

FULLY COUPLED FORWARD-BACKWARD SDES INVOLVING THE VALUE FUNCTION AND ASSOCIATED NONLOCAL HAMILTON–JACOBI–BELLMAN EQUATIONS*

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Abstract. A new type of controlled fully coupled forward-backward stochastic differential equations is discussed, namely those involving the value function. With a new iteration method, we prove an existence and uniqueness theorem of a solution for this type of equations. Using the notion of extended “backward semigroup”, we prove that the value function satisfies the dynamic programming principle and is a viscosity solution of the associated nonlocal Hamilton–Jacobi–Bellman equation.

Mathematics Subject Classification. 60H10, 60H30, 35K65.

Received February 24, 2014. Revised November 24, 2014
Published online April 6, 2016.

1. INTRODUCTION

The main purpose of this work is to establish a new type of controlled fully coupled forward-backward stochastic differential equations (FBSDEs), namely those involving the value function. We prove the existence and uniqueness of a solution for this new type of equations and we study the related nonlocal second-order quasi-linear Hamilton–Jacobi–Bellman (HJB) equations. Our work is inspired by the recent developments in mean-field stochastic differential equations (SDEs), mean-field backward SDEs (BSDEs) and fully coupled mean-field FBSDEs (see [5, 6, 12], *etc.*), and Hao and Li ([9]). Another motivation comes from the study of control problems in the mean-field framework and the investigation of the associated nonlocal partial differential equations of HJB type.

Models of large stochastic particle systems with mean-field interaction can be met in various fields such as statistical mechanics and physics, quantum mechanics and quantum chemistry. Mean-field problems have been studied by many authors (see, *e.g.*, [2, 3, 7, 11, 18, 19]). Due to these works we know that linear McKean–Vlasov PDEs can be interpreted stochastically. In 2009, Buckdahn *et al.* [6] investigated in a purely stochastic approach

Keywords and phrases. Fully coupled FBSDE involving value function, dynamic programming principle, fully coupled mean-field FBSDE, viscosity solution, nonlocal Hamilton–Jacobi–Bellman equation.

* Juan Li is the corresponding author. The work has been supported by the NSF of P.R.China (Nos. 11071144, 11171187, 11222110), Shandong Province (Nos. BS2011SF010, JQ201202), SRF for ROCS (SEM), Program for New Century Excellent Talents in University (No. NCET-12-0331), 111 Project (No. B12023).

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fully nonlinear BSDEs of mean-field type, called mean-field BSDEs. In [5], Buckdahn *et al.* deepened the study of mean-field BSDEs and gave a probability interpretation for the associated second-order quasi-linear nonlocal partial differential equations (PDEs).

On the other hand, Antonelli [1] was the first to study the existence and the uniqueness for fully coupled FBSDEs on a small time interval; he pointed out that the Lipschitz assumption is not sufficient for the existence of the solution of fully coupled FBSDEs over an arbitrarily large time interval. Since then, the theory of fully coupled FBSDEs has been developed very quickly. For instance, with PDE approaches combined with stochastic methods, Ma *et al.* [15] showed the existence and the uniqueness for a type of fully coupled FBSDEs which forward SDEs are non-degenerate. Another direction of generalization was considered by Hu and Peng [10]: Under suitable monotonicity conditions they obtained the existence and the uniqueness for fully coupled FBSDEs on an arbitrarily large time interval, but with forward and backward equation being of the same dimension. In 1999, Peng and Wu [17] overcame this strict restriction by considering a multiplicative full-rank matrix G with which one solution of the forward equation is multiplied, and they proved the existence and the uniqueness for general fully coupled FBSDEs under weaker monotonicity assumptions and discussed applications in optimal control. Li and Wei [13] considered optimal control problems for fully coupled FBSDEs and the associated second-order quasi-linear HJB equations. Unlike [13] we study the optimal control problem of fully coupled mean-field FBSDEs, and the associated non-local HJB equation is mean-field type. Recently, fully coupled mean-field FBSDEs were discussed by Min *et al.* [12]. The authors gave an existence and uniqueness theorem of a solution for this type of equations by using a continuation method.

In [5] their original motivation is to give the stochastic interpretation of nonlocal PDE of mean-field type. However, the formally associated SDE does not generate a flow. To overcome this difficulty in [5] the initial condition was partially frozen, which leads to a system composed of decoupled mean-field FBSDE and a classical decoupled FBSDE depending on the solution of the mean-field SDE. In this paper we adopt and extend this approach to control problems, in order to give stochastic interpretation to nonlocal Hamilton–Jacobi–Bellman equations of mean-field type. More precisely, let $x_0 \in \mathbb{R}^n$ and $\bar{v} \in \mathcal{V}_{0,T}$ be arbitrary but fixed, and let $x \in \mathbb{R}^n$, $t \in [0, T]$, $v \in \mathcal{V}_{t,T}$, we consider the following fully coupled FBSDE:

$$\left\{ \begin{array}{l} dX_s^{t,x;v} = E'[b(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)]ds \\ \quad + E'[\sigma(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, v_s)]dB_s, \\ dY_s^{t,x;v} = -E'[f(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)]ds \\ \quad + Z_s^{t,x;v}dB_s, \\ X_t^{t,x;v} = x, \quad Y_T^{t,x;v} = E'[\Phi((X_T^{0,x_0;\bar{v}})')', X_T^{t,x;v}], \\ W(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v}, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \end{array} \right. \quad (1.1)$$

where the essential infimum is taken over all admissible stochastic control processes v over the time interval $[t, T]$. By $(X^{0,x_0;\bar{v}})' = (X_s^{0,x_0;\bar{v}})'_{s \in [t,T]}$ we denote an independent and identically distributed copy of the process $X^{0,x_0;\bar{v}} = (X_s^{0,x_0;\bar{v}})_{s \in [t,T]}$ which is supposed to be independent of the driving Brownian motion B and, thus also of all processes $X^{0,x_0;\bar{v}}, X^{t,x;v}, Y^{t,x;v}, Z^{t,x;v}$, and \bar{v}, v adapted with respect to the filtration generated by B . By $E'[\cdot]$ we denote an expectation concerning only $(X^{0,x_0;\bar{v}})'$ but not the other processes (for more details the reader is referred to [5]). In order to solve (1.1) involving the value function of the control problem itself, we replace first W by any function $\tilde{W}(s, x)$, which is Lipschitz and of linear growth in x , and under suitable monotonicity conditions we show the existence and the uniqueness of a solution $(X^{t,x;v}, Y^{t,x;v}, Z^{t,x;v})$ of (1.1), for all (t, x, v) (and so, in particular, for $(t, x, v) = (0, x_0, \bar{v})$. In fact, beginning with choosing $(t, x, v) = (0, x_0, \bar{v})$ we find $(X^{0,x_0;\bar{v}}, Y^{0,x_0;\bar{v}}, Z^{0,x_0;\bar{v}})$ which allows then to consider arbitrary (t, x, v) in (1.1)). Moreover, we will show that there exists a constant $L_0 > 0$ depending only on L_b in (H3.3) and the Lipschitz constants of b, σ, f and Φ in (x, y, z) and x , such that the value function $W(t, x) \triangleq \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v}$ is Lipschitz and of linear growth in x , both with a constant L_0 which is independent of \tilde{W} . This means that, no matter what the Lipschitz and the linear

growth constants of \tilde{W} are, there is a common constant $L_0 > 0$ such that $W(t, x)$ is Lipschitz and of linear growth in x with a Lipschitz and growth constant L_0 . With a new iteration method which is different from [9], beginning with $\tilde{W} = W^0 = 0$, we get four convergent sequences $\{X^{0,x_0;\bar{v},i}\}, \{X^{t,x;v,i}\}, \{Y^{t,x;v,i}\}, \{Z^{t,x;v,i}\}$ and the associated $W^{i+1}(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v,i}$. We prove that all these sequences converge to the solution processes (Thm. 3.7). Once having the existence and the uniqueness of a solution for (1.1) we prove the DPP for the value function W . For this we use an extended “backward semigroup” (Thm. 4.2) generalizing Peng’s original one [16]. Using this DPP, it becomes easy to check that $W(t, x)$ is Hölder continuous in t (Prop. 4.4). Furthermore, we study the related nonlocal HJB equations associated with (1.1). Using a short and direct probabilistic argument based on results by Li and Wei [13], we prove that the value function W is a viscosity solution of our nonlocal HJB equation (5.1) (Thm. 5.2). In Theorem 5.3 we show the uniqueness of viscosity solution of our nonlocal HJB equation (5.1) when σ doesn’t depend on y with the help of the uniqueness of the solution of fully coupled FBSDE (1.1).

In comparison with [9] a main difficulty consists in the fact that we have to deal here with a fully coupled FBSDE: Any iteration procedure also involves the solution of the forward equation $X^{t,x;v,i}$, and thus also $(X^{0,x_0;\bar{v},i})'$ -the variable of W^{i-1} . In order to get the convergence of the iteration sequence we have to suppose that the coefficients $b(s, x', y', x, y, z, v), \sigma(s, x', y', x, y, v)$ and $f(s, x', y', x, y, z, v)$ are Lipschitz in y' with a sufficiently small Lipschitz constant.

Our paper is organized as follows. In Section 2, we recall some results on fully coupled mean-field FBSDEs, which are used frequently in what follows. Section 3 introduces the fully coupled FBSDEs involving value function and gives the proof of the existence and the uniqueness of solutions of equations. The DPP and the regularity of the value function are shown in Section 4. In Section 5, for this new type of equation we show that the value function is a unique viscosity solution of the associated nonlocal HJB equation. Some auxiliary lemmas are proved in Section 6.

2. FULLY COUPLED MEAN-FIELD FBSDES

Our setting is that of the classical Wiener space: Let $T > 0$ and $\Omega = C_0([0, T]; \mathbb{R}^d)$ be the set of all continuous functions from $[0, T]$ to \mathbb{R}^d , beginning from 0; P is the Wiener measure under which the coordinate process $B: B_s(\omega) = \omega_s, s \in [0, T], \omega \in \Omega$, is a d -dimensional Brownian motion. By \mathcal{F} we denote the Borel σ -field over Ω , completed by the set \mathcal{N} of all P -null sets. We denote by $\mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\}$ the natural filtration generated by $\{B_s\}_{0 \leq s \leq T}$ and completed by \mathcal{N} .

We introduce the following spaces which will be often used:

- $\mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^n) \triangleq \left\{ (\phi_t)_{0 \leq t \leq T} \text{ } \mathbb{R}^n\text{-valued } \mathbb{F}\text{-adapted càdlàg process} : E\left[\sup_{0 \leq t \leq T} |\phi_t|^2 \right] < +\infty \right\};$
- $\mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^{n \times d}) \triangleq \left\{ (l_t)_{t \in [0, T]} \text{ } \mathbb{R}^{n \times d}\text{-valued } \mathbb{F}\text{-progressively measurable process} : \|l\|^2 = E\left[\int_0^T |l_t|^2 dt \right] < +\infty \right\}.$

Let us first recall some notations used for the mean-field framework and for the fully coupled mean-field FBSDEs in [12]; see also [5, 6].

By $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P)$ we denote the product of (Ω, \mathcal{F}, P) with itself. The filtration in this space we consider is $\bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t = \mathcal{F} \otimes \mathcal{F}_t, 0 \leq t \leq T\}$. By setting $\xi'(\omega', \omega) = \xi(\omega'), (\omega', \omega) \in \bar{\Omega}$, we can extend the real-valued random variables ξ from Ω to $\bar{\Omega}$. For any integrable random variable $\theta \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, $\theta(\cdot, \omega) : \Omega \rightarrow \mathbb{R}$ is in $L^1(\Omega, \mathcal{F}, P)$, $P(d\omega)$ -a.s. Hence, we can define $E'[\theta(\cdot, \omega)] = \int_{\Omega} \theta(\omega', \omega) P(d\omega')$. Using Fubini’s Theorem, the expectation of θ can be computed as follows:

$$\bar{E}[\theta] = \int_{\bar{\Omega}} \theta d\bar{P} = \int_{\Omega} E'[\theta(\cdot, \omega)] P(d\omega) = E[E'[\theta]].$$

- Let $b(\bar{\omega}, t, x', y', z', x, y, z) : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^n,$
 $\sigma(\bar{\omega}, t, x', y', x, y) : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times d},$
 $f(\bar{\omega}, t, x', y', z', x, y, z) : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m,$
 $\Phi(\bar{\omega}, x', x) : \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$

satisfy the following assumptions:

(H2.1) (i) There exist $L, \tilde{L}_1, \tilde{L}_2, \tilde{L}_3 > 0$ such that, for all $t \in [0, T]$, $x_1, x'_1, x_2, x'_2 \in \mathbb{R}^n$, $y_1, y'_1, y_2, y'_2 \in \mathbb{R}^m$, $z_1, z'_1, z_2, z'_2 \in \mathbb{R}^{m \times d}$,

$$\begin{aligned} & |h(t, x'_1, y'_1, z'_1, x_1, y_1, z_1) - h(t, x'_2, y'_2, z'_2, x_2, y_2, z_2)| \\ & \leq \tilde{L}_1|x'_1 - x'_2| + \tilde{L}_2|y'_1 - y'_2| + \tilde{L}_3|z'_1 - z'_2| + L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \end{aligned}$$

where $h = b, \sigma, f$, respectively, and $b(\cdot, \lambda', \lambda) \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$, $\sigma(\cdot, \lambda', \lambda) \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^{n \times d})$, $f(\cdot, \lambda', \lambda) \in \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^m)$, $\lambda = (x, y, z)$, $\lambda' = (x', y', z')$;

(ii) For $(x'_i, x_i) \in \mathbb{R}^n \times \mathbb{R}^n$, $i = 1, 2$,

$$|\Phi(x'_1, x_1) - \Phi(x'_2, x_2)| \leq \tilde{L}_\Phi|x'_1 - x'_2| + L_\Phi|x_1 - x_2|,$$

and for any $x', x \in \mathbb{R}^n$, $\Phi(x', x) \in L^2(\Omega, \bar{\mathcal{F}}_T, \mathbb{R}^m)$.

