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# DELAUNAY TYPE DOMAINS FOR AN OVERDETERMINED ELLIPTIC PROBLEM IN $\mathbb{S}^n \times \mathbb{R}$ AND $\mathbb{H}^n \times \mathbb{R}$ *, **

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**Abstract.** We prove the existence of a countable family of Delaunay type domains

$$\Omega_t \subset \mathbb{M}^n \times \mathbb{R},$$

$t \in \mathbb{N}$, where $\mathbb{M}^n$ is the Riemannian manifold $\mathbb{S}^n$ or $\mathbb{H}^n$ and $n \geq 2$, bifurcating from the cylinder $B^n \times \mathbb{R}$ (where $B^n$ is a geodesic ball in $\mathbb{M}^n$) for which the first eigenfunction of the Laplace–Beltrami operator with zero Dirichlet boundary condition also has constant Neumann data at the boundary. In other words, the overdetermined problem

$$\begin{cases} \Delta_g u + \lambda u = 0 & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t \\ g(\nabla u, \nu) = \text{const.} & \text{on } \partial\Omega_t \end{cases}$$

has a bounded positive solution for some positive constant $\lambda$, where $g$ is the standard metric in $\mathbb{M}^n \times \mathbb{R}$. The domains $\Omega_t$ are rotationally symmetric and periodic with respect to the $\mathbb{R}$-axis of the cylinder and the sequence $\{\Omega_t\}_t$ converges to the cylinder $B^n \times \mathbb{R}$.

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## 1. Introduction and statement of the result

A long-standing open problem is to classify (smooth) domains $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, for which the overdetermined elliptic problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \langle \nabla u, \nu \rangle = \text{const.} & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

admits a solution $u \in C^2(\overline{\Omega})$, where $f$ is a given Lipschitz function, $\nu$ is the normal vector to $\partial\Omega$, and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product.

By Serrin's Theorem [28], if $\Omega$ is bounded, then $\Omega$ must be a ball and the solution $u$ is radial (see also [25]). Such a result has many applications to Physics. For example, problem (1.1), when $f$ is constant, describes a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross section $\Omega$ (see [28]), and Serrin's Theorem shows then that the tangential stress per unit area on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section. Problem (1.1) is used in the linear theory of torsion of a solid straight bar of cross section $\Omega$ (see [31]). In this setting Serrin's Theorem implies that when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of the position if and only if the bar has a circular cross section.

Overdetermined boundary conditions arise naturally also in free boundary problems, when the variational structure imposes suitable conditions on the separation interface (see for example [3]). In this context it is important to underline that several methods for studying locally the regularity of solutions of free boundary problems are often based on blow-up techniques applied to the intersection of $\Omega$ with a small ball centered in a point of $\partial\Omega$, which lead then to the study of an elliptic problem in an unbounded domain. Problem (1.1) in unbounded domains was considered by Berestycki *et al.* in [5].

For some types of functions $f$ the structure of the family of domains $\Omega$ where the overdetermined problem (1.1) can be solved shares many similarities with the class of embedded constant mean curvatures surfaces (CMC surfaces). For the bounded case, the analogy is very simple: the only compact embedded CMC surfaces in $\mathbb{R}^n$ are the round spheres (very well known result by Alexandrov [2]) and the only bounded domains in $\mathbb{R}^n$ where problem (1.1) can be solved are balls by Serrin's Theorem. For the unbounded case, a very well known family of CMC surfaces is the family of Delaunay onduloids, see [7]. In [29] Sicbaldi showed the existence of *Delaunay type domains*, *i.e.* perturbations of the straight solid cylinder in $\mathbb{R}^n$ which are rotationally symmetric and periodic in the vertical direction, where it is possible to solve problem (1.1) for the linear function $f(t) = \lambda\, t$. In [27], Schlenk and Sicbaldi showed that the previous unbounded domains belong in fact to a smooth 1-parameter family, a property enjoyed also by Delaunay onduloids.

In order to show that the analogy with the CMC surfaces is even deeper, we remark that domains where problem (1.1) with $f = 0$ can be solved arise as limits under scaling of sequences of domains where problem (1.1) with $f(t) = \lambda\, t$ can be solved, just like minimal surfaces arise as limits under scaling of sequences of CMC surfaces. In a recent paper, [32], M. Traizet shows a one-to-one correspondence, under some weak hypothesis, between 2-dimensional domains where problem (1.1) with $f = 0$ can be solved and a special class of minimal surfaces.

The analogy between problem (1.1) and CMC surfaces has been explored in a systematic way by Ros and Sicbaldi in [26]. In particular they obtain, for 2-dimensional domains where (1.1) can be solved, a half-space theorem and also, for some functions $f$, the boundedness of the ends of the domain, paralleling analogous results for CMC surfaces.

One of the most remarkable recent achievements in the field of Differential Geometry is the extension of the classical theory of CMC surfaces in the Euclidean space to other ambient spaces, and in particular to the the eight Thurston's 3-dimensional geometries: the Euclidean space $\mathbb{R}^3$, the round sphere $\mathbb{S}^3$, the hyperbolic space $\mathbb{H}^3$, the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, the Heisenberg group $Nil_3$, the universal covering of $PSL_2(\mathbb{R})$

and the Lie group $Sol_3$. The importance of the classification of CMC surfaces in such ambient spaces comes from the outstanding Thurston's Geometrization Conjecture (which includes in particular the Poincaré's Conjecture), proved finally by Perelman in 2003 [22–24] using Ricci flow with surgery, according to which every closed 3-dimensional manifold can be decomposed in a canonical way into pieces in order that each piece has one of the eight Thurston's geometric structures. For a survey on the Thurston's Geometrization Conjecture we refer to [4]. The number of results in the framework of CMC surfaces in Thurston's 3-dimensional geometries is very large, and we cite only the works by Abresch *et al.* [1, 17, 18] which have set the direction of the subsequent research in the field.

As for CMC surfaces, overdetermined problems can be considered also in a Riemannian manifold, and in this framework problem (1.1) becomes

$$\begin{cases} \Delta_g u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ g(\nabla u, \nu) = \text{const.} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $g$ denotes the metric of the manifold and $\Delta_g$ is the Laplace–Beltrami operator.

First, we remark that unlike CMC surfaces, where the lowest possible dimension for the ambient space is 3, in the case of overdetermined elliptic problems the lowest possible dimension for the ambient space is 2. In dimension 2 the equivalent of the Thurston's Geometrization Conjecture is the Riemann's Uniformization Theorem, according to which every 2-dimensional Riemannian manifold is a quotient of one of the following manifolds by a free action of a discrete subgroup of their isometries group: the round sphere $\mathbb{S}^2$, the Euclidean space $\mathbb{R}^2$ and the hyperbolic plane $\mathbb{H}^2$ (remark that in the case of dimension 2 it is not necessary to decompose the manifold in pieces and this is the reason why the 2-dimensional case is much simpler than the 3-dimensional one).

Serrin's Theorem for overdetermined elliptic problems in $\mathbb{R}^n$ has been generalized by Molzon [19] and Kumaresan and Prajapat [15] to the case of the round sphere $\mathbb{S}^n$ and the hyperbolic space $\mathbb{H}^n$, for every dimension $n \geq 2$: assuming that $\Omega$ is a bounded domain in $\mathbb{H}^n$ or that $\Omega$ is a domain contained in a hemisphere of $\mathbb{S}^n$, and that problem (1.2) has a solution $u \in C^2(\overline{\Omega})$, then $\Omega$ is a ball. In the round sphere $\mathbb{S}^n$ there exists nontrivial (bounded) domains (not contained in a hemisphere) where problem (1.2) can be solved, see [12]. Such results parallel analogous results about CMC surfaces in $\mathbb{S}^n$ and $\mathbb{H}^n$, see [2].

In 3-dimensional Riemannian manifolds, results on overdetermined elliptic problems are expected in particular for the remaining five Thurston's geometries: $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $Nil_3$, the universal covering of $PSL_2(\mathbb{R})$ and $Sol_3$. Up to now very few results are known.

In this paper we generalize the construction of Delaunay type domains of Sicbaldi in [29] to the case of the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ (and in particular our result holds in the two Thurston's 3-dimensional geometries $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$). In fact, we prove that the solid straight cylinder $B^n_R \times \mathbb{R}$ (where $B^n_R$ is a geodesic ball of radius $R$ properly contained in $\mathbb{S}^n$ or $\mathbb{H}^n$) can be perturbed in order to obtain new domains where problem (1.2) can be solved for the function $f(t) = \lambda t$ for some positive constant $\lambda$. The boundary of such domains is rotationally symmetric with respect to the $\mathbb{R}$-axis of the cylinder, and is periodic in the vertical direction. The parallel of our result in the framework of CMC surfaces is the construction of Delaunay surfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, done by Pedrosa and Ritoré [21].

In order to state our result, let $\mathbb{M}^n$ denote the Riemannian manifold $\mathbb{S}^n$ or $\mathbb{H}^n$, *i.e.* the $n$-dimensional manifold of constant sectional curvature equal to 1 or -1. Points of $\mathbb{M}^n \times \mathbb{R}$ are denoted by $(x, t)$, $x \in \mathbb{M}^n$ and $t \in \mathbb{R}$. Let us fix a point 0 (origin) in $\mathbb{M}^n$ and let $r(x)$ denote the distance of $x \in \mathbb{M}^n$ to the origin $0 \in \mathbb{M}^n$. Our main result is the following:

**Theorem 1.1.** *Let $R > 0$ and $B_R$ a geodesic ball of radius $R$ centered at $0$ such that $B_R \subsetneq \mathbb{M}^n$. There exist a real positive number $T_*$, a sequence of real positive numbers $T_j \longrightarrow T_*$ and a sequence of nonconstant functions $v_j \in C^{2,\alpha}(\mathbb{R})$ (of small norm, of period $T_j$, and converging to $0$ in $C^{2,\alpha}(\mathbb{R})$) such that the domains*

$$\Omega_j = \{(x,t) \in \mathbb{M}^n \times \mathbb{R},\, r(x) < R + v_j(t)\}$$

*have a positive solution $u_j \in C^{2,\alpha}(\Omega_j)$ to the problem (1.2). Moreover $\displaystyle\int_0^{T_j} v_j \,\mathrm{d}t = 0$.*

The reader will notice that the condition $B_R \subsetneq \mathbb{M}$ is an empty condition when $\mathbb{M} = \mathbb{H}$ and is equivalent to ask $R < \pi$ when $\mathbb{M} = \mathbb{S}$.

**Remark 1.2.** More generally, the same construction can be done in the spaces $\mathbb{M}^n(k) \times \mathbb{R}$, where $\mathbb{M}^n(k)$, $k \in \mathbb{R}$, is the $n$-dimensional space form of constant sectional curvature $k$. In other words, Theorem 1.1 still holds when we replace $\mathbb{M}^n$ with $\mathbb{M}^n(k)$. The case $k = 0$ corresponds to the Euclidean one, settled in [27, 29], and here we will consider only the cases $k \neq 0$. We recall that the condition $B_R \subsetneq \mathbb{M}^n(k)$ is again an empty condition when $k \leq 0$ and is equivalent to $R < \pi\sqrt{1/k}$ when $k > 0$. Sections 6 and 7, which play a crucial role in this paper, have been redacted using $\mathbb{M}^n(k)$ instead of $\mathbb{M}^n$. In the other sections we consider $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$, but we always point out the main changes to do in order to adapt the formulas to the ambient space $\mathbb{M}^n(k) \times \mathbb{R}$, $k \neq 0$.

The previous result leaves two open interesting questions:

(1) We do not have a smooth one-parameter family of domains, but only a sequence of domains converging to the straight cylinder. In view of the result which holds in $\mathbb{R}^n$ [27] and the existence of CMC surfaces in $\mathbb{M}^n \times \mathbb{R}$ [21], it is tempting to conjecture that the domains in Theorem 1.1 belong in fact to a smooth one-parameter family of domains.
(2) In the framework of $\mathbb{M}^n(k) \times \mathbb{R}$, it would be very interesting to study the dependence on $k$ of the domains $\Omega_j$, and understand their behavior as $k$ changes sign. We trust that this is a very nontrivial question.

In order to simplify the redaction, we will prove Theorem 1.1 in the case $R = 1$, and we will show, according to Remark 1.2, that the construction can be done also in the more general space $\mathbb{M}^n(k) \times \mathbb{R}$, for all $k \neq 0$. There is no loss of generality in choosing $R = 1$. Indeed, the problem of finding overdetermined domains does not depend on the value of $k$, and perturbations of $B_R \times \mathbb{R}$ in $\mathbb{M}^n(k) \times \mathbb{R}$ turn equivalently into perturbations of $B_1 \times \mathbb{R}$ in $\mathbb{M}^n(k') \times \mathbb{R}$, for some real number $k'$ of the same sign of $k$.

The strategy of the proof of our result is the one adopted in [29], and the real novelty here stays in the tools used to solve the central step of the proof. If $(x,t)$ are the points of $\mathbb{M}^n \times \mathbb{R}$, or more generally $\mathbb{M}^n(k) \times \mathbb{R}$ (where $k$ satisfies the condition that $B_1 \subsetneq \mathbb{M}^n(k)$), we construct the domain $C^T_{1+v}$ as the interior of the radial graph over the cylinder of radius 1 of a periodic function $v(t)$ with period $T$ (Sect. 2). We consider the operator that to the function $v$ associates the normal derivative of the first eigenfunction $\phi$ of the Laplace–Beltrami operator on $C^T_{1+v}$ with zero Dirichlet boundary condition. In order to find nontrivial solutions $v$ such that the normal derivative of $\phi$ at $\partial C^T_{1+v}$ is constant, we need to study the linearized operator with respect to the variable $v$ (Sects. 3 and 4) and show that for some value of the parameter $T$ it has a nontrivial kernel. In [29] such step could be easily solved because the study of the linearized operator led to solving a Bessel ODE. In our case, we have to handle a much more difficult situation, and we are able to study the linearized operator by using some large classes of complex valued functions known as Legendre and Ferrer's functions, with complex argument and depending on two parameters. For convenience of the reader, in Section 5 we recollect the basic facts about such classes of functions, their asymptotics and related differential equations, and this material will be used in the study of the linearized operator in Sections 6 and 7. The final step of the proof uses the Lyapunov−Schmidt reduction and a bifurcation argument (Sect. 8).

