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OPTIMAL STOCHASTIC CONTROL WITH RECURSIVE COST FUNCTIONALS OF STOCHASTIC DIFFERENTIAL SYSTEMS REFLECTED IN A DOMAIN *, **

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Abstract. The paper is concerned with optimal control of a stochastic differential system reflected in a domain. The cost functional is implicitly defined *via* a generalized backward stochastic differential equation developed by Pardoux and Zhang [*Probab. Theory Relat. Fields* **110** (1998) 535–558]. The value function is shown to be the unique viscosity solution to the associated Hamilton–Jacobi–Bellman equation, which is a fully nonlinear parabolic partial differential equation with a nonlinear Neumann boundary condition. The proof requires new estimates for the reflected stochastic differential system.

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1. Introduction

Let D be an open connected bounded convex subset of $\mathbb{R}^d$ such that $D = \{\phi > 0\}$, $\partial D = \{\phi = 0\}$ for some function $\phi \in C_b^2(\mathbb{R}^d)$ satisfying $|\nabla\phi(x)| = 1$ at any $x \in \partial D$. Note that at any $x \in \partial D$, $\nabla\phi(x)$ is a unit normal vector on the boundary point x, pointing towards the interior of D.

Let U be a metric space. An admissible control process is a U-valued $\mathbb{F}$-progressively measurable process. The set of all admissible control processes is denoted by $\mathcal{U}$. In this paper, for the initial data $(t,x) \in [0,T]\times\mathbb{R}^d$

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we consider the optimal control problem for the following stochastic differential equations (SDEs) reflected in domain D:

$$\begin{cases} X_s = x + \int_t^s b\left(r, X_r, u_r\right) \mathrm{d}r + \int_t^s \sigma\left(r, X_r, u_r\right)\, \mathrm{d}B_r + \int_t^s \nabla\phi\left(X_r\right)\, \mathrm{d}K_r,\ \ s \in [t,T]; \\ K_s = \int_t^s I_{\{X_r \in \partial D\}} \mathrm{d}K_r, \quad K \text{ is increasing.} \end{cases} \tag{1.1}$$

Here, $u(\cdot) \in \mathcal{U}$ is an admissible control, and the drift b: $[0,T]\times\mathbb{R}^d\times U \to \mathbb{R}^d$ and the diffusion $\sigma : [0,T]\times\mathbb{R}^d\times U \to \mathbb{R}^{d\times d}$ are uniformly Lipschitz continuous and grows linearly in the state variable x. For each $u(\cdot) \in \mathcal{U}$, in view of Proposition A.1 in the appendix, the above reflected SDE (1.1) has a unique strong solution, denoted by $(X^{t,x;u}, K^{t,x;u})$. Consider the following controlled generalized backward stochastic differential equation (GBSDE):

$$\begin{cases} -\mathrm{d}Y_s = f\left(s, X_s^{t,x;u}, Y_s, Z_s, u_s\right)\, \mathrm{d}s + g\left(s, X_s^{t,x;u}, Y_s\right)\, \mathrm{d}K_s^{t,x;u} - Z_s\, \mathrm{d}B_s, \quad s \in [0,T); \\ \quad Y_T = \Phi\left(X_T^{t,x;u}\right). \end{cases} \tag{1.2}$$

Under suitable conditions on the functions f, g and Φ (see (H3.2) in Sect. 3 for more details), it has a unique adapted solution (see Pardoux and Zhang [20]), denoted by $(Y^{t,x;u}, Z^{t,x;u})$ hereafter. The optimal control problem is to maximize the cost functional $J(t,x;u) := Y_t^{t,x;u}$ over all admissible controls $u \in \mathcal{U}$. The associated Hamilton–Jacobi–Bellman (HJB) equation turns out to have a nonlinear Neumann boundary condition, and reads as follows:

$$\begin{cases} \dfrac{\partial}{\partial t} W(t,x) + H\left(t,x,W,DW,D^2W\right) = 0, & (t,x) \in [0,T) \times D, \\ \dfrac{\partial}{\partial n} W(t,x) + g(t,x,W(t,x)) = 0, & 0 \le t < T, x \in \partial D; \\ W(T,x) = \Phi(x), & x \in \bar{D}, \end{cases} \tag{1.3}$$

where at a point $x \in \partial D$, $\frac{\partial}{\partial n} = \sum_{i=1}^d \frac{\partial}{\partial x_i}\phi(x)\frac{\partial}{\partial x_i}$, and the Hamiltonian H is given by

$$H(t,x,y,p,A) := \sup_{u\in U}\left\{\frac{1}{2}\mathrm{tr}\left(\sigma\sigma^T(t,x,u)A\right) + \langle p, b(t,x,u)\rangle + f(t,x,y,p.\sigma,u)\right\}$$

for $(t,x,y,p,A) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{S}^d$. We aim to show that the value function of our optimal control problem is the unique viscosity solution to above HJB equation (1.3).

BSDEs were initially studied by Bismut in 1973 (see Bismut [2–4]), and a general nonlinear version was studied by Pardoux and Peng [17] in 1990. Since then BSDE has received an extensive attention both in the theory and in the application. The reader is referred to, among others, El Karoui *et al.* [12], Darling and Pardoux [9], Pardoux and Peng [18], Peng [21, 22], Hu [13], and Delbaen and Tang [11]. Stochastic differential equations reflected in a domain are referred to Lions [14], Lions and Sznitman [15], Menaldi [16], Pardoux and Williams [19], Saisho [23], among others. Pardoux and Zhang [20] studied BSDEs (1.2), and gave a probabilistic formula for the solution of a semi-linear system of parabolic or elliptic partial differential equation (PDE) with a nonlinear Neumann boundary condition. Other related studies on a PDE with a nonlinear Neumann boundary condition include Boufoussia and Van Casterenb [5], who gave an approximation result to semi-linear parabolic PDEs with Neumann boundary conditions with the help of BSDEs, and Day [10], who studied the Neumann boundary conditions for viscosity solutions of Hamilton–Jacobi equations. In contrast to those works, we study optimal control of stochastic differential systems reflected in a domain, and give the stochastic representation for the solution of the associated HJB equation (1.3) with a nonlinear Neumann boundary condition.

In this paper, the generalized BSDE formulation of dynamic programming given by Peng [21,22] for optimally controlled SDEs, is extended to our controlled stochastic differential systems reflected in a domain. The relevant arguments of Buckdahn and Li [7] is generalized to show that our value function W (see (3.7)) is deterministic (see Prop. 3.1). Since our associated BSDE involves an increasing process which incorporates the reflection of the system state on the boundary of the given domain, we have to resolve some new issues, for example, a new estimate (see Prop. A.3) for the increasing process K, and the linear growth and locally Lipschitz continuity of the value of the system path Y at the initial time with respect to the initial (random) state (see Prop. A.2, which improves the estimates on GBSDE of Pardoux and Zhang [20]). Using these new results, we can prove that the value function is continuous (see Thm. 3.2) and moreover, it is the unique viscosity solution of the associated HJB equation (see Thm. 4.1). On the other hand, Proposition 3.1 allows us to prove the dynamic programming principle (DPP in short, see Thm. 3.1) in a straight forward way by adapting to GBSDEs the method of stochastic backward semigroups introduced by Peng [21]. Furthermore, our proof of Theorem 4.1 contains techniques so as to deal with the Neumann boundary condition, which differs heavily from the counterpart of either Buckdahn and Li [7] or Peng [21]. For more details, the reader is referred to among others Lemmas 4.2 and 4.3 and the constructions of BSDEs (4.10), (4.12), (4.23) and (4.24), *etc.*

The rest of the paper is organized as follows. In Section 2, we give some preliminary results on BSDEs and GBSDEs. In Section 3, we formulate the optimal stochastic control problem and define the value function W. We prove that W is deterministic and satisfies the DPP. Furthermore, we prove that W is continuous. In Section 4, we prove that W is the unique viscosity solution to the associated HJB equation. In the end, we give some properties on GBSDEs associated with forward reflected SDEs in the Appendix (Sect. A.1), where Propositions A.2 and A.3 contain new results on GBSDEs. For the reader's convenience, the proofs of Proposition 3.1 and Theorem 3.1 are given in Section A.2.

2. Preliminaries

We consider the Wiener space $(\Omega, \mathcal{F}, P)$, where Ω is the set of continuous functions from [0, T] to $\mathbb{R}^d$ starting from 0 ($\Omega = C_0([0,T];\mathbb{R}^d)$), $\mathcal{F}$ the completed Borel σ-algebra over Ω, and P the Wiener measure. Let B be the canonical process: $B_s(\omega) = \omega_s$, $s \in [0,T]$, $\omega \in \Omega$. By $\mathbb{F} = \{\mathcal{F}_s, 0 \le s \le T\}$ we denote the natural filtration generated by $\{B_s\}_{0\le s\le T}$ and augmented by all P-null sets, *i.e.*,

$$\mathcal{F}_s = \sigma\{B_r, r \le s\} \vee \mathcal{N}, \quad s \in [0,T],$$

where $\mathcal{N}$ is the set of all P-null subsets, and $T > 0$ a fixed real time horizon. For any $n \ge 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. We introduce the following two spaces of processes: $\mathcal{S}^2(0,T;\mathbb{R})$ is the collection of $(\psi_t)_{0\le t\le T}$ which is a real-valued adapted càdlàg process such that $E[\sup\limits_{0\le t\le T} |\psi_t|^2] < +\infty$; and $\mathcal{H}^2(0,T;\mathbb{R}^n)$ is the collection of $(\psi_t)_{0\le t\le T}$ which is an $\mathbb{R}^n$-valued progressively measurable process such that $||\psi||_2^2 = E[\int_0^T |\psi_t|^2 \mathrm{d}t] < +\infty$.