Given an $m \times n$ full-rank matrix G , we put

$$\lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \lambda' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \quad B(t, \lambda', \lambda) = \begin{pmatrix} -G^T f \\ Gb \\ G\sigma \end{pmatrix} (t, \lambda', \lambda).$$

From (H2.1) we know that $B(t, \lambda', \lambda)$ is Lipschitz in λ' and λ , uniformly with respect to t . Let us denote the two Lipschitz constants by \tilde{L}_B and L_B , respectively.

(H2.2) (i) $\langle B(t, \lambda', \lambda) - B(t, \lambda', \bar{\lambda}), \lambda - \bar{\lambda} \rangle \leq -\beta_1|\hat{x}|^2$;

(ii) $\langle \Phi(x', x) - \Phi(x', \bar{x}), G(x - \bar{x}) \rangle \geq \mu_1|\hat{x}|^2$,

where $\lambda = (x, y, z)$, $\lambda' = (x', y', z')$, $\bar{\lambda} = (\bar{x}, \bar{y}, \bar{z})$, $\hat{x} = x - \bar{x}$, $\hat{y} = y - \bar{y}$, $\hat{z} = z - \bar{z}$, and β_1, μ_1 are nonnegative constants with $\beta_1 > \tilde{L}_B(1 + 2e^{\hat{\beta}T})$; $\mu_1 > \tilde{L}_\Phi\lambda_1 + 4(\tilde{L}_\Phi^2 + L_\Phi^2)\tilde{L}_B e^{\hat{\beta}T}$, where $\hat{\beta} = 6L^2 + 4\tilde{L}_3^2 + 2\tilde{L}_1^2 + 2\tilde{L}_2 + 2L + 1$ (in particular, as $h = b, \sigma, f$ is independent of (y', z') , $\hat{\beta} = 4L^2 + 2\tilde{L}_1^2 + 2L + 1$), and $\lambda_1 > 0$ satisfies $|Gl| \leq \lambda_1|l|$, for any vector $l \in \mathbb{R}^n$.

For $a \in \mathbb{R}^n$, we consider the following fully coupled mean-field FBSDE:

$$\begin{cases} X_t = a + \int_0^t E'[b(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]ds + \int_0^t E'[\sigma(s, X'_s, Y'_s, X_s, Y_s)]dB_s, \\ Y_t = E'[\Phi((X_T)') + \int_t^T E'[f(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]ds - \int_t^T Z_s dB_s, \quad t \in [0, T]. \end{cases} \quad (2.1)$$

In [12], the authors proved that the fully coupled mean-field FBSDE (2.1) has a unique solution. Precisely speaking, we have:

Lemma 2.1. *Under the assumptions (H2.1) and (H2.2), the fully coupled mean-field FBSDE (2.1) admits a unique solution $(X, Y, Z) \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^m) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^{m \times d})$.*

Moreover, for any initial time $t \in [0, T]$, initial value $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ and given $x_0 \in \mathbb{R}^n$, also the following fully coupled FBSDE:

$$\begin{cases} X_s^{t, \zeta} = \zeta + \int_t^s E'[b(r, (X_r^{0, x_0})', (Y_r^{0, x_0})', (Z_r^{0, x_0})', X_r^{t, \zeta}, Y_r^{t, \zeta}, Z_r^{t, \zeta})]dr \\ \quad + \int_t^s E'[\sigma(r, (X_r^{0, x_0})', (Y_r^{0, x_0})', X_r^{t, \zeta}, Y_r^{t, \zeta})]dB_r, \\ Y_s^{t, \zeta} = E'[\Phi((X_T^{0, x_0})') + \int_s^T E'[f(r, (X_r^{0, x_0})', (Y_r^{0, x_0})', (Z_r^{0, x_0})', X_r^{t, \zeta}, Y_r^{t, \zeta}, Z_r^{t, \zeta})]dr \\ \quad - \int_s^T Z_r^{t, \zeta} dB_r, \quad s \in [t, T], \end{cases} \quad (2.2)$$

has a unique adapted solution, denoted by $(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})_{s \in [t,T]} \in \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^m) \times \mathcal{H}_{\mathbb{F}}^2(t, T; \mathbb{R}^{m \times d})$. In fact, if $(t, \zeta) = (0, x_0)$, then (2.2) changes to a fully coupled mean-field FBSDE. Thanks to Lemma 2.1, we see that this equation has a unique adapted solution, denoted by $(X_s^{0,x_0}, Y_s^{0,x_0}, Z_s^{0,x_0})_{s \in [0,T]} \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^m) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^{m \times d})$. Then, it can be checked that the coefficient $h^0(s, x, y, z) \triangleq E'[h(s, (X_s^{0,x_0})', (Y_s^{0,x_0})', (Z_s^{0,x_0})', x, y, z)]$ satisfies the assumptions for classical fully coupled FBSDEs, where $h = b, \sigma, f$, respectively. Consequently, according to the existence and uniqueness theorem of classical fully coupled FBSDEs, the above equation (2.2) admits a unique solution $(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})_{s \in [t,T]} \in \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^m) \times \mathcal{H}_{\mathbb{F}}^2(t, T; \mathbb{R}^{m \times d})$ (see, e.g., Thm. 2.2 in [17] or Thm. 3.1 in [10]).

From standard estimates for classical fully coupled FBSDEs, we have (see, e.g., Prop. 6.1 in [13]) the following:

Lemma 2.2. *Under the assumptions (H2.1) and (H2.2), there exists a constant $C > 0$ such that, for all $t \in [0, T]$ and $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, P -a.s.,*

$$\begin{aligned} \text{(i)} \quad & E \left[\sup_{t \leq s \leq T} |X_s^{t,\zeta} - X_s^{t,\zeta'}|^2 + \sup_{t \leq s \leq T} |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2 + \int_t^T |Z_s^{t,\zeta} - Z_s^{t,\zeta'}|^2 ds | \mathcal{F}_t \right] \leq C |\zeta - \zeta'|^2; \\ \text{(ii)} \quad & E \left[\sup_{t \leq s \leq T} |X_s^{t,\zeta}|^2 + \sup_{t \leq s \leq T} |Y_s^{t,\zeta}|^2 + \int_t^T |Z_s^{t,\zeta}|^2 ds | \mathcal{F}_t \right] \leq C(1 + |\zeta|^2). \end{aligned} \tag{2.3}$$

In particular,

$$\text{(iii)} \quad |Y_t^{t,\zeta}| \leq C(1 + |\zeta|); \quad \text{(iv)} \quad |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C|\zeta - \zeta'|.$$

3. FULLY COUPLED FBSDES INVOLVING THE VALUE FUNCTION

Let V be a compact metric space. An \mathbb{F} -progressively measurable process $v \in \{v_r, r \in [t, s]\}$ taking its values in V is called an admissible control process on the time interval $[t, s]$ ($0 \leq t < s \leq T$). By $\mathcal{V}_{t,s}$ we denote the set of all admissible controls over $[t, s]$.

In this section, we study the following fully coupled FBSDE involving the value function. For $x_0 \in \mathbb{R}^n$ and $\bar{v} \in \mathcal{V}_{0,T}$ arbitrarily chosen but fixed, we consider

$$\left\{ \begin{aligned} dX_s^{t,x;v} &= E'[b(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)] ds \\ &\quad + E'[\sigma(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, v_s)] dB_s, \\ dY_s^{t,x;v} &= -E'[f(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)] ds \\ &\quad + Z_s^{t,x;v} dB_s, \quad s \in [t, T], \\ X_t^{t,x;v} &= x, \quad Y_T^{t,x;v} = E'[\Phi((X_T^{0,x_0;\bar{v}})')', X_T^{t,x;v}], \\ W(t, x) &= \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v}, \quad (t, x) \in [0, T] \times \mathbb{R}^n. \end{aligned} \right. \tag{3.1}$$

Let us introduce now the assumptions on the coefficients which we need. We suppose that the functions

$$\begin{aligned} b(t, x', y', x, y, z, v) &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times V \rightarrow \mathbb{R}^n; \\ \sigma(t, x', y', x, y, v) &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times V \rightarrow \mathbb{R}^{n \times d}; \\ f(t, x', y', x, y, z, v) &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times V \rightarrow \mathbb{R}; \\ \Phi(x', x) &: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \end{aligned}$$

satisfy:

(H3.1) (i) There exist $\tilde{L}, L > 0$ and $\alpha > 0$ small enough (a precision for α will be given in Rem. 3.8) such that, for all $t \in [0, T]$, $x_1, x'_1, x_2, x'_2 \in \mathbb{R}^n$, $y_1, y'_1, y_2, y'_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$, $v \in V$,

$$\begin{aligned} & |h(t, x'_1, y'_1, x_1, y_1, z_1, v) - h(t, x'_2, y'_2, x_2, y_2, z_2, v)| \\ & \leq \tilde{L}|x'_1 - x'_2| + \alpha|y'_1 - y'_2| + L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \end{aligned}$$

where $h = b, \sigma, f$;

- (ii) For $h = b, \sigma, f$, $h(t, x', y', x, y, z, v)$ is continuous with respect to (t, v) , uniformly in (x', y', x, y, z) ;
 (iii) $\Phi(x', x)$ is uniformly Lipschitz in (x', x) , *i.e.*, there exist $\tilde{L}_\Phi, L_\Phi > 0$ such that, for $x_1, x'_1, x_2, x'_2 \in \mathbb{R}^n$,

$$|\Phi(x'_1, x_1) - \Phi(x'_2, x_2)| \leq \tilde{L}_\Phi |x'_1 - x'_2| + L_\Phi |x_1 - x_2|.$$

Further assumptions on the coefficients are made through the function

$$B(t, \lambda', \lambda, v) = \begin{pmatrix} -G^T f \\ Gb \\ G\sigma \end{pmatrix} (t, \lambda', \lambda, v), \quad t \in [0, T], \quad v \in V, \quad \lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \lambda' = \begin{pmatrix} x' \\ y' \end{pmatrix},$$

where $G^T \in \mathbb{R}^n \setminus \{0\}$, $|G^T| = 1$ is suitably chosen.

As indicated in Section 2, under the assumption (H3.1)-(i), $B(t, \lambda', \lambda, v)$ is Lipschitz in λ' and λ . By \tilde{L}_B and L_B we denote their respective Lipschitz constants. From Hölder inequality, we have

$$|B(t, \bar{\lambda}', \bar{\lambda}, v) - B(t, \lambda', \lambda, v)| \leq \sqrt{3} \sqrt{\tilde{L}^2 + \alpha^2} |\hat{\lambda}'| + 3L |\hat{\lambda}|,$$

where $\hat{\lambda}' = \bar{\lambda}' - \lambda'$, $\hat{\lambda} = \bar{\lambda} - \lambda$. Hence here we can set $\tilde{L}_B = \sqrt{3} \sqrt{\tilde{L}^2 + \alpha^2}$ and $L_B = 3L$.

We suppose the following monotonicity conditions

- (H3.2) (i) $\langle B(t, \lambda', \lambda, v) - B(t, \lambda', \bar{\lambda}, v), \lambda - \bar{\lambda} \rangle \leq -\beta_1 |\hat{x}|^2$;
 (ii) $\langle G^T (\Phi(x', x) - \Phi(x', \bar{x})), x - \bar{x} \rangle \geq \mu_1 |\hat{x}|^2$,

where $\lambda = (x, y, z)$, $\bar{\lambda} = (\bar{x}, \bar{y}, \bar{z})$, $\lambda' = (x', y', z')$, $\hat{x} = x - \bar{x}$, $\hat{y} = y - \bar{y}$, $\hat{z} = z - \bar{z}$, $x, \bar{x}, x' \in \mathbb{R}^n$, $y, \bar{y}, y' \in \mathbb{R}$, $z, \bar{z}, z' \in \mathbb{R}^d$, $\beta_1 \geq 0$, $\mu_1 \geq 0$, with $\beta_1 > \tilde{L}_B(1 + 2e^{\beta T})$, $\mu_1 > \tilde{L}_\Phi + 4\tilde{L}_B(\tilde{L}_\Phi^2 + L_\Phi^2)e^{\beta T}$.