## 2. Rephrasing the problem

Given a continuous function $v : \mathbb{R}/2\pi\mathbb{Z} \longmapsto (0, +\infty)$ whose $L^\infty$-norm is small enough, we define

$$C^T_{1+v} := \{(x,t) \in \mathbb{M}^n \times \mathbb{R}/T\mathbb{Z} \,:\, 0 \le r(x) < 1 + v(2\pi t/T)\}\,.$$

Our aim is to show that there exists a positive real number $T_*$, a sequence $T_j \to T_*$ and a sequence of nonconstant functions $v_j \in C^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ of mean value equal to zero and converging to the zero function in the $C^{2,\alpha}$-norm, such that the overdetermined problem

$$\begin{cases} \Delta_g\,\phi + \lambda\,\phi = \quad 0 \quad \text{in } C^T_{1+v} \\ \qquad\qquad \phi = \quad 0 \quad \text{on } \partial C^T_{1+v} \\ \quad g(\nabla\phi, \nu) = \text{const. on } \partial C^T_{1+v} \end{cases} \tag{2.1}$$

has a nontrivial positive solution $(\phi, \lambda) = (\phi_j, \lambda_j)$ for the sequence $(v_j, T_j)$. Here $\nu$ denote the normal vector field to $\partial C^T_{1+v}$, $\lambda$ is a positive constant, and $g$ is the product metric of $\mathbb{M}^n \times \mathbb{R}/T\,\mathbb{Z}$ (in particular the second factor is equipped with the metric induced by the standard metric of $\mathbb{R}$).

We remark that the symmetry of the problem allow us to require the function $v$ to be even.

Let $g_{\mathbb{M}^n}$ denote the usual metric on $\mathbb{M}^n$. Let $\lambda_1$ be the first eigenvalue of the Laplace–Beltrami operator with zero Dirichlet boundary condition in the unit geodesic ball

$$B_1 = \{x \in \mathbb{M}^n \,:\, r(x) < 1\}.$$

Let $\tilde\phi_1$ denote the associated eigenfunction

$$\begin{cases} \Delta_{g_{\mathbb{M}^n}}\,\tilde\phi_1 + \lambda_1\,\tilde\phi_1 = 0 \text{ in } B_1 \\ \qquad\qquad\quad \tilde\phi_1 = 0 \text{ on } \partial B_1 \end{cases} \tag{2.2}$$

which is normalized to have $L^2(B_1)$-norm equal to $1/2\pi$. Then $\phi_1(x,t) = \tilde\phi_1(x)$ solves the problem

$$\begin{cases} \Delta_g\phi_1 + \lambda_1\,\phi_1 = 0 \text{ in } C^T_1 \\ \qquad\qquad \phi_1 = 0 \text{ on } \partial C^T_1 \end{cases} \tag{2.3}$$

and

$$\int_{C_1^{2\pi}} \phi_1^2 \,\mathrm{dvol}_g = 1. \tag{2.4}$$

As $\phi_1$ do not depend on $t$, sometimes we will write simply $\phi_1(x)$.

Let $C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ be the set of even functions on $\mathbb{R}/2\pi\mathbb{Z}$ of mean value equal to zero. For all function $v \in C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ whose norm is small enough, the domain $C^T_{1+v}$ is well defined for all $T > 0$ and standard results on Dirichlet eigenvalue problem (see [6,10]) apply to give the existence, for all $T > 0$, of a unique positive function

$$\phi = \phi_{v,T} \in C^{2,\alpha}\left(C^T_{1+v}\right)$$

and a constant $\lambda = \lambda_{v,T} \in \mathbb{R}$ such that $\phi$ is a solution to the problem

$$\begin{cases} \Delta_g\,\phi + \lambda\,\phi = 0 \text{ in } C^T_{1+v} \\ \qquad\qquad \phi = 0 \text{ on } \partial C^T_{1+v} \end{cases} \tag{2.5}$$

which is normalized by

$$\int_{C^{2\pi}_{1+v}} \left(\phi\left(x, \frac{T}{2\pi}\, t\right)\right)^2 \mathrm{dvol}_g = 1 \tag{2.6}$$

In addition $\phi$ and $\lambda$ depend smoothly on the function $v$, and $\phi = \phi_1$, $\lambda = \lambda_1$ when $v \equiv 0$.

After canonical identification of $\partial C^T_{1+v}$ with $\mathbb{S}^{n-1} \times \mathbb{R}/T\mathbb{Z}$, we define the Dirichlet-to-Neumann operator $N$:

$$N(v, T) = g(\nabla\phi, \nu)\,|_{\partial C^T_{1+v}} - \frac{1}{\mathrm{Vol}_g(\partial C^T_{1+v})} \int_{\partial C^T_{1+v}} g(\nabla\phi, \nu)\,\mathrm{dvol}_g,$$

where $\nu$ denotes the unit normal vector field to $\partial C^T_{1+v}$ and $\phi$ is the solution of (2.5). *A priori* $N(v, t)$ is a function defined over $\mathbb{S}^{n-1} \times \mathbb{R}/T\mathbb{Z}$, but it is easy to see that it depends only on the variable $t \in \mathbb{R}/T\mathbb{Z}$ because $v$ has such a property. For the same reason it is an even function, and moreover it is clear that its mean vanishes. If now we operate a rescaling and we define

$$F(v, T)\,(t) = N(v, T)\left(\frac{T}{2\pi}\, t\right), \tag{2.7}$$

Schauder's estimates imply that $F$ is well defined in a neighborhood of $(0, T)$ in the space $C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$, and takes its values in $C^{1,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$. Our aim is to find a positive real number $T_*$, a sequence $T_j \to T_*$ and a sequence of nonconstant functions $v_j \in C^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ of mean equal to zero and converging to the zero function in the $C^{2,\alpha}$-norm, such that $F(v_j, T_j) = 0$. Observe that, with this condition, $\phi = \phi_{v_j, T_j}$ will be the solution to the problem (2.1) and our Theorem 1.1 will be proved.

## 3. The linearized operator

Let $k$ be the sectional curvature of the manifold $\mathbb{M}^n$ (*i.e.* $k = 1$ if $\mathbb{M}^n = \mathbb{S}^n$ and $k = -1$ if $\mathbb{M}^n = \mathbb{H}^n$). If we choose spherical coordinates $(r, \theta)$, with $\theta \in \mathbb{S}^{n-1}$ and $r \in [0, +\infty)$ if $k < 0$ and $r \in [0, \pi]$ if $k > 0$, the usual metric in $\mathbb{M}^n$ can be written as

$$g_{\mathbb{M}^n} = \mathrm{d}r^2 + S_k(r)^2\, \mathrm{d}\theta^2$$

where

$$S_k(r) = \begin{cases} \sinh r & \text{if } k = -1 \\ \sin r & \text{if } k = 1 \end{cases}$$

(see [6], Sect. II.5, Thm. 1).

**Remark 3.1.** According to Remark 1.2, when we consider $\mathbb{M}^n(k)$ instead of $\mathbb{M}^n$, we use spherical coordinates $(r, \theta)$, with $\theta \in \mathbb{S}^{n-1}$ and $r \in [0, +\infty)$ if $k < 0$ and $r \in [0, \pi/\sqrt{k})$ if $k > 0$, and then the usual metric in $\mathbb{M}^n(k)$ is

$$g_{\mathbb{M}^n(k)} = \mathrm{d}r^2 + S_k(r)^2\, \mathrm{d}\theta^2$$

where

$$S_k(r) = \begin{cases} \dfrac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\, r) & \text{if } k < 0 \\ \dfrac{1}{\sqrt{k}} \sin(\sqrt{k}\, r) & \text{if } k > 0 \end{cases}$$

(see [6], Sect. II.5, Thm. 1). The computations that follow are true in general for the manifold $\mathbb{M}^n(k) \times \mathbb{R}$, under the hypothesis that $\mathbb{M}^n(k)$ contains properly a geodesic ball $B_1$ of radius 1. For the convenience of the reader, we consider only the cases $k = 1$ and $-1$ till Section 5, but Sections 6 and 7, which are the crucial part of this paper, will be established for any $k \neq 0$.

For all $v \in C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ and all $T > 0$, let $\psi$ be the (unique) solution (periodic with respect to the variable $t$) of

$$\begin{cases} \Delta_g \psi + \lambda_1 \psi = 0 & \text{in} \quad C_1^T \\ \psi = -\partial_r \phi_1 \, v(2\pi t/T) & \text{on} \quad \partial C_1^T \end{cases} \tag{3.1}$$

which is $L^2(C_1^T)$-orthogonal to $\phi_1$. The function $\phi_1 = \phi_1(r)$ is the solution on $C_1^T$, for any $T > 0$, of (2.3) with $L^2$-norm equal to 1. We define

$$\tilde{H}_T(v) := \left( \partial_r \psi + \partial_r^2 \phi_1 \, v \left( \frac{2\pi t}{T} \right) \right) \Big|_{\partial C_1^T} . \tag{3.2}$$

By symmetry it is clear that $\tilde{H}_T(v)$ is a function only depending on $t$, then changing the variable we can define

$$H_T(v)(t) := \tilde{H}_T(v) \left( \frac{T}{2\pi} \, t \right). \tag{3.3}$$

The main result of this section is the:

**Proposition 3.2.** *The linearization of the operator $F$ with respect to $v$ computed at the point $(0, T)$ is given by $H_T$.*

*Proof.* To linearize the operator $F$ (see (2.7)) with respect to $v$ at $(0, T)$ we will compute

$$\lim_{s \to 0} \frac{F(s\, w, T) - F(0, T)}{s}.$$

Precisely we determine the first order approximation of $F(s\, w, T)$ with respect to the variable $s$. Let $\{e_1, \ldots, e_n\}$ denote an orthonormal basis of the tangent space to $\mathbb{M}^n$ at the origin 0. Suppose that $y = (y_1, y_2, \ldots, y_n)$ are geodesic normal coordinates at $0 \in \mathbb{M}^n$, and let $x$ denote the point of $\mathbb{M}^n$ whose geodesic coordinates are $y$. We parameterize $C^T_{1+sw}$ on $C_1^{2\pi}$ by

$$Y(x, t) := \left( \mathrm{Exp}_0 \left( (1 + s\, \chi(y)\, w) \sum_1^n y_i e_i \right), \frac{T t}{2\pi} \right)$$

for $x \in \mathbb{M}^n$ and $t \in \mathbb{R}$ and where $\chi$ is a cutoff function identically equal to 0 when $|y| \leq 1/4$ and identically equal to 1 when $|y| \geq 1/2$. If we use the coordinates $(r, \theta, t)$, being $(r, \theta)$ the coordinates introduced at the beginning of Section 3, the map $Y$ reduces to

$$(r, \theta, t) \to \left( (1 + s\, \chi(r)\, w)\ r\, ,\ \theta,\ \frac{T\, t}{2\pi} \right).$$

The metric induced by $Y$ will be denoted by

$$\hat{g} := Y^* g. \tag{3.4}$$

If $\phi$ solves (2.5) and (2.6), then $\hat{\phi} = Y^* \phi$ is solution (smoothly depending on the real parameter $s$) of

$$\begin{cases} \Delta_{\hat{g}}\, \hat{\phi} + \hat{\lambda}\, \hat{\phi} = 0 \ \text{ in } \ C_1^{2\pi} \\ \hat{\phi} = 0 \text{ on } \partial C_1^{2\pi} \end{cases}$$

with $\hat{\lambda} = \lambda$ and satisfying

$$\int_{C_1^{2\pi}} \hat{\phi}^2 \, \mathrm{dvol}_{\hat{g}} = 1.$$

Consider the function $\phi_1$ defined in (2.3) and (2.4). Clearly the function $\hat{\phi}_1 := Y^* \phi_1$ solves

$$\Delta_{\hat{g}}\, \hat{\phi}_1 + \lambda_1\, \hat{\phi}_1 = 0$$

and

$$\hat{\phi}_1(x,t) = \phi_1 \left( \mathrm{Exp}_0 \left( (1 + s\, w) \, \sum_1^n y_i e_i \right), \frac{Tt}{2\pi} \right) \tag{3.5}$$

for $|y| \geq \frac{1}{2}$. Writing $\hat{\phi} = \hat{\phi}_1 + \hat{\psi}$ and $\hat{\lambda} = \lambda_1 + \mu$, we find out that $\hat{\psi}$ solves

$$\begin{cases} \Delta_{\hat{g}}\, \hat{\psi} + (\lambda_1 + \mu)\, \hat{\psi} + \mu\, \hat{\phi}_1 = \quad 0 \quad \text{in } C_1^{2\pi} \\ \qquad\qquad\qquad\qquad \hat{\psi} = -\hat{\phi}_1 \text{ on } \partial C_1^{2\pi} \end{cases} \tag{3.6}$$

with

$$\int_{C_1^{2\pi}} (2\, \hat{\phi}_1\, \hat{\psi} + \hat{\psi}^2)\, \mathrm{dvol}_{\hat{g}} = \int_{C_1^{2\pi}} \phi_1^2\, \mathrm{dvol}_g - \int_{C_{1+sw}^{2\pi}} \phi_1^2\, \mathrm{dvol}_g. \tag{3.7}$$

Obviously $\hat{\psi}$ and $\mu$ are smooth functions of $s$. If $s = 0$, then $C_{1+sw}^T = C_1^T$ and in particular we have $\phi = \phi_1 = \hat{\phi}_1$, $\lambda = \lambda_1$, $\hat{\psi} \equiv 0$, $\mu = 0$ and $\hat{g} = g$. We set

$$\dot{\psi} := \partial_s \hat{\psi}|_{s=0} \qquad \text{and} \qquad \dot{\mu} := \partial_s \mu|_{s=0}.$$

Differentiating (3.6) with respect to $s$ and evaluating the result at $s = 0$, we obtain

$$\begin{cases} \Delta_g\, \dot{\psi} + \lambda_1\, \dot{\psi} + \dot{\mu}\, \phi_1 = 0 \qquad\quad \text{in } C_1^{2\pi} \\ \qquad\qquad\qquad \dot{\psi} = -\partial_r \phi_1\, w \text{ on } \partial C_1^{2\pi} \end{cases} \tag{3.8}$$

because from (3.5), differentiation with respect to $s$ at $s = 0$ yields $\partial_s \hat{\phi}_1|_{s=0} = \partial_r \phi_1\, w$, where $r = r(x)$.

