1)). We also recall a structure theorem for integral 1-currents (see Thm. 3.2) which allows one to better understand why such an abstract class of competitors is indeed a natural choice.

In Section 4, we describe how to associate to the given measures  $\mu_-$  and  $\mu_+$  a normed group  $G$  and a mass minimization problem in a class of rectifiable currents with coefficients in  $G$  (which we call problem (P2)). In Theorem 4.4 we prove the equivalence between problems (P1) and (P2). Then, we apply this result and the calibration technique introduced in Section 2 to prove the optimality of some simple transportation networks. We conclude explaining how one should modify the two problems (P1) and (P2) in order to keep the equivalence, when the measures  $\mu_-, \mu_+$ , given as initial datum, are rescaled.

In Section 5, we define an irrigation network in the separable Hilbert space  $\ell^2$ , having infinitely many branching points and a continuous number of endpoints. Then, we prove that such a network is a solution of the continuous version of the Gilbert–Steiner problem associated to the corresponding boundary. The proof is based on an argument of reduction to a finite dimensional framework, where we are able to exhibit a calibration and therefore to prove the optimality.

## 2. NOTATION AND PRELIMINARIES

In this first section, we briefly recall the main definitions and results concerning 1-dimensional rectifiable currents with coefficients in the abelian normed group  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ . In this paper we limit ourselves to the essential facts which are unavoidable for the statement and the development of the main result. See [20] for a more general theory and further details.

### 2.1. Differential forms with values in $\mathbb{R}^n$

The ambient space where we want to treat the irrigation problem is  $\mathbb{R}^d$ . Concerning the coefficients, let us fix a standard basis  $(g_1, \dots, g_n)$  for  $\mathbb{R}^n$ . We fix  $\alpha \in (0, 1)$  and consider the  $n$ -dimensional normed vector space  $(\mathbb{R}^n, \|\cdot\|_\alpha)$ , with

$$\|h\|_\alpha := \left( \sum_{j=1}^n |h_j|^{\frac{1}{\alpha}} \right)^\alpha$$

for every  $h = h_1g_1 + \dots + h_n g_n \in \mathbb{R}^n$ . Once for all, we fix the (canonical) basis  $(e_1, \dots, e_n)$  of the dual space of  $(\mathbb{R}^n, \|\cdot\|_\alpha)$  so that  $\langle e_i, g_j \rangle = \delta_{ij}$ . With this notation, the dual space of  $(\mathbb{R}^n, \|\cdot\|_\alpha)$  is the  $n$ -dimensional normed vector space  $(\mathbb{R}^n, \|\cdot\|_{1-\alpha})$ .

**Definition 2.1.** An  $\mathbb{R}^n$ -valued covector in  $\mathbb{R}^d$  is a map<sup>4</sup>  $\omega : A_1(\mathbb{R}^d) \times \mathbb{R}^n \rightarrow \mathbb{R}$  with the following properties<sup>5</sup>:

- (i)  $\langle \omega; \tau, \cdot \rangle \in \mathbb{R}^n$  for every vector  $\tau \in A_1(\mathbb{R}^d)$ ;
- (ii)  $\langle \omega; \cdot, h \rangle \in A^1(\mathbb{R}^d)$  is a standard 1-dimensional covector in  $\mathbb{R}^d$  for every element  $h \in \mathbb{R}^n$ .

The linear space of  $\mathbb{R}^n$ -valued covectors in  $\mathbb{R}^d$ , endowed with the comass norm

$$\|\omega\|_\alpha^* := \sup_{|\tau| \leq 1} \|\langle \omega; \tau, \cdot \rangle\|_{1-\alpha},$$

will be denoted by  $A_{(n,\alpha)}^1(\mathbb{R}^d)$ . An  $\mathbb{R}^n$ -valued (1-dimensional) differential form defined on  $\mathbb{R}^d$  is a map  $\omega : \mathbb{R}^d \rightarrow A_{(n,\alpha)}^1(\mathbb{R}^d)$ , its regularity is the one inherited from the components

$$\omega_j := \langle \omega; \cdot, g_j \rangle : \mathbb{R}^d \rightarrow A^1(\mathbb{R}^d),$$

for  $j = 1, \dots, n$ .

**Definition 2.2.** Consider a function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^n$  of class  $\mathcal{C}^1$ . We can compute the differential  $d\psi_j$  of each component  $\psi_j$  of  $\psi$  ( $j = 1 \dots, n$ ), thus we will denote

$$d\psi = d\psi_1 e_1 + \dots + d\psi_n e_n \in \mathcal{C}(\mathbb{R}^d; A_{(n,\alpha)}^1(\mathbb{R}^d)).$$

## 2.2. Rectifiable currents with coefficients in $\mathbb{R}^n$

**Definition 2.3.** Given an integer number  $n \geq 1$  and a real parameter  $\alpha \in (0, 1)$ , we say that a map  $Z : \mathcal{C}_c^\infty(\mathbb{R}^d; A_{(n,\alpha)}^1(\mathbb{R}^d)) \rightarrow \mathbb{R}$  is a *rectifiable current with coefficients in the normed abelian group*  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  if there exist a countably 1-rectifiable set  $\Sigma \subset \mathbb{R}^d$  (see [27], for instance, for the definition of countably 1-rectifiable set), a vector field  $\tau : \Sigma \rightarrow A_1(\mathbb{R}^d)$  which is  $\mathcal{H}^1$ -almost everywhere tangent to  $\Sigma$  and an integrable function  $\theta : \Sigma \rightarrow (\mathbb{Z}^n, \|\cdot\|_\alpha)$  such that

- (i) for every  $\omega \in \mathcal{C}_c^\infty(\mathbb{R}^d; A_{(n,\alpha)}^1(\mathbb{R}^d))$  we can write

$$Z(\omega) = \int_\Sigma \langle \omega(x); \tau(x), \theta(x) \rangle d\mathcal{H}^1(x); \tag{2.1}$$

- (ii) the *mass* of the current is finite, that is

$$\mathbf{M}^{(n,\alpha)}(Z) := \int_\Sigma \|\theta(x)\|_\alpha d\mathcal{H}^1(x) < +\infty.$$

Moreover, we define the *boundary* of  $Z$  as the linear functional  $\partial Z : \mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{R}^n) \rightarrow \mathbb{R}$  satisfying  $\partial Z(\psi) := Z(d\psi)$  for every  $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{R}^n)$ . We require that there exist a finite set of points  $X = \{x_1, \dots, x_m\} \subset \mathbb{R}^d$  and a function  $\eta : X \rightarrow (\mathbb{Z}^n, \|\cdot\|_\alpha)$  such that

- (iii) for every  $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{R}^n)$  we can write

$$\partial Z(\psi) = \sum_{k=1}^m \psi(x_k) \eta(x_k). \tag{2.2}$$

<sup>4</sup>As usual,  $A_1(\mathbb{R}^d)$  and  $A^1(\mathbb{R}^d)$  denote respectively the vector spaces of one-dimensional vectors and covectors in  $\mathbb{R}^d$ .

<sup>5</sup>In the following, we write  $\langle \omega; \tau, h \rangle$  for  $\omega(\tau, h)$ .

Under these conditions, we will often denote  $Z = \llbracket \Sigma, \tau, \theta \rrbracket$ ,  $\partial Z = \sum_{k=1}^m \eta(x_k) \delta_{x_k}$  and  $\mathbb{M}^{(n,\alpha)}(\partial Z) := \sum_{k=1}^m \|\eta(x_k)\|_\alpha$ . Moreover, we write  $Z = (Z_1, \dots, Z_n)$ , and we say that  $Z_j$  are the *components* of  $Z$ , if  $Z_j = \llbracket \Sigma_j, \tau_j, \theta_j \rrbracket$  ( $j = 1, \dots, n$ ) are classical<sup>6</sup> integral currents satisfying

$$Z(\omega) = \sum_{j=1}^n \int_{\Sigma_j} \langle \omega_j(x); \tau_j(x) \rangle \theta_j(x) \, d\mathcal{H}^1(x),$$

for every  $\omega \in \mathcal{C}_c^\infty(\mathbb{R}^d; \Lambda_{(n,\alpha)}^1(\mathbb{R}^d))$ .

The appropriate notion of weak\*-topology for the set of rectifiable currents with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  is dual to the suitable locally convex topology on the space  $\mathcal{C}_c^\infty(\mathbb{R}^d; \Lambda_{(n,\alpha)}^1(\mathbb{R}^d))$ , built in analogy with the topology on  $\mathcal{C}_c^\infty(\mathbb{R}^d)$  with respect to which distributions are dual. We write  $Z_i \xrightarrow{*} Z$  when the sequence  $(Z_i)_{i \geq 1}$  is weakly\*-converging to  $Z$ .