Here $\bar{\beta} = 4L^2 + 2\tilde{L}^2 + 2L + 2\alpha + 1$, and L, \tilde{L} and α are the Lipschitz constants of $h = b, \sigma, f$ introduced in (H3.1)-(i).

Moreover, we need the following assumption

- (H3.3) $h(t, x', y', 0, 0, 0, v)$ and $\Phi(x', 0)$ are bounded, *i.e.*, there is a constant $L_b > 0$ such that, for all $t \in [0, T]$, $x' \in \mathbb{R}^n$, $y' \in \mathbb{R}$, $v \in V$, $|\Phi(x', 0)| \leq L_b$; $|h(t, x', y', 0, 0, 0, v)| \leq L_b$, for $h = b, \sigma, f$, respectively.

Remark 3.1. Under the assumptions (H3.1) and (H3.3), $h = b, \sigma, f$ and Φ are of linear growth with respect to (x, y, z) and x , uniformly in (t, x', y', v) and x' , respectively: For $t \in [0, T]$, $x', x \in \mathbb{R}^n$, $y', y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $v \in V$, $|\Phi(x', x)| \leq (L_\Phi + L_b)(1 + |x|)$; $|h(t, x', y', x, y, z, v)| \leq (L + L_b)(1 + |x| + |y| + |z|)$.

Let $K > 0$ be a given constant and $\tilde{W} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a deterministic Borel function such that, $|\tilde{W}(t, x)| \leq K(1 + |x|)$, and $|\tilde{W}(t, x) - \tilde{W}(t, \bar{x})| \leq K|x - \bar{x}|$, for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^n$. For arbitrarily chosen but fixed $x_0 \in \mathbb{R}^n$, $\bar{v} \in \mathcal{V}_{0,T}$, and for $(t, x) \in [0, T] \times \mathbb{R}^n$, $v \in \mathcal{V}_{t,T}$, let us consider the following fully coupled FBSDE:

$$\begin{cases} dX_s^{t,x;v} = E'[b(s, (X_s^{0,x_0;\bar{v}})')', \tilde{W}(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)] ds \\ \quad + E'[\sigma(s, (X_s^{0,x_0;\bar{v}})')', \tilde{W}(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, v_s)] dB_s, \\ dY_s^{t,x;v} = -E'[f(s, (X_s^{0,x_0;\bar{v}})')', \tilde{W}(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)] ds \\ \quad + Z_s^{t,x;v} dB_s, \\ X_t^{t,x;v} = x, \quad Y_T^{t,x;v} = E'[\Phi((X_T^{0,x_0;\bar{v}})')', X_T^{t,x;v}]. \end{cases} \quad (3.2)$$

Lemma 3.2. *We suppose that the assumptions (H3.1)–(H3.3) hold true. Then the fully coupled FBSDE (3.2) admits a unique solution $(X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v})_{s \in [t,T]} \in \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(t, T; \mathbb{R}^d)$.*

Proof. For $(s, x', x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$, we define $\tilde{h}(s, x', x, y, z) \triangleq h(s, x', \tilde{W}(s, x'), x, y, z, v_s)$, $h = b, \sigma, f$, respectively. Then, for $s \in [0, T]$, $x'_1, x'_2, x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,

$$|\tilde{h}(s, x'_1, x_1, y_1, z_1) - \tilde{h}(s, x'_2, x_2, y_2, z_2)| \leq (\tilde{L} + \alpha K)|x'_1 - x'_2| + L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|),$$

where L, \tilde{L} and α are the Lipschitz constants given in (H3.1)-(i) and K is the Lipschitz and the linear growth constant of \tilde{W} .

We put

$$\tilde{B}(s, \lambda', \lambda) \triangleq \begin{pmatrix} -G^T \tilde{f} \\ G \tilde{b} \\ G \tilde{\sigma} \end{pmatrix} (s, \lambda', \lambda), \quad \tilde{\Phi}(x', x) \triangleq \Phi(x', x),$$

where $\lambda = (x, y, z)$, $\lambda' = (x', y', z')$.

By $\tilde{L}_{\tilde{B}}, L_{\tilde{B}}$ and $\tilde{L}_{\tilde{\Phi}}, L_{\tilde{\Phi}}$ we denote the Lipschitz constants of \tilde{B} and $\tilde{\Phi}$ with respect to λ', λ and x', x , respectively. From Hölder's inequality we can obtain

$$|\tilde{B}(s, \bar{\lambda}', \bar{\lambda}) - \tilde{B}(s, \lambda', \lambda)| \leq \sqrt{3}(\tilde{L} + \alpha K)|\hat{\lambda}'| + 3L|\hat{\lambda}|,$$

where $\hat{k} = \bar{k} - k$, $k = \lambda', \lambda$. Consequently, we can take $\tilde{L}_{\tilde{B}} = \sqrt{3}(\tilde{L} + \alpha K)$ and $L_{\tilde{B}} = 3L$. Notice that α is small enough such that $\beta_1 > \tilde{L}_{\tilde{B}}(1 + 2e^{\beta T})$, $\mu_1 > \tilde{L}_{\tilde{\Phi}} + 4\tilde{L}_{\tilde{B}}(\tilde{L}_{\tilde{\Phi}}^2 + L_{\tilde{\Phi}}^2)e^{\beta T}$, where $\beta = 4L^2 + 2(\tilde{L} + \alpha K)^2 + 2L + 1$, $\tilde{L}_{\tilde{\Phi}} = \tilde{L}_{\Phi}$, $L_{\tilde{\Phi}} = L_{\Phi}$ (see Rem. 3.8). According to Lemma 2.1, for $(t, x) = (0, x_0)$ and given $\bar{v} \in \mathcal{V}_{0,T}$, the fully coupled FBSDE (3.2) with coefficients $\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{\Phi}$ admits a unique solution, denoted by $(X_s^0, Y_s^0, Z_s^0)_{s \in [0,T]} \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(0, T; \mathbb{R}^d)$.

Hence we can define

$$h^0(s, x, y, z) \triangleq E'[h(s, (X_s^0)', \tilde{W}(s, (X_s^0)'), x, y, z, v_s)]; \quad \Phi^0(x) \triangleq E'[\Phi((X_T^0)', x)], \tag{3.3}$$

where $h = b, \sigma, f$, respectively. Obviously, b^0, σ^0, f^0 and Φ^0 are Lipschitz with respect to (x, y, z) and x with the same Lipschitz constants as b, σ, f and Φ in (x, y, z) and x , respectively, and satisfy the monotonicity condition for classical fully coupled FBSDEs with the constants β_1 and μ_1 which do not depend on \tilde{W} . Consequently, for $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{V}_{t,T}$, there is a unique triple $(X_s^{t,x,v}, Y_s^{t,x,v}, Z_s^{t,x,v})_{s \in [t,T]} \in \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(t, T; \mathbb{R}^d)$ satisfying the fully coupled FBSDE (3.2) (see, Thm. 3.1 in [10] or Thm. 2.2 in [17]). \square

Lemma 3.3. *Let us suppose that (H3.1)–(H3.3) hold true and $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^n$, $v \in \mathcal{V}_{t,T}$. By $(X_s^{t,x,v}, Y_s^{t,x,v}, Z_s^{t,x,v})_{s \in [t,T]}$ (resp., $(X_s^{t,\bar{x},v}, Y_s^{t,\bar{x},v}, Z_s^{t,\bar{x},v})_{s \in [t,T]}$) we denote the solution of the fully coupled FBSDE (3.2) associated with initial value $(t, x) \in [0, T] \times \mathbb{R}^n$ (resp., $(t, \bar{x}) \in [0, T] \times \mathbb{R}^n$) and $v \in \mathcal{V}_{t,T}$. Then, for any $p \geq 2$, there exists a constant $L_0 > 0$ depending on L, L_{Φ}, L_b and p such that P-a.s.,*

$$\begin{aligned} \text{(i)} \quad & E \left[\sup_{t \leq s \leq T} (|X_s^{t,x,v} - X_s^{t,\bar{x},v}|^2 + |Y_s^{t,x,v} - Y_s^{t,\bar{x},v}|^2) + \int_t^T |Z_s^{t,x,v} - Z_s^{t,\bar{x},v}|^2 ds | \mathcal{F}_t \right] \leq L_0^2 |x - \bar{x}|^2; \\ \text{(ii)} \quad & E \left[\sup_{t \leq s \leq T} (|X_s^{t,x,v}|^p + |Y_s^{t,x,v}|^p) + \left(\int_t^T |Z_s^{t,x,v}|^2 ds \right)^{\frac{p}{2}} | \mathcal{F}_t \right] \leq L_0^p (1 + |x|^p); \end{aligned} \tag{3.4}$$

and, in particular,

$$\text{(iii)} \quad |Y_t^{t,x,v} - Y_t^{t,\bar{x},v}| \leq L_0 |x - \bar{x}|; \quad \text{(iv)} \quad |Y_t^{t,x,v}| \leq L_0 (1 + |x|).$$

Proof. From the proof of Lemma 3.2, we know that the fully coupled FBSDE (3.2) can be treated as a classical fully coupled FBSDE with coefficients b^0, σ^0, f^0 and Φ^0 . Thanks to the assumptions (H3.1) and (H3.3), it is easy to get that b^0, σ^0, f^0 (resp., Φ^0) have the same Lipschitz and linear growth constants in (x, y, z) (resp., x) as b, σ, f (resp., Φ), and these Lipschitz and linear growth constants do not depend on \tilde{W} . According to standard estimates of classical fully coupled FBSDEs (see [13]), we have that there exists a constant L_0 depending only on L, L_{Φ}, L_b and p such that (3.4) holds. \square

Let us define $W(t, x) \triangleq \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v}$, $(t, x) \in [0, T] \times \mathbb{R}^n$, where $(X^{t,x;v}, Y^{t,x;v}, Z^{t,x;v})$ is the solution of the fully coupled FBSDE (3.2). Obviously, for the constant L_0 given in Lemma 3.3, we have, for $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^n$, P-a.s.,

$$\begin{aligned} \text{(i)} \quad & |W(t, x)| \leq L_0(1 + |x|); \\ \text{(ii)} \quad & |W(t, x) - W(t, \bar{x})| \leq L_0|x - \bar{x}|. \end{aligned} \tag{3.5}$$

This means, no matter what the Lipschitz and the linear growth constants of \tilde{W} in fully coupled FBSDE (3.2) are, the value function $W(t, x)$ defined by the solution of fully coupled FBSDE (3.2) with \tilde{W} always has the same Lipschitz and linear growth constant L_0 depending only on L , L_Φ and L_b .

Notice that due to the definition as essential infimum over a family of \mathcal{F}_t -measurable random variables, $W(t, x)$ is *a priori* a \mathcal{F}_t -measurable random variable. However, we can even prove that the value function W is deterministic with the Girsanov transformation.

Lemma 3.4. *We suppose the assumptions (H3.1)–(H3.3) hold true. The value function W is deterministic and can be characterized by $E[W]$, i.e., $W(t, x) = E[W(t, x)]$, $(t, x) \in [0, T] \times \mathbb{R}^n$, P-a.s.*

The proof is similar to Proposition 3.1 in Li and Wei [13]. Thus we omit it.

Remark 3.5. The two inequalities for parameters β_1, μ_1 in assumption (H3.2) can be reduced to

$$\beta_1 > \sqrt{3}\tilde{L}(1 + 2e^{\bar{\beta}T}), \quad \mu_1 > \tilde{L}_\Phi + 4\sqrt{3}\tilde{L}(\tilde{L}_\Phi^2 + L_\Phi^2)e^{\bar{\beta}T}, \tag{3.6}$$

where $\bar{\beta} = 4L^2 + 2\tilde{L}^2 + 2L + 1$. In fact, what we really use in the proof of Lemma 3.2 are the two inequalities:

$$\begin{aligned} \beta_1 &> \sqrt{3}(\tilde{L} + \alpha K)(1 + 2e^{T\{4L^2 + 2(\tilde{L} + \alpha K)^2 + 2L + 1\}}), \\ \mu_1 &> \tilde{L}_\Phi + 4\sqrt{3}(\tilde{L} + \alpha K)(\tilde{L}_\Phi^2 + L_\Phi^2)e^{T\{4L^2 + 2(\tilde{L} + \alpha K)^2 + 2L + 1\}}. \end{aligned}$$

Moreover, from Lemma 3.3, we also see that the Lipschitz and the linear growth constant L_0 of W does not depend on the choice of K . Consequently, let $\tilde{W} \equiv 0$, i.e., $K = 0$, the same constant L_0 can still be obtained. Hence, the two inequalities for the parameters β_1, μ_1 in (H3.2) have only to satisfy (3.6).