Differentiating (3.7) with respect to $s$ and evaluating the result at $s = 0$, we obtain

$$\int_{C_1^{2\pi}} \phi_1\, \dot{\psi}\, \mathrm{dvol}_g = 0. \tag{3.9}$$

Indeed, the derivative of the right hand side of (3.7) with respect to $s$ vanishes when $s = 0$ since $\phi_1$ vanishes identically on $\partial C_1^{2\pi}$.

If we multiply the first equation of (3.8) by $\phi_1$ and we integrate it over $C_1^{2\pi}$, using (3.9) we get:

$$\int_{C_1^{2\pi}} (\phi_1\, \Delta_g \dot{\psi} + \dot{\mu}\, \phi_1^2)\, \mathrm{dvol}_g = 0.$$

By Gauss−Green Theorem and the boundary conditions $\phi_1 = 0, \dot{\psi} = -\partial_r \phi_1 w$, we deduce the following identity

$$\int_{C_1^{2\pi}} \phi_1\, \Delta_g \dot{\psi}\, \mathrm{dvol}_g = \int_{C_1^{2\pi}} \dot{\psi}\, \Delta_g \phi_1\, \mathrm{dvol}_g + \int_{\partial C_1^{2\pi}} w\, \partial_\nu \phi_1\, \partial_r \phi_1\, \mathrm{dvol}_g,$$

where $\partial_\nu \phi_1$ is the normal derivative of $\phi_1$ and $\nu$ is the unit normal vector to $\partial C_1^{2\pi}$. The first term of the right hand side is easily seen to vanish by multiplying by $\dot{\psi}$ the equation satisfied by $\phi_1$ and integrating. As $s = 0$, then $\partial_\nu \phi_1 = \partial_r \phi_1$ on $\partial C_1^{2\pi}$. Since on this set $\partial_\nu \phi_1$ is constant and the average of $w$ is 0 we conclude that $\dot{\mu} = 0$. Consequently the $2\pi$-periodic function $\dot{\psi}(x,t)$ is related to the solution $\psi(x,t)$ of (3.1) by the identity $\psi(x,t) := \dot{\psi}(x, 2\pi t/T)$, using $v = w$.

We proved that

$$\hat{\phi}(x,t) = \hat{\phi}_1(x,t) + s\, \psi(x, Tt/2\pi) + \mathcal{O}(s^2).$$

In particular, in $C_1^{2\pi} \setminus C_{3/4}^{2\pi}$, we have

$$\begin{aligned}\hat\phi(x,t) &= \phi_1\left(\mathrm{Exp}_0\left((1+s\,w)\,\sum_1^n y_i e_i\right),\,Tt/2\pi\right) + s\,\psi(x,Tt/2\pi) + \mathcal{O}(s^2)\\ &= \phi_1\left(x,Tt/2\pi\right) + s\left(w\,r(x)\,\partial_r\phi_1 + \psi(x,Tt/2\pi)\right) + \mathcal{O}(s^2).\end{aligned}$$

To complete the proof of the result, we will compute $\hat g(\nabla\hat\phi,\hat\nu)$ on the boundary of $C_1^{2\pi}$. Such a function is the normal derivative of $\hat\phi$ when the normal is computed with respect to the metric $\hat g$. We now use the coordinates $(r,\theta,t)$. In $C_1^{2\pi}\setminus C_{3/4}^{2\pi}$ the metric $\hat g$ equals

$$\hat g = (1+sw)^2\mathrm{d}r^2 + 2sr\,w'\,(1+sw)\,\mathrm{d}r\mathrm{d}t + \left(\left(\frac{T}{2\pi}\right)^2 + s^2\,r^2\,(w')^2\right)\mathrm{d}t^2 + S_k^2((1+sw)r)\,\mathrm{d}\theta^2.$$

It follows from this expression that the unit normal vector field to $\partial C_1^{2\pi}$ for the metric $\hat g$ is given by

$$\hat\nu = \left((1+s\,w)^{-1} + \mathcal{O}(s^2)\right)\,\partial_r + \mathcal{O}(s)\,\partial_t. \tag{3.10}$$

Hence, on $\partial C_1^{2\pi}$,

$$\hat g(\nabla\hat\phi,\hat\nu) = \partial_r\phi_1 + s\,\left(w\,\partial_r^2\phi_1 + \partial_r\psi(x,Tt/2\pi)\right) + \mathcal{O}(s^2).$$

On $\partial C_1^{2\pi}$ the term $w\,\partial_r^2\phi_1 + \partial_r\psi(x,Tt/2\pi)$ has mean equal to zero and $\partial_r\phi_1$ is constant. Using $\hat g(\nabla\hat\phi,\hat\nu)$ to compute $F(sw)$, we get that the linearized of $F$ is $H_T$. □

## 4. The structure of the linearized operator

Let $v \in C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$. Recalling that the mean of $v$ is zero and the fact that $v$ is even, by Fourier expansion $v$ can be written as

$$v = \sum_{j\ge 1} a_j\,\cos(jt). \tag{4.1}$$

Observe that in principle $\phi_1$ is only defined in the cylindrical domain $C_1^{2\pi}$, however, this function being radial in the first $n$ variables and not depending on $t$, it is a solution of a second order ordinary differential equation and then it can be extended at least in a neighborhood of $\partial C_1^{2\pi}$.

We will need the following:

**Lemma 4.1.** *Assume that $v \in C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ and write $v$ as in (4.1). For $T>0$ we define*

$$\phi_0(x,t) = \partial_r\phi_1(x)\,v(2\pi t/T)$$

*where $r = r(x)$. Then*

$$\Delta_g\phi_0 + \lambda_1\,\phi_0 = \partial_r\phi_1\sum_{j\ge1} a_j\,\frac{1}{S_k(r)^2}\,\cos\left(\frac{2\pi jt}{T}\right)\left[n-1-\left(\frac{2\pi j}{T}\right)^2 S_k(r)^2\right].$$

*Proof.* The Laplace–Beltrami operator for the metric $g$ can be written as

$$\Delta_g = \partial_r^2 + (n-1)\frac{C_k(r)}{S_k(r)}\,\partial_r + \frac{1}{S_k(r)^2}\,\Delta_{\mathbb{S}^{n-1}} + \partial_t^2$$

where

$$C_k(r) = \begin{cases}\cosh r & \text{if } k=-1\\ \cos r & \text{if } k=1\end{cases}$$

(see [6], Sect. II.5, Thm. 1). Then it is easy to obtain

$$\Delta_g \, \partial_r \phi_1 = -\lambda_1 \, \partial_r \phi_1 + \frac{n-1}{S_k^2(r)} \, \partial_r \phi_1$$

and

$$\Delta_g \phi_0 = -\lambda_1 \, \phi_0 + \partial_r \phi_1 \sum_{j \geq 1} a_j \, \frac{1}{S_k(r)^2} \, \cos\left(\frac{2\pi j t}{T}\right) \left[ n - 1 - \left(\frac{2\pi j}{T}\right)^2 S_k(r)^2 \right].$$

This completes the proof of the result. □

**Remark 4.2.** With respect to Remark 3.1 we give the formula of $C_k(r)$ when $k \notin \{-1, 1\}$. In fact, we have

$$C_k(r) = \begin{cases} \cosh(\sqrt{-k}\, r) & \text{if } k < 0 \\ \cos(\sqrt{k}\, r) & \text{if } k > 0. \end{cases}$$

We investigate now the structure of the linearized operator $H_T$. The main result of this section is the:

**Proposition 4.3.** *For all $T > 0$, the operator*

$$H_T : C^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \longrightarrow \mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}),$$

*defined by* (3.3)*, is a self-adjoint, first order elliptic operator preserving, for all $j \in \mathbb{N} \backslash \{0\}$, the eigenspace $V_j$ spanned by the function* $\cos(jt)$*.*

*Proof.* The fact that $H_T$ is a first order elliptic operator is standard since it is the sum of the Dirichlet-to-Neumann operator for $\Delta_g + \lambda_1$ and a constant times the identity. In particular, elliptic estimates yield

$$\|H_T(w)\|_{\mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})} \leq c \, \|w\|_{C^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})}.$$

The fact that the operator $H_T$ is (formally) self-adjoint is easy. Let $\psi_1$ (resp. $\psi_2$) the solution of (3.1) corresponding to the function $w_1$ (resp. $w_2$). Let $\tilde{\psi}_i(x,t) = \psi_i(x, Tt/2\pi)$. We compute

$$\begin{aligned} \partial_r \phi_1(1) \int_0^{2\pi} (H_T(w_1)\, w_2 - w_1 \, H_T(w_2)) \, \mathrm{d}t &= \partial_r \phi_1(1) \int_0^{2\pi} (\partial_r \tilde{\psi}_1 \, w_2 - \partial_r \tilde{\psi}_2 \, w_1) \, \mathrm{d}t \\ &= \int_0^{2\pi} (\tilde{\psi}_1 \, \partial_r \tilde{\psi}_2 - \tilde{\psi}_2 \, \partial_r \tilde{\psi}_1) \, \mathrm{d}t \\ &= \frac{1}{\mathrm{Vol}_g(\mathbb{S}^{n-1})} \int_{C_1^{2\pi}} (\tilde{\psi}_1 \, \Delta_g \tilde{\psi}_2 - \tilde{\psi}_2 \Delta_g \, \tilde{\psi}_1) \, \mathrm{dvol}_g \\ &= 0. \end{aligned}$$

To prove the other statements, we define for all $v \in C^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ written as in (4.1), $\Psi$ to be the continuous solution of

$$\begin{cases} \Delta_g \Psi + \lambda_1 \, \Psi = \partial_r \phi_1 \displaystyle\sum_{j \geq 1} a_j \, \frac{1}{S_k(r)^2} \, \cos\left(\frac{2\pi j t}{T}\right) \left[ n - 1 - \left(\frac{2\pi j}{T}\right)^2 S_k(r)^2 \right] & \text{in } C_1^T \\ \Psi = 0 & \text{on } \partial C_1^T. \end{cases} \tag{4.2}$$

Observe that $\partial_r \phi_1$ vanishes at first order at $r = 0$ and hence the right hand side is smaller than a constant times $r^{-1}$ near the origin. Standard elliptic estimates then imply that the solution $\Psi$ is at least continuous near the origin (the right side of (4.2) belongs to the space $L^p(C_1^T)$ for each $p < n$, then the solution $\Psi$ belongs to the Sobolev space $W^{2,p}(C_1^T)$ for each $p<n$, and by the Sobolev embedding Theorem for a compact domain $\Omega$ we have $W^{2,p}(\Omega)\subseteq C^{0,\alpha}(\Omega)$ for $p\geq \frac{n}{2-\alpha}$). A straightforward computation using the result of Lemma 4.1 and writing $\Psi(x,t)=\psi(x,t)+\partial_r\phi_1(x)\,v(2\pi t/T)$, shows that

$$\tilde{H}_T(v)=\partial_r\Psi|_{\partial C_1^T}.$$

By this alternative definition, it is clear that $H_T$ preserves the eigenspaces $V_j$ and in particular, $H_T$ maps into the space of functions whose mean is zero. □

By the previous proposition

$$\tilde{H}_T(v)=\sum_{j\geq 1}\sigma_j(T)\,a_j\,\cos\left(\frac{2\pi jt}{T}\right), \tag{4.3}$$

where $\sigma_j(T)$ are the eigenvalues of $H_T$ with respect to the eigenfunctions $\cos(jt)$. From (3.2), (4.3) and (3.1) we deduce that

$$\psi=\sum_{j\geq 1}c_j(r)\,a_j\,\cos\left(\frac{2\pi jt}{T}\right),$$

where $c_j$ is the continuous solution on $[0,1]$ of

$$\left(\partial_r^2+(n-1)\frac{C_k(r)}{S_k(r)}\,\partial_r+\lambda_1\right)c_j-\left(\frac{2\pi j}{T}\right)^2 c_j=0, \tag{4.4}$$

with $c_j(1)=-\partial_r\phi_1(1)$. Then

$$\sigma_j(T)=\partial_r c_j(1)+\partial_r^2\phi_1(1). \tag{4.5}$$

Our next task is to find the kernel of the operator $H_T$. For this it is enough to study the eigenvalues $\sigma_j$. We remark that if we set

$$\frac{j}{T}=\frac{1}{D},$$

for $T>0$, from (4.4) we obtain that

$$\sigma_j(T)=\sigma_1(D).$$

Then, in order to study the kernel of the linearized operator, it suffices to consider only the first eigenvalue $\sigma_1$. For this aim we will use Legendre and Ferrers functions.

To simplify the notation, in the sequel we will drop the lower index $_1$, and we set $\sigma_1=\sigma$.

## 5. Recollection on Legendre and Ferrers functions

In what follows we shall use several properties of associated Legendre and Ferrers functions. For the convenience of the reader, we recall their definitions and some properties. This section can be skipped by the reader who is familiar with these functions. For more details we refer to [8, 16, 20].