Let $\{A_t, t \ge 0\}$ be a continuous increasing $\mathbb{F}$-progressively measurable scalar process, satisfying $A_0 = 0$ and $E[\mathrm{e}^{\mu A_T}] < \infty$ for all $\mu > 0$. We are given a final condition $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ such that $E(\mathrm{e}^{\mu A_T}|\xi|^2) < \infty$ for all $\mu > 0$, and two random fields $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying,

(H2.1)
(i) The processes $f(\cdot, y, z)$ and $g(\cdot, y)$ are $\mathbb{F}$-progressively measurable and
$$E\left[\int_0^T \mathrm{e}^{\mu A_t}|f(t,0,0)|^2\,\mathrm{d}t\right] + E\left[\int_0^T \mathrm{e}^{\mu A_t}|g(t,0)|^2\mathrm{d}A_t\right] < \infty, \text{ for all } \mu > 0;$$
(ii) There is a constant C such that, for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$,
$$|f(t,y,z) - f(t,y',z')| \le C(|y-y'| + |z-z'|);$$
(iii) There is a constant C such that, for all $(t,y) \in [0,T] \times \mathbb{R}$,
$$|g(t,y) - g(t,y')| \le C|y-y'|.$$

A solution to the following GBSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T g(s, Y_s)\,\mathrm{d}A_s - \int_t^T Z_s\,\mathrm{d}B_s, \quad 0 \le t \le T, \tag{2.1}$$

is a pair of $\mathbb{F}$-progressively measurable processes $(Y_t, Z_t)_{0\le t\le T}$ taking values in $\mathbb{R}\times\mathbb{R}^d$ which satisfies equation (2.1) and

$$E\left[\sup_{0\le t\le T}|Y_t|^2\right]+E\left[\int_0^T|Z_t|^2\,\mathrm{d}t\right]<\infty,\quad 0\le t\le T. \tag{2.2}$$

From Theorem 1.6 and Proposition 1.1 of [20], we have the following two lemmas.

Lemma 2.1. *Let* (H2.1) *be satisfied. Then GBSDE* (2.1) *has a unique solution* (Y,Z).

Lemma 2.2. *Under the assumption* (H2.1)*, we have for any* $\mu>0$

$$E\left[\sup_{0\le t\le T}\mathrm{e}^{\mu A_t}|Y_t|^2+\int_0^T\mathrm{e}^{\mu A_t}|Y_t|^2\,\mathrm{d}A_t+\int_0^T\mathrm{e}^{\mu A_t}|Z_t|^2\,\mathrm{d}t\right]$$
$$\le CE\left[\mathrm{e}^{\mu A_T}|\xi|^2+\int_0^T\mathrm{e}^{\mu A_t}|f(t,0,0)|^2\,\mathrm{d}t+\int_0^T\mathrm{e}^{\mu A_t}|g(t,0)|^2\,\mathrm{d}A_t\right] \tag{2.3}$$

for a positive constant C*, which depends on the Lipschitz constant of* f *and* g*,* μ*, and* T.

Let two sets of data (ξ,f,g,A) and (ξ',f',g',A') satisfy assumption (H2.1). Let (Y,Z) be a solution to GBSDE (2.1) for data (ξ,f,g,A) and (Y',Z') for data (ξ',f',g',A'). We define

$$\left(\bar{Y},\bar{Z},\bar{\xi},\bar{f},\bar{g},\bar{A}\right)=\left(Y-Y',Z-Z',\xi-\xi',f-f',g-g',A-A'\right).$$

The following two lemmas are borrowed from Proposition 1.2 and Theorem 1.4 of Pardoux and Zhang [20], respectively.

Lemma 2.3. *For any* $\mu>0$*, there exists a constant* C *such that*

$$E\left[\sup_{0\le t\le T}\mathrm{e}^{\mu k_t}\left|\overline{Y}_t\right|^2+\int_0^T\mathrm{e}^{\mu k_t}\left|\bar{Z}_t\right|^2\mathrm{d}t\right]$$
$$\le CE\left[\mathrm{e}^{\mu k_T}\left|\bar{\xi}\right|^2+\int_0^T\mathrm{e}^{\mu k_t}\left|\bar{f}\left(t,Y_t,Z_t\right)\right|^2\mathrm{d}t+\int_0^T\mathrm{e}^{\mu k_t}\left|\bar{g}\left(t,Y_t\right)\right|^2\mathrm{d}A_t'+\int_0^T\mathrm{e}^{\mu k_t}\left|g\left(t,Y_t\right)\right|^2 d\left|\left|\bar{A}\right|\right|_t\right], \tag{2.4}$$

where $k_t:=||\bar{A}||_t+A_t'$*, and* $||\bar{A}||_t$ *is the total variation of the process* $\bar{A}$ *on the interval* $[0,t]$.

For the particular case $A\equiv A'$, we have

Lemma 2.4 (Comparison Theorem)**.** *Assume that* $\xi\le\xi'$*,* $f(t,y,z)\le f'(t,y,z)$*, and* $g(t,y)\le g'(t,y)$*, for all* $(y,z)\in\mathbb{R}\times\mathbb{R}^d$*,* $\mathrm{d}P\times\mathrm{d}t$*, a.s. Then* $Y_t\le Y_t'$*,* $0\le t\le T$*, a.s.*
Moreover, if $Y_0=Y_0'$*, then* $Y_t=Y_t'$*,* $0\le t\le T$*, a.s. In particular, if in addition either* $P(\xi<\xi')>0$ *or* $f(t,y,z)<f'(t,y,z)$ *for any* $(y,z)\in\mathbb{R}\times\mathbb{R}^d$ *holds on a set of positive* $\mathrm{d}t\times\mathrm{d}P$ *measure, or* $g(t,y)<g'(t,y)$ *for any* $y\in\mathbb{R}$ *holds on a set of positive* $\mathrm{d}A_t\times\mathrm{d}P$ *measure, then* $Y_0<Y_0'$.

3. Formulation of the problem and related DPP

We assume that the two functions $b:[0,T]\times\mathbb{R}^d\times U\to\mathbb{R}^d$ and $\sigma:[0,T]\times\mathbb{R}^d\times U\to\mathbb{R}^{d\times d}$ satisfy the following three conditions:

(H3.1)
- (i) The two functions b and σ are uniformly continuous in (t,u);
- (ii) There is a constatnt $C>0$ such that, for all $(t,u)\in[0,T]\times U$ and $x,x'\in\mathbb{R}^n$,
  $|b(t,x,u)-b\left(t,x',u\right)|+|\sigma(t,x,u)-\sigma\left(t,x',u\right)|\le C\left|x-x'\right|;$
- (iii) There is a constant $C>0$ such that, for all $(t,x,u)\in[0,T]\times\mathbb{R}^n\times U$,
  $|b(t,x,u)|+|\sigma(t,x,u)|\le C(1+|x|).$

For $u \in \mathcal{U}$, the corresponding state process starting from $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$ at the initial time t, is governed by the following reflected SDE:

$$\begin{cases} X_s^{t,\zeta;u} = \zeta + \int_t^s b\left(r, X_r^{t,\zeta;u}, u_r\right) \mathrm{d}r + \int_t^s \sigma\left(r, X_r^{t,\zeta;u}, u_r\right) \mathrm{d}B_r \\ \qquad\qquad + \int_t^s \nabla\phi\left(X_r^{t,\zeta;u}\right) \mathrm{d}K_r^{t,\zeta;u},\ s \in [t,T], \\ K_s^{t,\zeta;u} = \int_t^s I_{\left\{X_r^{t,\zeta;u} \in \partial D\right\}} \mathrm{d}K_r^{t,\zeta;u},\ K^{t,\zeta;u} \text{ is increasing.} \end{cases} \tag{3.1}$$

In view of Proposition A.1 in the Appendix, SDE (3.1) has a unique strong solution $(X^{t,\zeta;u},\ K^{t,\zeta;u})$. Moreover, for any $(t,u) \in [0,T] \times \mathcal{U}$ and $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$, we have

$$\begin{aligned} &E\left[\sup_{s\in[t,T]} |X_s^{t,\zeta;u} - X_s^{t,\zeta';u}|^4 | \mathcal{F}_t\right] \le C\,|\zeta - \zeta'|^4, \\ &E\left[\sup_{s\in[t,T]} |X_s^{t,\zeta;u}|^4 | \mathcal{F}_t\right] \le C\left(1 + |\zeta|^4\right). \end{aligned} \tag{3.2}$$

Here, the constant C depends only on the Lipschitz and the linear growth constants of b and σ with respect to x.