However, in what follows, we use the following assumption instead of (H3.2).

(H3.2L₀) (i) $\langle B(t, \lambda', \lambda, v) - B(t, \lambda', \bar{\lambda}, v), \lambda - \bar{\lambda} \rangle \leq -\beta_1|\hat{x}|^2$;
 (ii) $\langle G^T(\Phi(x', x) - \Phi(x', \bar{x})), x - \bar{x} \rangle \geq \mu_1|\hat{x}|^2$,
 where $\lambda = (x, y, z)$, $\bar{\lambda} = (\bar{x}, \bar{y}, \bar{z})$, $\lambda' = (x', y', z')$, $\hat{x} = x - \bar{x}$, $\hat{y} = y - \bar{y}$, $\hat{z} = z - \bar{z}$, $x, \bar{x}, x' \in \mathbb{R}^n$, $y, \bar{y}, y' \in \mathbb{R}$, $z, \bar{z}, z' \in \mathbb{R}^d$, $\beta_1 \geq 0$, $\mu_1 \geq 0$, with $\beta_1 > 2(\tilde{L} + \alpha L_0 + \alpha)(1 + 2e^{\beta T})$, $\mu_1 > \tilde{L}_\Phi + 8(\tilde{L} + \alpha L_0 + \alpha)(\tilde{L}_\Phi^2 + L_\Phi^2)e^{\beta T}$, where $\beta = 4L^2 + 2(\tilde{L} + \alpha L_0)^2 + 2L + 2\alpha + 1$, and L, \tilde{L} and α are the Lipschitz constants of $h = b, \sigma, f$ introduced in (H3.1)-(i), and $L_0 > 0$ is given in Lemma 3.3.

Obviously, under the assumption (H3.2L₀), (3.6) holds true.

Let $\mathcal{W} \triangleq \{W : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable} \mid |W(t, 0)| \leq L_0, |W(t, x) - W(t, x')| \leq L_0|x - x'|, \text{ for all } t \in [0, T], x, x' \in \mathbb{R}^n\}$.

Let b, σ, f and Φ satisfy (H3.1), (H3.2L₀) and (H3.3). From Lemma 3.2 we know that, for any $\tilde{W} \in \mathcal{W}$, the fully coupled FBSDE (3.2) has a unique adapted solution $(X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v})_{s \in [t, T]} \in \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(t, T; \mathbb{R}^d)$. We define $\hat{W}(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v}$, $(t, x) \in [0, T] \times \mathbb{R}^n$. Due to Lemma 3.4, \hat{W} is deterministic. Moreover, from (3.4), for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^n$,

$$\text{(i)} \quad |\hat{W}(t, x) - \hat{W}(t, \bar{x})| \leq L_0|x - \bar{x}|; \quad \text{(ii)} \quad |\hat{W}(t, x)| \leq L_0(1 + |x|).$$

Putting now $\psi(\tilde{W}) \triangleq \hat{W}$, we define a mapping $\psi : \mathcal{W} \rightarrow \mathcal{W}$. Let $M > 0$, $K_M \triangleq [0, T] \times \overline{B_M(0)}$, and let us consider the following supremum norm of the restriction of $W \in \mathcal{W}$ to K_M :

Definition 3.6. For $W \in \mathcal{W}$ we define

$$\|W\|_{K_M} \triangleq \sup_{t \in [0, T], |x| \leq M} |W(t, x)|.$$

Now we state the existence and the uniqueness of the fully coupled FBSDE (3.1) involving the value function.

Theorem 3.7. *We suppose that the assumptions (H3.1), (H3.2L₀), (H3.3) hold true. Then the fully coupled FBSDE (3.1) involving the value function has a unique solution $\{(X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v})_{s \in [t, T]} \in \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(t, T; \mathbb{R}^d)$, $(t, x) \in [0, T] \times \mathbb{R}^n$, $W \in \mathcal{W}\}$.*

Proof. Let $W^0(t, x) \equiv 0$, $(t, x) \in [0, T] \times \mathbb{R}^n$. Obviously, $W^0 \in \mathcal{W}$. Moreover, given $W^i \in \mathcal{W}$, $W^{i+1} \triangleq \psi(W^i)$ belongs again to \mathcal{W} . Thus, we get by iteration a sequence $(W^i)_{i \geq 0} \subset \mathcal{W}$.

For $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{V}_{t, T}$, let $(X^{t,x;v,i}, Y^{t,x;v,i}, Z^{t,x;v,i})$ be the solution of the fully coupled FBSDE (3.2) associated with W^i :

$$\begin{cases} dX_s^{t,x;v,i} = E'[b(s, (X_s^{0,x_0;\bar{v},i})', W^i(s, (X_s^{0,x_0;\bar{v},i})'), X_s^{t,x;v,i}, Y_s^{t,x;v,i}, Z_s^{t,x;v,i}, v_s)]ds \\ \quad + E'[\sigma(s, (X_s^{0,x_0;\bar{v},i})', W^i(s, (X_s^{0,x_0;\bar{v},i})'), X_s^{t,x;v,i}, Y_s^{t,x;v,i}, v_s)]dB_s, \\ dY_s^{t,x;v,i} = -E'[f(s, (X_s^{0,x_0;\bar{v},i})', W^i(s, (X_s^{0,x_0;\bar{v},i})'), X_s^{t,x;v,i}, Y_s^{t,x;v,i}, Z_s^{t,x;v,i}, v_s)]ds \\ \quad + Z_s^{t,x;v,i}dB_s, \\ X_t^{t,x;v,i} = x, \quad Y_T^{t,x;v,i} = E'[\Phi((X_T^{0,x_0;\bar{v},i})', X_T^{t,x;v,i})]. \end{cases} \quad (3.7)$$

In the particular case $(t, x, v) = (0, x_0, \bar{v})$, we write $(X^{0,x_0;\bar{v},i}, Y^{0,x_0;\bar{v},i}, Z^{0,x_0;\bar{v},i})$ instead of $(X^{t,x;v,i}, Y^{t,x;v,i}, Z^{t,x;v,i})$.

For any $(t, x) \in [0, T] \times \mathbb{R}^n$ and $v \in \mathcal{V}_{t, T}$, we put, $\hat{X}^{t,x;v,i} \triangleq X^{t,x;v,i+1} - X^{t,x;v,i}$, $\hat{Y}^{t,x;v,i} \triangleq Y^{t,x;v,i+1} - Y^{t,x;v,i}$, $\hat{Z}^{t,x;v,i} \triangleq Z^{t,x;v,i+1} - Z^{t,x;v,i}$, $\hat{W}^i(t, x) \triangleq W^{i+1}(t, x) - W^i(t, x)$, $i \geq 0$. In particular, $\hat{X}^{0,x_0;\bar{v},i} \triangleq X^{0,x_0;\bar{v},i+1} - X^{0,x_0;\bar{v},i}$, $\hat{Y}^{0,x_0;\bar{v},i} \triangleq Y^{0,x_0;\bar{v},i+1} - Y^{0,x_0;\bar{v},i}$, $\hat{Z}^{0,x_0;\bar{v},i} \triangleq Z^{0,x_0;\bar{v},i+1} - Z^{0,x_0;\bar{v},i}$.

We split the proof into the following two steps.

Step 1. We prove that the sequence $(X^{t,x;v,i}, Y^{t,x;v,i}, Z^{t,x;v,i}, W^i)$ converge in $\mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(t, T; \mathbb{R}^d) \times \mathcal{W}$ to some quadruple $(X^{t,x;v}, Y^{t,x;v}, Z^{t,x;v}, W)$, such that $(X^{t,x;v}, Y^{t,x;v}, Z^{t,x;v})$ is the unique solution of the fully coupled FBSDE (3.2) associated with W instead of \tilde{W} .

We prove the existence of the solution with the help of an iteration method.

From Lemma A.3, we know that there exists a constant $C_0 > 0$ independent of $(t, x) \in [0, T] \times \mathbb{R}^n$ (Indeed, C_0 depends only on \tilde{L}, L, L_0 and $L_\Phi, \tilde{L}_\Phi, \alpha$) such that

$$E[|Y_t^{t,x;v,i+1} - Y_t^{t,x;v,i}|^2] \leq \alpha C_0 (\|\hat{W}^i\|_{K_M}^2 + \varepsilon).$$

From the Girsanov transformation argument (see Prop. 3.3 in [4]), we have

$$W^{i+1}(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t, T}} Y_t^{t,x;v,i} = \inf_{v \in \mathcal{V}_{t, T}} E[Y_t^{t,x;v,i}], \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad i \geq 0.$$

Hence, for $(t, x) \in [0, T] \times \mathbb{R}^n$, using that $\hat{W}^{i+1}(t, x) = W^{i+2}(t, x) - W^{i+1}(t, x)$, we get, by Lemma A.3

$$|\hat{W}^{i+1}(t, x)|^2 = |W^{i+2}(t, x) - W^{i+1}(t, x)|^2 \leq \sup_{v \in \mathcal{V}_{t, T}} E|\hat{Y}_t^{t,x;v,i}|^2 \leq \alpha C_0 (\|\hat{W}^i\|_{K_M}^2 + \varepsilon),$$

where C_0 depends only on \tilde{L}, L, L_0 and $L_\Phi, \tilde{L}_\Phi, \alpha$. Therefore, for all $i \geq 0$,

$$\|\hat{W}^{i+1}\|_\infty^2 \leq \alpha C_0 (\|\hat{W}^i\|_{K_M}^2 + \varepsilon) \leq \alpha C_0 (\|\hat{W}^i\|_\infty^2 + \varepsilon).$$

Letting $\varepsilon \downarrow 0$, we get

$$\|\hat{W}^{i+1}\|_\infty^2 \leq \alpha C_0 \|\hat{W}^i\|_\infty^2, \quad i \geq 0. \quad (3.8)$$

Notice that α is small sufficiently such that $\alpha C_0 < 1$ (see Rem. 3.8). Consequently, there exists a deterministic $W : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\|W^i - W\|_\infty \rightarrow 0$, as $i \rightarrow \infty$. Moreover, as $(W^i)_{i \geq 0} \subset \mathcal{W}$, also $W \in \mathcal{W}$.

Using the argument of Lemmas A.1–A.3 again, but now with $E \int_0^T |\hat{W}^i(s, X_s^{0,x_0;\bar{v},i})|^2 ds \leq T \|\hat{W}^i\|_\infty^2$ instead of $E \int_0^T |\hat{W}^i(s, X_s^{0,x_0;\bar{v},i})|^2 ds \leq T \|\hat{W}^i\|_{K_M}^2 + \varepsilon$, we get, for all $i \geq 0$,

$$E \left[\sup_{0 \leq s \leq T} |\hat{X}_s^{0,x_0;\bar{v},i}|^2 + \sup_{t \leq s \leq T} (|\hat{X}_s^{t,x;v,i}|^2 + |\hat{Y}_s^{t,x;v,i}|^2) + \int_t^T |\hat{Z}_s^{t,x;v,i}|^2 ds \right] \leq \alpha C_0 \|\hat{W}^i\|_\infty^2. \quad (3.9)$$

From (3.8) and (3.9) we see that

$$E \left[\sup_{0 \leq s \leq T} |\hat{X}_s^{0,x_0;\bar{v},i}|^2 + \sup_{t \leq s \leq T} (|\hat{X}_s^{t,x;v,i}|^2 + |\hat{Y}_s^{t,x;v,i}|^2) + \int_t^T |\hat{Z}_s^{t,x;v,i}|^2 ds \right] \leq (\alpha C_0)^i \|\hat{W}^0\|_\infty^2. \quad (3.10)$$

Notice that α is small enough such that $\alpha C_0 < 1$, thus there exists $(X^{0,x_0;\bar{v}}, X^{t,x;v}, Y^{t,x;v}, Z^{t,x;v}) \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, T; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(t, T; \mathbb{R}^d)$ such that, as $i \rightarrow \infty$,

$$E \left[\sup_{0 \leq s \leq T} |X_s^{0,x_0;\bar{v},i} - X_s^{0,x_0;\bar{v}}|^2 + \sup_{t \leq s \leq T} (|X_s^{t,x;v,i} - X_s^{t,x;v}|^2 + |Y_s^{t,x;v,i} - Y_s^{t,x;v}|^2) + \int_t^T |Z_s^{t,x;v,i} - Z_s^{t,x;v}|^2 ds \right] \rightarrow 0, \text{ uniformly in } \bar{v} \in \mathcal{V}_{0,T}, v \in \mathcal{V}_{t,T}.$$

Taking the limit in the fully coupled forward-backward SDE (3.7) we get

$$\begin{cases} dX_s^{t,x;v} = E' [b(s, (X_s^{0,x_0;\bar{v}})', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)] ds \\ \quad + E' [\sigma(s, (X_s^{0,x_0;\bar{v}})', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, v_s)] dB_s, \\ dY_s^{t,x;v} = -E' [f(s, (X_s^{0,x_0;\bar{v}})', W(s, (X_s^{0,x_0;\bar{v}})'), X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)] ds \\ \quad + Z_s^{t,x;v} dB_s, \\ X_t^{t,x;v} = x, \quad Y_T^{t,x;v} = E' [\Phi((X_T^{0,x_0;\bar{v}})', X_T^{t,x;v})]. \end{cases} \quad (3.11)$$

Finally, by Lemma 3.2 we know that the solution of the system (3.11) is unique.