Let us remark that these definitions are consistent with the classical theory of currents and currents with coefficients in a (normed, abelian, discrete) group. In particular, rectifiable currents with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  belong to the larger linear space of currents with coefficients in  $(\mathbb{R}^n, \|\cdot\|_\alpha)$  and the mass of a current  $Z$  is the supremum of  $|Z(\omega)|$  when  $\omega \in \mathcal{C}_c^\infty(\mathbb{R}^d; \Lambda_{(n,\alpha)}^1(\mathbb{R}^d))$  has norm

$$\sup_{x \in \mathbb{R}^d} \|\omega(x)\|_\alpha^* \leq 1.$$

Despite the complexity of the notation, this framework is not redundant for the objects we would like to represent and analyze. Let us clarify, firstly, how we endow a curve  $\gamma$  with a canonical structure of current in the following example.

**Example 2.4.** We associate to a Lipschitz path  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$  and a coefficient  $\theta \in \mathbb{R}^n$ , the 1-dimensional rectifiable current  $Z = \llbracket \Gamma, \tau, \theta \rrbracket$  with coefficients in  $\mathbb{R}^n$ , where  $\Gamma$  is the support of the curve  $\gamma([0, 1])$  and the orientation  $\tau$  is defined by  $\tau(\gamma(t)) := \gamma'(t)/|\gamma'(t)|$  for a.e.  $t \in [0, 1]$ . The boundary of such current is  $\partial T = \theta(\delta_{\gamma(1)} - \delta_{\gamma(0)})$ .

The next theorem provides a reasonable structure for a rectifiable current with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  as a countable sum of loops plus finitely many Lipschitz curves in  $\mathbb{R}^d$  with constant multiplicity.

**Theorem 2.5.** *Let  $Z$  be a rectifiable current in  $\mathbb{R}^d$  with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ , then*

$$Z = \sum_{i=1}^m \tilde{Z}_i + \sum_{\ell=1}^\infty \hat{Z}_\ell.$$

Here  $\tilde{Z}_i = \llbracket \tilde{\Gamma}_i, \tilde{\tau}_i, \tilde{\theta}_i \rrbracket$  are rectifiable currents with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ ,  $\tilde{\Gamma}_i$  being the support of an injective Lipschitz curve and  $\tilde{\theta}_i$  a constant on  $\tilde{\Gamma}_i$ . Similarly  $\hat{Z}_\ell = \llbracket \hat{\Gamma}_\ell, \hat{\tau}_\ell, \hat{\theta}_\ell \rrbracket$  are rectifiable currents with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ ,  $\hat{\Gamma}_\ell$  being the support of a Lipschitz closed curve  $\gamma_\ell : [0, 1] \rightarrow \mathbb{R}^d$ , which is injective on  $(0, 1)$ . Again,  $\hat{\theta}_\ell$  is constant on  $\hat{\Gamma}_\ell$ .

For the proof of this theorem and a more detailed statement see 2.3 in [11] (the proof revisits the argument employed for classical integral currents in 4.2.25 of [14]).

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<sup>6</sup>Throughout the paper we refer to “classical” currents to indicate currents with coefficients in  $\mathbb{R}$  or  $\mathbb{Z}$  (that is,  $n = 1$ ). An *integral* 1-current is a classical rectifiable current with integer coefficients. We refer to [14, 27] or [16] for the classical theory of currents.

### 2.3. Compactness

We recall the fundamental compactness result for rectifiable currents with coefficients in a group.

**Theorem 2.6.** *Consider a sequence  $(Z_i)_{i \geq 1}$  of rectifiable currents with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  such that*

$$\sup_{i \geq 1} \left\{ \mathbf{M}^{(n,\alpha)}(Z_i) + \mathbf{M}^{(n,\alpha)}(\partial Z_i) \right\} < +\infty.$$

*Then there exists a rectifiable current  $Z$ , with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ , and a subsequence  $(Z_{i_h})_{h \geq 1}$  such that*

$$Z_{i_h} \xrightarrow{*} Z.$$

The proof of this theorem can be carried out componentwise, using the analogous celebrated result for classical integral currents (see [14] or [1]). Since the mass is lower semicontinuous with respect to the same topology, we directly get the existence of a mass-minimizing rectifiable current for a given boundary.

**Corollary 2.7.** *Let  $\hat{Z}$  be a rectifiable current with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ , then there exists a rectifiable current  $\bar{Z}$  such that*

$$\mathbf{M}^{(n,\alpha)}(\bar{Z}) = \min_{\partial Z = \partial \hat{Z}} \mathbf{M}^{(n,\alpha)}(Z),$$

*where the minimum is computed among rectifiable currents with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ .*

### 2.4. Calibrations

The main reason why we want to treat the transportation problem as a mass minimization problem for rectifiable currents with coefficients in a group is the availability of the calibration technique as a tool to prove optimality.

**Definition 2.8.** Consider a rectifiable current  $Z = \llbracket \Sigma, \tau, \theta \rrbracket$  with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ . A smooth  $\mathbb{R}^n$ -valued differential form  $\omega : \mathbb{R}^d \rightarrow \Lambda^1_{(n,\alpha)}(\mathbb{R}^d)$  is a *calibration* for  $Z$  if the following conditions hold:

- (i) for every  $x \in \Sigma$  we have that  $\langle \omega(x); \tau(x), \theta(x) \rangle = \|\theta(x)\|_\alpha$ ;
- (ii) the form is closed<sup>7</sup>, that is,  $d\omega = 0$ ;
- (iii) for every  $x \in \mathbb{R}^d$ , every unit vector  $\tau \in \Lambda_1(\mathbb{R}^d)$  and every  $h \in \mathbb{R}^n$  we have that

$$\langle \omega(x); \tau, h \rangle \leq \|h\|_\alpha.$$

**Theorem 2.9.** *Let  $Z = \llbracket \Sigma, \tau, \theta \rrbracket$  be a calibrated rectifiable current with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ , then  $Z$  minimizes the mass among the rectifiable currents with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  with boundary  $\partial Z$ .*

*Proof.* Let  $\omega$  be a calibration for  $Z$ . Consider a competitor  $Z' = \llbracket \Sigma', \tau', \theta' \rrbracket$  with boundary  $\partial Z' = \partial Z$ , then, as a consequence of Stokes' Theorem and Definition 2.8, we have that

$$\begin{aligned} \mathbf{M}^{(n,\alpha)}(Z) &= \int_{\Sigma} \|\theta(x)\|_\alpha \, d\mathcal{H}^1(x) \\ &\stackrel{(i)}{=} \int_{\Sigma} \langle \omega(x); \tau(x), \theta(x) \rangle \, d\mathcal{H}^1(x) \\ &\stackrel{(ii)}{=} \int_{\Sigma'} \langle \omega(x); \tau'(x), \theta'(x) \rangle \, d\mathcal{H}^1(x) \\ &\stackrel{(iii)}{\leq} \int_{\Sigma'} \|\theta'(x)\|_\alpha \, d\mathcal{H}^1(x) = \mathbf{M}^{(n,\alpha)}(Z'), \end{aligned}$$

where each equality (resp. inequality) is motivated by the corresponding relation in Definition 2.8. □

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<sup>7</sup>For the sake of readability, in Definition 2.2 we did not define what the differential of a 1-dimensional  $\mathbb{R}^n$ -valued form is. On the one hand, all the examples we will provide have a constant calibration, so closedness is trivial. On the other hand, it is not difficult to extend the definition of differential of a classical  $k$ -form to  $\mathbb{R}^n$ -valued  $k$ -forms, componentwise.

**Remark 2.10.** In Section 5, we need to allow also the possibility that the competitor  $Z'$  is a rectifiable current with coefficients in  $(\mathbb{R}^n, \|\cdot\|_\alpha)$ . The existence of a calibration for  $Z$  guarantees the optimality of the mass of  $Z$  even in this larger class. The proof of this fact is the same as that given above.

### 3. THE GILBERT–STEINER PROBLEM

#### 3.1. The energy minimization problem

Let  $X = (x_1, \dots, x_m)$  and  $Y = (y_1, \dots, y_m) \in (\mathbb{R}^d)^m$ , with  $x_i \neq y_j$  for every  $i, j = 1, \dots, m$ . Denote by  $B_{X,Y}^{(1,1)}$  the integral 0-current

$$B_{X,Y}^{(1,1)} := \sum_{i=1}^m (\delta_{y_i} - \delta_{x_i}). \tag{3.1}$$

Consider the following problem.