Assume that three functions $\Phi : \mathbb{R}^d \to \mathbb{R},\ f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}$, and $g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(H3.2)
(i) f is uniformly continuous in (t,u); $g(\cdot) \in C^{1,2,2}([0,T] \times \mathbb{R}^d \times \mathbb{R})$; and there exists a constant $C > 0$ such that, for all $t \in [0,T],\ u \in U,\ (x,y,z), (x',y',z') \in \mathbb{R}^{d+1+d}$,
$$|f(t,x,y,z,u) - f(t,x',y',z',u)| + |g(t,x,y) - g(t,x',y')| \le C\left(|x - x'| + |y - y'| + |z - z'|\right);$$
(ii) There is a constant $C > 0$ such that, for all $x,\ x' \in \mathbb{R}^d$,
$$|\Phi(x) - \Phi(x')| \le C\,|x - x'|;$$
(iii) There is some $C > 0$ such that, for all $(t,u) \in [0,T] \times U$ and $x \in \mathbb{R}^n$,
$$|f(t,x,0,0,u)| \le C(1 + |x|).$$

Note that conditions (i) and (ii) of assumption (H3.2) imply the globally linear growth in the state variable of the two functions g and Φ: for some $C > 0$, $|g(t,x,0)| + |\Phi(x)| \le C(1 + |x|)$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

For any $u(\cdot) \in \mathcal{U}$, and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$, the mappings $\xi := \Phi(X_T^{t,\zeta;u})$, $\widetilde{g}(s,y) := g(s, X_s^{t,\zeta;u}, y)$ and $\widetilde{f}(s,y,z) := f(s, X_s^{t,\zeta;u}, y, z, u_s)$ satisfy the conditions (H2.1) on the interval $[t,T]$. Therefore, there is a unique solution to the following GBSDE:

$$\begin{cases} -\mathrm{d}Y_s^{t,\zeta;u} = f\left(s, X_s^{t,\zeta;u}, Y_s^{t,\zeta;u}, Z_s^{t,\zeta;u}, u_s\right) \mathrm{d}s \\ \qquad\qquad + g\left(s, X_s^{t,\zeta;u}, Y_s^{t,\zeta;u}\right) \mathrm{d}K_s^{t,\zeta;u} - Z_s^{t,\zeta;u}\,\mathrm{d}B_s, \\ \quad Y_T^{t,\zeta;u} = \Phi\left(X_T^{t,\zeta;u}\right), \end{cases} \tag{3.3}$$

where $(X^{t,\zeta;u},\ K^{t,\zeta;u})$ solves the reflected SDE (3.1).

Moreover, similar to Proposition A.2, there exists some constant $C > 0$ such that, for all $t \in [0,T], \zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D}), u \in \mathcal{U}$, P-a.s.,

$$\begin{aligned} &\text{(i)}\ \left|Y_t^{t,\zeta;u} - Y_t^{t,\zeta';u}\right| \le C\left(|\zeta - \zeta'| + |\zeta - \zeta'|^{\frac{1}{2}}\right); \\ &\text{(ii)}\ \left|Y_t^{t,\zeta;u}\right| \le C(1 + |\zeta|). \end{aligned} \tag{3.4}$$

We now define our admissible controls.

Definition 3.1. An admissible control process $u=\{u_r, r\in[t,s]\}$ on $[t,s]$ (with $s\in(t,T]$) is an $\mathcal{F}_r$-progressively measurable process taking values in U. The set of all admissible controls on $[t,s]$ is denoted by $\mathcal{U}_{t,s}$. We identify two processes u and $\bar{u}$ in $\mathcal{U}_{t,s}$ and write $u\equiv\bar{u}$ on $[t,s]$, if $P\{u=\bar{u}$ a.e. in $[t,s]\}=1$.

At $u\in\mathcal{U}_{t,T}$, the value of the cost functional is given by

$$J(t,x;u):=Y_t^{t,x;u},\ (t,x)\in[0,T]\times\bar{D}, \tag{3.5}$$

where the process $Y^{t,x;u}$ is defined by GBSDE (3.3).

From Theorem A.7, we have

$$J(t,\zeta;u)=Y_t^{t,\zeta;u},\quad (t,\zeta)\in[0,T]\times L^2\left(\Omega,\mathcal{F}_t,P;\bar{D}\right). \tag{3.6}$$

We define the value function of our stochastic control problem as follows:

$$W(t,x):=\operatorname*{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u),\quad (t,x)\in[0,T]\times\bar{D}. \tag{3.7}$$

Under assumptions (H3.1) and (H3.2), the value function W is well-defined on $[0,T]\times D$, and its values at time t are bounded and $\mathcal{F}_t$-measurable random variables. In fact, they are all deterministic. We have

Proposition 3.2. *For any $(t,x)\in[0,T]\times\bar{D}$, we have $W(t,x)=E[W(t,x)]$, P-a.s. Let $W(t,x)$ equal to its deterministic version $E[W(t,x)]$. Then $W:[0,T]\times\bar{D}\to\mathbb{R}$ is a deterministic function.*

The proof is an adaptation of relevant arguments of Buckdahn and Li [7]. For the readers' convenience we give it in the Section A.2 of Appendix.

As an immediate result of (3.4) and (3.7), the value function W has the following property.

Lemma 3.3. *There exists a constant $C>0$ such that, for all $(t,x,x')\in[0,T]\times\bar{D}\times\bar{D}$,*

$$\begin{aligned}&\text{(i)}\ \ |W(t,x)-W(t,x')|\le C\left[|x-x'|+|x-x'|^{\frac12}\right];\\&\text{(ii)}\ |W(t,x)|\le C(1+|x|).\end{aligned} \tag{3.8}$$

We now study the (generalized) DPP for our stochastic control problem (3.1), (3.3), and (3.7). For this we have to define the family of (backward) semigroups related with GBSDE (3.3). Peng [21] first introduced the notion of backward stochastic semigroups to study the DPP for the optimal stochastic control of SDEs. In what follows, it is adapted to the optimal control problem of stochastic differential systems *reflected* in a domain.

Given the initial data (t,x), a positive number $\delta\le T-t$, an admissible control $u(\cdot)\in\mathcal{U}_{t,t+\delta}$, and a random variable $\eta\in L^2(\Omega,\mathcal{F}_{t+\delta},P;\mathbb{R})$, we define

$$G_{s,t+\delta}^{t,x;u}[\eta]:=\tilde{Y}_s^{t,x;u},\qquad s\in[t,t+\delta], \tag{3.9}$$

where $(\tilde{Y}_s^{t,x;u},\tilde{Z}_s^{t,x;u})_{t\le s\le t+\delta}$ is the solution of the following GBSDE on the time interval $[t,t+\delta]$:

$$\begin{cases}-\mathrm{d}\tilde{Y}_s^{t,x;u}=f\left(s,X_s^{t,x;u},\tilde{Y}_s^{t,x;u},\tilde{Z}_s^{t,x;u},u_s\right)\,\mathrm{d}s+g\left(s,X_s^{t,x;u},\tilde{Y}_s^{t,x;u}\right)\,\mathrm{d}K_s^{t,x;u}\\ \qquad\qquad\qquad -\tilde{Z}_s^{t,x;u}\mathrm{d}B_s,\quad s\in[t,t+\delta];\\ \tilde{Y}_{t+\delta}^{t,x;u}=\eta,\end{cases}$$

and $(X^{t,x;u},K^{t,x;u})$ is the solution of reflected SDE (3.1). Then, obviously, for the solution $(Y^{t,x;u},Z^{t,x;u})$ of GBSDE (3.3), we have

$$G_{t,T}^{t,x;u}\left[\Phi\left(X_T^{t,x;u}\right)\right]=G_{t,t+\delta}^{t,x;u}\left[Y_{t+\delta}^{t,x;u}\right]. \tag{3.10}$$

Furthermore,

$$J(t,x;u)=Y_t^{t,x;u}=G_{t,T}^{t,x;u}\left[\Phi\left(X_T^{t,x;u}\right)\right]=G_{t,t+\delta}^{t,x;u}\left[Y_{t+\delta}^{t,x;u}\right]=G_{t,t+\delta}^{t,x;u}\left[J\left(t+\delta,X_{t+\delta}^{t,x;u};u\right)\right].$$

Remark 3.4. If both f and g do not depend on (y,z), we have

$$G^{t,x;u}_{s,t+\delta}[\eta]=E\left[\eta+\int_s^{t+\delta} f\left(r,X^{t,x;u}_r,u_r\right)\mathrm{d}r+\int_s^{t+\delta} g\left(r,X^{t,x;u}_r\right)\mathrm{d}K^{t,x;u}_r|\mathcal{F}_s\right],\quad s\in[t,t+\delta].$$

Theorem 3.5. *Under assumptions* (H3.1) *and* (H3.2), *the value function W satisfies the following DPP: For any $0\le t<t+\delta\le T$, $x\in\bar{D}$,*

$$W(t,x)=\operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u}_{t,t+\delta}\left[W\left(t+\delta,X^{t,x;u}_{t+\delta}\right)\right].\tag{3.11}$$

The proof is similar to [4]. For the readers's convenience we give it in Section A.2.

Lemma 3.3 shows that the value function $W(t,x)$ is continuous in x, uniformly in t. From Theorem 3.5 we can get the continuity of $W(t,x)$ in t.