### 5.1. Legendre functions

The (general) Legendre equation in the variable $z\in\mathbb{C}$ (see [20], 5.12) is

$$(1-z^2)\,\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-2z\,\frac{\mathrm{d}w}{\mathrm{d}z}+\left[\nu(\nu+1)-\frac{\mu^2}{1-z^2}\right]w=0 \tag{5.1}$$

where $\mu,\nu$ are complex parameters. To solve this equation one considers special solutions to the hypergeometric equation:

$$z(1-z)\frac{\mathrm{d}^2u}{\mathrm{d}z^2}+\{c-(a+b+1)z\}\frac{\mathrm{d}u}{\mathrm{d}z}-abu=0,$$

where $a, b, c \in \mathbb{C}$. The solutions to this equation can be found by the power series method. If we consider a series centered at $z = 0$ we find a series which is convergent for $|z| < 1$ and whose sum is known as hypergeometric function:

$$F(a, b; c; z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s} \frac{z^s}{s!},$$

where $c > 0$ (see [20], 9.02, p. 159). Let $\Gamma$ be the Gamma function and let $(\cdot)_s$ denote the Pochammer symbol

$$(q)_n = \begin{cases} 1 & \text{if } n = 0 \\ q\,(q+1)\,(q+2)\ldots(q+n-1) & \text{if } n \geq 1. \end{cases}$$

The Olver hypergeometric function $\mathbf{F}$ (see [20], 9.03, p. 159) is defined by

$$\mathbf{F}(a, b; c; z) = \frac{F(a, b; c; z)}{\Gamma(c)} = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{\Gamma(c+s)} \frac{z^s}{s!}$$

for $|z| < 1$ and extended to $|z| \geq 1$ by analytic continuation. Such a function presents the advantage of being defined for all values of $c$. Using the Olver hypergeometric function we can construct a first solution of (5.1):

$$\mathcal{P}_\nu^{-\mu}(z) = \left(\frac{z-1}{z+1}\right)^{\mu/2} \mathbf{F}\left(\nu+1\,,\, -\nu\,;\, \mu+1\,;\, \frac{1-z}{2}\right). \tag{5.2}$$

A second solution can be built from the first one by using the fact that also

$$(-z)^a\, \mathbf{F}\left(a, 1+a-c; 1+a-b; \frac{1}{z}\right)$$

is a solution to the hypergeometric equation and replacing $a = \nu + 1$, $b = -\nu$, $c = \mu + 1$ and $z$ by $\frac{1-z}{2}$. We get (after multiplication by $2^\nu\, \Gamma(\nu+1)$):

$$\mathbf{Q}_\nu^\mu(z) = 2^\nu\, \Gamma(\nu+1) \frac{(z-1)^{\mu/2-\nu-1}}{(z+1)^{\mu/2}}\, \mathbf{F}\left(\nu+1\,,\, \nu-\mu+1\,;\, 2\nu+2\,;\, \frac{2}{1-z}\right). \tag{5.3}$$

Because the Legendre equation is unchanged by replacing $\mu$ by $-\mu$ or $\nu$ by $-\nu-1$, the functions

$$\mathcal{P}_\nu^{\pm\mu}(z), \mathcal{P}_{-\nu-1}^{\pm\mu}(z), \mathbf{Q}_\nu^{\pm\mu}(z), \mathbf{Q}_{-\nu-1}^{\pm\mu}(z)\,,$$

are all solutions, but only the following four of them are distinct:

$$\mathcal{P}_\nu^{\pm\mu}(z), \mathbf{Q}_\nu^\mu(z), \mathbf{Q}_{-\nu-1}^\mu(z).$$

Moreover only two of them are linearly independent, as one can see by the two following connection formulas:

$$\begin{aligned} \frac{2\sin(\mu\pi)}{\pi}\, \mathbf{Q}_\nu^\mu(z) &= \frac{\mathcal{P}_\nu^\mu(z)}{\Gamma(\nu+\mu+1)} - \frac{\mathcal{P}_\nu^{-\mu}(z)}{\Gamma(\nu-\mu+1)} \\ \cos(\nu\pi)\, \mathcal{P}_\nu^{-\mu}(z) &= \frac{\mathbf{Q}_{-\nu-1}^\mu(z)}{\Gamma(\nu+\mu+1)} - \frac{\mathbf{Q}_\nu^\mu(z)}{\Gamma(\mu-\nu)}. \end{aligned} \tag{5.4}$$

The functions $\mathcal{P}_\nu^{\pm\mu}(z)$ are called associated Legendre functions of first kind. The functions $\mathbf{Q}_\nu^{\pm\mu}(z)$ are called associated Legendre functions of second kind[4]. Such functions exist for all values of $\nu, \mu, z$, except possibly the singular points $z = \pm 1$ and $\infty$. They are multi-valued functions of $z$ with branch points at $z = \pm 1$ and $\infty$. The principal branches of both solutions are obtained by introducing a cut along the real axis from $z = -\infty$ to $z = +1$, and assigning the principal value to each function.

[4]For more clarity the associated Legendre functions of first kind are denoted by $\mathcal{P}_\nu^{\pm\mu}(x)$. We do not adopt the standard notation $P_\nu^{\pm\mu}(x)$ which is very similar to $\mathrm{P}_\nu^{\pm\mu}(x)$, that denotes the associated Ferrers function of first kind.

### 5.2. Ferrers functions

Suppose that $\mathcal{P}_\nu^{-\mu}(z)$ and $\mathbf{Q}_\nu^\mu(z)$ are real valued on the real interval $[1,+\infty)$ (it is the case when $\nu,\mu\in\mathbb{R}$). On the cut from $-\infty$ to 1 there are two possible values for each function, depending whether the cut is approached from the upper or lower side. Replacing $z$ by $x$, these values are denoted by

$$\mathcal{P}_\nu^{-\mu}(x+i0),\ \mathcal{P}_\nu^{-\mu}(x-i0),\ \mathbf{Q}_\nu^\mu(x+i0),\ \mathbf{Q}_\nu^\mu(x-i0).$$

For $|x|<1$, it is possible to define four real valued functions if $\nu$ and $\mu$ are real. They are known as associated Ferrers functions. Two of such functions are defined as follows under the assumption $-(\nu+\mu)\notin\mathbb{N}^*$ (here $\mathbb{N}^*=\{1,2,3,\dots\}$):

$$\mathrm{P}_\nu^\mu(x)=\mathrm{e}^{i\mu\pi/2}\,\mathcal{P}_\nu^\mu(x+i0)=\mathrm{e}^{-i\mu\pi/2}\,\mathcal{P}_\nu^\mu(x-i0)$$

$$\mathrm{Q}_\nu^\mu(x)=\frac{1}{2}\,\Gamma(\nu+\mu+1)\left[\mathrm{e}^{-i\mu\pi/2}\,\mathbf{Q}_\nu^\mu(x+i0)+\mathrm{e}^{i\mu\pi/2}\,\mathbf{Q}_\nu^\mu(x-i0)\right]. \tag{5.5}$$

The two other associated Ferrers functions are $\mathrm{P}_\nu^{-\mu}(x)$ and $\mathrm{Q}_\nu^{-\mu}(x)$. It is possible to show that

$$\mathrm{P}_\nu^\mu(x)=\left(\frac{1+x}{1-x}\right)^{\mu/2}\mathbf{F}\left(\nu+1\,,\,-\nu\,;\,1-\mu\,;\,\frac{1-x}{2}\right).$$

Such a formula allows to extend the definition of $\mathrm{P}_\nu^\mu(x)$ to complex values of $\nu,\mu$ and $x$ : cuts are introduced along the real intervals $(-\infty,-1]$ and $[1,+\infty)$. The expression for other Ferrers functions can be derived using the connection formulas:

$$\begin{aligned}\mathrm{P}_\nu^\mu&=\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left[\cos(\mu\pi)\,\mathrm{P}_\nu^{-\mu}+\frac{2\sin(\mu\pi)}{\pi}\,\mathrm{Q}_\nu^{-\mu}\right]\\ \mathrm{Q}_\nu^\mu&=\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left[\cos(\mu\pi)\,\mathrm{Q}_\nu^{-\mu}-\frac{\pi\sin(\mu\pi)}{2}\,\mathrm{P}_\nu^{-\mu}\right].\end{aligned} \tag{5.6}$$

In particular the formula we get for $\mathrm{Q}_\nu^\mu$ is used to extend $\mathrm{Q}_\nu^\mu(x)$ to complex values of $\nu,\mu$ and $x$ in the same way as for $\mathrm{P}_\nu^\mu(x)$.

### 5.3. Asymptotics

We recall now some asymptotics about Legendre and Ferrers functions that we will need through the paper.

**Lemma 5.1** (see [8], Sect. 14.8, or [20] p. 186 and [9] p. 163)**.** *The associated Legendre functions $\mathcal{P}_\nu^\mu(x)$, $\mathbf{Q}_\nu^\mu(x)$ defined on $(1,+\infty)$ have the following asymptotic behaviour for $x\to1^+$:*

$$\mathcal{P}_\nu^\mu(x)\sim\frac{1}{\Gamma(1-\mu)}\left(\frac{2}{x-1}\right)^{\frac{\mu}{2}}\qquad \textit{if } \mu\notin\mathbb{N}^* \tag{5.7}$$

$$\mathcal{P}_\nu^\mu(x)\sim\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)\mu!}\left(\frac{x-1}{2}\right)^{\frac{\mu}{2}}\qquad \textit{if } \mu\in\mathbb{N}^*,-(\nu\pm\mu)\notin\mathbb{N}^* \tag{5.8}$$

$$\mathbf{Q}_\nu^\mu(x)\sim\frac{\Gamma(\mu)}{2\Gamma(\nu+\mu+1)}\left(\frac{2}{x-1}\right)^{\frac{\mu}{2}}\qquad \textit{if } Re(\mu)>0,-(\nu+\mu)\notin\mathbb{N}^* \tag{5.9}$$

$$\mathbf{Q}_\nu^0(x)=-\frac{\ln(x-1)}{2\Gamma(\nu+1)}+\frac{\ln\sqrt{2}-\gamma-\psi(\nu+1)}{\Gamma(\nu+1)}+O\left(x-1\right)\qquad \textit{if } -\nu\notin\mathbb{N}^*, \tag{5.10}$$

*where $\gamma$ is the Euler–Mascheroni constant and $\psi(x) = \Gamma'(x)/\Gamma(x)$. The associated Ferrers functions $\mathrm{P}^\mu_\nu(x)$, $\mathrm{Q}^\mu_\nu(x)$ have the following asymptotic behaviour for $x \to 1^-$:*

$$\mathrm{P}^\mu_\nu(x) \sim \frac{1}{\Gamma(1-\mu)}\left(\frac{2}{1-x}\right)^{\frac{\mu}{2}}, \ \mu \notin \mathbb{N}^* \tag{5.11}$$

$$\mathrm{P}^\mu_\nu(x) \sim \frac{\Gamma(\nu+\mu+1)\,(-1)^\mu}{\Gamma(\nu-\mu+1)\,\mu!}\left(\frac{1-x}{2}\right)^{\frac{\mu}{2}}, \ \mu \in \mathbb{N}^*, \nu \neq \mu-1, \mu-2, \ldots, -\mu \tag{5.12}$$

$$\mathrm{Q}^\mu_\nu(x) \sim \frac{1}{2}\cos(\pi\mu)\Gamma(\mu)\left(\frac{2}{1-x}\right)^{\mu/2}, \mu \notin \left(\mathbb{N}^* - \frac{1}{2}\right) \tag{5.13}$$

$$\mathrm{Q}^\mu_\nu(x) \sim \frac{\pi\,\Gamma(\nu+\mu+1)\,(-1)^{\mu+\frac{1}{2}}}{2\,\Gamma(\mu+1)\Gamma(\nu-\mu+1)}\left(\frac{1-x}{2}\right)^{\frac{\mu}{2}}, \ \mu \in \left(\mathbb{N}^* - \frac{1}{2}\right), -(\nu \pm \mu) \notin \mathbb{N}^* \tag{5.14}$$

*where* $\mathbb{N}^* - \dfrac{1}{2} = \left\{\dfrac{1}{2}, \dfrac{3}{2}, \dfrac{5}{2}, \ldots\right\}$

$$\mathrm{Q}^0_\nu(x) = \frac{1}{2}\ln\left(\frac{2}{1-x}\right) - \gamma - \psi(\nu+1) + O\left(1-x\right) \quad \textit{if } -\nu \notin \mathbb{N}^*. \tag{5.15}$$

## 6. Finding a formula for $\sigma(T)$ *via* Legendre and Ferrers functions

We are going now to study the first eigenvalue $\sigma_1(T) = \sigma(T)$ of the linearized operator $H_T$. For this we need a formula for $\sigma(T)$. Recall that

$$\sigma(T) \ = \ c'(1) + \phi''(1), \tag{6.1}$$

where $\phi(r)$ is the bounded solution of the ordinary differential equation

$$u''(r) + (n-1)\,\frac{C_k(r)}{S_k(r)}\,u'(r) + \lambda_1\,u(r) = 0 \tag{6.2}$$

such that $\phi(1) = 0$ and $\phi(r) > 0$ on $[0,1)$, and normalized by (2.4), and $c(r)$ is the continuous solution on $[0,1]$ of the ordinary differential equation

$$u''(r) + (n-1)\,\frac{C_k(r)}{S_k(r)}\,u'(r) + \left[\lambda_1 - \left(\frac{2\pi}{T}\right)^2\right]\,u(r) = 0 \tag{6.3}$$

such that $c(1) = -\phi'(1)$. We observe that $\phi'(1) \neq 0$ otherwise $\phi(r) \equiv 0$. Indeed the solution of (6.2) satisfying also $\phi(1) = \phi'(1) = 0$ is the function identically equal to zero.

The general solution of (6.2) can be found as follows. The function

$$p(r) := S_k(r)^{\frac{n}{2}-1}\,u(r)$$

satisfies:

$$p''(r) + \frac{C_k(r)}{S_k(r)}\,p'(r) + \left\{\lambda_1 + k\left(\frac{n}{2}-1\right) + \left[\left(\frac{n}{2}-1\right)\frac{C_k(r)}{S_k(r)}\right]^2\right\}\,p(r) = 0.$$

By the change of variable $x = x(r) = C_k(r)$, we get that the function

$$w(x) = p(r(x))$$

satisfies (5.1) after replacing $z$ by the real variable $x$ and setting

$$\mu = \frac{n-2}{2}, \quad \nu = -\frac{1}{2} + \sqrt{\frac{(n-1)^2}{4} + \frac{\lambda_1}{k}}.$$

When $\frac{(n-1)^2}{4} + \frac{\lambda_1}{k} < 0$ then we will always consider the square root having positive imaginary part. In other terms $Im(\nu) > 0$. The general solution to (5.1) can be expressed as linear combination of $\mathcal{P}^\mu_\nu(x), \mathbf{Q}^\mu_\nu(x)$ if $k < 0$ and of $\mathrm{P}^\mu_\nu(x), \mathrm{Q}^\mu_\nu(x)$ if $k > 0$. Consequently the general solution to (6.2) is:

$$u(r) = \begin{cases} a\,(S_k(r))^{1-\frac{n}{2}}\,\mathrm{P}^\mu_\nu(C_k(r)) + b\,(S_k(r))^{1-\frac{n}{2}}\,\mathrm{Q}^\mu_\nu(C_k(r)) & \text{if } k > 0 \\ a\,(S_k(r))^{1-\frac{n}{2}}\,\mathcal{P}^\mu_\nu(C_k(r)) + b\,(S_k(r))^{1-\frac{n}{2}}\,\mathbf{Q}^\mu_\nu(C_k(r)) & \text{if } k < 0. \end{cases}$$

Lemma 5.1 says that such functions are, in some cases, unbounded on $[0,1]$. They can diverge as $r$ tends to 0, as specified below. $\mathbf{Q}^\mu_\nu(C_k(r))$ is unbounded for: (a) $\mathrm{R}e(\mu) > 0$ and $\mu + \nu \neq -1, -2, -3, \ldots$; (b) $\mu = 0$. $\mathcal{P}^\mu_\nu(C_k(r))$ is unbounded if $\mu$ is half-integer (that is $n$ is odd). $\mathrm{Q}^\mu_\nu(C_k(r))$ is unbounded if $\mu$ is integer (that is $n$ is even). $\mathrm{P}^\mu_\nu(C_k(r))$ is unbounded if $\mu$ is half-integer (that is $n$ is odd). Furthermore, the function $\mathrm{Q}^\mu_\nu(C_k(r))$ is bounded if $\mu$ is half-integer, but it is a complex valued function.