(P1) Among all (classical) integral 1-currents  $T = \llbracket \Sigma, \tau, \theta \rrbracket$  in  $\mathbb{R}^d$  with  $\partial T = B_{X,Y}^{(1,1)}$ , find one which minimizes the *Gilbert–Steiner energy*

$$\mathbb{E}^\alpha(T) = \int_\Sigma |\theta(x)|^\alpha d\mathcal{H}^1(x).$$

The problem (P1) is better known as the *Gilbert–Steiner problem*<sup>8</sup>, that is, a transportation problem with sources in  $x_1, \dots, x_m$  and wells in  $y_1, \dots, y_m$ . The cost of the transportation network is precisely the energy  $\mathbb{E}^\alpha$ . Note that it is possible that  $x_i = x_j$  (respectively:  $y_i = y_j$ ) for some  $i \neq j$ : indeed a point with higher multiplicity represents a source generating (respectively: a well absorbing) a more intense flow.

**Remark 3.1.** The existence of a solution for (P1) is again a consequence of the direct method. Which we outline in the following steps.

- (i) The class of competitors is non-empty, in fact the current  $B_{X,Y}^{(1,1)}$  is the boundary of a current with finite energy. Take, for instance, the integral 1-current  $\hat{T} = \sum_{i=1}^m \llbracket \sigma_i, \tau_i, 1 \rrbracket$ , where  $\sigma_i$  is the segment between  $x_i$  and  $y_i$  and  $\tau_i = (y_i - x_i)/|y_i - x_i|$ , then it is trivial to check that  $\partial \hat{T} = B_{X,Y}^{(1,1)}$  and  $\mathbb{E}^\alpha(\hat{T}) < +\infty$ .
- (ii) The set of integral currents in  $\mathbb{R}^d$  with boundary  $B_{X,Y}^{(1,1)}$  and energy bounded by a constant is contained in a weak\* compact set.
- (iii) The energy  $\mathbb{E}^\alpha$  is lower semicontinuous with respect to the weak\* topology.

#### 3.2. The structure of integral 1-currents

Let us recall the analogue of Theorem 2.5 in the simpler setting of 1-dimensional integral currents in  $\mathbb{R}^d$ , with the double aim of an easier picture and a better understanding of the Gilbert–Steiner problem.

**Theorem 3.2.** *Let  $T$  be an integral 1-current in  $\mathbb{R}^d$ , with<sup>9</sup>  $\mathbb{M}(\partial T) = 2m$ . Then*

$$T = \sum_{i=1}^m \tilde{T}_i + \sum_{\ell=1}^\infty \hat{T}_\ell, \tag{3.2}$$

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<sup>8</sup>The actual version of the Gilbert–Steiner problem (as described *e.g.* in Sect. 2.2 of [4]) allows as competitors also polyhedral currents with real multiplicities. For the purposes of this paper, such issue is not fundamental, indeed the method we use to prove that a certain object is a solution of problem (P1) is strong enough to prove also the optimality among polyhedral chains with real multiplicities (see Rem. 2.10). Moreover it is not difficult to prove that the energy of a minimizer of (P1) coincides with the energy of a minimizer among polyhedral currents.

<sup>9</sup>We will denote by  $\mathbb{M}$  the mass of a classical currents.



where  $\tilde{T}_i$  are integral 1-currents associated to injective Lipschitz paths, for every  $i = 1, \dots, m$  and  $\mathring{T}_\ell$  are integral 1-currents associated to Lipschitz paths  $\gamma_\ell : [0, 1] \rightarrow \mathbb{R}^d$  satisfying  $\gamma_\ell(0) = \gamma_\ell(1)$  and injective on  $(0, 1)$ , for every  $\ell \geq 1$ . In particular  $\partial \mathring{T}_\ell = 0$  for every  $\ell \geq 1$ . Moreover

$$\mathbb{M}(T) = \sum_{i=1}^m \mathbb{M}(\tilde{T}_i) + \sum_{\ell=1}^{\infty} \mathbb{M}(\mathring{T}_\ell). \tag{3.3}$$

**Remark 3.3.** Since  $\mathbb{M}(\partial T) = 2m$  we can represent  $\partial T = \sum_{i=1}^m (\delta_{y_i} - \delta_{x_i})$ , where  $(x_1, \dots, x_m)$  and  $(y_1, \dots, y_m) \in (\mathbb{R}^d)^m$ , satisfy  $x_i \neq y_j$  for every  $i, j = 1, \dots, m$ . Without loss of generality, we may assume that there exists a permutation  $\sigma \in \mathcal{S}_m$  such that in the decomposition (3.2) we have  $\partial \tilde{T}_i = \delta_{y_i} - \delta_{x_{\sigma(i)}}$ .

**Remark 3.4.** In Theorem 3.2, if we represent  $\tilde{T}_i = \llbracket \tilde{I}_i, \tilde{\tau}_i, \tilde{\theta}_i \rrbracket$ , we may assume that the multiplicity  $\tilde{\theta}_i$  is always non negative, hence if  $\mathcal{H}^1(\tilde{I}_i \cap \tilde{I}_j) > 0$ , then by (3.3)  $\tilde{\tau}_i = \tilde{\tau}_j$   $\mathcal{H}^1$ -a.e. on  $\tilde{I}_i \cap \tilde{I}_j$ . This implies that the multiplicity of the current  $\tilde{T}_1 + \dots + \tilde{T}_m$  is  $\theta_1 + \dots + \theta_m$ .

#### 4. GILBERT–STEINER PROBLEM AND CURRENTS WITH COEFFICIENTS IN $\mathbb{R}^n$

##### 4.1. The mass minimization problem

In this section we show that the Gilbert–Steiner problem (P1) can be rephrased as a mass minimization problem for rectifiable currents with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ , where  $n$  is the number of sources (resp. wells), counted with multiplicity, in the boundary datum  $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in (\mathbb{R}^d)^n$ .

(P2) Among all permutations  $\sigma \in \mathcal{S}_n$  and among all rectifiable currents  $Z = \llbracket \Sigma, \tau, \theta \rrbracket$  in  $\mathbb{R}^d$  with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  and boundary

$$B_{\sigma(X), Y}^{(n, \alpha)} = \sum_{j=1}^n g_j (\delta_{y_j} - \delta_{\sigma(x_j)}),$$

find one which minimizes the mass

$$\mathbb{M}^{(n, \alpha)}(Z) = \int_{\Sigma} \|\theta(x)\|_\alpha d\mathcal{H}^1(x).$$

It will be clear from Definition 4.1 that fixing the permutation  $\sigma$  in (P2) corresponds somehow to prescribe which source goes into which well (see also the description of the *who goes where* problem given in [2]). This is not prescribed, instead, when we fix the boundary datum in (P1). This is the reason why in (P2) we need to minimize also among all permutations  $\sigma \in \mathcal{S}_n$ . Of course we may drop this condition when the set of sources contains only one point with multiplicity  $n$ , as in the case of the irrigation problem.

##### 4.2. The equivalence of (P1) and (P2)

In the next theorem we state the equivalence between problems (P1) and (P2). Let us first describe a canonical way to associate to a competitor for the problem (P1) a competitor for the problem (P2) and *vice versa*.

**Definition 4.1.** Let  $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in (\mathbb{R}^d)^n$ , with  $x_i \neq y_j$  for every  $i, j = 1, \dots, n$ .

(1) Given an integral current  $T$  in  $\mathbb{R}^d$  with boundary  $\partial T = B_{X, Y}^{(1, 1)}$  as in (3.1), consider a decomposition according to Theorem 3.2 (indeed,  $\mathbb{M}(\partial T) = 2n$ ), satisfying the assumption of Remark 3.3, that is

$$T = \sum_{i=1}^n \tilde{T}_i + \sum_{\ell=1}^{\infty} \mathring{T}_\ell$$



and there exists  $\sigma \in \mathcal{S}_n$  such that

$$\partial \tilde{T}_i = \delta_{y_i} - \delta_{x_{\sigma(i)}}.$$

We set

$$Z[T] := (\tilde{T}_1, \dots, \tilde{T}_n),$$

(according to the basis  $(g_1, \dots, g_n)$ ) which is a rectifiable current with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  and boundary

$$\partial(Z[T]) = B_{\sigma(X), Y}^{(n, \alpha)}.$$

Notice that  $Z[T]$  depends on the decomposition chosen for  $T$ , therefore it may be non-unique.