Theorem 3.6. *Let assumptions* (H3.1) *and* (H3.2) *be satisfied. Then the value function $W(t,x)$ is continuous in t.*

Proof. Let $(t,x)\in[0,T]\times\bar{D}$ and $\delta\in(0,T-t]$. We want to prove that W is continuous in t. For this we notice that from (A.33), for an arbitrarily small $\varepsilon>0$,

$$I^1_\delta+I^2_\delta\le W(t,x)-W(t+\delta,x)\le I^1_\delta+I^2_\delta+C\varepsilon,\tag{3.12}$$

where

$$I^1_\delta:=G^{t,x;u^\varepsilon}_{t,t+\delta}\left[W\left(t+\delta,X^{t,x;u^\varepsilon}_{t+\delta}\right)\right]-G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta,x)],$$
$$I^2_\delta:=G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta,x)]-W(t+\delta,x),$$

for $u^\varepsilon\in\mathcal{U}_{t,t+\delta}$ such that (A.33) holds. From Lemma 2.3 (taking $\mu=1$) and the estimates (A.4), in Appendix, (3.8) we get that, for some constant C which does not depend on the controls u^ε,

$$\begin{aligned}\left|I^1_\delta\right|&\le C\left(E\left[\left|W\left(t+\delta,X^{t,x;u^\varepsilon}_{t+\delta}\right)-W(t+\delta,x)\right|^4\Big|\mathcal{F}_t\right]\right)^{\frac14}\\&\le C\left(E\left[\left|X^{t,x;u^\varepsilon}_{t+\delta}-x\right|^4+\left|X^{t,x;u^\varepsilon}_{t+\delta}-x\right|^2\Big|\mathcal{F}_t\right]\right)^{\frac14},\end{aligned}$$

and since $E[|X^{t,x;u^\varepsilon}_{t+\delta}-x|^8|\mathcal{F}_t]\le C\delta^4$ (refer to (A.17) in Appendix) we get that $|I^1_\delta|\le C\delta^{\frac14}$. From the definition of $G^{t,x;u^\varepsilon}_{t,t+\delta}[\cdot]$ (see (3.9)),

$$\begin{aligned}I^2_\delta&=E\Big[W(t+\delta,x)+\int_t^{t+\delta} f\left(s,X^{t,x;u^\varepsilon}_s,\tilde{Y}^{t,x;u^\varepsilon}_s,\tilde{Z}^{t,x;u^\varepsilon}_s,u^\varepsilon_s\right)\mathrm{d}s\\&\quad+\int_t^{t+\delta} g\left(s,X^{t,x;u^\varepsilon}_s,\tilde{Y}^{t,x;u^\varepsilon}_s\right)\mathrm{d}K^{t,x;u^\varepsilon}_s-\int_t^{t+\delta}\tilde{Z}^{t,x;u^\varepsilon}_s\mathrm{d}B_s|\mathcal{F}_t\Big]-W(t+\delta,x)\\&=E\left[\int_t^{t+\delta} f\left(s,X^{t,x;u^\varepsilon}_s,\tilde{Y}^{t,x;u^\varepsilon}_s,\tilde{Z}^{t,x;u^\varepsilon}_s,u^\varepsilon_s\right)\mathrm{d}s+\int_t^{t+\delta} g\left(s,X^{t,x;u^\varepsilon}_s,\tilde{Y}^{t,x;u^\varepsilon}_s\right)\mathrm{d}K^{t,x;u^\varepsilon}_s|\mathcal{F}_t\right].\end{aligned}$$

From the Schwartz inequality, Propositions A.2 and A.3 in Appendix and (3.2), we then get

$$\begin{aligned}
\left|I_\delta^2\right| &\le \delta^{\frac{1}{2}} E\left[\int_t^{t+\delta} \left|f\left(s, X_s^{t,x;u^\varepsilon}, \tilde{Y}_s^{t,x;u^\varepsilon}, \tilde{Z}_s^{t,x;u^\varepsilon}, u_s^\varepsilon\right)\right|^2 \mathrm{d}s|\mathcal{F}_t\right]^{\frac{1}{2}} \\
&\quad + E\left[K_{t+\delta}^{t,x;u^\varepsilon}\middle|\mathcal{F}_t\right]^{\frac{1}{2}} E\left[\int_t^{t+\delta} \left|g\left(s, X_s^{t,x;u^\varepsilon}, \tilde{Y}_s^{t,x;u^\varepsilon}\right)\right|^2 \mathrm{d}K_s^{t,x;u^\varepsilon}\middle|\mathcal{F}_t\right]^{\frac{1}{2}} \\
&\le \delta^{\frac{1}{2}} E\left[\int_t^{t+\delta} \left(\left|f\left(s, X_s^{t,x;u^\varepsilon}, 0, 0, u_s^\varepsilon\right)\right| + C\left|\tilde{Y}_s^{t,x;u^\varepsilon}\right| + C\left|\tilde{Z}_s^{t,x;u^\varepsilon}\right|\right)^2 \mathrm{d}s|\mathcal{F}_t\right]^{\frac{1}{2}} \\
&\quad + E\left[K_{t+\delta}^{t,x;u^\varepsilon}\middle|\mathcal{F}_t\right]^{\frac{1}{2}} E\left[\int_t^{t+\delta} \left(\left|g\left(s, X_s^{t,x;u^\varepsilon}, 0\right)\right| + C\left|\tilde{Y}_s^{t,x;u^\varepsilon}\right|\right)^2 \mathrm{d}K_s^{t,x;u^\varepsilon}\middle|\mathcal{F}_t\right]^{\frac{1}{2}} \\
&\le C\delta^{\frac{1}{2}} E\left[\int_t^{t+\delta} \left(\left|1 + \left|X_s^{t,x;u^\varepsilon}\right| + \left|\tilde{Y}_s^{t,x;u^\varepsilon}\right| + \left|\tilde{Z}_s^{t,x;u^\varepsilon}\right|\right)^2 \mathrm{d}s|\mathcal{F}_t\right]^{\frac{1}{2}} \\
&\quad + CE\left[K_{t+\delta}^{t,x;u^\varepsilon}\middle|\mathcal{F}_t\right]^{\frac{1}{2}} E\left[\int_t^{t+\delta} \left(\left|1 + \left|X_s^{t,x;u^\varepsilon}\right| + \left|\tilde{Y}_s^{t,x;u^\varepsilon}\right|\right)^2 \mathrm{d}K_s^{t,x;u^\varepsilon}\middle|\mathcal{F}_t\right]^{\frac{1}{2}} \\
&\le C\delta^{\frac{1}{2}} + C\left(E\left[\left|K_{t+\delta}^{t,x;u^\varepsilon}\right|^2\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \le C\delta^{\frac{1}{2}}.
\end{aligned}$$

Then, from (3.12), $|W(t,x) - W(t+\delta,x)| \le C\delta^{\frac{1}{4}} + C\delta^{\frac{1}{2}} + C\varepsilon$, and letting $\varepsilon \downarrow 0$ we get $W(t,x)$ is continuous in t. The proof is complete. □

4. Viscosity solutions of related HJB equations

We consider the following PDE:

$$\begin{cases}
\dfrac{\partial}{\partial t}W(t,x) + H\left(t,x,W,DW,D^2W\right) = 0, & (t,x) \in [0,T) \times D, \\
\dfrac{\partial}{\partial n}W(t,x) + g(t,x,W(t,x)) = 0, & 0 \le t < T,\ x \in \partial D; \\
W(T,x) = \Phi(x), & x \in \bar{D},
\end{cases} \tag{4.1}$$

where at a point $x \in \partial D$, $\frac{\partial}{\partial n} = \sum_{i=1}^d \frac{\partial}{\partial x_i}\phi(x)\frac{\partial}{\partial x_i}$, and the Hamiltonian H is defined by

$$H(t,x,y,p,A) := \sup_{u\in U}\left\{\frac{1}{2}\mathrm{tr}\left(\sigma\sigma^T(t,x,u)A\right) + \langle p, b(t,x,u)\rangle + f(t,x,y,p\sigma,u)\right\},$$

where $(t,x,y,p,A) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{S}^d$ with $\mathbf{S}^d$ being the set of all $d \times d$ symmetric matrices.

In this section we shall prove that the value function W defined by (3.7) is the unique viscosity solution of (4.1). The interested reader is referred to Crandall, Ishii, and Lions [8] for a detailed introduction to viscosity solutions. Let $C_{l,b}^3([0,T] \times \bar{D})$ be the set of the real-valued functions that are continuously differentiable up to the third order and whose derivatives of order from 1 to 3 are bounded.