Step 2. We prove $W = \bar{W} \triangleq \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v}$.

Recall that $W^{i+1}(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v,i} = \inf_{v \in \mathcal{V}_{t,T}} E[Y_t^{t,x;v,i}]$, $i \geq 0$. We conclude by observing that, for $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$\begin{aligned} |W(t, x) - \inf_{v \in \mathcal{V}_{t,T}} E[Y_t^{t,x;v}]| &\leq \sup_{v \in \mathcal{V}_{t,T}} E[|Y_t^{t,x;v} - Y_t^{t,x;v,i}|] + |W^{i+1}(t, x) - W(t, x)| \\ &\leq \sup_{v \in \mathcal{V}_{t,T}} E[|Y_t^{t,x;v} - Y_t^{t,x;v,i}|] + \|W^{i+1} - W\|_\infty \rightarrow 0, \text{ as } i \rightarrow \infty. \quad \square \end{aligned}$$

Remark 3.8. Indeed, from the proof of Lemma 3.2 and that of Theorem 3.7, we know that α in (H3.1) small enough to satisfy the following conditions:

$$\begin{cases} 1 > \alpha C_0, \\ \beta_1 > \sqrt{3}(\tilde{L} + \alpha K)(1 + 2e^{(4L^2 + 2(\tilde{L} + \alpha K)^2 + 2L + 1)T}), \\ \mu_1 > \tilde{L}_\Phi + 4\sqrt{3}(\tilde{L} + \alpha K)(\tilde{L}_\Phi^2 + L_\Phi^2)e^{(4L^2 + 2(\tilde{L} + \alpha K)^2 + 2L + 1)T}, \end{cases}$$

where C_0 equals to C_{14} in Lemma A.3, \tilde{L} , L , \tilde{L}_Φ , L_Φ are the Lipschitz constants of $h = b$, σ , f and Φ with respect to x' and x , respectively, and K is the Lipschitz constant of \tilde{W} .

4. THE DYNAMIC PROGRAMMING PRINCIPLE

Peng [16] was the first to introduce the notion of “backward semigroup”. From then on, it has been widely used to solve stochastic control problems. Here we extend Peng’s concept of “backward semigroups” to controlled fully coupled FBSDEs involving their value function. Let $x_0 \in \mathbb{R}^n$, $\bar{v} \in \mathcal{V}_{0,T}$ be chosen arbitrarily but fixed. For $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{V}_{t,T}$ and $0 < \delta \leq T - t$, we consider the following fully coupled FBSDE with time horizon $t + \delta$,

$$\begin{cases} d\tilde{X}_s^{t,x;v} = E'[b(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), \tilde{X}_s^{t,x;v}, \tilde{Y}_s^{t,x;v}, \tilde{Z}_s^{t,x;v}, v_s)]ds \\ \quad + E'[\sigma(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), \tilde{X}_s^{t,x;v}, \tilde{Y}_s^{t,x;v}, v_s)]dB_s, \\ d\tilde{Y}_s^{t,x;v} = -E'[f(s, (X_s^{0,x_0;\bar{v}})')', W(s, (X_s^{0,x_0;\bar{v}})'), \tilde{X}_s^{t,x;v}, \tilde{Y}_s^{t,x;v}, \tilde{Z}_s^{t,x;v}, v_s)]ds \\ \quad + \tilde{Z}_s^{t,x;v}dB_s, \\ \tilde{X}_t^{t,x;v} = x, \quad \tilde{Y}_{t+\delta}^{t,x;v} = E'[\Psi((X_{t+\delta}^{0,x_0;\bar{v}})')', \tilde{X}_{t+\delta}^{t,x;v}], \end{cases} \tag{4.1}$$

where $\Psi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lipschitz function with $\langle G^T(\Psi(x', x) - \Psi(x', \bar{x})), x - \bar{x} \rangle \geq 0$, and W is defined as in Theorem 3.7, and $(X^{0,x_0;\bar{v}}, Y^{0,x_0;\bar{v}}, Z^{0,x_0;\bar{v}})$ is the solution of the fully coupled FBSDE (3.1) involving the value function, for our given $x_0 \in \mathbb{R}^n$ and $\bar{v} \in \mathcal{V}_{0,T}$ with time horizon T .

Lemma 4.1. *Under the assumptions (H3.1), (H3.2L₀) and (H3.3), there is a positive constant δ_0 depending on the Lipschitz constant L such that, for $0 \leq \delta \leq \delta_0$, the fully coupled FBSDE (4.1) admits a unique solution $(\tilde{X}_s^{t,x;v}, \tilde{Y}_s^{t,x;v}, \tilde{Z}_s^{t,x;v})_{s \in [t, t+\delta]} \in \mathcal{S}_{\mathbb{F}}^2(t, t + \delta; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, t + \delta; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(t, t + \delta; \mathbb{R}^d)$.*

Proof. From Theorem 3.7 we have the existence of W and $(X_s^{0,x_0;\bar{v}})_{s \in [0, T]} \in \mathcal{S}_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$. Hence, we can define the (deterministic) functions $h^W(s, x, y, z, v) = E[h(s, X_s^{0,x_0;\bar{v}}, W(s, X_s^{0,x_0;\bar{v}}), x, y, z, v)]$, $h = b, \sigma, f$, respectively, $\Psi^W(x) = E[\Psi(X_{t+\delta}^{0,x_0;\bar{v}}, x)]$.

Obviously, h^W is Lipschitz in (x, y, z) , uniformly with respect to (s, v) , with the same Lipschitz constant as that of the functions b, σ, f ; and Ψ^W is Lipschitz in x . According to standard estimates for classical fully coupled FBSDEs (or see Prop. 6.4 in [13] for the general case where σ depends on z), we know that there is a $\delta_0 > 0$ such that, for $s \in [t, t + \delta]$ with $0 \leq \delta \leq \delta_0$, the fully coupled FBSDE (4.1) has a unique adapted solution on time horizon $[t, t + \delta]$, denoted by $(\tilde{X}_s^{t,x;v}, \tilde{Y}_s^{t,x;v}, \tilde{Z}_s^{t,x;v})_{s \in [t, t+\delta]} \in \mathcal{S}_{\mathbb{F}}^2(t, t + \delta; \mathbb{R}^n) \times \mathcal{S}_{\mathbb{F}}^2(t, t + \delta; \mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^2(t, t + \delta; \mathbb{R}^d)$. \square

Now we can define the backward semigroup:

$$G_{s,t+\delta}^{t,x;v}(E'[\Psi((X_{t+\delta}^{0,x_0;\bar{v}})')', \tilde{X}_{t+\delta}^{t,x;v})]) \triangleq \tilde{Y}_s^{t,x;v}, \quad s \in [t, t + \delta],$$

where $(\tilde{X}_s^{t,x;v}, \tilde{Y}_s^{t,x;v}, \tilde{Z}_s^{t,x;v})_{s \in [t, t+\delta]}$ is the solution of FBSDE (4.1).

Inspired by the work of Li, Wei [13], with the help of the notion of extended “backward semigroup”, we can establish the DPP for the value function W defined through the fully coupled FBSDE (3.1).

Theorem 4.2. (DPP) *Suppose the assumptions (H3.1), (H3.2L₀) and (H3.3) hold. Then there is a positive δ_0 depending on L such that, for all $0 \leq t \leq T - \delta$ with $0 < \delta \leq \delta_0$,*

$$W(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,t+\delta}} G_{t,t+\delta}^{t,x;v}(W(t + \delta, \tilde{X}_{t+\delta}^{t,x;v})).$$

The proof of Theorem 4.2 uses the same argument as that of Buckdahn and Li [4] and is omitted here.

Remark 4.3. According to the definition of backward semigroup, we get, for $t \leq s \leq t + \delta$, $v \in \mathcal{V}_{t,T}$, $x \in \mathbb{R}^n$,

$$G_{s,t+\delta}^{t,x;v}(W(t + \delta, \tilde{X}_{t+\delta}^{t,x;v})) = \tilde{Y}_s^{t,x;v},$$

where $(\tilde{X}_s^{t,x;v}, \tilde{Y}_s^{t,x;v}, \tilde{Z}_s^{t,x;v})_{s \in [t, t+\delta]}$ is the solution of (4.1) with $\Psi(x', x) = W(t + \delta, x)$, $(x', x) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proposition 4.4. *The value function W introduced in Theorem 3.7 has for some suitable constant \bar{L} the following property of Hölder continuity in t :*

$$|W(t, x) - W(\bar{t}, x)| \leq \bar{L}(1 + |x|)|t - \bar{t}|^{\frac{1}{2}}, \quad t, \bar{t} \in [0, T], \quad x \in \mathbb{R}^n.$$

Proof. The functions $b^W, \sigma^W, f^W, \Psi^W$ defined in the proof of Lemma 4.1 are jointly continuous and Lipschitz with respect to (x, y, z) and x , respectively, with the same Lipschitz constants as those of b, σ, f and Ψ . Moreover, from (H3.3) we know that for $s \in [0, T], v \in V, |h^W(s, 0, 0, 0, v)| \leq L_b$. Hence, from standard conclusions for classical fully coupled FBSDEs (see Thm. 3.2 in [13]), we have the above estimate. \square

5. VISCOSITY SOLUTION OF THE ASSOCIATED NONLOCAL HJB EQUATION

In this section, we discuss the associated nonlocal HJB equation and give its probability interpretation in term of the solution of FBSDE (3.1).

Let us consider the PDE

$$\begin{cases} \partial_t W(t, x) + H(t, x, W(t, \cdot), W(t, x), DW(t, x), D^2W(t, x)) = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ W(T, x) = E[\Phi(X_T^{0, x_0; \bar{v}}), x], & x \in \mathbb{R}^n, \end{cases} \quad (5.1)$$

where the Hamiltonian H is defined by

$$\begin{aligned} & H(t, x, \psi(t, \cdot), y, p, X) \\ &= \inf_{v \in V} \left\{ \frac{1}{2} \text{tr}(\bar{\sigma} \bar{\sigma}^T(t, x, \psi(t, \cdot), y, v) X) + p \cdot \bar{b}(t, x, \psi(t, \cdot), y, p \cdot \bar{\sigma}, v) + \bar{f}(t, x, \psi(t, \cdot), y, p \cdot \bar{\sigma}, v) \right\}, \end{aligned} \quad (5.2)$$

for $t \in [0, T], y \in \mathbb{R}, x, p \in \mathbb{R}^n, X \in \mathbf{S}^d$ (by \mathbf{S}^d we denote all the $d \times d$ symmetric matrix), and $\psi(\cdot)$ is a deterministic Lipschitz function, and

$$\begin{aligned} \bar{b}(t, x, \psi(t, \cdot), y, z, v) &= E[b(t, X_t^{0, x_0; \bar{v}}, \psi(t, X_t^{0, x_0; \bar{v}}), x, y, z, v)]; \\ \bar{\sigma}(t, x, \psi(t, \cdot), y, v) &= E[\sigma(t, X_t^{0, x_0; \bar{v}}, \psi(t, X_t^{0, x_0; \bar{v}}), x, y, v)]; \\ \bar{f}(t, x, \psi(t, \cdot), y, z, v) &= E[f(t, X_t^{0, x_0; \bar{v}}, \psi(t, X_t^{0, x_0; \bar{v}}), x, y, z, v)]. \end{aligned} \quad (5.3)$$

Here $(X^{0, x_0; \bar{v}}, Y^{0, x_0; \bar{v}}, Z^{0, x_0; \bar{v}})$ is the solution of the fully coupled FBSDE (3.1) involving the value function for initial time $t = 0$, initial value $x = x_0$ and fixed $\bar{v} \in \mathcal{V}_{0, T}$.

Let us recall the definition of a viscosity solution of the associated HJB equation (5.1) (see [8] for details).