- (2) *Vice versa*, given a rectifiable current  $Z$  with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ , we can write it componentwise as  $Z = (Z_1, \dots, Z_n)$  and we set

$$T[Z] := Z_1 + \dots + Z_n,$$

which is an integral current with boundary  $B_{X, Y}^{(1, 1)}$ .

**Remark 4.2.** Consider a rectifiable current  $Z = \llbracket \Sigma, \tau, \theta \rrbracket$  with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$ . Then

$$\mathbb{E}^\alpha(T[Z]) \leq \mathbb{M}^{n, \alpha}(Z).$$

Indeed, the integral current  $T[Z]$  has multiplicity  $\theta_1 + \dots + \theta_n$ , where  $\theta = \theta_1 g_1 + \dots + \theta_n g_n$  is the multiplicity of  $Z$ . Moreover,

$$\left| \sum_{j=1}^n \theta_j \right|^\alpha \leq \left( \sum_{j=1}^n |\theta_j|^{\frac{1}{\alpha}} \right)^\alpha = \|\theta\|_\alpha, \tag{4.1}$$

because  $\theta_j \in \mathbb{Z}$  for any  $j = 1, \dots, n$  and  $0 < \alpha < 1$ . Notice that the equality holds in (4.1) if and only if  $\theta_j \in \{0, 1\}$  for every  $j = 1, \dots, n$ .

**Proposition 4.3.** Consider an integral current  $T$  in  $\mathbb{R}^d$  with  $\mathbb{M}(\partial T) = 2n$  and a decomposition of type (3.2) as in Theorem 3.2. Then,

$$\mathbb{M}^{n, \alpha}(Z[T]) = \mathbb{E}^\alpha \left( \sum_{i=1}^n \tilde{T}_i \right).$$

*Proof.* Let us denote by  $\theta = \tilde{\theta}_1 g_1 + \dots + \tilde{\theta}_n g_n$  the multiplicity of  $Z[T]$ . By definition of  $Z[T]$ , each  $\tilde{\theta}_j$  is the multiplicity of  $\tilde{T}_j = \llbracket \tilde{T}_j, \tilde{\tau}_j, \tilde{\theta}_j \rrbracket$ . Since  $\mathbb{M}(\partial T) = 2n$ , then  $\tilde{\theta}_j \in \{0, 1\}$  almost everywhere on  $\bigcup_j \tilde{T}_j$  (in particular,  $\tilde{\theta}_j = 1$  a.e. on  $\tilde{T}_j$ ). Thus, we can infer that

$$\|\theta\|_\alpha = |\tilde{\theta}_1 + \dots + \tilde{\theta}_n|^\alpha$$

almost everywhere on  $\bigcup_j \tilde{T}_j$ . The conclusion follows, because the multiplicity of  $\sum_{j=1}^n \tilde{T}_j$  is precisely  $\tilde{\theta}_1 + \dots + \tilde{\theta}_n$  (see Rem. 3.4).  $\square$

**Theorem 4.4.** Let  $X$  and  $Y$  be as in Definition 4.1. Then problems (P1) and (P2) are equivalent. More precisely, the following results hold.

- (1) If  $T$  is an energy minimizer for the Gilbert–Steiner problem (P1) with boundary  $B_{X, Y}^{(1, 1)} = \sum_{j=1}^n (\delta_{y_j} - \delta_{x_j})$ , then  $Z[T]$  is a mass minimizer for the problem (P2). Moreover  $\mathbb{E}^\alpha(T) = \mathbb{M}^{n, \alpha}(Z[T])$ .
- (2) If  $Z$  is a solution for problem (P2) with the datum  $(X, Y)$ , then  $T[Z]$  minimizes the energy  $\mathbb{E}^\alpha$  among all integral currents with boundary  $B_{X, Y}^{(1, 1)}$ . Moreover  $\mathbb{M}^{n, \alpha}(Z) = \mathbb{E}^\alpha(T[Z])$ .

*Proof.*

- (1) Let  $T$  be a solution of the Gilbert–Steiner problem (P1). Since we want to show that  $Z[T]$  is a solution for the problem (P2), then we compare its mass with the mass of an admissible competitor for (P2), that is, a rectifiable current  $Z'$  with coefficients in  $(\mathbb{Z}^n, \|\cdot\|_\alpha)$  and boundary  $B_{\sigma'(X),Y}^{(n,\alpha)}$  for some  $\sigma' \in \mathcal{S}_n$ . Then, we have that

$$\mathbb{M}^{n,\alpha}(Z') \stackrel{4.2}{\geq} \mathbb{E}^\alpha(T[Z']) \geq \mathbb{E}^\alpha(T) \stackrel{(3.3)}{\geq} \mathbb{E}^\alpha\left(\sum_{i=1}^n \tilde{T}_i\right) \stackrel{4.3}{=} \mathbb{M}^{n,\alpha}(Z[T]).$$

Let us explain that the first inequality is due to Remark 4.2, the second one holds because  $T$  is a minimizer for the energy  $\mathbb{E}^\alpha$  with boundary  $B_{X,Y}^{(1,1)}$  and the third one holds because of (3.3). Finally, the last equality has already been proved in Proposition 4.3. Notice also that the third inequality must be in fact an equality, because  $\sum_{i=1}^n \tilde{T}_i$  is a competitor of  $T$  in (P1).

- (2) Let  $Z$  be a solution of the problem (P2). We claim that  $T[Z]$  is a solution of (P1): indeed, if  $T'$  is an integral current with boundary  $B_{X,Y}^{(1,1)}$ , then we have that

$$\mathbb{E}^\alpha(T') \stackrel{(3.3)}{\geq} \mathbb{E}^\alpha\left(\sum_{i=1}^n \tilde{T}'_i\right) \stackrel{4.3}{=} \mathbb{M}^{n,\alpha}(Z[T']) \geq \mathbb{M}^{n,\alpha}(Z) \stackrel{4.2}{\geq} \mathbb{E}^\alpha(T[Z]).$$

As before, let us remind that the first inequality is a consequence of (3.3) and the equality is due to Proposition 4.3. Finally, the last inequality relies on Remark 4.2. This last is an equality, indeed. This follows from the fact that the multiplicity of every component  $Z_j = \llbracket \Sigma_j, \tau_j, \theta_j \rrbracket$  of  $Z$  is 1 ( $\mathcal{H}^1$ -a.e. on  $\Sigma_j$ ) and the inequality in (4.1) is an equality when every  $\theta_j$  is either 0 or 1.  $\square$

### 4.3. Examples of calibrated minima

We now show some examples of calibrations for the problem (P2) with the only aim to make the reader confident with this technique and to throw light on some issues that one may not notice immediately in the general formulation. In the first example there are two sources and two wells with unit multiplicity on the vertices of a rectangle in  $\mathbb{R}^2$ . The second example is an irrigation problem with only two wells, but where we allow also higher multiplicities.

**Example 4.5.** Let  $\alpha = 1/2$  and consider the following points in the plane

$$x_1 := (-3, -2), \quad x_2 := (-2, -3), \quad y_1 := (1, 0), \quad y_2 := (0, 1), \quad z := (-2, -2).$$

Consider the 0-dimensional current in  $\mathbb{R}^2$  with coefficients in  $\mathbb{Z}^2$  given by

$$B_{(x_1,x_2),(y_1,y_2)}^{(2,1/2)} = g_1(\delta_{y_1} - \delta_{x_1}) + g_2(\delta_{y_2} - \delta_{x_2}).$$

We claim that a solution of problem (P2) with boundary  $B_{(x_1,x_2),(y_1,y_2)}^{(2,1/2)}$  is

$$Z = \llbracket \sigma_1, \tau_1, g_1 \rrbracket + \llbracket \sigma_2, \tau_2, g_2 \rrbracket + \llbracket \sigma_3, \tau_3, g_1 + g_2 \rrbracket + \llbracket \sigma_4, \tau_1, g_1 \rrbracket + \llbracket \sigma_5, \tau_2, g_2 \rrbracket$$

where  $\sigma_1$  is the segment from  $x_1$  to  $z$  with orientation  $\tau_1 = (1, 0)$ ,  $\sigma_2$  is the segment from  $x_2$  to  $z$  with orientation  $\tau_2 = (0, 1)$ ,  $\sigma_3$  is the segment from  $z$  to 0 with orientation  $\tau_3 = (\sqrt{2}/2, \sqrt{2}/2)$ ,  $\sigma_4$  is the segment from 0 to  $y_1$  and  $\sigma_5$  is the segment from 0 to  $y_2$  (see Fig. 2).