Definition 4.1. A real-valued continuous function $W \in C([0,T]\times\bar{D})$ is called

(i) a viscosity subsolution of (4.1) if $W(T,x) \le \Phi(x)$, for all $x \in \bar{D}$, and if for all functions $\varphi \in C^3_{l,b}([0,T]\times\bar{D})$ and $(t,x) \in [0,T)\times\bar{D}$ such that $W-\varphi$ attains its local maximum at (t,x):

$$\frac{\partial\varphi}{\partial t}(t,x) + H\left(t,x,W,D\varphi,D^2\varphi\right) \ge 0, \quad \text{if } x \in D;$$
$$\max\left\{\frac{\partial\varphi}{\partial t}(t,x) + H\left(t,x,W,D\varphi,D^2\varphi\right), \quad \frac{\partial\varphi}{\partial n}(t,x) + g(t,x,W)\right\} \ge 0, \text{ if } x \in \partial D;$$

(ii) a viscosity supersolution of (4.1) if $W(T,x) \ge \Phi(x)$, for all $x \in \bar{D}$, and if for all functions $\varphi \in C^3_{l,b}([0,T]\times\bar{D})$ and $(t,x) \in [0,T)\times\bar{D}$ such that $W-\varphi$ attains its local minimum at (t,x):

$$\frac{\partial\varphi}{\partial t}(t,x) + H\left(t,x,W,D\varphi,D^2\varphi\right) \le 0, \quad \text{if } \ x \in D;$$
$$\min\left\{\frac{\partial\varphi}{\partial t}(t,x) + H\left(t,x,W,D\varphi,D^2\varphi\right), \quad \frac{\partial\varphi}{\partial n}(t,x) + g(t,x,W)\right\} \le 0, \text{ if } x \in \partial D;$$

(iii) a viscosity solution of (4.1) if it is both a viscosity sub- and a supersolution of (4.1).

For simplicity of notations, we define for $\varphi \in C^3_{l,b}([0,T]\times\bar{D})$,

$$\begin{aligned} F(s,x,y,z,u) &= \frac{\partial}{\partial s}\varphi(s,x) + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^T(s,x,u)D^2\varphi\right) + D\varphi.b(s,x,u) \\ &\quad + f(s,x,y+\varphi(s,x), z + D\varphi(s,x).\sigma(s,x,u),u), \\ G(s,x,y) &= \frac{\partial}{\partial n}\varphi(s,x) + g(s,x,y+\varphi(s,x)), \end{aligned} \tag{4.2}$$

for $(s,x,y,z,u) \in [0,T]\times\bar{D}\times\mathbb{R}\times\mathbb{R}^d\times U$.

Proposition 4.2. *Under the assumptions* (H3.1) *and* (H3.2) *the value function W is a viscosity subsolution to* (4.1).

Proof. Obviously, $W(T,x) = \Phi(x)$, $x \in \bar{D}$. Suppose that $\varphi \in C^3_{l,b}([0,T]\times\bar{D})$ and $(t,x) \in [0,T)\times\bar{D}$ is such that $W-\varphi$ attains its maximum at (t,x). Without loss of generality, we assume that $\varphi(t,x) = W(t,x)$.

We first consider the case $x \in D$. We shall prove that

$$\sup_{u\in U} F(t,x,0,0,u) \ge 0.$$

If this is not true, then there exists some $\theta > 0$ such that

$$F_0(t,x) := \sup_{u\in U} F(t,x,0,0,u) \le -\theta < 0. \tag{4.3}$$

Therefore, $F(t,x,0,0,u) \le -\theta$, for all $u \in U$.

Since F_0 is continuous at (t,x), we can choose $\bar{\alpha} \in (0, T-t]$ such that

$$O_{\bar{\alpha}}(x) := \{\tilde{y} : |\tilde{y} - x| \le \bar{\alpha}\} \subset D, \tag{4.4}$$

$$F\left(s,\tilde{y},0,0,u\right) \le -\tfrac{1}{2}\theta, \quad \text{for all } (s,\tilde{y},u) \in [t,t+\bar{\alpha}]\times O_{\bar{\alpha}}(x)\times U. \tag{4.5}$$

For any $\alpha \in (0, \bar{\alpha}]$, we consider the following BSDE:

$$\begin{cases} -\mathrm{d}Y_s^{1,u} = F\left(s, X_s^{t,x;u}, Y_s^{1,u}, Z_s^{1,u}, u_s\right) \mathrm{d}s + G\left(s, X_s^{t,x;u}, Y_s^{1,u}\right) \mathrm{d}K_s^{t,x;u} \\ \qquad\qquad -Z_s^{1,u}\mathrm{d}B_s, \quad s \in [t, t+\alpha]; \\ Y_{t+\alpha}^{1,u} = 0, \end{cases} \tag{4.6}$$

where the pair of processes $(X^{t,x,u}, K^{t,x,u})$ are given by (3.1) and $u(\cdot) \in \mathcal{U}_{t,t+\alpha}$. It is not hard to check that $F(s, X_s^{t,x;u}, y, z, u_s)$ and $G(s, X_s^{t,x;u}, y)$ satisfy (H2.1). Thus, due to Lemma 2.1, GBSDE (4.6) has a unique solution. We have the following observation.

Lemma 4.3. *For every $s \in [t, t+\alpha]$, we have the following relationship:*

$$Y_s^{1,u} = G_{s,t+\alpha}^{t,x;u}\left[\varphi\left(t+\alpha, X_{t+\alpha}^{t,x;u}\right)\right] - \varphi\left(s, X_s^{t,x;u}\right), \quad P\text{-a.s.} \tag{4.7}$$

□

Proof. We recall that $G_{s,t+\alpha}^{t,x;u}[\varphi(t+\alpha, X_{t+\alpha}^{t,x;u})]$ is defined by the solution of the GBSDE

$$\begin{cases} -\mathrm{d}Y_s^u = f\left(s, X_s^{t,x;u}, Y_s^u, Z_s^u, u_s\right) \mathrm{d}s + g\left(s, X_s^{t,x;u}, Y_s^u\right) \mathrm{d}K_s^{t,x;u} \\ \qquad\qquad -Z_s^u \,\mathrm{d}B_s, \quad s \in [t, t+\alpha]; \\ Y_{t+\alpha}^u = \varphi\left(t+\alpha, X_{t+\alpha}^{t,x;u}\right), \end{cases}$$

with the following formula:

$$G_{s,t+\alpha}^{t,x;u}\left[\varphi\left(t+\alpha, X_{t+\alpha}^{t,x;u}\right)\right] = Y_s^u, \qquad s \in [t, t+\alpha], \tag{4.8}$$

(see (3.9)). Hence, we only need to show that $Y_s^u - \varphi(s, X_s^{t,x;u}) \equiv Y_s^{1,u}$ for $s \in [t, t+\alpha]$. This can be verified directly by applying Itô's formula to $\varphi(s, X_s^{t,x;u})$. Indeed, the stochastic differentials of $Y_s^u - \varphi(s, X_s^{t,x;u})$ and $Y_s^{1,u}$ equal, and with the same terminal condition $Y_{t+\alpha}^u - \varphi(t+\alpha, X_{t+\alpha}^{t,x;u}) = 0 = Y_{t+\alpha}^{1,u}$. □

Remark 4.4. For $x \in \partial D$ Lemma 4.1 still holds.

On the other hand, from the DPP (see Thm. 3.5), for every α,

$$\varphi(t, x) = W(t, x) = \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} G_{t,t+\alpha}^{t,x;u}\left[W\left(t+\alpha, X_{t+\alpha}^{t,x;u}\right)\right],$$

and from $W \le \varphi$ and the monotonicity property of $G_{t,t+\delta}^{t,x;u}[\cdot]$ (see Lem. 2.4) we get

$$\operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} \left\{ G_{t,t+\delta}^{t,x;u}\left[\varphi\left(t+\alpha, X_{t+\alpha}^{t,x;u}\right)\right] - \varphi(t, x) \right\} \ge 0, \; P\text{-a.s.}$$

Thus, from Lemma 4.3, we have $\operatorname{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} Y_t^{1,u} \ge 0$, P-a.s.

Hence, for arbitrary $\varepsilon > 0$, similar to that of inequality (A.33), there is $u^\varepsilon \in \mathcal{U}_{t,t+\alpha}$ such that

$$Y_t^{1,u^\varepsilon} \ge -\varepsilon\alpha, \quad P\text{-a.s.} \tag{4.9}$$

Remark 4.5. Similarly, (4.9) is still true for $x \in \partial D$.

For $u^\varepsilon \in \mathcal{U}_{t,t+\alpha}$ we define $\tau = \inf\{s \geq t : |X_s^{t,x;u^\varepsilon} - x| \geq \bar{\alpha}\} \wedge (t+\alpha)$. Consequently, on $[t,\tau]$ the process $(K^{t,x;u})$ is zero and, hence

$$Y_s^{1;u^\varepsilon} = Y_\tau^{1;u^\varepsilon} + \int_s^\tau F\left(r, X_r^{t,x;u^\varepsilon}, Y_r^{1;u^\varepsilon}, Z_r^{1;u^\varepsilon}, u_r^\varepsilon\right) \mathrm{d}r - \int_s^\tau Z_r^{1;u^\varepsilon} \mathrm{d}B_r.$$

We consider the following two BSDEs:

$$\begin{cases} -\mathrm{d}Y_s^2 = \left(C^*\left(\left|Y_s^2\right| + \left|Z_s^2\right|\right) - \frac{1}{2}\theta\right) \mathrm{d}s - Z_s^2 \,\mathrm{d}B_s, \\ Y_{t+\alpha}^2 = 0, \end{cases} \tag{4.10}$$

whose unique solution is given by

$$Y_s^2 = -\frac{\theta}{2C^*}\left(1 - \mathrm{e}^{C^*(s-(t+\alpha))}\right), \ Z_s^2 = 0, \quad s \in [t, t+\alpha], \tag{4.11}$$

and

$$\begin{cases} -\mathrm{d}Y_s^3 = \left(C^*\left(\left|Y_s^3\right| + \left|Z_s^3\right|\right) - \frac{1}{2}\theta\right) \mathrm{d}s - Z_s^3 \,\mathrm{d}B_s, \quad s \in [t,\tau]; \\ Y_\tau^3 = Y_\tau^{1;u^\varepsilon}. \end{cases} \tag{4.12}$$

Here, C^* is the Lipschitz constant of F with respect to y, z; also the Lipschitz constant of G with respect to y, in order to be different from the constant C which may vary from lines to lines. We have the following lemma.