Definition 5.1. A real-valued continuous function $V \in C([0, T] \times \mathbb{R}^n)$ is called

(i) a viscosity subsolution of equation (5.1), if $V(T, x) \leq E[\Phi(X_T^{0, x_0; \bar{v}}), x]$, for all $x \in \mathbb{R}^n$, and if for all function $\phi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$ such that $V - \phi$ attains a local maximum at (t, x) ,

$$\partial_t \phi(t, x) + H(t, x, V(t, \cdot), V(t, x), D\phi(t, x), D^2\phi(t, x)) \geq 0;$$

(ii) a viscosity supersolution of equation (5.1), if $V(T, x) \geq E[\Phi(X_T^{0, x_0; \bar{v}}), x]$, for all $x \in \mathbb{R}^n$, and if for all function $\phi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$ such that $V - \phi$ attains a local minimum at (t, x) ,

$$\partial_t \phi(t, x) + H(t, x, V(t, \cdot), V(t, x), D\phi(t, x), D^2\phi(t, x)) \leq 0;$$

(iii) a viscosity solution of equation (5.1) if it is both a viscosity subsolution and supersolution of equation (5.1).

Now we state the main result of this section.

Theorem 5.2. *Assume that the assumptions (H3.1), (H3.2L₀) and (H3.3) hold true. Then the value function $W(t, x) \in C([0, T] \times \mathbb{R}^n)$ given in Theorem 3.7 is a viscosity solution of HJB equation (5.1).*

Proof. According to Theorem 3.7 and Proposition 4.4, we know that the deterministic function W is Lipschitz in x and $\frac{1}{2}$ -Hölder continuous in t . More precisely, there exists a positive constant \tilde{L} such that, for $x, \bar{x} \in \mathbb{R}^n$, $t, \bar{t} \in [0, T]$,

$$(i) |W(t, x) - W(t, \bar{x})| \leq \tilde{L}|x - \bar{x}|; \quad (ii) |W(t, x) - W(\bar{t}, x)| \leq \tilde{L}(1 + |x|)|t - \bar{t}|^{\frac{1}{2}}.$$

Consequently, $W(t, x) \in C([0, T] \times \mathbb{R}^n)$ is of linear growth.

The coefficients $\bar{b}(t, x, W(t, \cdot), y, z, v)$, $\bar{\sigma}(t, x, W(t, \cdot), y, z, v)$, $\bar{f}(t, x, W(t, \cdot), y, z, v)$ and $\bar{\Phi}(x) = E[\Phi(X_T^{0, x_0; \bar{v}}), x]$ are Lipschitz in (x, y, z) , uniformly in (t, v) , and x , respectively, and satisfy the monotonicity condition of classical fully coupled FBSDEs. Hence, our fully coupled FBSDE (3.1) involving its value function W can be written as

$$\begin{cases} dX_s^{t,x;v} = \bar{b}(s, X_s^{t,x;v}, W(s, \cdot), Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)ds + \bar{\sigma}(s, X_s^{t,x;v}, W(s, \cdot), Y_s^{t,x;v}, v_s)dB_s, \\ dY_s^{t,x;v} = -\bar{f}(s, X_s^{t,x;v}, W(s, \cdot), Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)ds + Z_s^{t,x;v}dB_s, \\ X_t^{t,x;v} = x, \quad Y_T^{t,x;v} = \bar{\Phi}(X_T^{t,x;v}), \\ W(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v}. \end{cases}$$

From the probability interpretation for HJB equations associated with controlled fully coupled FBSDEs (see Thm. 4.1 in Li and Wei [13]), we know that W is a viscosity solution of HJB equation (5.1). \square

Theorem 5.3. *When σ also does not depend on y , under the assumptions (H3.1), (H3.2L₀), (H3.3) the value function $W(t, x)$ defined in Theorem 3.7 is the unique viscosity solution of HJB equation (5.1) in the class $\Theta = \{\varphi \in C([0, T] \times \mathbb{R}^n) : \exists \tilde{A} > 0 \text{ such that } \lim_{|x| \rightarrow \infty} |\varphi(t, x)| \exp\{-\tilde{A}[\log(|x|^2 + 1)]^{\frac{1}{2}}\} = 0, \text{ uniformly in } t \in [0, T]\}$.*

Proof. Let $W^i \in \Theta$, $i = 1, 2$, be two viscosity solutions of HJB equation (5.1). We define

$$\begin{aligned} \tilde{b}(s, x, W^i(s, \cdot), y, z, v) &= E[b(s, X_s^{0, x_0; \bar{v}}, x, W^i(s, X_s^{0, x_0; \bar{v}}), y, z, v)], \\ \tilde{\sigma}(s, x, W^i(s, \cdot), v) &= E[\sigma(s, X_s^{0, x_0; \bar{v}}, x, W^i(s, X_s^{0, x_0; \bar{v}}), v)], \\ \tilde{f}(s, x, W^i(s, \cdot), y, z, v) &= E[f(s, X_s^{0, x_0; \bar{v}}, x, W^i(s, X_s^{0, x_0; \bar{v}}), y, z, v)], \\ \tilde{\Phi}(x) &= E[\Phi(X_T^{0, x_0; \bar{v}}, x)], \end{aligned}$$

$(s, x, y, z, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times V$. Hence $W^i(t, x) \in \Theta$ is a viscosity solution of the following HJB equation

$$\begin{cases} \partial_t W^i(t, x) + \inf_{v \in V} \left\{ \frac{1}{2} \operatorname{tr}(\tilde{\sigma} \tilde{\sigma}^T(t, x, W^i(t, \cdot), v) D^2 W^i(t, x)) \right. \\ \quad + DW^i(t, x) \cdot \tilde{b}(t, x, W^i(t, \cdot), W^i(t, x), DW^i(t, x) \cdot \tilde{\sigma}(t, x, W^i(t, \cdot), v), v) \\ \quad \left. + \tilde{f}(t, x, W^i(t, \cdot), W^i(t, x), DW^i(t, x) \cdot \tilde{\sigma}(t, x, W^i(t, \cdot), v), v) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \\ W^i(T, x) = \tilde{\Phi}(x), \quad x \in \mathbb{R}^n. \end{cases}$$

Since \tilde{b} , $\tilde{\sigma}$, \tilde{f} and $\tilde{\Phi}$ satisfy all the assumptions for classical FBSDEs and $W^i(t, x) \in \Theta$, *i.e.*, $W^i(t, x)$ is of at most linear growth, $W^i(t, x)$ possesses the stochastic interpretation as the value function

$$W^i(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{i,t,x;v}, \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

of the optimal control problem for system driven by

$$\begin{cases} dX_s^{i,t,x;v} = \tilde{b}(s, X_s^{i,t,x;v}, W^i(s, \cdot), Y_s^{i,t,x;v}, Z_s^{i,t,x;v}, v_s)ds + \tilde{\sigma}(s, X_s^{i,t,x;v}, W^i(s, \cdot), v_s)dB_s, \\ dY_s^{i,t,x;v} = -\tilde{f}(s, X_s^{i,t,x;v}, W^i(s, \cdot), Y_s^{i,t,x;v}, Z_s^{i,t,x;v}, v_s)ds + Z_s^{i,t,x;v}dB_s, \quad s \in [t, T], \\ X_t^{i,t,x;v} = x, \quad Y_T^{i,t,x;v} = \tilde{\Phi}(X_T^{i,t,x;v}), \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad v \in \mathcal{V}_{t,T} \end{cases} \quad (5.4)$$

(see, e.g., Thms. 4.1 and 5.1 in Li and Wei [14]). Recall the definitions of \tilde{b} , $\tilde{\sigma}$, \tilde{f} and $\tilde{\Phi}$, and $W^i(t, x)$ is Lipschitz and of linear growth in x , (5.4) is in fact the following equation:

$$\begin{cases} dX_s^{i,t,x;v} = E'[b(s, (X_s^{i,0,x_0;\bar{v}})^\prime, X_s^{i,t,x;v}, W^i(s, (X_s^{i,0,x_0;\bar{v}})^\prime), Y_s^{i,t,x;v}, Z_s^{i,t,x;v}, v_s)]ds \\ \quad + E'[\sigma(s, (X_s^{i,0,x_0;\bar{v}})^\prime, X_s^{i,t,x;v}, W^i(s, (X_s^{i,0,x_0;\bar{v}})^\prime), v_s)]dB_s, \\ dY_s^{i,t,x;v} = -E'[f(s, (X_s^{i,0,x_0;\bar{v}})^\prime, X_s^{i,t,x;v}, W^i(s, (X_s^{i,0,x_0;\bar{v}})^\prime), Y_s^{i,t,x;v}, Z_s^{i,t,x;v}, v_s)]ds \\ \quad + Z_s^{i,t,x;v}dB_s, \quad s \in [t, T], \\ X_t^{i,t,x;v} = x, \quad Y_T^{i,t,x;v} = E'[\Phi((X_T^{i,0,x_0;\bar{v}})^\prime, X_T^{i,t,x;v})], \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad v \in \mathcal{V}_{t,T}, \\ W^i(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{i,t,x;v}, \quad (t, x) \in [0, T] \times \mathbb{R}^n. \end{cases} \quad (5.5)$$

However, from Theorem 3.7 we know that (5.5) has a unique solution. It means $W^1(t, x) = W^2(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^n$. The proof is complete. \square

Example 5.4. Let $n = d = 1, G = 1$ and $V = [0, \frac{\pi}{2}]$. Frozen $x_0 \in \mathbb{R}$ and $\bar{v} \in \mathcal{V}_{0,T}$, for $(t, x) \in [0, T] \times \mathbb{R}$ and $v \in \mathcal{V}_{t,T}$, we first consider the controlled fully coupled mean-field FBSDE:

$$\begin{cases} d\bar{X}_s^{t,x;v} = (-\beta\bar{Y}_s^{t,x;v} + v_s)ds - (\beta\bar{X}_s^{t,x;v} + v_s)dB_s, \\ -d\bar{Y}_s^{t,x;v} = (\alpha E[\arctan\bar{X}_s^{0,x_0;\bar{v}}] + \beta\bar{X}_s^{t,x;v} - \beta\bar{Z}_s^{t,x;v} - v_s)ds - \bar{Z}_s^{t,x;v}dB_s, \\ \bar{X}_t^{t,x;v} = x, \\ \bar{Y}_T^{t,x;v} = \bar{X}_T^{t,x;v}, \end{cases} \quad (5.6)$$

where β and α satisfy

$$\begin{cases} \beta > 4\alpha \left(1 + 2e^{T\{4\beta^2+2\alpha^2+2\beta+2\alpha+1\}}\right), \\ 1 > 16\alpha e^{T\{4\beta^2+2\alpha^2+2\beta+2\alpha+1\}}. \end{cases} \quad (5.7)$$

We remark that (5.6) does not involve the value function itself. According to Remark 3.5, we know that under the assumption (5.7), the fully coupled FBSDE (5.6) has a unique solution $(\bar{X}^{t,x;v}, \bar{Y}^{t,x;v}, \bar{Z}^{t,x;v})$. We define $\bar{W}(t, x) \triangleq \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} \bar{Y}_t^{t,x;v}$. Applying standard estimate technique for classical fully coupled FBSDEs, we obtain for $t \in [0, T], x, \bar{x} \in \mathbb{R}$,

$$(i) |\bar{W}(t, x) - \bar{W}(t, \bar{x})| \leq L_1|x - \bar{x}|; \quad (ii) |\bar{W}(t, x)| \leq L_2(1 + |x|),$$

where $L_1 = (C_1C_3e^{C_2C_3T})^{\frac{1}{2}}$, $L_2 = \{(C_4C_8) \vee (C_5C_8 + C_7)\}^{\frac{1}{2}}e^{\frac{1}{2}C_6C_8T}$ and

$$\begin{aligned} C_1 &= 3 + 36T\beta^2e^{3T\beta^2}, \quad C_2 = 3T\beta^2 + 36T^2\beta^4e^{3T\beta^2}, \quad C_3 = \frac{1}{2}(2 + T)e^{(4\beta^2+1)T}, \\ C_4 &= 3 + 72T\beta^2e^{6T\beta^2}, \quad C_5 = \frac{3}{2}\pi^2T^2 + 6\pi^2T + 36\beta^2\pi^2(T^3 + T^2)e^{6T\beta^2}, \quad C_6 = 144T^2\beta^4e^{6T\beta^2} + 6T\beta^2, \\ C_7 &= Te^{(4\beta^2+2\pi^2+1)T}, \quad C_8 = \frac{1}{2}(2 + T)e^{(4\beta^2+2\pi^2+1)T}. \end{aligned}$$

Taking $L_0 = L_1 \vee L_2$ and choosing α and β with

$$\begin{cases} \beta > 2(2 + L_0)\alpha \left(1 + 2e^{T\{4\beta^2 + 2\alpha^2(1+L_0)^2 + 2\beta + 2\alpha + 1\}}\right), \\ 1 > 8(2 + L_0)\alpha e^{T\{4\beta^2 + 2\alpha^2(1+L_0)^2 + 2\beta + 2\alpha + 1\}}, \end{cases} \quad (5.8)$$

from Theorem 3.7, the following FBSDE involving the value function:

$$\begin{cases} dX_s^{t,x;v} = (-\alpha E[\arctan W(s, X_s^{0,x_0;\bar{v}})] - \beta Y_s^{t,x;v} + v_s)ds - (\beta X_s^{t,x;v} + v_s)dB_s, \\ -dY_s^{t,x;v} = (\alpha E[\arctan X_s^{0,x_0;\bar{v}}] + \beta X_s^{t,x;v} - \beta Z_s^{t,x;v} - v_s)ds - Z_s^{t,x;v}dB_s, \\ X_t^{t,x;v} = x, \quad Y_T^{t,x;v} = X_T^{t,x;v}, \\ W(t, x) = \operatorname{ess\,inf}_{v \in \mathcal{V}_{t,T}} Y_t^{t,x;v}, \end{cases} \quad (5.9)$$

has a unique solution, denoted by $(X^{t,x;v}, Y^{t,x;v}, Z^{t,x;v}, W)$. Moreover, from Theorem 5.2 the value function W is a viscosity solution of the nonlocal HJB equation:

$$\begin{cases} \partial_t W + \inf_{v \in [0, \frac{\pi}{2}]} \left\{ \frac{1}{2}(\beta x + v)^2 \partial_{xx} W - (\alpha E[\arctan W(t, X_t^{0,x_0;\bar{v}})] + \beta W - \beta^2 x - (\beta + 1)v) \partial_x W \right. \\ \left. + \alpha E[\arctan X_t^{0,x_0;\bar{v}}] + \beta x - v \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \\ W(T, x) = x, \quad x \in \mathbb{R}. \end{cases}$$

APPENDIX

In this section we prove some auxiliary lemmas for Theorem 3.7.