If $\mu$ is half-integer, then a bounded real valued solution to equation (6.2) is $\mathcal{P}^{-\mu}_\nu(x)$ if $k < 0$, and $\mathrm{P}^{-\mu}_\nu(x)$ if $k > 0$ (see (5.7), (5.12) with $\mu$ replaced by $-\mu$). Formulas (5.4) and (5.6) show that the function $\mathcal{P}^{-\mu}_\nu(x)$ is a linear combination of $\mathcal{P}^\mu_\nu(x), \mathbf{Q}^\mu_\nu(x)$, and $\mathrm{P}^{-\mu}_\nu(x)$ is a linear combination of $\mathrm{P}^\mu_\nu(x)$, $\mathrm{Q}^\mu_\nu(x)$. Consequently:

$$\phi(r) = \begin{cases} s\,(S_k(r))^{1-\frac{n}{2}}\,\mathrm{P}^\mu_\nu(C_k(r)), \text{ if } k > 0, \mu \text{ integer} \\ s\,(S_k(r))^{1-\frac{n}{2}}\,\mathrm{P}^{-\mu}_\nu(C_k(r)), \text{ if } k > 0, \mu \text{ half-integer} \\ s\,(S_k(r))^{1-\frac{n}{2}}\,\mathcal{P}^\mu_\nu(C_k(r)), \text{ if } k < 0, \mu \text{ integer} \\ s\,(S_k(r))^{1-\frac{n}{2}}\,\mathcal{P}^{-\mu}_\nu(C_k(r)), \text{ if } k < 0, \mu \text{ half-integer} \end{cases}$$

where $s$ is a constant chosen in order to ensure the conditions $\phi(r) > 0$ for $r \in [0,1)$ and (2.4). The value of the eigenvalue $\lambda_1$ which appears in $\nu$ is the smallest positive real number so that $\phi(1) = 0$.

In order to find the function $c(r)$ we set

$$\nu^* = -\frac{1}{2} + \sqrt{\frac{(n-1)^2}{4} + \frac{\lambda_1 - \frac{4\pi^2}{T^2}}{k}}.$$

When $\frac{(n-1)^2}{4} + \frac{\lambda_1 - \frac{4\pi^2}{T^2}}{k} < 0$ then we will always suppose that the imaginary part of $\nu^*$ is positive. By the same reasoning we did for $\phi$, we find that the solution of (6.3) is given by

$$c(r) = \begin{cases} A\,(S_k(r))^{1-\frac{n}{2}}\,\mathrm{P}^\mu_{\nu^*}(C_k(r)), \text{ if } k > 0, \mu \text{ integer} \\ A\,(S_k(r))^{1-\frac{n}{2}}\,\mathrm{P}^{-\mu}_{\nu^*}(C_k(r)), \text{ if } k > 0, \mu \text{ half-integer} \\ A\,(S_k(r))^{1-\frac{n}{2}}\,\mathcal{P}^\mu_{\nu^*}(C_k(r)), \text{ if } k < 0, \mu \text{ integer} \\ A\,(S_k(r))^{1-\frac{n}{2}}\,\mathcal{P}^{-\mu}_{\nu^*}(C_k(r)), \text{ if } k < 0, \mu \text{ half-integer} \end{cases} \tag{6.4}$$

where $A$ is a constant that can be determined using the boundary condition $c(1) = -\phi'(1)$.

In the next two sections we will study $\sigma(T)$. An essential ingredient will be the following:

**Proposition 6.1.** *The following facts hold:*

(1) *Let $r_0 > 0$ be the $n$-th zero of the associated Legendre function $\mathcal{P}^\mu_{-\frac{1}{2}+i\tau}(C_k(r))$. If $\tau \in \mathbb{R}^+$, then $r_0$ is a decreasing function of $\tau$.*

(2) *Let $r_0 \in (0,\pi)$ be the $n$-th zero of the associated Ferrers function $\mathrm{P}^\mu_{-\frac{1}{2}+i\tau}(C_k(r))$. If $\tau \in \mathbb{R}^+$, then $r_0$ is a decreasing function of $\tau$.*

*Proof.* We follow the proof of Theorem 7.6.4 in [20]. Suppose that $z_0 = \cosh(r_0)$ and $\nu = -1/2 + i\tau$. If we differentiate $\mathcal{P}^{\mu}_{-1/2+i\tau}(z_0) = 0$, we get

$$\left(\mathcal{P}^{\mu}_{-\frac{1}{2}+i\tau}\right)'(z_0)\frac{\mathrm{d}z_0}{\mathrm{d}\tau} + \frac{\partial \mathcal{P}^{\mu}_{-\frac{1}{2}+i\tau}}{\partial \tau}(z_0) = 0. \tag{6.5}$$

The differential equation satisfied by the function $\mathcal{P}^{\mu}_{\nu}$ is

$$[(1-x^2)(\mathcal{P}^{\mu}_{\nu})']' + \left(\nu(\nu+1) - \frac{\mu}{1-x^2}\right)\mathcal{P}^{\mu}_{\nu} = 0.$$

We multiply it by $\mathcal{P}^{\mu}_{\eta}$, with $\eta \neq \nu$, and we subtract from the expression we get this way, the differential equation satisfied by $\mathcal{P}^{\mu}_{\eta}$ multiplied by $\mathcal{P}^{\mu}_{\nu}$. We get:

$$[(1-x^2)((\mathcal{P}^{\mu}_{\nu})'\mathcal{P}^{\mu}_{\eta} - (\mathcal{P}^{\mu}_{\eta})'\mathcal{P}^{\mu}_{\nu})]' + (\nu(\nu+1) - \eta(\eta+1))\mathcal{P}^{\mu}_{\nu}\mathcal{P}^{\mu}_{\eta} = 0.$$

If $\eta = -\frac{1}{2} + i\rho$, then $\nu(\nu+1) - \eta(\eta+1) = \rho^2 - \tau^2$. In conclusion, if $\rho \neq \tau$,

$$\int \mathcal{P}^{\mu}_{\nu}\mathcal{P}^{\mu}_{\eta}\mathrm{d}x = \frac{(x^2-1)((\mathcal{P}^{\mu}_{\nu})'\mathcal{P}^{\mu}_{\eta} - (\mathcal{P}^{\mu}_{\eta})'\mathcal{P}^{\mu}_{\nu})}{\rho^2 - \tau^2}\cdot$$

If we let $\rho$ tend to $\tau$, then using the l'Hôpital rule, we get:

$$\int (\mathcal{P}^{\mu}_{\nu})^2\mathrm{d}x = \frac{(x^2-1)}{2\tau}\left((\mathcal{P}^{\mu}_{\nu})'\frac{\partial(\mathcal{P}^{\mu}_{\nu})}{\partial\tau} - \frac{\partial(\mathcal{P}^{\mu}_{\nu})'}{\partial\tau}\mathcal{P}^{\mu}_{\nu}\right)\cdot$$

If we set the integration bounds equal to 1 and $z_0$ then

$$\int_1^{z_0} (\mathcal{P}^{\mu}_{\nu})^2\mathrm{d}x = \frac{(z_0^2-1)}{2\tau}(\mathcal{P}^{\mu}_{\nu})'(z_0)\frac{\partial(\mathcal{P}^{\mu}_{\nu})}{\partial\tau}(z_0). \tag{6.6}$$

In other terms:

$$\frac{\partial(\mathcal{P}^{\mu}_{\nu})}{\partial\tau}(z_0) = \frac{2\tau}{(z_0^2-1)}\frac{1}{(\mathcal{P}^{\mu}_{\nu})'(z_0)}\int_1^{z_0}(\mathcal{P}^{\mu}_{\nu})^2\mathrm{d}x$$

which replaced in (6.5) gives:

$$\frac{\mathrm{d}z_0}{\mathrm{d}\tau} = -\frac{2\tau}{(z_0^2-1)}\frac{1}{((\mathcal{P}^{\mu}_{\nu})'(z_0))^2}\int_1^{z_0}(\mathcal{P}^{\mu}_{\nu})^2\mathrm{d}x < 0.$$

As $z_0 = \cosh(r_0)$ then

$$\frac{\mathrm{d}z_0}{\mathrm{d}\tau} = \frac{\mathrm{d}z_0}{\mathrm{d}r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\tau} = \sinh(r_0)\frac{\mathrm{d}r_0}{\mathrm{d}\tau}\cdot$$

So

$$\frac{\mathrm{d}r_0}{\mathrm{d}\tau} = \frac{1}{\sinh(r_0)}\frac{\mathrm{d}z_0}{\mathrm{d}\tau} < 0.$$

The proof of the monotonicity for the zeros of $\mathrm{P}^{\mu}_{\nu}$ is essentially the same. Suppose that $z_0 = \cos(r_0)$. As $z_0 \in (-1,1)$, we set the bounds of integration equal to $-1$ and $z_0$. In this case instead of (6.6) we have:

$$\int_{-1}^{z_0} (\mathrm{P}^{\mu}_{\nu})^2\mathrm{d}x = \frac{(z_0^2-1)}{2\tau}(\mathrm{P}^{\mu}_{\nu})'(z_0)\frac{\partial(\mathrm{P}^{\mu}_{\nu})}{\partial\tau}(z_0).$$

Plugging it into (6.5) (which is true also for $\mathrm{P}_\nu^\mu$), we get:

$$\frac{\mathrm{d}z_0}{\mathrm{d}\tau} = -\frac{2\tau}{(z_0^2-1)}\frac{1}{((\mathrm{P}_\nu^\mu)'(z_0))^2}\int_{-1}^{z_0}(\mathrm{P}_\nu^\mu)^2\mathrm{d}x > 0.$$

Now we consider the identity $z_0 = \cos(r_0)$ then

$$\frac{\mathrm{d}z_0}{\mathrm{d}\tau} = \frac{\mathrm{d}z_0}{\mathrm{d}r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\tau} = -\sin(r_0)\frac{\mathrm{d}r_0}{\mathrm{d}\tau}\cdot$$

As a consequence:

$$\frac{\mathrm{d}r_0}{\mathrm{d}\tau} = -\frac{1}{\sin(r_0)}\frac{\mathrm{d}z_0}{\mathrm{d}\tau} < 0.$$

This completes the proof of the proposition. □

## 7. Study of $\sigma(T)$

It is easy to see that $\sigma(T)$ is analytic. This fact comes from the following remark: if $K$ is an invertible operator and $I$ is the identity, then for $T>0$ and any continuous function $v$, the solution $u$ of

$$\left(K - \frac{1}{T^2}\,\rho\, I\right) u = v$$

is analytic with respect to $T$ for each constant $\rho$ (this follows from the equality

$$(I - sK)^{-1} = \sum_{n\geq 0} s^n\, K^n$$

for each $s\in\mathbb{R}$). Then to prove that $c$ is analytic it suffices to take

$$K = \left(\partial_r^2 + (n-1)\frac{C_k(r)}{S_k(r)}\,\partial_r + \lambda_1\right), \qquad v = 0, \qquad \rho = (2\pi)^2\,.$$

We conclude that $c'(1)$ is analytic with respect to $T$, and from (4.5) the analyticity of $\sigma$ follows. The following proposition shows the behavior of $\sigma$ at $0^+$ and $+\infty$.

**Proposition 7.1.** *The function $\sigma(T)$ satisfies*

$$\lim_{T\to 0^+}\sigma(T) = +\infty \qquad \textit{and} \qquad \lim_{T\to+\infty}\sigma(T) = -\infty.$$

*Proof.* We consider four cases, depending on the dimension $n$ is odd or even and if the curvature $k$ of $\mathbb{M}^n$ is positive ($\mathbb{S}^n$) or negative ($\mathbb{H}^n$). According to Remark 3.1, we could use $k\neq 0$ instead of $k\in\{-1,1\}$. For this reason, in the following computation we will distinguish the case $k<0$ from the case $k>0$ and we do not replace $k$ by its value. Furthermore, as $\sigma(T) = c'(1)+\phi''(1)$ and $\phi''(1)$ does not depend on $T$ it suffices to study the behavior of $c'(1)$.