Lemma 4.6. *We have $Y_t^{1,u^\varepsilon} \leq Y_t^3$ and $|Y_t^2 - Y_t^3| \leq C\alpha^{\frac{3}{2}}$, P-a.s. Here $C > 0$ is independent of both the control u and α.*

Proof.

(1) We observe from (4.5) and the definition of τ that, for all $(s,y,z,u) \in [t,\tau] \times \mathbb{R} \times \mathbb{R}^d \times U$,

$$\begin{aligned} F\left(s, X_s^{t,x;u^\varepsilon}, y, z, u^\varepsilon\right) &\leq C^*\left(|y| + |z|\right) + F\left(s, X_s^{t,x;u^\varepsilon}, 0, 0, u^\varepsilon\right) \\ &\leq C^*(|y| + |z|) - \frac{1}{2}\theta. \end{aligned}$$

Consequently, from Lemma 2.2 in [7] (the comparison result for BSDEs) we have that

$$Y_s^{1,u^\varepsilon} \leq Y_s^3, \ s \in [t,\tau], \ \text{P-a.s.},$$

where (Y^3, Z^3) is the solution of BSDE (4.12).

(2) From equation (4.6), Proposition A.1 and Proposition A.2 in the Appendix, we have

$$\left|Y_\tau^{1;u^\varepsilon}\right| \leq C(t+\alpha-\tau)^{\frac{1}{2}} + C\left(E\left[\left(K_{t+\alpha}^{t,x;u^\varepsilon} - K_\tau^{t,x;u^\varepsilon}\right)^2 \middle| \mathcal{F}_\tau\right]\right)^{\frac{1}{2}},$$

where C is independent of controls, and $K_{t+\alpha}^{t,x;u^\varepsilon} - K_\tau^{t,x;u^\varepsilon} = K_{t+\alpha}^{\tau, X_\tau^{t,x;u^\varepsilon};u^\varepsilon}$ by means of the uniqueness of solution of reflected SDE (3.1). Therefore, we have

$$E\left[\left|Y_\tau^{1;u^\varepsilon}\right|^2 \middle| \mathcal{F}_t\right] \leq CE\left[(t+\alpha-\tau)|\mathcal{F}_t\right] + CE\left[\left|K_{t+\alpha}^{\tau, X_\tau^{t,x;u^\varepsilon};u^\varepsilon}\right|^2 \middle| \mathcal{F}_t\right].$$

From Proposition A.3 in Appendix, we have

$$E\left[\left|K_{t+\alpha}^{\tau, X_\tau^{t,x;u^\varepsilon};u^\varepsilon}\right|^2 \middle| \mathcal{F}_t\right] \leq C\left(E\left[(t+\alpha-\tau)^2|\mathcal{F}_t\right]\right)^{\frac{1}{2}}. \tag{4.13}$$

Therefore, we get

$$E\left[\left|Y_\tau^{1;u^\varepsilon}\right|^2\left|\mathcal{F}_t\right.\right]\le C\left(E\left[(t+\alpha-\tau)^2|\mathcal{F}_t\right]\right)^{\frac{1}{2}}. \tag{4.14}$$

On the other hand, we consider the following SDE:

$$\mathrm{d}\bar{X}_s^{t,x;u^\varepsilon}=b\left(s,\bar{X}_s^{t,x;u^\varepsilon},u_s^\varepsilon\right)\mathrm{d}s+\sigma\left(s,\bar{X}_s^{t,x;u^\varepsilon},u_s^\varepsilon\right)\mathrm{d}B_s,\ s\ge t;\quad \bar{X}_t^{t,x;u^\varepsilon}=x. \tag{4.15}$$

Then we know on $[t,\tau]$, P-a.s., $X^{t,x;u^\varepsilon}=\bar{X}^{t,x;u^\varepsilon}$. For $\bar{X}^{t,x;u^\varepsilon}$ we have the classical estimate

$$E\left[\sup_{t\le s\le t+\alpha}\left|\bar{X}_s^{t,x;u^\varepsilon}-x\right|^8\left|\mathcal{F}_t\right.\right]\le C\alpha^4,\ \text{P-a.s.}$$

Therefore, we have

$$P\left\{\tau<t+\alpha|\mathcal{F}_t\right\}\le P\left\{\sup_{s\in[t,t+\alpha]}|\bar{X}_s^{t,x;u^\varepsilon}-x|\ge\bar{\alpha}|\mathcal{F}_t\right\}\le\frac{C}{\bar{\alpha}^8}\alpha^4. \tag{4.16}$$

Hence,

$$E\left[\left|Y_\tau^{1;u^\varepsilon}\right|^2\left|\mathcal{F}_t\right.\right]\le C\alpha\left(P\left\{\tau<t+\alpha|\mathcal{F}_t\right\}\right)^{\frac{1}{2}}\le\frac{C}{\bar{\alpha}^4}\alpha^3. \tag{4.17}$$

Furthermore, from Lemma 2.3 in [7],

$$\begin{aligned}\left|Y_t^2-Y_t^3\right|&\le C\left(E\left[\left|Y_\tau^2-Y_\tau^3\right|^2|\mathcal{F}_t\right]\right)^{\frac{1}{2}}\le C\left(E\left[\left|Y_\tau^2\right|^2|\mathcal{F}_t\right]\right)^{\frac{1}{2}}+C\left(E\left[\left|Y_\tau^3\right|^2|\mathcal{F}_t\right]\right)^{\frac{1}{2}}\\&\le C\frac{\theta}{2}\left(1-\mathrm{e}^{-C^*\alpha}\right)\left(P\left\{\tau<t+\alpha|\mathcal{F}_t\right\}\right)^{\frac{1}{2}}+C\left(E\left[\left|Y_\tau^{1;u^\varepsilon}\right|^2\left|\mathcal{F}_t\right.\right]\right)^{\frac{1}{2}}\\&\le C\frac{\theta}{2}\left(1-\mathrm{e}^{-C^*\alpha}\right)\frac{1}{\bar{\alpha}^4}\alpha^2+\frac{C}{\bar{\alpha}^2}\alpha^{\frac{3}{2}}\le C\alpha^{\frac{3}{2}},\end{aligned} \tag{4.18}$$

for any $\alpha\in(0,\bar{\alpha}]$. □

Proof of Proposition 4.1 (sequel).

By combining (4.9) with Lemma 4.2 we then obtain

$$-\varepsilon\alpha\le Y_t^{1,u^\varepsilon}\le Y_t^3\le Y_t^2+\left|Y_t^2-Y_t^3\right|\le Y_t^2+C\alpha^{\frac{3}{2}},\ \text{P-a.s.}$$

i.e., $-\varepsilon\alpha\le Y_t^{1,u^\varepsilon}\le-\frac{\theta}{2C^*}(1-\mathrm{e}^{-C^*\alpha})+C\alpha^{\frac{3}{2}}$, P-a.s. Therefore,

$$-\varepsilon\le-\frac{\theta}{2C^*}\frac{1-\mathrm{e}^{-C^*\alpha}}{\alpha}+C\alpha^{\frac{1}{2}}.$$

Letting $\alpha\to0+$ and $\varepsilon\to0+$, we get $0\le-\frac{\theta}{2}$, which contradicts our assumption that $\theta>0$. Therefore, we have $\sup_{u\in U}F(t,x,0,0,u)\ge0$, which implies by the definition of F that

$$\frac{\partial\varphi}{\partial t}(t,x)+H\left(t,x,W,D\varphi,D^2\varphi\right)\ge0,\quad\text{if}\quad x\in D.$$

We now consider the case $x\in\partial D$. We must prove that

$$\max\left\{\frac{\partial\varphi}{\partial t}(t,x)+H\left(t,x,\varphi,D\varphi,D^2\varphi\right),\ \ \frac{\partial\varphi}{\partial n}(t,x)+g(t,x,\varphi)\right\}\ge0.$$

If this is not true, then there exists some $\theta > 0$ such that

$$\sup_{u\in U} F(t,x,0,0,u) \le -\theta < 0, \quad G(t,x,0) \le -\theta < 0, \tag{4.19}$$

therefore, $F(t,x,0,0,u) \le -\theta$ for all $u \in U$; and $G(t,x,0) \le -\theta$ for all $u \in U$.