Firstly, we exhibit a constant<sup>10</sup> calibration  $\omega(x) \equiv \omega \in A^1_{(2,1/2)}(\mathbb{R}^2)$ , which proves that  $Z$  is a mass minimizer for the boundary  $B_{(x_1,x_2),(y_1,y_2)}^{(2,1/2)}$ . Then, we will prove that the achieved minimum (that is, the value  $\mathbb{M}^{(2,1/2)}(Z)$ )

<sup>10</sup>Since  $\omega$  is constant, condition (ii) in Definition 2.8 is automatically satisfied.

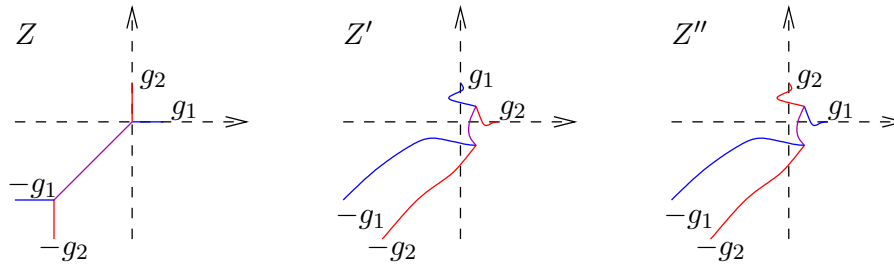


FIGURE 2. Example 4.5.

is less or equal then the minimum for the boundary  $B_{(x_1, x_2), (y_2, y_1)}^{(2, 1/2)}$  (remember that in (P2) we require to minimize among all possible permutations).

Let us remind that an  $\mathbb{R}^2$ -valued covector acts bilinearly on the pair  $(\tau, g) \in \Lambda_1(\mathbb{R}^2) \times \mathbb{Z}^2$  and we can represent it as a  $2 \times 2$  matrix with real coefficients. We let

$$\omega(x) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we can easily check that  $\langle \omega; \tau_i, g_i \rangle = 1$  for  $i = 1, 2$ , while

$$\langle \omega; \tau_3, g_1 + g_2 \rangle = \sqrt{2} = \|g_1 + g_2\|_\alpha,$$

so condition (i) in Definition 2.8 is fulfilled. Condition (iii), concerning the norm of  $\omega$ , is trivially verified (for  $\alpha = 1/2$  it is sufficient to check that the Euclidean norm of the matrix is less or equal then 1).

To prove that the minimum  $\mathbb{M}^{(2, 1/2)}(Z)$  is less or equal then the minimum for the boundary  $B_{(x_1, x_2), (y_2, y_1)}^{(2, 1/2)}$ , we fix a competitor  $Z'$  for this second problem. Without loss of generality, we can consider  $Z'$  as the sum of two curves, one with multiplicity  $g_1$  and one with multiplicity  $g_2$ . There are two cases: either the two curves in  $Z'$  do not intersect, in this case

$$\mathbb{M}^{(2, 1/2)}(Z') \geq |x_1 - y_2| + |x_2 - y_1| = 6\sqrt{2} > 8 = \mathbb{M}^{(2, 1/2)}(Z), \tag{4.2}$$

or the curves intersect at some point  $z'$ . In this case, it is sufficient to “switch” the multiplicity  $g_1$  with  $g_2$  from that point to the part of the boundary supported on  $\{y_1, y_2\}$ .

In this way, we created a competitor  $Z''$  for the problem with boundary  $B_{(x_1, x_2), (y_1, y_2)}^{(2, 1/2)}$  with the property that

$$\mathbb{M}^{(2, 1/2)}(Z') = \mathbb{M}^{(2, 1/2)}(Z'').$$

Therefore we may conclude

$$\mathbb{M}^{(2, 1/2)}(Z') = \mathbb{M}^{(2, 1/2)}(Z'') \geq \mathbb{M}^{(2, 1/2)}(Z).$$

**Example 4.6.** Fix  $\alpha = 1/2$  and  $\theta := \arccos(1/\sqrt{3}) \in (0, \pi/2)$ . Fix  $x, y_1, y_2 \in \mathbb{R}^2$  given by  $x = (-\cos\theta, -\sin\theta)$ ,  $y_1 = (1, 0)$  and  $y_2 = (0, 1)$ . Denote

$$X = (x, x, x) \in (\mathbb{R}^2)^3$$

and

$$Y = (y_1, y_2, y_2) \in (\mathbb{R}^2)^3.$$

We claim that a mass-minimizing current with coefficients in  $(\mathbb{Z}^2, \|\cdot\|_{1/2})$  among those with boundary  $B_{X, Y}^{(2, 1/2)}$  is

$$T = [\hat{\sigma}_0, \hat{\tau}_0, g_1 + g_2 + g_3] + [\hat{\sigma}_1, \hat{\tau}_1, g_1] + [\hat{\sigma}_2, \hat{\tau}_2, g_2 + g_3],$$

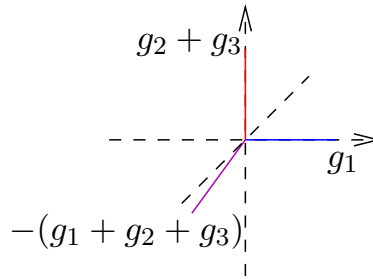


FIGURE 3. Example 4.6.

where  $\hat{\sigma}_0$  is the segment from  $x$  to 0 and  $\hat{\sigma}_i$  is the segment from 0 to  $y_i$  for  $i = 1, 2$ , as one can see in Figure 3. The orientation  $\hat{\tau}_i$  on each segment goes from the first to the second extreme point.

The constant  $\mathbb{R}^3$ -valued form

$$\hat{\omega}(x) \equiv \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 \end{pmatrix}$$

is a calibration for  $T$ . Indeed

$$\langle \hat{\omega}; \hat{\tau}_0, g_1 + g_2 + g_3 \rangle = \sqrt{3} = \|g_1 + g_2 + g_3\|_{1/2};$$

$$\langle \hat{\omega}; \hat{\tau}_1, g_1 \rangle = 1 = \|g_1\|_{1/2};$$

$$\langle \hat{\omega}; \hat{\tau}_2, g_2 + g_3 \rangle = \sqrt{2} = \|g_2 + g_3\|_{1/2},$$

therefore condition (i) is satisfied. Moreover  $\|\langle \hat{\omega}; \tau, \cdot \rangle\|_{1/2} = 1$  for every  $\tau \in \mathbb{R}^2$ , hence (iii) is also satisfied. As before, condition (ii) is trivially verified because  $\hat{\omega}$  is constant.

#### 4.4. Rescaled problems

In Section 5 we want to deal with a boundary made of infinitely many points. To this aim it is worthwhile mentioning the approach by Xia in [31], where the continuous version of the Gilbert–Steiner problem (*i.e.*  $\mu_-$  and  $\mu_+$  are not necessarily atomic measures) is defined through the relaxation of the Gilbert–Steiner energy. Given a normal 1-dimensional current  $T$  one considers all possible approximating sequences  $T_n$  of polyedral currents with real coefficients (where the approximation is meant with respect to the flat norm), then one defines the energy of  $T$  as

$$\mathbb{E}^\alpha(T) := \inf_{T_n} \left\{ \liminf_n \mathbb{E}^\alpha(T_n) \right\}.$$

As shown in Theorem 2.7 of [32], only rectifiable currents enter in the competition, since non-rectifiable normal currents have infinite energy.

As a preliminary step to prove the optimality of our irrigation network in Section 5, it is convenient to introduce a renormalized version of the problem (P1), in which the boundary is rescaled to have mass equal to 2. More precisely, consider the following problem, where  $B_{X,Y}^{(1,1)}$  and  $n$  are as in Section 3.1.

(P1') Among all rectifiable 1-currents  $T' = \llbracket \Sigma, \tau, \theta \rrbracket$  in  $\mathbb{R}^d$  with real coefficients, with  $T' = n^{-1}T$  and  $T$  is an integral current with  $\partial T = B_{X,Y}^{(1,1)}$ , find one which minimizes the extended Gilbert–Steiner energy

$$\mathbb{E}^\alpha(T') = \int_\Sigma \|\theta\|^\alpha d\mathcal{H}^1.$$

There is a clear bijection between the competitors of problems (P1) and (P1'). The feature we want to underline is how the energy rescales. Indeed

$$\mathbb{E}^\alpha(n^{-1}T) = n^{-\alpha}\mathbb{E}^\alpha(T), \tag{4.3}$$

for every  $T$  which is a competitor for (P1).