**First case: $n$ even and $k$ negative.** If $n$ is even then $\mu$ is integer. In the case $k<0$ and $\mu$ integer the derivative of $c(r)$ is

$$c'(r) = A\left(1-\frac{n}{2}\right)S_k^{-\frac{n}{2}}(r)\,C_k(r)\,\mathcal{P}_{\nu^*}^\mu(C_k(r)) - k\,A\,S_k^{2-\frac{n}{2}}(r)\,(\mathcal{P}_{\nu^*}^\mu)'(C_k(r)).$$

The last summand can be expressed in terms of $\mathcal{P}_{\nu^*}^\mu$ and $\mathcal{P}_{\nu^*}^{\mu+1}$ using formula (7.12.17, p. 195, [16]):

$$(\mathcal{P}_{\nu^*}^\mu(x))' = \frac{1}{x^2-1}\left[\sqrt{(x^2-1)}\,\mathcal{P}_{\nu^*}^{\mu+1}(x) + \mu\, x\,\mathcal{P}_{\nu^*}^\mu(x)\right]. \tag{7.1}$$

If $x = C_k(r)$ then $C_k^2(r) - 1 = -k\, S_k^2(r)$ and $\sqrt{C_k^2(r) - 1} = \sqrt{-k}\, S_k(r)$. As a consequence

$$(\mathcal{P}^{\mu}_{\nu^*})'(C_k(r)) = \frac{1}{-k\, S_k^2(r)} \left[\sqrt{-k}\, S_k(r)\, \mathcal{P}^{\mu+1}_{\nu^*}(C_k(r)) + \mu\, C_k(r)\, \mathcal{P}^{\mu}_{\nu^*}(C_k(r))\right]$$

and

$$\begin{aligned} c'(r) &= A\left(1 - \frac{n}{2}\right) S_k^{-\frac{n}{2}}(r)\, C_k(r)\, \mathcal{P}^{\mu}_{\nu^*}(C_k(r)) \\ &\quad + A\, S_k^{-\frac{n}{2}}(r) \left[\sqrt{-k}\, S_k(r)\, \mathcal{P}^{\mu+1}_{\nu^*}(C_k(r)) + \mu\, C_k(r)\, \mathcal{P}^{\mu}_{\nu^*}(C_k(r))\right] = \\ &= A\, S_k^{-\frac{n}{2}}(r) \left[C_k(r)\, \mathcal{P}^{\mu}_{\nu^*}(C_k(r)) \left(1 - \frac{n}{2} + \mu\right) + \sqrt{-k}\, S_k(r)\, \mathcal{P}^{\mu+1}_{\nu^*}(C_k(r))\right] \\ &= A\, \sqrt{-k}\, S_k^{1-\frac{n}{2}}(r)\, \mathcal{P}^{\mu+1}_{\nu^*}(C_k(r)). \end{aligned}$$

If we replace $\nu^*$ by $\nu$ and $A$ by $s$, then $c(r)$ reduces to $\phi(r)$. So the computation above shows also that

$$\phi'(r) = s\, \sqrt{-k}\, S_k^{1-\frac{n}{2}}(r)\, \mathcal{P}^{\mu+1}_{\nu}(C_k(r)).$$

As

$$A = -\frac{\phi'(1)}{S_k^{1-\frac{n}{2}}(1) \mathcal{P}^{\mu}_{\nu^*}(C_k(1))} = -\frac{s\, \sqrt{-k}\, \mathcal{P}^{\mu+1}_{\nu}(C_k(1))}{\mathcal{P}^{\mu}_{\nu^*}(C_k(1))},$$

then the function $c'(r)$ is

$$c'(r) = \frac{s\, k\, \mathcal{P}^{\mu+1}_{\nu}(C_k(1))}{\mathcal{P}^{\mu}_{\nu^*}(C_k(1))}\, S_k^{1-\frac{n}{2}}(r)\, \mathcal{P}^{\mu+1}_{\nu^*}(C_k(r)).$$

Consequently

$$c'(1) + \phi''(1) = s\, k\, S_k^{1-\frac{n}{2}}(1)\, \frac{\mathcal{P}^{\mu+1}_{\nu}(C_k(1))}{\mathcal{P}^{\mu}_{\nu^*}(C_k(1))}\, \mathcal{P}^{\mu+1}_{\nu^*}(C_k(1)) + \phi''(1).$$

We remark that

$$\lim_{T \to +\infty} \nu^* = \nu.$$

Consequently the numerator of $c'(1)$ tends to

$$s\, k\, S_k^{1-\frac{n}{2}}(1)\, (\mathcal{P}^{\mu+1}_{\nu}(C_k(1)))^2$$

when $T$ goes to $+\infty$. We observe that

$$\nu = -\frac{1}{2} + \sqrt{\frac{(n-1)^2}{4} + \frac{\lambda_1}{k}}$$

has a non-vanishing imaginary part because $k < 0$ and $\lambda_1 > -k\frac{(n-1)^2}{4}$. As $\mathrm{Im}(\nu^*) < \mathrm{Im}(\nu)$, then Proposition 6.1 ensures that the first positive zero of $\mathcal{P}^{\mu}_{\nu^*}(C_k(r))$ is bigger than 1. Indeed from the definition of $\phi$ and $\phi(1) = 0$, it is easily seen that $\mathcal{P}^{\mu}_{\nu}(C_k(1)) = 0$. Furthermore $\mathcal{P}^{\mu}_{\nu^*}(C_k(1)) > 0$ if $\mathcal{P}^{\mu}_{\nu}(C_k(r)) > 0$ for $r \in [0, 1)$ or $\mathcal{P}^{\mu}_{\nu^*}(C_k(1)) < 0$ if $\mathcal{P}^{\mu}_{\nu}(C_k(r)) < 0$ for $r \in [0, 1)$. By definition, $s$ has the same sign as $\mathcal{P}^{\mu}_{\nu}(C_k(r))$ on $r \in [0, 1)$. Then, if $s > 0$,

$$\lim_{\nu^* \to \nu} \mathcal{P}^{\mu}_{\nu^*}(C_k(1)) = 0^+ (= \mathcal{P}^{\mu}_{\nu}(C_k(1))),$$

*i.e.*

$$\lim_{\nu^* \to \nu} \frac{1}{\mathcal{P}^{\mu}_{\nu^*}(C_k(1))} = +\infty.$$

Similarly

$$\lim_{\nu^*\to\nu} \frac{1}{\mathcal{P}^{\mu}_{\nu^*}(C_k(1))} = -\infty\,,$$

if $s<0$. We conclude that

$$\lim_{T\to+\infty} \sigma(T) = \lim_{T\to+\infty} [c'(1)+\phi''(1)] = -\infty.$$

Now we consider the limit of $\sigma(T)$ as $T\to 0^+$. As $k<0$ then

$$\lim_{T\to 0^+} \nu^* = \lim_{T\to 0^+} -\frac{1}{2} + \sqrt{\frac{(n-1)^2}{4} + \frac{\lambda_1 - \frac{4\pi^2}{T^2}}{k}} = +\infty.$$

That says also that for $T$ small enough, $\nu^*$ is real. Let us observe that $c'(1)$ can be written in the following form:

$$c'(1) = -\phi'(1)\sqrt{-k}\,\frac{\mathcal{P}^{\mu+1}_{\nu^*}(C_k(1))}{\mathcal{P}^{\mu}_{\nu^*}(C_k(1))}.$$

Formula 14.15.13 [8] provides the asymptotic behaviour of $\mathcal{P}^{-\mu}_{\nu^*}$ with respect to $\nu^*$:

$$\mathcal{P}^{-\mu}_{\nu^*}(C_k(1)) \sim \frac{1}{(\nu^*)^{\mu}} \sqrt{\frac{1}{\sinh(1)}}\, I_{\mu}\left(\nu^* + \frac{1}{2}\right) \tag{7.2}$$

where $I_\mu$ denotes the modified Bessel function of first kind (we refer to [16] for basic facts about Bessel functions). To get the asymptotic expression for $\mathcal{P}^{\mu}_{\nu^*}(C_k(1))$ we use the following identity

$$\mathcal{P}^{\mu}_{\nu^*} = \frac{\Gamma(\nu^*+\mu+1)}{\Gamma(\nu^*-\mu+1)}\,\mathcal{P}^{-\mu}_{\nu^*} \tag{7.3}$$

which follows from (5.4) using the fact that $\mu$ is integer. Notice that $\frac{\Gamma(\nu^*+\mu+1)}{\Gamma(\nu^*-\mu+1)} \sim (\nu^*)^t$ for $\nu^*$ big, where $t=2\mu$ if $\mu$ is integer and $t=2\mu+1$ if $\mu$ is not integer. We are considering the case where $\mu$ is integer, then from (7.2) and (7.3) we get

$$\mathcal{P}^{\mu}_{\nu^*}(C_k(1)) \sim (\nu^*)^{\mu} \sqrt{\frac{1}{\sinh(1)}}\, I_{\mu}\left(\nu^* + \frac{1}{2}\right)$$

for $\nu^*$ big. Observe that

$$I_{\mu}\left(\nu^* + \frac{1}{2}\right) \sim \frac{\mathrm{e}^{\nu^*+\frac{1}{2}}}{\sqrt{\pi(2\nu^*+1)}} \tag{7.4}$$

for $\nu^*$ big. This implies that

$$\frac{\mathcal{P}^{\mu+1}_{\nu^*}(C_k(1))}{\mathcal{P}^{\mu}_{\nu^*}(C_k(1))} \sim \nu^*\,\frac{I_{\mu+1}\left(\nu^*+\frac{1}{2}\right)}{I_{\mu}\left(\nu^*+\frac{1}{2}\right)} \sim \nu^*$$

for $\nu^*$ big, and in conclusion

$$c'(1) \sim -\phi'(1)\sqrt{-k}\,\nu^*$$

for $\nu^*$ big. As $\phi'(1)<0$, we conclude that

$$\lim_{T\to 0^+} \sigma(T) = \lim_{T\to 0^+} [c'(1)+\phi''(1)] = +\infty.$$

**Second case: $n$ odd and $k$ negative.** If $n$ is odd, then $\mu$ is half-integer. If $k<0$ and $\mu$ is half-integer, then $c(r)$ is given by

$$c(r) = A\,S_k^{1-\frac{n}{2}}(r)\,\mathcal{P}^{-\mu}_{\nu^*}(C_k(r))$$

where $A$ is the constant such that $c(1) = -\phi'(1)$. Moreover

$$\phi(r) = s\, S_k^{1-\frac{n}{2}}(r)\, \mathcal{P}_\nu^{-\mu}(C_k(r))$$

where $s$ is a constant such that $\phi(r) > 0$ for $r \in [0, 1)$ and (2.4). Moreover we have $\phi(1) = 0$. Using (7.1) with $\mu$ replaced by $-\mu$, we get:

$$(\mathcal{P}_{\nu^*}^{-\mu})'(C_k(r)) = \frac{1}{-k\, S_k^2(r)} \left[\sqrt{-k}\, S_k(r)\, \mathcal{P}_{\nu^*}^{-\mu+1}(C_k(r)) - \mu\, C_k(r)\, \mathcal{P}_{\nu^*}^{-\mu}(C_k(r))\right]$$

and

$$\begin{aligned}
c'(r) &= A\left(1 - \frac{n}{2}\right) S_k^{-\frac{n}{2}}(r)\, C_k(r) \mathcal{P}_{\nu^*}^{-\mu}(C_k(r)) \\
&\quad + A\, S_k^{-\frac{n}{2}}(r) \left[\sqrt{-k}\, S_k(r)\, \mathcal{P}_{\nu^*}^{-\mu+1}(C_k(r)) - \mu\, C_k(r)\, \mathcal{P}_{\nu^*}^{-\mu}(C_k(r))\right] \\
&= A\, S_k^{-\frac{n}{2}}(r) \left[C_k(r)\, \mathcal{P}_{\nu^*}^{-\mu}(C_k(r)) \left(1 - \frac{n}{2} - \mu\right) + \sqrt{-k}\, S_k(r)\, \mathcal{P}_{\nu^*}^{-\mu+1}(C_k(r))\right] \\
&= A \left[\sqrt{-k}\, S_k^{1-\frac{n}{2}}(r)\, \mathcal{P}_{\nu^*}^{-\mu+1}(C_k(r)) - 2\mu\, C_k(r)\, S_k^{-\frac{n}{2}}(r)\, \mathcal{P}_{\nu^*}^{-\mu}(C_k(r))\right].
\end{aligned}$$

If we replace $\nu^*$ by $\nu$ and $A$ by $s$, then $c(r)$ reduces to $\phi(r)$. So the computation above shows also that

$$\phi'(r) = s \left[\sqrt{-k}\, S_k^{1-\frac{n}{2}}(r)\, \mathcal{P}_\nu^{-\mu+1}(C_k(r)) - 2\mu\, C_k(r) S_k^{-\frac{n}{2}}(r)\, \mathcal{P}_\nu^{-\mu}(C_k(r))\right].$$

As a consequence

$$\phi'(1) = s\, \sqrt{-k}\, S_k^{1-\frac{n}{2}}(1)\, \mathcal{P}_\nu^{-\mu+1}(C_k(1))$$

because $\mathcal{P}_\nu^{-\mu}(C_k(1)) = 0$. From $c(1) = -\phi'(1)$, we get the value of the constant $A$:

$$A = -\frac{\phi'(1)}{S_k^{1-\frac{n}{2}}(1)\, \mathcal{P}_{\nu^*}^{-\mu}(C_k(1))} = -s\, \sqrt{-k}\, \frac{\mathcal{P}_\nu^{-\mu+1}(C_k(1))}{\mathcal{P}_{\nu^*}^{-\mu}(C_k(1))}.$$

If $T \to 0^+$, then $\nu^* \to +\infty$. If $\nu^*$ is big enough, then (7.2) gives the asymptotic behaviour for $\nu^*$ big:

$$\mathcal{P}_{\nu^*}^{-\mu}(C_k(1)) \sim \frac{1}{(\nu^*)^\mu} \sqrt{\frac{1}{\sinh(1)}}\, I_\mu \left(\nu^* + \frac{1}{2}\right).$$

The asymptotic behaviour of $I_\mu$ is described by (7.4). Consequently

$$\begin{aligned}
c'(1) &= A \left[\sqrt{-k}\, S_k^{1-\frac{n}{2}}(1)\, \mathcal{P}_{\nu^*}^{-\mu+1}(C_k(1)) - 2\mu\, C_k(1)\, S_k^{-\frac{n}{2}}(1) \mathcal{P}_{\nu^*}^{-\mu}(C_k(1))\right] \\
&= \frac{-\phi'(1)}{S_k^{1-\frac{n}{2}}(1)\, \mathcal{P}_{\nu^*}^{-\mu}(C_k(1))} \sqrt{-k}\, S_k^{1-\frac{n}{2}}(1)\, \mathcal{P}_{\nu^*}^{-\mu+1}(C_k(1)) \\
&\sim -\phi'(1)\, \sqrt{-k}\, \frac{(\nu^*)^\mu}{(\nu^*)^{\mu-1}} \frac{I_\mu\left(\nu^* + \frac{1}{2}\right)}{I_{\mu-1}\left(\nu^* + \frac{1}{2}\right)} \\
&\sim -\phi'(1)\, \sqrt{-k}\, \nu^*.
\end{aligned}$$