Let $W^i \in \mathcal{W}$. In order to make notations concisely, we denote by (X^i, Y^i, Z^i) (resp., $(X^{0,i}, Y^{0,i}, Z^{0,i})$) the solution of the fully coupled FBSDE (3.7) for $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{V}_{t,T}$ (resp., for $(t, x, v) = (0, x_0, \bar{v})$).

For any $(t, x) \in [0, T] \times \mathbb{R}^n$ and $v \in \mathcal{V}_{t,T}$, we set, $\hat{X}^i \triangleq X^{i+1} - X^i$, $\hat{Y}^i \triangleq Y^{i+1} - Y^i$, $\hat{Z}^i \triangleq Z^{i+1} - Z^i$, $\hat{W}^i(t, x) \triangleq W^{i+1}(t, x) - W^i(t, x)$, $i \geq 0$. Accordingly, $\hat{X}^{0,i} \triangleq X^{0,i+1} - X^{0,i}$, $\hat{Y}^{0,i} \triangleq Y^{0,i+1} - Y^{0,i}$, $\hat{Z}^{0,i} \triangleq Z^{0,i+1} - Z^{0,i}$.

Lemma A.1. *Under the assumptions (H3.1), (H3.2L₀) and (H3.3), for any $\varepsilon > 0$, there exists some M large enough such that, for all $i \in \mathbb{N}$,*

$$E \int_0^T |\hat{W}^i(s, X_s^{0,i})|^2 ds \leq T \|\hat{W}^i\|_{K_M}^2 + \varepsilon.$$

Proof. For arbitrarily given $M > 0$, we have, for all $i \geq 0$,

$$\begin{aligned} E \int_0^T |\hat{W}^i(s, X_s^{0,i})|^2 ds &\leq \int_0^T \|\hat{W}^i\|_{K_M}^2 ds + E \int_0^T |\hat{W}^i(s, X_s^{0,i})|^2 I_{\{|X_s^{0,i}| > M\}} ds \\ &\leq T \|\hat{W}^i\|_{K_M}^2 + 8L_0^2 E \int_0^T (1 + |X_s^{0,i}|^2) I_{\{|X_s^{0,i}| > M\}} ds \leq T \|\hat{W}^i\|_{K_M}^2 + \frac{8L_0^2}{M} E \int_0^T (|X_s^{0,i}| + |X_s^{0,i}|^3) ds. \end{aligned} \quad (A.1)$$

From Lemma 3.3 (ii), for $(t, x, v) = (0, x_0, \bar{v})$, $\sup_{i \geq 0} E[\int_0^T (|X_s^{0,i}| + |X_s^{0,i}|^3) ds] \leq TL_0^3(1 + |x_0| + |x_0|^3)$.

Hence, for any $\varepsilon > 0$, there exists some M enough large such that, for all $i \in \mathbb{N}$,

$$E \int_0^T |\hat{W}^i(s, X_s^{0,i})|^2 ds \leq T \|\hat{W}^i\|_{K_M}^2 + \varepsilon. \quad \square$$

Lemma A.2. *We suppose that the assumptions (H3.1), (H3.2L₀), (H3.3) hold true. Then there exists a constant $C_0 > 0$ depending only on $\tilde{L}, L, L_0, L_\Phi, \tilde{L}_\Phi, \alpha$ such that*

$$E \left[\sup_{0 \leq s \leq T} (|\hat{X}_s^{0,i}|^2 + |\hat{Y}_s^{0,i}|^2) + \int_0^T |\hat{Z}_s^{0,i}|^2 ds \right] \leq \alpha C_0 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right).$$

Proof. For the forward SDE of (3.7) and $(t, x, v) = (0, x_0, \bar{v})$, $i \geq 0$, we have from standard estimates and Lemma A.1 that

$$E \left[\sup_{0 \leq s \leq T} |\hat{X}_s^{0,i}|^2 \right] \leq C_0 T \alpha^2 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right) + C_0 L^2 E \int_0^T (|\hat{Y}_s^{0,i}|^2 + |\hat{Z}_s^{0,i}|^2) ds, \tag{A.2}$$

where $C_0 = (10T + 32)(1 + \kappa e^\kappa)$ and $\kappa = (10T + 8)\{(\tilde{L} + \alpha L_0)^2 + L^2\}T$.

To estimate the backward SDE of (3.7) for $(t, x, v) = (0, x_0, \bar{v})$, we apply the Itô's formula to $e^{\beta s} |\hat{Y}_s^{0,i}|^2$ (Recall that $\beta = 4L^2 + 2(\tilde{L} + \alpha L_0)^2 + 2L + 2\alpha + 1$). Then, thanks to Lemma (A.1) and to standard estimates we have, for $0 \leq s \leq T$, $i \geq 0$,

$$E[|\hat{Y}_s^{0,i}|^2 + \int_s^T (|\hat{Y}_r^{0,i}|^2 + |\hat{Z}_r^{0,i}|^2) dr] \leq 4(\tilde{L}_\Phi^2 + L_\Phi^2) e^{\beta T} E|\hat{X}_T^{0,i}|^2 + 2e^{\beta T} E \int_s^T |\hat{X}_r^{0,i}|^2 dr + \alpha T e^{\beta T} \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right). \tag{A.3}$$

On the other hand,

$$\begin{aligned} E \left[\left\langle \hat{X}_T^{0,i}, G^T \hat{Y}_T^{0,i} \right\rangle \right] &= E \left[E' \left[\left\langle \hat{X}_T^{0,i}, G^T (\Phi((X_T^{0,i+1})', X_T^{0,i+1}) - \Phi((X_T^{0,i})', X_T^{0,i})) \right\rangle \right] \right] \\ &\geq (\mu_1 - \tilde{L}_\Phi) E \left| \hat{X}_T^{0,i} \right|^2. \end{aligned}$$

We denote

$$\hat{B}^{0,i}(s) = \begin{pmatrix} -G^T \hat{f}_s^{0,i} \\ G \hat{b}_s^{0,i} \\ G \hat{\sigma}_s^{0,i} \end{pmatrix},$$

where

$$\begin{aligned} \hat{f}_s^{0,i} &= E'[f(s, (X_s^{0,i+1})', W^{i+1}(s, (X_s^{0,i+1})'), X_s^{0,i+1}, Y_s^{0,i+1}, Z_s^{0,i+1}, \bar{v}_s)] \\ &\quad - E'[f(s, (X_s^{0,i})', W^i(s, (X_s^{0,i})'), X_s^{0,i}, Y_s^{0,i}, Z_s^{0,i}, \bar{v}_s)], \\ \hat{b}_s^{0,i} &= E'[b(s, (X_s^{0,i+1})', W^{i+1}(s, (X_s^{0,i+1})'), X_s^{0,i+1}, Y_s^{0,i+1}, Z_s^{0,i+1}, \bar{v}_s)] \\ &\quad - E'[b(s, (X_s^{0,i})', W^i(s, (X_s^{0,i})'), X_s^{0,i}, Y_s^{0,i}, Z_s^{0,i}, \bar{v}_s)], \\ \hat{\sigma}_s^{0,i} &= E'[\sigma(s, (X_s^{0,i+1})', W^{i+1}(s, (X_s^{0,i+1})'), X_s^{0,i+1}, Y_s^{0,i+1}, \bar{v}_s)] \\ &\quad - E'[\sigma(s, (X_s^{0,i})', W^i(s, (X_s^{0,i})'), X_s^{0,i}, Y_s^{0,i}, \bar{v}_s)]. \end{aligned}$$

Then, due to (H3.1) and (H3.2L₀),

$$\begin{aligned} &\left| E \left[\left\langle \hat{B}^{0,i}(s), \left(\hat{X}_s^{0,i}, \hat{Y}_s^{0,i}, \hat{Z}_s^{0,i} \right) \right\rangle \right] \right| \\ &\leq E \left[-\beta_1 |\hat{X}_s^{0,i}|^2 \right] + E \left[\sqrt{3} \left\{ (\tilde{L} + \alpha L_0) E' |\widehat{(X_s^{0,i})'}| + \alpha E' |\hat{W}^i(s, (X_s^{0,i})')| \right\} \sqrt{|\hat{X}_s^{0,i}|^2 + |\hat{Y}_s^{0,i}|^2 + |\hat{Z}_s^{0,i}|^2} \right]. \end{aligned}$$

Hence, we have, for $0 \leq s \leq T$,

$$\begin{aligned} E \left[\left\langle \hat{X}_s^{0,i}, G^T \hat{Y}_s^{0,i} \right\rangle \right] &= E \left[\left\langle \hat{X}_T^{0,i}, G^T \hat{Y}_T^{0,i} \right\rangle \right] - E \left[\int_s^T \left\langle \hat{B}^{0,i}(r), \left(\hat{X}_r^{0,i}, \hat{Y}_r^{0,i}, \hat{Z}_r^{0,i} \right) \right\rangle dr \right] \\ &\geq (\mu_1 - \tilde{L}_\Phi) E |\hat{X}_T^{0,i}|^2 + (\beta_1 - 2(\tilde{L} + \alpha L_0 + \alpha)) E \int_s^T |\hat{X}_r^{0,i}|^2 dr \\ &\quad - \frac{1}{2} (\tilde{L} + \alpha L_0 + \alpha) E \int_s^T (|\hat{Y}_r^{0,i}|^2 + |\hat{Z}_r^{0,i}|^2) dr - \frac{3}{2} \alpha T \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right). \tag{A.4} \end{aligned}$$

From (A.3) and (A.4),

$$\begin{aligned}
 E \left[\left\langle \hat{X}_s^{0,i}, G^T \hat{Y}_s^{0,i} \right\rangle \right] &\geq \left(\mu_1 - \tilde{L}_\Phi - 2(\tilde{L} + \alpha L_0 + \alpha)(\tilde{L}_\Phi^2 + L_\Phi^2) e^{\beta T} \right) E |\hat{X}_T^{0,i}|^2 \\
 &+ \left\{ \beta_1 - 2(\tilde{L} + \alpha L_0 + \alpha) \left(1 + \frac{1}{2} e^{\beta T} \right) \right\} E \int_s^T |\hat{X}_r^{0,i}|^2 dr - \alpha \left(\frac{3}{2} T + \frac{1}{2} T (\tilde{L} + \alpha L_0 + \alpha) e^{\beta T} \right) \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right),
 \end{aligned}
 \tag{A.5}$$

where β is given in (H3.2L₀).