Choose $\bar{\alpha} \in (0, T-t]$ such that

$$F(s,y,0,0,u) \le -\tfrac{1}{2}\theta, \tag{4.20}$$

$$G(s,y,0) \le -\tfrac{1}{2}\theta, \text{ for all } u \in U, t \le s \le t+\bar{\alpha}, \ |y-x| \le \bar{\alpha}. \tag{4.21}$$

Now we fix $\bar{\alpha}$, and we consider any $\alpha \in (0,\bar{\alpha}]$. Similarly, we consider GBSDE (4.6) with $x \in \partial D$, then we also can get (4.7) and (4.9). For $u^\varepsilon \in \mathcal{U}_{t,t+\alpha}$ in (4.9) we define

$$\tau = \inf\left\{ s \ge t : \left| X_s^{t,x;u^\varepsilon} - x \right| \ge \bar{\alpha} \right\} \wedge (t+\alpha).$$

We observe that, for all $(s,y,z) \in [t,\tau] \times \mathbb{R} \times \mathbb{R}^d$, from (4.20), (4.21) and the definition of τ

$$\begin{aligned} F\left(s, X_s^{t,x;u^\varepsilon}, y, z, u_s^\varepsilon\right) &\le C^*\left(|y|+|z|\right) + F\left(s, X_s^{t,x;u^\varepsilon}, 0, 0, u_s^\varepsilon\right) \\ &\le C^*(|y|+|z|) - \tfrac{1}{2}\theta; \end{aligned}$$

$$G\left(s, X_s^{t,x;u^\varepsilon}, y\right) \le C^*|y| + G\left(s, X_s^{t,x;u^\varepsilon}, 0\right) \le C^*|y| - \tfrac{1}{2}\theta.$$

Consequently, applying the comparison result for GBSDEs (Lem. 2.7, or Rem. 1.5 in Pardoux and Zhang [20]) to GBSDEs (4.6) and (4.23) we have that

$$Y_s^{1,u^\varepsilon} \le Y_s^4, \ s \in [t,\tau], \ \text{P-a.s.}, \tag{4.22}$$

where Y^4 is defined by the following BSDE:

$$\begin{cases} -\mathrm{d}Y_s^4 = \left(C^*\left(\left|Y_s^4\right| + \left|Z_s^4\right|\right) - \tfrac{1}{2}\theta\right)\mathrm{d}s + \left(C^*\left|Y_s^4\right| - \tfrac{1}{2}\theta\right)\mathrm{d}K_s^{t,x;u^\varepsilon} - Z_s^4 \mathrm{d}B_s, \\ Y_\tau^4 = Y_\tau^{1;u^\varepsilon}. \end{cases} \tag{4.23}$$

On the other hand, we also have to introduce the following BSDE:

$$\begin{cases} -\mathrm{d}Y_s^5 = \left(C^*\left(\left|Y_s^5\right| + \left|Z_s^5\right|\right) - \tfrac{1}{2}\theta\right)\mathrm{d}s + \left(C^*\left|Y_s^5\right| - \tfrac{1}{2}\theta\right)\mathrm{d}K_s^{t,x;u^\varepsilon} - Z_s^5 \mathrm{d}B_s, \\ Y_{t+\alpha}^5 = 0. \end{cases} \tag{4.24}$$

Notice that $C^*|Y_s^2| - \frac{1}{2}\theta < 0$, therefore $Y_s^5 \le Y_s^2$, $s \in [t, t+\alpha]$, P-a.s., from the comparison theorem-Lemma 2.4. From Lemma 2.3 we have

$$\begin{aligned} \left|Y_t^4 - Y_t^5\right| &\le C\left(E\left[\left|Y_\tau^4 - Y_\tau^5\right|^2 |\mathcal{F}_t\right]\right)^{\frac{1}{2}} \le C\left(E\left[\left|Y_\tau^4\right|^2 |\mathcal{F}_t\right]\right)^{\frac{1}{2}} + C\left(E\left[\left|Y_\tau^5\right|^2 |\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\ &\le C\left(E\left[\left|Y_\tau^{1;u^\varepsilon}\right|^2 |\mathcal{F}_t\right]\right)^{\frac{1}{2}} + C\left(E\left[\left|Y_\tau^2\right|^2 |\mathcal{F}_t\right]\right)^{\frac{1}{2}} + C\left(E\left[\left|Y_\tau^5 - Y_\tau^2\right|^2 |\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\ &\le C\alpha^{\frac{3}{2}} + C\left(E\left[\left|Y_\tau^5 - Y_\tau^2\right|^2 |\mathcal{F}_t\right]\right)^{\frac{1}{2}} \ \text{(from the proof of (4.18))}, \end{aligned}$$

for any $\alpha \in (0,\bar{\alpha}]$. From (A.17) of Remark A.3 in Appendix, similarly we also have

$$P\left\{\tau < t+\alpha\,|\mathcal{F}_t\right\} \le P\left\{\sup_{s\in[t,t+\alpha]}\left|X_s^{t,x;u^\varepsilon} - x\right| \ge \bar{\alpha}|\mathcal{F}_t\right\} \le \frac{C}{\bar{\alpha}^8}\alpha^4. \tag{4.25}$$

On the other hand, from Lemma 2.3 (taking $\mu = 1$)

$$\begin{aligned} E\left[\left|Y_\tau^5 - Y_\tau^2\right|^2 \middle| \mathcal{F}_t\right] &\le CE\left[\int_\tau^{t+\alpha} \mathrm{e}^{2K_s^{t,x;u^\varepsilon}} \left(C^* \left|Y_s^2\right| - \frac{1}{2}\theta\right)^2 \mathrm{d}K_s^{t,x;u^\varepsilon} \middle| \mathcal{F}_t\right] \\ &= CE\left[\int_\tau^{t+\alpha} \mathrm{e}^{2K_s^{t,x;u^\varepsilon}} \frac{\theta^2}{4} \mathrm{e}^{2C^*(s-(t+\alpha))} \mathrm{d}K_s^{t,x;u^\varepsilon} \middle| \mathcal{F}_t\right] \\ &\le C\frac{\theta^2}{4} E\left[I_{\{\tau<t+\alpha\}}\left(\mathrm{e}^{2K_{t+\alpha}^{t,x;u^\varepsilon}} - \mathrm{e}^{2K_\tau^{t,x;u^\varepsilon}}\right) \middle| \mathcal{F}_t\right] \\ &\le C\frac{\theta^2}{4} E\left[I_{\{\tau<t+\alpha\}} \mathrm{e}^{2K_{t+\alpha}^{t,x;u^\varepsilon}} \left(K_{t+\alpha}^{t,x;u^\varepsilon} - K_\tau^{t,x;u^\varepsilon}\right) \middle| \mathcal{F}_t\right] \\ &\le C\frac{\theta^2}{4} \left(P\left[\tau < t+\alpha | \mathcal{F}_t\right]\right)^{\frac{1}{4}} \left(E\left[\mathrm{e}^{8K_{t+\alpha}^{t,x;u^\varepsilon}} \middle| \mathcal{F}_t\right]\right)^{\frac{1}{4}} \left(E\left[\left|K_{t+\alpha}^{\tau, X_\tau^{t,x;u^\varepsilon}; u^\varepsilon}\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \\ &\le C\frac{\theta^2}{4} \left(P\left\{\tau < t+\alpha | \mathcal{F}_t\right\}\right)^{\frac{1}{4}} \left(E\left[(t+\alpha-\tau)^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{4}} \text{ (from Props. A.1 and A.3.)} \\ &\le C\theta^2 \left(P\left\{\tau < t+\alpha | \mathcal{F}_t\right\}\right)^{\frac{1}{4}} \left(\alpha^2 P\left\{\tau < t+\alpha \,| \mathcal{F}_t\right\}\right)^{\frac{1}{4}} \\ &\le C\theta^2 \alpha^{\frac{5}{2}}. \end{aligned} \tag{4.26}$$

Therefore,

$$\left|Y_t^4 - Y_t^5\right| \le C\alpha^{\frac{3}{2}} + C\theta\alpha^{\frac{5}{4}}. \tag{4.27}$$

Now we obtain

$$-\varepsilon\alpha \le Y_t^{1,u^\varepsilon} \le Y_t^4 \le Y_t^5 + \left|Y_t^4 - Y_t^5\right| \le Y_t^2 + C\alpha^{\frac{3}{2}} + C\theta\alpha^{\frac{5}{4}}, \quad \text{P-a.s.}$$

i.e., $-\varepsilon\alpha \le Y_t^{1,u^\varepsilon} \le -\frac{\theta}{2C^*}(1 - \mathrm{e}^{-C^*\alpha}) + C\alpha^{\frac{3}{2}} + C\theta\alpha^{\frac{5}{4}}$, P-a.s. Therefore,

$$-\varepsilon \le -\frac{\theta}{2C^*}\frac{1 - \mathrm{e}^{-C^*\alpha}}{\alpha} + C\alpha^{\frac{1}{2}} + C\theta\alpha^{\frac{1}{4}},$$

and by taking the limit as $\alpha \downarrow 0, \varepsilon \downarrow 0$ we get $0 \le -\frac{\theta}{2}$ which contradicts our assumption that $\theta > 0$. Therefore, it must hold

$$\max\left\{\frac{\partial\varphi}{\partial t}(t,x) + H\left(t,x,W,D\varphi,D^2\varphi\right), \ \frac{\partial\varphi}{\partial n}(t,x) + g(t,x,W)\right\} \ge 0. \qquad \square$$

In an identical way, we can show

Proposition 4.7. *Under the assumptions* (H4.1) *and* (H4.2)*, the value function* W *is a viscosity supersolution to* (4.1)*.*

Proof. Obviously, $W(T,x) = \Phi(x),\ x \in \bar{D}$. Suppose that $\varphi \in C_{l,b}^3([0,T] \times \bar{D})$ and $(t,x) \in [0,T) \times \bar{D}$ is such that $W - \varphi$ attains its minimum at (t,x). Without loss of generality, assume that $\varphi(t,x) = W(t,x)$.