The quantity we minimize in the problem (P2), instead, has naturally a linear rescaling, therefore if we want to maintain the equality between the minima, we have to rescale the problem (P2) differently. It turns out that the correct rescaled version of the problem is the following.

(P2') Among all permutations  $\sigma \in \mathcal{S}_n$  and among all the rectifiable currents  $Z' = \llbracket \Sigma, \tau, \theta \rrbracket$  in  $\mathbb{R}^d$  with coefficients in  $n^{-\alpha}\mathbb{Z}^n$  and with boundary

$$\partial Z' = n^{-\alpha}B_{\sigma(X),Y}^{(n,\alpha)} = n^{-\alpha} \sum_{j=1}^n g_j(\delta_{y_j} - \delta_{x_{\sigma(j)}}),$$

find one which minimizes the mass

$$\mathbb{M}^{n,\alpha}(Z') = \int_{\Sigma} \|\theta\|_{\alpha} \, d\mathcal{H}^1.$$

Note that, if  $Z$  is a competitor for (P2), then  $n^{-\alpha}Z$  is a competitor for (P2') and naturally

$$\mathbb{M}^{n,\alpha}(n^{-\alpha}Z) = n^{-\alpha}\mathbb{M}^{n,\alpha}(Z). \tag{4.4}$$

The canonical way to associate to a competitor for the problem (P1') a competitor for the problem (P2') and *vice versa* is a straightforward modification of the method explained in Definition 4.1.

(1) To a competitor  $T'$  for (P1') we associate the current

$$Z'[T'] := n^{-\alpha}Z[nT'] \tag{4.5}$$

(note that  $nT$  is a competitor for (P1)).

(2) To a competitor  $Z'$  for (P2') we associate the current

$$T'[Z'] := n^{-1}T[n^{\alpha}Z'].$$

It is easy to observe that with this rescaling the analog of Remark 4.2 and of Proposition 4.3 still hold, therefore we have the following corollary.

**Corollary 4.7.** *Problems (P1') and (P2') are equivalent in the sense of Theorem 4.4.*

*Proof.* By construction, the solutions of the problem (P1') are in bijection with the solutions of the problem (P1). Analogously the solutions of the problem (P2') are in bijection with the solutions of the problem (P2). Thus the equivalence follows from Theorem 4.4 and from (4.3) and (4.4).  $\square$

### 4.5. Convex problems and existence of calibrations

Even if we have found a description of the Gilbert–Steiner problem which consists in a minimization of a convex functional<sup>11</sup>, we cannot really claim that we are solving a “convex problem” in which case a powerful machinery for the computation of minima would be available. Indeed our class of competitors is a non-convex set. While for the classical formulation of the Gilbert–Steiner problem (see footnote 5) one can safely extend

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<sup>11</sup>Actually we did it only for the irrigation problem or the “who goes where” problem. Indeed the general version of the Gilbert–Steiner problem has been described as a superposition of  $n!$  convex problems (being  $n$  the number of transported unit Dirac masses), which is computationally hard to accept.

the minimization problem to a convex and compact class of objects (being the energy infinite for non-rectifiable normal currents), this is not the case for problem (P2). Moreover, in (P2) we cannot guarantee that there is no gap of mass between the minimizers among rectifiable currents with coefficients in  $\mathbb{Z}^n$  and the minimizers among normal currents with coefficients in  $\mathbb{R}^n$ . An interesting occurrence of this gap phenomenon in a non-Euclidean setting is given in [20]. We conjecture that in the Euclidean setting there is no such gap. The validity of this conjecture would imply that the Gilbert–Steiner problem can be viewed as a convex problem and moreover the existence of a (sort of) calibration would be a necessary and sufficient condition for the minimality (see Sect. 4 of [20] for more details).

## 5. A MINIMIZER WITH INFINITELY MANY BRANCHING POINTS

### 5.1. Currents in metric spaces

In this section we prove the optimality of a certain irrigation network having an infinite boundary datum, as mentioned in Section 4.4. The main tools to prove its optimality have already been introduced in Sections 2 and 4, more precisely in Theorem 2.9 (the existence of a calibration is a sufficient condition for minimality) and in Corollary 4.7 (the equivalence of problems (P1') and (P2')). Nevertheless, since the ambient space for this irrigation network is the separable Hilbert space<sup>12</sup>  $\ell^2$ , a short digression on currents in metric spaces is needed. In order to keep the discussion here as incisive as possible, we recall only those facts that are strictly unavoidable to define a certain irrigation problem and to prove the optimality of our configuration. The reader is referred to [1] for the theory of currents in a metric space and for all the relevant definitions. Throughout this section, we will assume  $\alpha = 1/2$ .

Let  $T$  be a rectifiable 1-current in  $\ell^2$  (see Sect. 4 in [1]) and let  $\|T\|$  be the associated *mass* (in the sense of Definition 2.6 of [1]). In particular,  $T$  is supported on a countably  $\mathcal{H}^1$ -rectifiable set  $\Sigma$  and  $\|T\| = \theta \mathcal{H}^1 \llcorner \Sigma$  for some non-negative  $\theta \in L^1(\Sigma, \mathbb{R})$ . We call the *extended Gilbert–Steiner energy*<sup>13</sup> of  $T$  the quantity

$$\mathbb{E}(T) := \int_{\Sigma} \sqrt{\theta(x)} \, d\mathcal{H}^1(x), \quad (5.1)$$

if the integral makes sense, or  $+\infty$  otherwise. One may notice immediately that the extended Gilbert–Steiner energy is still subadditive, *i.e.*,

$$\mathbb{E}(T_1 + T_2) \leq \mathbb{E}(T_1) + \mathbb{E}(T_2).$$

Since the projection on a finite dimensional space is going to be a fundamental simplification in the handling of our problem, we recall the following proposition, which just relies on the concavity of the square root. An analogous statement for the mass is given in equation (2.4) of [1].

**Proposition 5.1.** *Let  $\pi : \ell^2 \rightarrow V$  be the orthogonal projection of  $\ell^2$  onto a subspace  $V$ . Then, for every rectifiable 1-current  $T$  in  $\ell^2$ , there holds<sup>14</sup>  $\mathbb{E}(\pi_{\#}T) \leq \mathbb{E}(T)$ .*

### 5.2. Construction of the irrigation tree

Let  $(e_i)_{i \in \mathbb{N}}$  be the standard orthonormal basis of  $\ell^2$ . Roughly speaking, we are going to define a rectifiable 1-dimensional current  $T$  in  $\ell^2$  (with coefficients in  $\mathbb{Q}$ ) in the following way: we start from a “trunk”, which is the oriented segment from the origin  $O$  to  $e_1 \in \ell^2$ , with unit multiplicity, then, for every  $n \in \mathbb{N}$ , we are going to build a family of  $2^n$  “branches”, that is,  $2^n$  currents supported on segments connected to the main “tree”, with suitable length, orientation and multiplicity in  $\mathbb{Q}$ . The irrigation tree will be the sum, up to infinity, of these currents.

<sup>12</sup>We choose this setting instead of the more natural Euclidean space, because the possibility to have infinitely many mutually orthogonal directions allows us to find a very simple calibration for the problem we will introduce.

<sup>13</sup>We denote the extended Gilbert–Steiner energy by  $\mathbb{E}$ , omitting  $\alpha$ , because  $\alpha = 1/2$  throughout all the current section.

<sup>14</sup>For the definition of the pushforward  $F_{\#}T$  of a current  $T$  through a map  $F$ , see Definition 2.4 of [1].

To begin with, we deal with the orientations of the segments. If it is not obvious from the context, we always denote by  $\sigma(p, q)$  a segment in  $\ell^2$  from  $p$  to  $q$  and by  $x_{\sigma(p,q)}$  the (unit) orientation of  $\sigma(p, q)$  from  $p$  to  $q$ . We now establish the directions  $y(x), z(x)$  according to which we will assign the orientation of the segments of a new generation. For every  $x \in \ell^2$  of the form  $x = \sum_{i=1}^l a_i e_i$ , with  $a_l \neq 0$ , we put

$$y(x) := \frac{\sqrt{2}}{2} \sum_{i=1}^l a_i (e_i + e_{l+i})$$

and

$$z(x) := \frac{\sqrt{2}}{2} \sum_{i=1}^l a_i (e_i - e_{l+i}).$$

Notice that  $y(x)$  and  $z(x)$  are orthogonal and  $y(x) + z(x)$  is parallel to  $x$ . Moreover, given a set of mutually orthonormal directions  $\{x_1, \dots, x_m\}$ , then  $\{y(x_1), z(x_1), \dots, y(x_m), z(x_m)\}$  are mutually orthonormal, too<sup>15</sup>.