As $\phi'(1) < 0$, $k < 0$, we conclude that

$$\lim_{T \to 0^+} \sigma(T) = \lim_{T \to 0^+} [c'(1) + \phi''(1)] = +\infty.$$

It remains to study the behaviour of $\sigma(T)$ as $T \to +\infty$. In this case $\nu^* \to \nu$. Proposition 6.1 ensures that the first positive zero of $\mathcal{P}^{-\mu}_{\nu^*}(C_k(r))$ is bigger than 1. Consequently

$$\lim_{\nu^* \to \nu} \frac{1}{\mathcal{P}^{-\mu}_{\nu^*}(C_k(1))} = +\infty$$

if $\mathcal{P}^{-\mu}_{\nu}(C_k(r)) > 0$ on $[0,1)$ (that is $s > 0$) and

$$\lim_{\nu^* \to \nu} \frac{1}{\mathcal{P}^{-\mu}_{\nu^*}(C_k(1))} = -\infty$$

if $\mathcal{P}^{-\mu}_{\nu}(C_k(r)) < 0$ on $[0,1)$ (that is $s > 0$). In other terms such a limit has the same sign as $s$. Moreover the numerator of $c'(1)$ tends to

$$-s\, S_k^{1-\frac{n}{2}}(1)\, [\sqrt{-k}\, \mathcal{P}^{-\mu+1}_{\nu}(C_k(1))]^2.$$

Then

$$\lim_{T \to +\infty} \sigma(T) = \lim_{T \to +\infty} [c'(1) + \phi''(1)] = -\infty.$$

**Third case: $n$ even and $k$ positive.** If $n$ is even then $\mu$ is integer. If $k > 0$ and $\mu$ is integer then the function $c(r)$ is given by the first line of (6.4). As a consequence

$$c'(r) = A\left(1 - \frac{n}{2}\right) S_k^{-\frac{n}{2}}(r)\, C_k(r)\, \mathrm{P}^{\mu}_{\nu^*}(C_k(r)) - k\, A\, S_k^{2-\frac{n}{2}}(r)\, (\mathrm{P}^{\mu}_{\nu^*})'(C_k(r)).$$

The derivative $(\mathrm{P}^{\mu}_{\nu^*})'(x)$ is expressed in terms of $\mathrm{P}^{\mu+1}_{\nu^*}(x)$ and $\mathrm{P}^{\mu}_{\nu^*}(x)$ using

$$(\mathrm{P}^{\mu}_{\nu}(x))' = \frac{1}{x^2 - 1}\left(\sqrt{1 - x^2}\, \mathrm{P}^{\mu+1}_{\nu}(x) + x\, \mu\, \mathrm{P}^{\mu}_{\nu}(x)\right). \tag{7.5}$$

Replacing $x$ by $C_k(r)$ we get:

$$(\mathrm{P}^{\mu}_{\nu})'(C_k(r)) = \frac{1}{-k\, S_k^2(r)}\left[\sqrt{k}\, S_k(r)\, \mathrm{P}^{\mu+1}_{\nu}(C_k(r)) + C_k(r)\, \mu\, \mathrm{P}^{\mu}_{\nu}(C_k(r))\right].$$

from which it follows:

$$\begin{aligned} c'(r) &= A\left(1 - \frac{n}{2}\right) S_k^{-\frac{n}{2}}(r)\, C_k(r)\, \mathrm{P}^{\mu}_{\nu^*}(C_k(r)) \\ &\quad + A\, S_k^{-\frac{n}{2}}(r)\left[\sqrt{k}\, S_k(r)\, \mathrm{P}^{\mu+1}_{\nu^*}(C_k(r)) + \mu\, C_k(r)\, \mathrm{P}^{\mu}_{\nu^*}(C_k(r))\right] \\ &= A\, S_k^{-\frac{n}{2}}(r)\left[C_k(r)\, \mathrm{P}^{\mu}_{\nu^*}(C_k(r))\, (1 - \frac{n}{2} + \mu) + \sqrt{k}\, S_k(r)\, \mathrm{P}^{\mu+1}_{\nu^*}(C_k(r))\right] \\ &= A\, \sqrt{k}\, S_k^{1-\frac{n}{2}}(r)\, \mathrm{P}^{\mu+1}_{\nu^*}(C_k(r)). \end{aligned}$$

The constant $A$ is determined in order to have $c(1) = -\phi'(1)$. The function $\phi$ is defined by

$$\phi(r) = s\, S_k^{1-\frac{n}{2}}(r)\, \mathrm{P}^{\mu}_{\nu}(C_k(r)),$$

where $s$ is the constant such that $\phi(r) > 0$ for $r \in [0,1)$. To get the expression of its derivative, we replace $A$ by $s$ and $\nu^*$ by $\nu$ in the expression of $c'(r)$ :

$$\phi'(r) = s\, \sqrt{k}\, S_k^{1-\frac{n}{2}}(r)\, \mathrm{P}^{\mu+1}_{\nu}(C_k(r)).$$

The value of the constant $A$ is given by

$$A = -\frac{\phi'(1)}{S_k^{1-\frac{n}{2}}(1)\, \mathrm{P}^{\mu}_{\nu^*}(C_k(1))} = -s\, \sqrt{k}\, \frac{\mathrm{P}^{\mu+1}_{\nu}(C_k(1))}{\mathrm{P}^{\mu}_{\nu^*}(C_k(1))}.$$

So $c'(1)$ is given by

$$c'(1) = -\phi'(1)\,\sqrt{k}\,\frac{\mathrm{P}^{\mu+1}_{\nu^*}(C_k(1))}{\mathrm{P}^{\mu}_{\nu^*}(C_k(1))} = -s\,k\,\frac{\mathrm{P}^{\mu+1}_{\nu}(C_k(1))}{\mathrm{P}^{\mu}_{\nu^*}(C_k(1))}\,S_k^{1-\frac{n}{2}}(1)\,\mathrm{P}^{\mu+1}_{\nu^*}(C_k(1)).$$

In conclusion, if $k > 0$ and $\mu$ is integer, then $\sigma(T) = c'(1) + \phi''(1)$ equals

$$-s\,k\,S_k^{1-\frac{n}{2}}(1)\,\frac{\mathrm{P}^{\mu+1}_{\nu}(C_k(1))}{\mathrm{P}^{\mu}_{\nu^*}(C_k(1))}\,\mathrm{P}^{\mu+1}_{\nu^*}(C_k(1)) + \phi''(1).$$

For $T$ big enough $\nu^*$ is a real valued increasing function of $T$. If $T \to +\infty$, then $\nu^* \to \nu$ (which is a real number in this case). If $\mathrm{P}^{\mu}_{\nu}(C_k(r)) > 0$ for $r \in [0,1)$, (that is $s > 0$) then, from Proposition 6.1, we get

$$\lim_{\nu^* \to \nu} \mathrm{P}^{\mu}_{\nu^*}(C_k(1)) = 0^+ (= \mathrm{P}^{\mu}_{\nu}(C_k(1))).$$

So

$$\lim_{T \to +\infty} \frac{1}{\mathrm{P}^{\mu}_{\nu^*}(C_k(1))} = +\infty.$$

Similarly, if $\mathrm{P}^{\mu}_{\nu}(C_k(r)) < 0$ for $r \in [0,1)$, (that is $s < 0$), then

$$\lim_{T \to +\infty} \frac{1}{\mathrm{P}^{\mu}_{\nu^*}(C_k(1))} = -\infty.$$

In other terms the sign of such a limit is the same as $s$. When $T \to +\infty$, the numerator of $c'(1)$ tends to

$$-s\,k\,S_k^{1-\frac{n}{2}}(1)\,[\mathrm{P}^{\mu+1}_{\nu}(C_k(1))]^2.$$

Consequently, as $k > 0$,

$$\lim_{T \to +\infty} \sigma(T) = \lim_{T \to +\infty} [c'(1) + \phi''(1)] = -\infty.$$

Now we study the limit of $\sigma(T)$ as $T \to 0^+$. If $T \to 0^+$ then $\nu^* \to -1/2 + i\infty$. We set $\nu^* = -1/2 + i\tau$. In this case we use the following asymptotic formula (exercise 13.4, p. 73 [20]):

$$\mathrm{P}^{-\mu}_{-\frac{1}{2}+i\tau}(C_k(1)) = \frac{1}{\tau^{\mu}}\sqrt{\frac{1}{\sin(1)}}\,I_{\mu}(\tau)\,\left(1 + \mathcal{O}\left(\frac{1}{\tau}\right)\right)$$

when $\tau$ goes to infinity. To get the corresponding formula for $\mathrm{P}^{\mu}_{-\frac{1}{2}+i\tau}(C_k(1))$ we use the following identity (formula 14.9.2 [8]):

$$\mathrm{P}^{\mu}_{\nu^*} = \frac{\Gamma(\nu^* + \mu + 1)}{\Gamma(\nu^* - \mu + 1)}\left[\cos(\mu\pi)\,\mathrm{P}^{-\mu}_{\nu^*} + \frac{2\sin(\mu\pi)}{\pi}\,\mathrm{Q}^{-\mu}_{\nu^*}\right]. \tag{7.6}$$

As $\mu$ is integer, then (7.6) reduces to:

$$\mathrm{P}^{\mu}_{\nu^*}(x) = (-1)^{\mu}\,\frac{\Gamma(\nu^* + \mu + 1)}{\Gamma(\nu^* - \mu + 1)}\,\mathrm{P}^{-\mu}_{\nu^*}(x).$$

We need to estimate the limit as $T \to 0^+$ of

$$\begin{aligned}\frac{\mathrm{P}^{\mu+1}_{\nu^*}(x)}{\mathrm{P}^{\mu}_{\nu^*}(x)} &= -\frac{\Gamma(\nu^* + \mu + 2)}{\Gamma(\nu^* - \mu)}\,\frac{\Gamma(\nu^* - \mu + 1)}{\Gamma(\nu^* + \mu + 1)}\,\frac{\mathrm{P}^{-\mu-1}_{\nu^*}(x)}{\mathrm{P}^{-\mu}_{\nu^*}(x)}\\ &= -(\nu^* + \mu + 1)(\nu^* - \mu)\frac{\mathrm{P}^{-\mu-1}_{\nu^*}(x)}{\mathrm{P}^{-\mu}_{\nu^*}(x)}.\end{aligned}$$

Observe that

$$(\nu^* + \mu + 1)(\nu^* - \mu) = (\nu^*)^2 + \nu^* - \mu - \mu^2 \left(-\frac{1}{2} + i\tau\right)^2 - \frac{1}{2} + i\tau - \mu - \mu^2$$
$$= -\frac{1}{4} - \tau^2 - \mu - \mu^2 < 0.$$

This implies that

$$\frac{\mathrm{P}^{\mu+1}_{\nu^*}(C_k(1))}{\mathrm{P}^{\mu}_{\nu^*}(C_k(1))} \sim \tau^2 \frac{\tau^\mu}{\tau^{\mu+1}} \frac{I_{\mu+1}(\tau)}{I_\mu(\tau)} \sim \tau,$$

for $\tau$ big, since

$$I_\mu(\tau) \sim \frac{\mathrm{e}^\tau}{\sqrt{2\pi\tau}}$$

(formula 5.16.5 [16]). In conclusion for $\tau$ big,

$$c'(1) \sim -\phi'(1)\sqrt{k}\,\frac{\mathrm{P}^{\mu+1}_{\nu^*}(C_k(1))}{\mathrm{P}^{\mu}_{\nu^*}(C_k(1))} \sim -\phi'(1)\sqrt{k}\,\tau.$$

As $\phi'(1) < 0$, then we conclude that

$$\lim_{T\to 0^+} \sigma(T) = \lim_{T\to 0^+} [c'(1) + \phi''(1)] = +\infty.$$

**Fourth case: $n$ odd and $k$ positive.** If $n$ is odd then $\mu$ is half-integer. If $k > 0$ and $\mu$ is half-integer, then $c(r)$ is given by

$$c(r) = A\, S_k^{1-\frac{n}{2}}(r)\, \mathrm{P}^{-\mu}_{\nu^*}(C_k(r))$$

where $A$ is the constant such that $c(1) = -\phi'(1)$. Moreover

$$\phi(r) = s\, S_k^{1-\frac{n}{2}}(r)\, \mathrm{P}^{-\mu}_{\nu}(C_k(r))$$

where $s$ is the constant such that $\phi(r) > 0$ for $r \in (0,1)$ and (2.4). Moreover we have $\phi(1) = 0$. Using (7.5) with $\mu$ replaced by $-\mu$, we get:

$$(\mathrm{P}^{-\mu}_{\nu^*})'(C_k(r)) = \frac{1}{-k\, S_k^2(r)} \left[\sqrt{k}\, S_k(r)\, \mathrm{P}^{-\mu+1}_{\nu^*}(C_k(r)) - \mu\, C_k(r)\, \mathrm{P}^{-\mu}_{\nu^*}(C_k(r))\right]$$

and

$$\begin{aligned}
c'(r) &= A\left(1 - \frac{n}{2}\right) S_k^{-\frac{n}{2}}(r)\, C_k(r)\, \mathrm{P}^{-\mu}_{\nu^*}(C_k(r)) \\
&\quad + A\, S_k^{-\frac{n}{2}}(r) \left[\sqrt{k}\, S_k(r)\, \mathrm{P}^{-\mu+1}_{\nu^*}(C_k(r)) - \mu\, C_k(r)\, \mathrm{P}^{-\mu}_{\nu^*}(C_k(r))\right] \\
&= A\, S_k^{-\frac{n}{2}}(r) \left[C_k(r)\, \mathrm{P}^{-\mu}_{\nu^*}(C_k(r))(1 - \frac{n}{2} - \mu) + \sqrt{k}\, S_k(r)\, \mathrm{P}^{-\mu+1}_{\nu^*}(C_k(r))\right] \\
&= A\, [\sqrt{k}\, S_k^{1-\frac{n}{2}}(r)\, \mathrm{P}^{-\mu+1}_{\nu^*}(C_k(r)) - 2\mu\, C_k(r)\, S_k^{-\frac{n}{2}}(r)\, \mathrm{P}^{-\mu}_{\nu^*}(C_k(r))].
\end{aligned}$$

If we replace $\nu^*$ by $\nu$ and $A$ by $s$, then $c(r)$ reduces to $\phi(r)$. So the computation above shows also that

$$\phi'(r) = s[\sqrt{k}\, S_k^{1-\frac{n}{2}}(r)\, \mathrm{P}^{-\mu+1}_{\nu}(C_k(r)) - 2\mu\, C_k(r)\, S_k^{-\frac{n}{2}}(r)\, \mathrm{P}^{-\mu}_{\nu}(C_k(r))]$$

from which

$$\phi'(1) = s\sqrt{k}\, S_k^{1-\frac{n}{2}}(1)\, \mathrm{P}^{-\mu+1}_{\nu}(C_k(1)).$$