Since β_1 and μ_1 satisfy

$$\begin{aligned}
 &\beta_1 \left\langle 2(\tilde{L} + \alpha L_0 + \alpha)(1 + 2e^{\beta T}) \right\rangle 2(\tilde{L} + \alpha L_0 + \alpha) \left(1 + \frac{1}{2} e^{\beta T} \right), \\
 &\mu_1 \tilde{L}_\Phi + 8 \left(\tilde{L} + \alpha L_0 + \alpha \right) \left(\tilde{L}_\Phi^2 + L_\Phi^2 \right) e^{\beta T} \tilde{L}_\Phi + 2 \left(\tilde{L} + \alpha L_0 + \alpha \right) \left(\tilde{L}_\Phi^2 + L_\Phi^2 \right) e^{\beta T},
 \end{aligned}$$

we have, for $s = 0$,

$$E \int_0^T |\hat{X}_r^{0,i}|^2 dr \leq \alpha C_1 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right)
 \tag{A.6}$$

and

$$E |\hat{X}_T^{0,i}|^2 \leq \alpha C_2 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right),
 \tag{A.7}$$

where

$$C_1 = \frac{\frac{3}{2} T + \frac{1}{2} T (\tilde{L} + \alpha L_0 + \alpha) e^{\beta T}}{\beta_1 - 2(\tilde{L} + \alpha L_0 + \alpha) \left(1 + \frac{1}{2} e^{\beta T} \right)}, \quad C_2 = \frac{\frac{3}{2} T + \frac{1}{2} T (\tilde{L} + \alpha L_0 + \alpha) e^{\beta T}}{\mu_1 - \tilde{L}_\Phi - 2(\tilde{L} + \alpha L_0 + \alpha)(\tilde{L}_\Phi^2 + L_\Phi^2) e^{\beta T}}.$$

According to (A.6), (A.7) and (A.3), for $0 \leq s \leq T$,

$$E |\hat{Y}_s^{0,i}|^2 + E \int_s^T (|\hat{Y}_r^{0,i}|^2 + |\hat{Z}_r^{0,i}|^2) dr \leq \alpha C_3 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right),
 \tag{A.8}$$

where $C_3 = 4(\tilde{L}_\Phi^2 + L_\Phi^2) e^{\beta T} C_2 + 2e^{\beta T} C_1 + T e^{\beta T}$.

From (A.2) it follows

$$E \left[\sup_{0 \leq s \leq T} |\hat{X}_s^{0,i}|^2 \right] \leq \alpha C_4 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right),
 \tag{A.9}$$

where $C_4 = C_0(T\alpha + C_3L^2)$. Estimating the backward SDE in (3.7), we have

$$\begin{aligned}
 E \left[\sup_{0 \leq s \leq T} |\hat{Y}_s^{0,i}|^2 + \int_0^T |\hat{Z}_s^{0,i}|^2 ds \right] &\leq 6(\tilde{L}_\Phi^2 + L_\Phi^2) E |\hat{X}_T^{0,i}|^2 + 15T \{ (\tilde{L} + \alpha L_0)^2 + L^2 \} E \left[\int_0^T |\hat{X}_s^{0,i}|^2 ds \right] \\
 &+ (15TL^2 + 13) E \left[\int_0^T (|\hat{Y}_s^{0,i}|^2 + |\hat{Z}_s^{0,i}|^2) ds \right] + 15T^2 \alpha^2 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right).
 \end{aligned}
 \tag{A.10}$$

From (A.6)–(A.10),

$$E \left[\sup_{0 \leq s \leq T} (|\hat{X}_s^{0,i}|^2 + |\hat{Y}_s^{0,i}|^2) + \int_0^T |\hat{Z}_s^{0,i}|^2 ds \right] \leq \alpha C_5 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right),
 \tag{A.11}$$

where $C_5 = 6(\tilde{L}_\Phi^2 + L_\Phi^2) C_2 + 15T \{ (\tilde{L} + \alpha L_0)^2 + L^2 \} C_1 + (15TL^2 + 13) C_3 + 15T^2 \alpha + C_4$. □

For the difference of solution (X^i, Y^i, Z^i) , we have the similar estimate.

In the following lemma, we still use the notations C_1 to C_5 in Lemma A.2.

Lemma A.3. *We make the same assumptions as in Lemma A.2. Then there exists a $C_0 > 0$ depending only on $\tilde{L}, L, L_0, L_\Phi, \tilde{L}_\Phi, \alpha$ such that, for $(t, x) \in [0, T] \times \mathbb{R}^n$ and $v \in \mathcal{V}_{t,T}$,*

$$E \left[\sup_{t \leq s \leq T} (|\hat{X}_s^i|^2 + |\hat{Y}_s^i|^2) + \int_t^T |\hat{Z}_s^i|^2 ds \right] \leq \alpha C_0 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right).$$

Proof. From (3.7) and Lemma A.2 we have, for $0 \leq t \leq s \leq T$, $i \geq 0$,

$$E \left[\sup_{t \leq s \leq T} |\hat{X}_s^i|^2 \right] \leq \alpha C_6 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right) + C_7 E \int_t^T (|\hat{Y}_r^i|^2 + |\hat{Z}_r^i|^2) dr, \quad (\text{A.12})$$

and

$$E|\hat{Y}_s^i|^2 + E \int_s^T (|\hat{Y}_r^i|^2 + |\hat{Z}_r^i|^2) dr \leq 4e^{\beta T} L_\Phi^2 E|\hat{X}_T^i|^2 + 2e^{\beta T} E \int_s^T |\hat{X}_r^i|^2 dr + \alpha C_8 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right), \quad (\text{A.13})$$

where

$$C_6 = (10T + 32)T\{(\tilde{L} + \alpha L_0)^2 C_4 + \alpha + L^2 e^{(10T+8)L^2 T} (10T + 8)T[(\tilde{L} + \alpha L_0)^2 C_4 + \alpha]\},$$

$$C_7 = (10T + 32)L^2[1 + (10T + 8)L^2 T e^{(10T+8)L^2 T}], \quad C_8 = T e^{\beta T} + 4e^{\beta T} \tilde{L}_\Phi^2 C_2 + e^{\beta T} C_4 T.$$

Furthermore, from our assumption (H3.1) and (H3.2L₀) as well as (A.7) we obtain, for $i \geq 0$,

$$\begin{aligned} E \left[\langle \hat{X}_T^i, \hat{Y}_T^i \rangle \right] &= E \left[E' \left[\langle \hat{X}_T^i, \Phi((X_T^{0,i+1})', X_T^{i+1}) - \Phi((X_T^{0,i})', X_T^i) \rangle \right] \right] \\ &\geq \mu_1 E[|\hat{X}_T^i|^2] - \tilde{L}_\Phi E[|\hat{X}_T^{0,i}|] E[|\hat{X}_T^i|] \geq (\mu_1 - \tilde{L}_\Phi) E[|\hat{X}_T^i|^2] - \frac{1}{4} \tilde{L}_\Phi E[|\hat{X}_T^{0,i}|^2] \\ &\geq (\mu_1 - \tilde{L}_\Phi) E[|\hat{X}_T^i|^2] - \frac{1}{4} \tilde{L}_\Phi C_2 \alpha \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right). \end{aligned} \quad (\text{A.14})$$

Let us put

$$\hat{B}^i(s) \triangleq \begin{pmatrix} -G^T \hat{f}_s^i \\ G \hat{b}_s^i \\ G \hat{\sigma}_s^i \end{pmatrix},$$

where $\hat{f}^i, \hat{b}^i, \hat{\sigma}^i$ are defined as follows:

$$\begin{aligned} \hat{f}_s^i &= E' [f(s, (X_s^{0,i+1})', W^{i+1}(s, (X_s^{0,i+1})'), X_s^{i+1}, Y_s^{i+1}, Z_s^{i+1}, v_s)] \\ &\quad - E' [f(s, (X_s^{0,i})', W^i(s, (X_s^{0,i})'), X_s^i, Y_s^i, Z_s^i, v_s)], \\ \hat{b}_s^i &= E' [b(s, (X_s^{0,i+1})', W^{i+1}(s, (X_s^{0,i+1})'), X_s^{i+1}, Y_s^{i+1}, Z_s^{i+1}, v_s)] \\ &\quad - E' [b(s, (X_s^{0,i})', W^i(s, (X_s^{0,i})'), X_s^i, Y_s^i, Z_s^i, v_s)], \\ \hat{\sigma}_s^i &= E' [\sigma(s, (X_s^{0,i+1})', W^{i+1}(s, (X_s^{0,i+1})'), X_s^{i+1}, Y_s^{i+1}, v_s)] \\ &\quad - E' [\sigma(s, (X_s^{0,i})', W^i(s, (X_s^{0,i})'), X_s^i, Y_s^i, v_s)]. \end{aligned}$$

Then, due to assumptions (H3.2L₀) and (H3.1) (Recall that $W^i \in \mathcal{W}$, $i \geq 0$),

$$\begin{aligned} \left| E \left[\langle \hat{B}^i(s), (\hat{X}_s^i, \hat{Y}_s^i, \hat{Z}_s^i) \rangle \right] \right| &\leq -\beta_1 E[|\hat{X}_s^i|^2] + E \left[\sqrt{3} \{ (\tilde{L} + \alpha L_0) E' \left[|\widehat{(X_s^{0,i})'}| \right] \right. \right. \\ &\quad \left. \left. + \alpha E' |\hat{W}^i(s, (X_s^{0,i})')| \} \sqrt{|X_s^i|^2 + |Y_s^i|^2 + |Z_s^i|^2} \right]. \end{aligned}$$

Consequently, from (A.14) we have, for $t \leq s \leq T$, $i \geq 0$,

$$\begin{aligned} E \left[\langle \hat{X}_s^i, \hat{Y}_s^i \rangle \right] &= E \left[\langle \hat{X}_T^i, \hat{Y}_T^i \rangle \right] - E \left[\int_s^T \langle \hat{B}^i(r), (\hat{X}_r^i, \hat{Y}_r^i, \hat{Z}_r^i) \rangle dr \right] \\ &\geq (\mu_1 - \tilde{L}_\Phi) E |\hat{X}_T^i|^2 + (\beta_1 - 2(\tilde{L} + \alpha L_0 + \alpha)) E \int_s^T |\hat{X}_r^i|^2 dr \\ &\quad - 2(\tilde{L} + \alpha L_0 + \alpha) E \int_s^T (|\hat{Y}_r^i|^2 + |\hat{Z}_r^i|^2) dr - \alpha C_9 \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right), \end{aligned} \tag{A.15}$$

where $C_9 = \frac{1}{4} \tilde{L}_\Phi C_2 + \frac{3}{8} (\tilde{L} + \alpha L_0) C_1 + \frac{3}{8} T$.

Thanks to (A.13) and (A.15), it follows

$$\begin{aligned} E[\langle \hat{X}_s^i, \hat{Y}_s^i \rangle] &\geq \left(\mu_1 - \tilde{L}_\Phi - 8(\tilde{L} + \alpha L_0 + \alpha) L_\Phi^2 e^{\beta T} \right) E |\hat{X}_T^i|^2 \\ &\quad + \{ \beta_1 - 2(\tilde{L} + \alpha L_0 + \alpha) (1 + 2e^{\beta T}) \} E \int_s^T |\hat{X}_r^i|^2 dr - \alpha C_{10} \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right), \end{aligned} \tag{A.16}$$

where β and β_1 , μ_1 are given in (H3.2L₀), respectively, and $C_{10} = 2(\tilde{L} + \alpha L_0 + \alpha) C_8 + C_9$.

Therefore, as $s = t$, for $i \geq 0$,

$$\begin{cases} E |\hat{X}_T^i|^2 \leq \alpha C_{11} \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right), \\ E \int_t^T |\hat{X}_r^i|^2 dr \leq \alpha C_{12} \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right), \end{cases} \tag{A.17}$$

where

$$C_{11} = \frac{C_{10}}{\mu_1 - \tilde{L}_\Phi - 8(\tilde{L} + \alpha L_0 + \alpha) L_\Phi^2 e^{\beta T}}, \quad C_{12} = \frac{C_{10}}{\beta_1 - 2(\tilde{L} + \alpha L_0 + \alpha) (1 + 2e^{\beta T})}.$$

From the backward SDE of (3.7), we obtain

$$\begin{aligned} &E \left[\sup_{t \leq s \leq T} |\hat{Y}_s^i|^2 + \int_t^T |\hat{Z}_s^i|^2 ds \right] \\ &\leq 6L_\Phi^2 E |\hat{X}_T^i|^2 + \alpha C_{13} \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right) + 15TL^2 E \left[\int_t^T |\hat{X}_s^i|^2 ds \right] + (15TL^2 + 13) E \left[\int_t^T |\hat{Y}_s^i|^2 + |\hat{Z}_s^i|^2 ds \right], \end{aligned} \tag{A.18}$$

where $C_{13} = 6\tilde{L}_\Phi^2 C_2 + 15T(\tilde{L} + \alpha L_0)^2 C_1 + 15T^2 \alpha$. Finally, combining (A.12), (A.13), (A.17) and (A.18), we conclude that, for $i \geq 0$,

$$E \left[\sup_{t \leq s \leq T} (|\hat{X}_s^i|^2 + |\hat{Y}_s^i|^2) + \int_t^T |\hat{Z}_s^i|^2 ds \right] \leq \alpha C_{14} \left(\|\hat{W}^i\|_{K_M}^2 + \varepsilon \right),$$

where $C_{14} = (15TL^2 + 13 + C_7)(C_8 + 4L_\Phi^2 e^{\beta T} C_{11} + 2e^{\beta T} C_{12}) + 6L_\Phi^2 C_{11} + 15TL^2 C_{12} + C_{13} + C_6$. □

Acknowledgements. The authors thank the associate editor and the referees for their helpful comments.

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