We consider the following iterative construction.

- We set

$$T_0 = \llbracket \sigma(O, e_1), x_0, 1 \rrbracket,$$

where  $\sigma(O, e_1)$  is the segment from the origin to  $e_1$  and  $x_0 = e_1$  is the outgoing orientation of  $\sigma(O, e_1)$ . Moreover, we define  $E_0 := \{\sigma(O, e_1)\}$ .

- For every line segment  $\sigma(p, q) \in E_{n-1}$  with orientation  $x = x_{\sigma(p,q)}$ , we put in the set  $E_n$  the pair of segments

$$\sigma(q, q + 4^{-n}y(x)) \quad \text{and} \quad \sigma(q, q + 4^{-n}z(x)).$$

Finally, we define

$$T_n := \sum_{i \leq n} \sum_{\sigma \in E_i} \llbracket \sigma, x_\sigma, 2^{-i} \rrbracket.$$

Notice that<sup>16</sup>  $\mathbb{M}(\partial T_n) = 2$  for every  $n \in \mathbb{N}$ .

If we call

$$E^n := \bigcup_{\sigma \in E_n} \sigma,$$

then the support of the irrigation tree is defined as the set

$$E := \bigcup_{n \in \mathbb{N}} E^n.$$

Notice that the segments are growing in orthogonal directions, hence  $\mathcal{H}^1$ -a.e.  $p \in E$  belongs to a unique segment  $\sigma(p) \in E_{n(p)}$  and we can define an orientation  $\tau : E \rightarrow \ell^2$  setting  $\tau(p) = x_{\sigma(p)}$ . Analogously, we can define a multiplicity  $m : E \rightarrow \mathbb{Q}$  setting  $m(p) = 2^{-n}$  for every  $p \in E^n$ . The irrigation tree  $T$  is defined as

$$T := \llbracket E, \tau, m \rrbracket.$$

The truncated currents  $T_n$  play a fundamental role in the proof of optimality of the irrigation tree  $T$ , because they are a good approximation for  $T$  (as we underline below in Rem. 5.2) and they are embedded in a finite dimensional space (indeed,  $\text{supp } T_n \subset \text{span}\{e_1, \dots, e_{2^n}\}$ ). Since the segments in  $E$  are essentially disjoint, we can also write  $T_n = T \llcorner \left( \bigcup_{i \leq n} E^i \right) := \llbracket \bigcup_{i \leq n} E^i, \tau, m \rrbracket$ .

<sup>15</sup>To check that they are mutually orthogonal, observe the interior products satisfy  $\langle y(x_i); y(x_j) \rangle = \langle z(x_i); z(x_j) \rangle = 2\langle x_i; x_j \rangle$  and  $\langle y(x_i); z(x_j) \rangle = \langle x_i; x_j \rangle - \langle x_i; x_j \rangle$ . This is due to the fact that the coordinates of  $y(x_i)$  are obtained by repeating twice the coordinates of  $x_i$  (analogously with  $z(x_i)$ , except for the change of sign).

<sup>16</sup>Here and in the following,  $\mathbb{M}(T)$  denotes the total mass of the mass measure  $\|T\|$ .



**Remark 5.2.** By construction, the following facts hold.

- (i)  $\mathbb{E}(T - T_n) = \sum_{j>n} 2^j 2^{-\frac{j}{2}} 4^{-j} = \sum_{j>n} 2^{-\frac{3}{2}j}$ .
- (ii)  $\mathbf{M}(T - T_n) = \sum_{j>n} 2^j 2^{-j} 4^{-j} = \sum_{j>n} 2^{-2j}$ .
- (iii) If  $\pi_n : \ell^2 \rightarrow \text{span}\{e_1, \dots, e_{2^n}\}$  are the orthogonal projections and if  $\{q_1, \dots, q_{2^n}\}$  are the second extreme points of the segments in  $E_n$ , then  $(\pi_n)_\#(T_n)$  is a candidate for the minimization problem (P1') in 4.4, where

$$X = (0, \dots, 0), \quad Y = (q_1, \dots, q_{2^n}).$$

Indeed  $2^n(\pi_n)_\#T_n$  is an integral current.

We are now ready to prove the optimality of  $T$ .

**Theorem 5.3.** *The rectifiable 1-current  $T$  in  $\ell^2$  minimizes the generalized Gilbert–Steiner energy (5.1) among all rectifiable currents in  $\ell^2$  with boundary  $\partial T$ .*

*Proof.* We split the proof into several steps.

**Step 1. General strategy.** Let  $S$  be an admissible competitor, *i.e.*,  $S$  is a rectifiable 1-current in  $\ell^2$  with  $\partial S = \partial T$ .

We want to reduce the problem to an energy minimization problem for rectifiable 1-currents in a finite dimensional space and then exploit the tools developed in the previous sections. For every  $n \in \mathbb{N}$ , let  $S_n := S - T + T_n$  and notice that  $\partial S_n = \partial T_n$ . By Proposition 5.1, we have that

$$\mathbb{E}((\pi_n)_\#S_n) \leq \mathbb{E}(S_n) \quad \forall n \in \mathbb{N}.$$

Fix  $\varepsilon > 0$ . Since, by subadditivity of  $\mathbb{E}$ , we have that

$$\mathbb{E}(S_n) \leq \mathbb{E}(S) + \mathbb{E}(T_n - T) \quad \forall n \in \mathbb{N}$$

and the energy  $\mathbb{E}(T_n - T)$  is going to 0 (see Rem. 5.2), then there exists  $n(\varepsilon) \in \mathbb{N}$  such that  $\mathbb{E}(T_{n(\varepsilon)} - T) \leq \varepsilon$ , thus  $\mathbb{E}(S) \geq \mathbb{E}(S_{n(\varepsilon)}) - \varepsilon$ . If we can prove that

$$\mathbb{E}(T_n) = \mathbb{E}((\pi_n)_\#T_n) \leq \mathbb{E}((\pi_n)_\#S_n) \tag{5.2}$$

for every  $n \in \mathbb{N}$ , then we have that

$$\mathbb{E}(T) \leq \mathbb{E}(T_{n(\varepsilon)}) + \mathbb{E}(T_{n(\varepsilon)} - T) \leq \mathbb{E}(T_{n(\varepsilon)}) + \varepsilon \leq \mathbb{E}((\pi_{n(\varepsilon)})_\#S_{n(\varepsilon)}) + \varepsilon \leq \mathbb{E}(S_{n(\varepsilon)}) + \varepsilon \leq \mathbb{E}(S) + 2\varepsilon,$$

which proves the theorem, since  $\varepsilon$  can be chosen arbitrarily small.

**Step 2. Reduction to problem (P1').** In order to prove the inequality in (5.2), we have already noticed that  $(\pi_n)_\#T_n$  is an admissible candidate for the problem (P1'). The issue is that, in general,  $(\pi_n)_\#S_n$  is not an admissible competitor and we cannot immediately look for a calibration for the current  $Z'[(\pi_n)_\#T_n]$ <sup>17</sup>. Indeed  $(\pi_n)_\#S_n$  is a rectifiable current, with admissible boundary datum, but its coefficients are in  $\mathbb{R}$  and not necessarily in  $2^{-n}\mathbb{Z}$ , as required in (P1')<sup>18</sup>.