From $c(1) = -\phi'(1)$, we get the value of the constant $A$:

$$A = \frac{-\phi'(1)}{S_k^{1-\frac{n}{2}}(r)\,\mathrm{P}_{\nu^*}^{-\mu}(C_k(r))} = -s\,\sqrt{k}\,\frac{\mathrm{P}_{\nu}^{-\mu+1}(C_k(1))}{\mathrm{P}_{\nu^*}^{-\mu}(C_k(1))}.$$

If $T \to 0^+$ then $Im(\nu^*) \to +\infty$. For $T$ big enough, $\nu^*$ is a real number and if $T \to +\infty$, then $\nu^* \to \nu \in \mathbb{R}$. If $\nu^*$ is big enough, then (7.2) gives the asymptotic behaviour for $\nu^*$ big:

$$P_{\nu^*}^{-\mu}(C_k(1)) \sim \frac{1}{(\nu^*)^\mu}\sqrt{\frac{1}{\sinh(1)}}\,I_\mu((\nu^*+1/2)).$$

The asymptotic behaviour of $I_\mu$ is described by (7.4). Consequently

$$\begin{aligned} c'(1) &= A[\sqrt{k}S_k^{1-\frac{n}{2}}(1)\mathrm{P}_{\nu^*}^{-\mu+1}(C_k(1)) - 2\mu C_k(1)S_k^{-\frac{n}{2}}(1)\mathrm{P}_{\nu^*}^{-\mu}(C_k(1))] \\ &\sim -\phi'(1)\sqrt{k}\frac{\mathrm{P}_{\nu^*}^{-\mu+1}(C_k(1))}{\mathrm{P}_{\nu^*}^{-\mu}(C_k(1))} \\ &\sim -\phi'(1)\sqrt{k}\frac{(\nu^*)^\mu}{(\nu^*)^{\mu-1}}\frac{I_\mu((\nu^*+1/2))}{I_{\mu-1}((\nu^*+1/2))} \sim -\phi'(1)\sqrt{k}\nu^*. \end{aligned}$$

As $-\phi'(1) > 0$, then we conclude that

$$\lim_{T\to 0}[c'(1) + \phi''(1)] = +\infty.$$

It remains to study the behaviour of $\sigma(T)$ as $T \to +\infty$. If $T \to +\infty$ then $\nu^* \to \nu$ increasing (for $T$ big enough $\nu^*$ is real). Proposition 6.1 ensures that, the first positive zero of $\mathrm{P}_{\nu^*}^{-\mu}(C_k(r))$ is bigger than 1. Consequently

$$\lim_{\nu^*\to\nu}\frac{1}{\mathrm{P}_{\nu^*}^{-\mu}(C_k(1))} = +\infty$$

if $\mathrm{P}_{\nu}^{-\mu}(C_k(r)) > 0$ on $[0,1)$ (that is $s > 0$) and

$$\lim_{\nu^*\to\nu}\frac{1}{\mathrm{P}_{\nu^*}^{-\mu}(C_k(1))} = -\infty$$

if $\mathrm{P}_{\nu}^{-\mu}(C_k(r)) < 0$ on $[0,1)$ (that is $s < 0$). In other terms such a limit has the same sign as $s$. The numerator of $c'(1)$ tends to

$$-s\,k\,S_k^{1-\frac{n}{2}}(1)\,[\mathrm{P}_{\nu}^{-\mu+1}(C_k(1))]^2.$$

As a conclusion

$$\lim_{T\to+\infty}\sigma(T) = \lim_{T\to+\infty}[c'(1) + \phi''(1)] = -\infty.$$

This completes the proof of the proposition. □

## 8. Lyapunov–Schmidt reduction and bifurcation

In view of the analyticity of $\sigma$ (showed in Sect. 7) and Proposition 7.1, $\sigma$ has at least one zero where it changes sign, and the set of the zeros of $\sigma$ is finite. Let $\{0_1, 0_2, \ldots, 0_p\}$ denotes the set of the zeros of $\sigma$, and let $T_*$ be the smallest zero such that $\sigma$ changes sign at $T_*$, say $T_* = 0_q$. It is clear then the eigenspace $V_1$ (defined in Prop. 4.3) belongs to the kernel of $H_{T_*}$. As $\sigma_j(T) = \sigma(T/j)$ we obtain that $\sigma_j$ is analytic on $T$ and the set of the zeros of $\sigma_j$ is $\{j\,0_1, j\,0_2, \ldots, j\,0_p\}$. It is clear that if $j$ is big enough then $T_* \notin \{j\,0_1, j\,0_2, \ldots, j\,0_p\}$, and this means that $V_j$ does not belong to the kernel of $H_{T_*}$ for almost all $j$. This implies that the kernel of $H_{T_*}$ is of the form $V_{j_1} \oplus \ldots \oplus V_{j_l}$ with $1 = j_1 < \ldots < j_l$. Moreover if $V_{j_i} \subset \mathrm{Ker}(H_{T_*})$ and $j_i \neq 1$ then the function $\sigma_{j_i}(T)$ does not change sign at $T_*$ by the definition of $T_*$.

We summarize such facts in the following proposition, where we use also the ellipticity of the linearized operator $H_T$ given by Proposition 4.3.

**Proposition 8.1.** *There exists a positive real number $T_*$ such that the kernel of $H_{T_*}$ is given by $V_{j_1} \oplus \ldots \oplus V_{j_l}$, with $1 = j_1 < \ldots < j_l$. Moreover the eigenvalue associated to the eigenspace $V_1$, considered as a function on $T$, changes sign at $T_*$, and the eigenvalues associated to the other eigenspaces $V_{j_2}, \ldots, V_{j_l}$, always considered as functions on $T$, do not change sign at $T_*$. There exists a constant $c > 0$ such that*

$$\|w\|_{C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})} \leq c\, \|H_{T_*}(w)\|_{\mathcal{C}^{1,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})},$$

*provided $w$ is $L^2(\mathbb{R}/2\pi\mathbb{Z})$-orthogonal to $V_0 \oplus V_{j_1} \oplus \ldots \oplus V_{j_l}$, where $V_0$ is the space of constant functions.*

Such proposition says us that the operator $H_{T_*}$ has finite-dimensional kernel, and that it is an isomorphism from the orthogonal to its kernel over its image (see also Prop. 4.3 and its proof). We are going to use now these two properties.

Consider the space $C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \times (0, +\infty)$. Clearly the curve

$$\Xi = \{(v, T) \quad : \quad v \equiv 0\}$$

in $C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \times (0, +\infty)$ belongs to the zero level set of the operator $F$, *i.e.* its points solve the equation

$$F(v, T) = 0.$$

In this section we prove that $(0, T_*)$ is a bifurcation point of $\Xi$ for the zero level set of the operator $F$.

Proposition 8.1 ensures that the kernel of the operator $H_{T_*}$ is finite-dimensional and it equals $V_{j_1} \oplus \ldots \oplus V_{j_l}$. Let $Q$ be the projection operator onto the image of $H_{T_*}$ and $Q \circ F$ the composition of operators $F$ and $Q$. We write a function $v \in C^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ as $v = v^{\|} + v^{\perp}$ with $v^{\|} \in \mathrm{Ker} H_{T_*}$ and $v^{\perp} \in (\mathrm{Ker} H_{T_*})^{\perp}$. The next result (that represent the classical Lyapunov$-$Schmidt reduction for our problem) follows from the implicit function Theorem:

**Proposition 8.2.** *For all $v^{\|} \in \mathrm{Ker} H_{T_*}$ whose norm is small enough and for all $T$ sufficiently close to $T_*$ there exists a unique function $v^{\perp} = v^{\perp}(v^{\|}, T)$ such that*

$$Q \circ F\left(v^{\|} + v^{\perp}, T\right) = 0.$$

*Proof.* Define the operator $J$ as follows:

$$J(v^{\|}, v^{\perp}, T) = Q \circ F\left(v^{\|} + v^{\perp}, T\right)$$

from $\mathrm{Ker} H_{T_*} \times (\mathrm{Ker} H_{T_*})^{\perp} \times (0, +\infty)$ into the image of $H_{T_*}$. By Proposition 8.1 the implicit function theorem applies to get the existence of a unique function

$$v^{\perp}(v^{\|}, T) \in (\mathrm{Ker} H_{T_*})^{\perp}$$

smoothly depending on $v^{\|}$ and $T$ in a neighborhood of $(0, T_*)$ such that

$$J(v^{\|}, v^{\perp}(v^{\|}, T), T) = 0.$$

This completes the proof of the proposition. $\square$

Now we can define the operator

$$G(v^{\|}, T) = (I - Q) \circ F\left(v^{\|} + v^{\perp}(v^{\|}, T), T\right) = 0,$$

where $I$ is the identity operator and $v^{\perp}(v^{\|}, T)$ is the function given by Proposition 8.2. $G$ is a finite-dimensional operator from $\mathrm{Ker}H_{T_*} \times (0, +\infty)$ into the space orthogonal to the image of $H_{T_*}$. We remark that our main theorem 1.1 will be proved if we show that $(0, T_*)$ is a bifurcation point for the zero level set of $G$. In fact, it is easy to prove that the curve

$$\Gamma = \{(v^{\|}, T) \in \mathrm{Ker}H_{T_*} \times (0, +\infty) \quad : \quad v^{\|} = 0\}$$

is a solution of $G(v^{\|}, T) = 0$ with $v^{\perp}(0, T) = 0$. Then, the fact that $(0, T_*)$ is a bifurcation point of $\Gamma$ for the zero level set of $G$ means that every neighborhood of $(0, T_*)$ in $\mathrm{Ker}H_{T_*} \times (0, +\infty)$ contains solutions of the equation $G(v^{\|}, T) = 0$ which are not in $\Gamma$, *i.e.* there exists a sequence $(v_i^{\|}, T_i) \in \mathrm{Ker}H_{T_*} \times (0, +\infty)$ with $v_i^{\|} \neq 0$ such that $G(v_i^{\|}, T_i) = 0$. Hence

$$Q \circ F\left(v_i^{\|} + v^{\perp}(v_i^{\|}, T_i), T_i\right) = 0$$

and

$$(I - Q) \circ F\left(v_i^{\|} + v^{\perp}(v_i^{\|}, T_i), T_i\right) = 0$$

that imply

$$F\left(v_i^{\|} + v^{\perp}(v_i^{\|}, T_i), T_i\right) = 0$$

and $v_i := v_i^{\|} + v^{\perp}(v_i^{\|}, T_i) \neq 0$.

Let us prove that $(0, T_*)$ is a bifurcation point of $\Gamma$ for the zero level set of $G$. We start by recalling a useful result about bifurcation (see [14, 30] for details). Let $L$ be an operator on $\mathbb{B}_1 \times \Lambda$ into $\mathbb{B}_2$, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces (or subspaces) and $\Lambda$ is an interval of $\mathbb{R}$. Thus suppose that $\Gamma = (0, s)$ is a curve of solutions of the equation $L(x, s) = 0$. A necessary condition for bifurcation at $(0, s_0)$ is that 0 is an isolated eigenvalue of finite algebraic multiplicity, say $l$, of the operator obtained by linearizing $L$ with respect to $x$ at $(0, s_0)$, which can be denoted by $D_x L(0, s_0)$. It is possible to show (see [13]) that the generalized eigenspace $E_{s_0}$ of the eigenvalue 0 of $D_x L(0, s_0)$ having dimension $l$ is perturbed to an invariant space $E_s$ of $D_x L(0, s)$ of dimension $l$ too, and all perturbed eigenvalues near 0 (the so-called 0-group) are eigenvalues of the finite-dimensional operator $D_x L(0, s)$ restricted to the $l$-dimensional invariant space $E_s$. Moreover the eigenvalues in that 0-group depend continuously on $s$. Let us give the definition of *odd crossing number*:

**Definition 8.3.** We set $\Theta(s)$ to be equal to 1 if there are no negative real eigenvalues in the 0-group of $D_x L(0, s)$. Otherwise

$$\Theta(s) = (-1)^{l_1 + \ldots + l_h}$$

if $\mu_1, \ldots, \mu_h$ are all the negative real eigenvalues of the 0-group having algebraic multiplicity $l_1, \ldots, l_h$, respectively. If $D_x L(0, s)$ is regular in a neighborhood of $s_0$ (naturally except in the point $s_0$) and $\Theta(s)$ changes the sign at $s_0$ then $D_x L(0, s)$ is said to have an odd crossing number at $s_0$.

In presence of an odd crossing number, a standard result known as the Krasnosel'skii Bifurcation Theorem (see [14] for the proof) applies:

**Theorem 8.4.** *If $D_x L(0,s)$ has an odd crossing number at $s_0$, then $(0,s_0)$ is a bifurcation point for $L(x,s)=0$ with respect to the curve* $\{(0,s) \mid s \text{ in a neighborhood of } s_0\}$.

In our case, $L$ is given by the operator $G$ defined above. The fact that $(0,T_*)$ is a bifurcation point for the operator $G$ follows then from the Krasnosel'skii Bifurcation Theorem and the following:

**Proposition 8.5.** *$D_{v^\parallel} G(0,T)$ has an odd crossing number at $T_*$.*

*Proof.* We observe that we can write

$$v^\parallel = \sum_{i=1}^{l} a_{k_i} \cos(k_i\, t)$$

where $1 = k_1 < \ldots < k_l$. It is clear, from the definition of $G$, that $D_{v^\parallel} G(0,T)$ preserves the eigenspaces, and

$$D_{v^\parallel} G(0,T) = H_T|_{V_{j_1} \oplus \ldots \oplus V_{j_l}}.$$

Then the 0-group of eigenvalues is given by $\sigma_{j_1}(T), \ldots, \sigma_{j_l}(T)$, where $\sigma_{j_1}(T) = \sigma(T)$. For $T = T_*$ they are all equal to 0. Moreover, by the Proposition 8.1 only $\sigma_{j_1}(T)$ changes sign at $T_*$, and the corresponding eigenspace has dimension 1. This means that $D_{v^\parallel} G(0,T)$ has a crossing number at $T_*$ and completes the proof of the proposition. □

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