In order to overcome this issue, we have to perform a second approximation. Fix  $\delta > 0$ . We associate to  $(\pi_n)_\#S_n$  a polyhedral current  $\hat{S} = \frac{1}{2^n N}R$ , where  $N \in \mathbb{N}$  and  $R$  has integer multiplicity. We can choose  $\hat{S}$  in such a way that the boundary is unchanged, *i.e.*,

$$\partial \hat{S} = \partial(\pi_n)_\#S_n = \partial(\pi_n)_\#T_n,$$

<sup>17</sup>The definition of  $Z'[(\pi_n)_\#T_n]$  is given in (4.5). The role of the basis  $(g_1, \dots, g_{2^n})$  here is played by  $(\pi_n(x_1), \dots, \pi_n(x_{2^n}))$ , where  $x_1, \dots, x_{2^n}$  are the (mutually orthogonal) orientations of the segments in the family  $E_n$

<sup>18</sup>In particular, we do not have a canonical way to associate to  $(\pi_n)_\#S_n$  a current with coefficients in  $\mathbb{R}^{2^n}$  whose mass is less or equal then the energy of  $(\pi_n)_\#S_n$  (see Prop. 4.3).

and

$$\mathbb{E}(\hat{S}) \leq \mathbb{E}((\pi_n)_\# S_n) + \delta. \tag{5.3}$$

The existence of  $\hat{S}$  is a consequence of Proposition 4.4 of [32]: every rectifiable current with finite energy can be approximated in energy by polyhedral currents and in particular by rational multiples of integral currents. Note that, technically speaking, not even  $\hat{S}$  is an admissible competitor for the problem (P1'), but we will see in the next step how to associate to  $\hat{S}$  a current  $\hat{Z}$  with coefficients in  $\mathbb{R}^{2^n}$ , whose mass is smaller or equal than the energy of  $\hat{S}$ . Then the calibration technique will allow us to compare the energy of  $\hat{S}$  with the energy of  $T_n$ .

**Step 3. Construction of a competitor for (P2').** Since we want to move the problem of energy minimization in the setting of (rational multiples of) rectifiable currents with coefficients in  $\mathbb{Z}^{2^n}$ , we associate to  $\hat{S}$  a current  $\hat{Z}$  in  $\mathbb{R}^{2^n}$  with coefficients in  $\mathbb{R}^{2^n}$ . We can choose  $\hat{Z}$  in such a way that the boundary is

$$\partial \hat{Z} = \partial Z'[(\pi_n)_\# T_n]$$

and

$$\mathbb{M}^{(2^n, 1/2)}(\hat{Z}) \leq \mathbb{E}(\hat{S}). \tag{5.4}$$

The construction of  $\hat{Z}$ , starting from  $\hat{S}$ , is analogous to that presented in Definition 4.1. Let  $x_1, \dots, x_{2^n}$  be the (mutually orthogonal) orientations of the segments in the family  $E_n$ . Let us write the polyhedral integral 1-current  $R = 2^n N \hat{S}$  as

$$R = \sum_{i \in I} [\sigma_i, \tau_i, a_i],$$

where  $\sigma_i$  is an oriented segment with orientation  $\tau_i$  and  $a_i \in \{1, \dots, 2^n N\}$ . Decomposing<sup>19</sup>  $R$  as in (3.2), one may notice that the current  $Z[R]$  (according to the basis  $(\pi_n(x_1), \dots, \pi_n(x_{2^n}))$ ) has the form

$$Z[R] = \sum_{i \in I} [\sigma_i, \tau_i, b_i],$$

where

- $b_i = c_{i,1} \pi_n(x_1) + \dots + c_{i,2^n} \pi_n(x_{2^n})$ ;
- $c_{i,j} \in \mathbb{N} \cup \{0\}$  and  $c_{i,j} \leq N$  for every  $j = 1, \dots, 2^n$ <sup>20</sup>;
- $c_{i,1} + \dots + c_{i,2^n} = a_i$ .

We can compute

$$\|b_i\| = \sqrt{c_{i,1}^2 + \dots + c_{i,2^n}^2} \leq \sqrt{c_{i,1} N + \dots + c_{i,2^n} N} = \sqrt{N a_i}.$$

Denote  $\hat{Z} := 2^{-n/2} N^{-1} Z[R]$ . We have

$$\begin{aligned} \partial \hat{Z} &= \partial(2^{-n/2} N^{-1} Z[R]) = 2^{-n/2} N^{-1} \partial Z[N 2^n \hat{S}] \\ &= 2^{-n/2} N^{-1} \partial Z[N 2^n (\pi_n)_\# T_n] = \partial(2^{-n/2} Z[2^n (\pi_n)_\# T_n]) = \partial Z'[(\pi_n)_\# T_n]. \end{aligned}$$

Moreover

$$\begin{aligned} \mathbb{M}^{(2^n, 1/2)}(\hat{Z}) &= 2^{-n/2} N^{-1} \mathbb{M}^{(2^n, 1/2)}(Z[R]) = 2^{-n/2} N^{-1} \sum_{i \in I} \|b_i\| \mathcal{H}^1(\sigma_i) \\ &\leq 2^{-n/2} N^{-1/2} \sum_{i \in I} \sqrt{a_i} \mathcal{H}^1(\sigma_i) = 2^{-n/2} N^{-1/2} \mathbb{E}(R) \\ &= \mathbb{E}(2^{-n} N^{-1} R) = \mathbb{E}(\hat{S}). \end{aligned}$$

<sup>19</sup>Without loss of generality we can assume  $R$  to be acyclic, i.e.,  $\hat{R}_\ell = 0$  for every  $\ell \in \mathbb{N}$ .

<sup>20</sup>Roughly speaking,  $c_{i,j}$  is the portion of the multiplicity  $a_i$  of  $\sigma_i$ , which is due to all paths ending in the second extreme of the segment of  $E_n$  having orientation  $x_j$ .

**Step 4. Proof of the minimality of  $Z'[(\pi_n)_\#T_n]$ .** Even if  $\hat{Z}$  does not belong to the set of competitors for (P2'), we can still prove that

$$\mathbb{M}^{(2^n, 1/2)}(Z'[(\pi_n)_\#T_n]) \leq \mathbb{M}^{(2^n, 1/2)}(\hat{Z}) \quad (5.5)$$

via the calibration technique (see Rem. 2.10).

Consider the basis of  $\mathbb{R}^{2^n}$   $(\pi_n(x_1), \dots, \pi_n(x_{2^n}))$ . The current  $Z'[(\pi_n)_\#T_n]$  has the following property: the multiplicity on each line segment of its support is a multiple of the orientation of the segment itself<sup>21</sup> (the orientation being defined above). This implies that, if we denote by  $\omega : \mathbb{R}^{2^n} \rightarrow \mathbb{R}^{2^n}$  the identity map, then  $\omega$  can be seen as a  $\mathbb{R}^{2^n}$ -valued (constant) differential 1-form, which in particular is a calibration for  $Z'[(\pi_n)_\#T_n]$ : the verification of conditions (i), (ii) and (iii) of Definition 2.8 is elementary, as in Example 4.5. Theorem 2.9 and Remark 2.10 allow us to conclude that (5.5) holds.

**Step 5. Conclusion.** Eventually we can prove the claim (5.2), indeed

$$\mathbb{E}((\pi_n)_\#S_n) + \delta \stackrel{(5.3)}{\geq} \mathbb{E}(\hat{S}) \stackrel{(5.4)}{\geq} \mathbb{M}^{(2^n, 1/2)}(\hat{Z}) \stackrel{(5.5)}{\geq} \mathbb{M}^{(2^n, 1/2)}(Z'[(\pi_n)_\#T_n])$$

and, from the arbitrariness of  $\delta$ , from Corollary 4.7, together with the fact that  $\text{supp}(T_n)$  is contained in  $\text{span}\{e_1, \dots, e_{2^n}\}$ , we finally get that

$$\mathbb{E}((\pi_n)_\#S_n) \geq \mathbb{M}^{(2^n, 1/2)}(Z'[(\pi_n)_\#T_n]) \stackrel{4.7}{=} \mathbb{E}((\pi_n)_\#T_n) = \mathbb{E}(T_n). \quad \square$$

**Remark 5.4.** In [12] the authors combine the notion of currents in metric spaces of [1] and currents with coefficients in a group of [30]. In this setting we can give a meaning to the problem of minimizing the mass in a class of currents in  $\ell^2$  with coefficients in  $\ell^2$  with given boundary. It is possible to associate to the current  $T$ , defined in Section 5.2, a current  $Z$  in  $\ell^2$  with coefficients in  $\ell^2$ , in a way which is similar to that explained in Definition 4.1. In particular the mass of  $Z$  would be equal to the energy of  $T$ . Via a simple calibration argument (the calibration being again the “identity” with respect to suitable coordinates) one can prove that  $Z$  minimizes the mass among all currents in  $\ell^2$  with coefficients in  $\ell^2$ , with boundary  $\partial Z$ .

The latter remark suggests the possibility to describe, in general, the continuous version of the Gilbert–Steiner problem (see Sect. 4.4) as a mass minimization problem among currents in the Euclidean space, with coefficients in  $\ell^2$ . The advantage of this formulation would be, for example, the availability of well known numerical methods for the minimization of a convex functional.

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<sup>21</sup>This is due to the fact that the sum of  $y(x)$  and  $z(x)$ , defined in Section 5.2, is parallel to  $x$ .

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