We first consider the case $x \in D$. We shall prove that

$$\sup_{u\in U} F(t,x,0,0,u) \le 0.$$

If this is not true, then there exists some $\theta > 0$ such that

$$F_0(t,x) := \sup_{u\in U} F(t,x,0,0,u) \ge \theta > 0. \tag{4.28}$$

Therefore, there exists a $u^* = u^*(t,x) \in U$ such that $F(t,x,0,0,u^*) \ge \frac{2\theta}{3}$.

Since F_0 is continuous at (t,x), we can choose $\bar{\alpha} \in (0, T-t]$ (for simplifying the notation, we still use $\bar{\alpha}$) such that

$$O_{\bar{\alpha}}(x) := \{\tilde{y} : |\tilde{y} - x| \leq \bar{\alpha}\} \subset D, \tag{4.29}$$

$$F\left(s, \tilde{y}, 0, 0, u^*\right) \geq \frac{1}{2}\theta \text{ for all } (s, \tilde{y}) \in [t, t+\bar{\alpha}] \times O_{\bar{\alpha}}(x). \tag{4.30}$$

For any $\alpha \in (0, \bar{\alpha}]$, we still consider the BSDE (4.6):

$$\begin{cases} -\mathrm{d}Y_s^{1,u} = F\left(s, X_s^{t,x;u}, Y_s^{1,u}, Z_s^{1,u}, u_s\right) \mathrm{d}s + G\left(s, X_s^{t,x;u}, Y_s^{1,u}\right) \mathrm{d}K_s^{t,x;u} \\ \qquad\qquad -Z_s^{1,u}\mathrm{d}B_s, \quad s \in [t, t+\alpha]; \\ Y_{t+\alpha}^{1,u} = 0, \end{cases} \tag{4.31}$$

where the pair of processes $(X^{t,x,u}, K^{t,x,u})$ are given by (3.1) and $u(\cdot) \in \mathcal{U}_{t,t+\alpha}$. Therefore, Lemma 4.1 still holds for $x \in \bar{D}$. On the other hand, from the DPP (Thm. 3.1), for every α,

$$\varphi(t,x) = W(t,x) = \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} G_{t,t+\alpha}^{t,x;u}\left[W\left(t+\alpha, X_{t+\alpha}^{t,x;u}\right)\right],$$

and from $W \geq \varphi$ and the monotonicity property of $G_{t,t+\delta}^{t,x;u}[\cdot]$ (see Lem. 2.4) we have

$$\operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} \left\{G_{t,t+\delta}^{t,x;u}\left[\varphi\left(t+\alpha, X_{t+\alpha}^{t,x;u}\right)\right] - \varphi(t,x)\right\} \leq 0, \; P\text{-a.s.}$$

Thus, from Lemma 4.3, we get $\operatorname{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} Y_t^{1,u} \leq 0$, P-a.s., which implies that

$$Y_t^{1,u^*} \leq 0, \quad P\text{-a.s.} \tag{4.32}$$

Remark 4.8. Similarly, the inequality (4.32) holds true for $x \in \partial D$.

For $u^* \in \mathcal{U}_{t,t+\alpha}$ we define $\tau = \inf\{s \geq t : |X_s^{t,x;u^*} - x| \geq \bar{\alpha}\} \wedge (t+\alpha)$. Consequently, on $[t,\tau]$ the process $(K^{t,x;u^*})$ is zero and, hence

$$Y_s^{1;u^*} = Y_\tau^{1;u^*} + \int_s^\tau F\left(r, X_r^{t,x;u^*}, Y_r^{1;u^*}, Z_r^{1;u^*}, u^*\right) \mathrm{d}r - \int_s^\tau Z_r^{1;u^*} \mathrm{d}B_r.$$

We consider the following two BSDEs:

$$\begin{cases} -\mathrm{d}\widehat{Y}_s^2 = \left(-C^*\left(\left|\widehat{Y}_s^2\right| + \left|\widehat{Z}_s^2\right|\right) + \frac{1}{2}\theta\right) \mathrm{d}s - \widehat{Z}_s^2 \,\mathrm{d}B_s, \\ \widehat{Y}_{t+\alpha}^2 = 0, \end{cases} \tag{4.33}$$

whose unique solution is given by

$$\widehat{Y}_s^2 = \frac{\theta}{2C^*}(1 - \mathrm{e}^{C^*(s-(t+\alpha))}), \quad \widehat{Z}_s^2 = 0, \quad s \in [t, t+\alpha], \tag{4.34}$$

and

$$\begin{cases} -\mathrm{d}\widehat{Y}_s^3 = \left(-C^*\left(\left|\widehat{Y}_s^3\right| + \left|\widehat{Z}_s^3\right|\right) + \frac{1}{2}\theta\right) \mathrm{d}s - \widehat{Z}_s^3 \,\mathrm{d}B_s, \quad s \in [t,\tau]; \\ \widehat{Y}_\tau^3 = Y_\tau^{1;u^*}. \end{cases} \tag{4.35}$$

We have the following lemma.

Lemma 4.9. *We have* $Y_t^{1,u^*} \geq \widehat{Y}_t^3$ *and* $|\widehat{Y}_t^2 - \widehat{Y}_t^3| \leq C\alpha^{\frac{3}{2}}$, *P-a.s. Here* $C > 0$ *is independent of both the control* u *and* α. □

Proof.

(1) We observe from (4.30) and the definition of τ that, for all $(s, y, z, u) \in [t, \tau] \times \mathbb{R} \times \mathbb{R}^d \times U$,

$$F\left(s, X_s^{t,x;u^*}, y, z, u^*\right) \geq -C^* (|y| + |z|) + F\left(s, X_s^{t,x;u^*}, 0, 0, u^*\right) \geq -C^* (|y| + |z|) + \frac{1}{2}\theta.$$

Consequently, from Lemma 2.2 in [7] we have that $Y_s^{1,u^*} \geq \widehat{Y}_s^3$, $s \in [t, \tau]$, where $\widehat{Y}^3$ is defined by BSDE (4.35).

(2) From the equation (4.31), Propositions A.1 and A.2

$$\left|Y_\tau^{1;u^*}\right| \leq C(t + \alpha - \tau)^{\frac{1}{2}} + C\left(E\left[\left(K_{t+\alpha}^{t,x;u^*} - K_\tau^{t,x;u^*}\right)^2 \middle| \mathcal{F}_\tau\right]\right)^{\frac{1}{2}},$$

where C is independent of controls. Then similar to the proof of estimate (4.14), we have

$$E\left[\left|Y_\tau^{1;u^*}\right|^2 \middle| \mathcal{F}_t\right] \leq C\left(E\left[(t + \alpha - \tau)^2 | \mathcal{F}_t\right]\right)^{\frac{1}{2}}. \tag{4.36}$$

Similar to (4.16), we still have

$$P\{\tau < t + \alpha \,|\mathcal{F}_t\} \leq \frac{C}{\bar{\alpha}^8}\alpha^4. \tag{4.37}$$

Therefore,

$$E\left[\left|Y_\tau^{1;u^*}\right|^2 \middle| \mathcal{F}_t\right] \leq C\alpha\left(P\{\tau < t + \alpha | \mathcal{F}_t\}\right)^{\frac{1}{2}} \leq \frac{C}{\bar{\alpha}^4}\alpha^3. \tag{4.38}$$

Furthermore, from Lemma 2.3 in [7],

$$\begin{aligned}\left|\widehat{Y}_t^2 - \widehat{Y}_t^3\right| &\leq C\left(E\left[\left|\widehat{Y}_\tau^2 - \widehat{Y}_\tau^3\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \leq C\left(E\left[\left|\widehat{Y}_\tau^2\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} + C\left(E\left[\left|\widehat{Y}_\tau^3\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \\ &\leq C\frac{\theta}{2}\left(1 - \mathrm{e}^{-C^*\alpha}\right)\left(P\{\tau < t + \alpha | \mathcal{F}_t\}\right)^{\frac{1}{2}} + C\left(E\left[\left|Y_\tau^{1;u^*}\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \\ &\leq C\frac{\theta}{2}\left(1 - \mathrm{e}^{-C^*\alpha}\right)\frac{1}{\bar{\alpha}^4}\alpha^2 + \frac{C}{\bar{\alpha}^2}\alpha^{\frac{3}{2}} \leq C\alpha^{\frac{3}{2}},\end{aligned} \tag{4.39}$$

for any $\alpha \in (0, \bar{\alpha}]$. □

Proof of Proposition 4.7 (sequel).
By combining (4.32) with Lemma 4.3 we then obtain

$$0 \geq Y_t^{1,u^*} \geq \widehat{Y}_t^3 \geq \widehat{Y}_t^2 - \left|\widehat{Y}_t^2 - \widehat{Y}_t^3\right| \geq \widehat{Y}_t^2 - C\alpha^{\frac{3}{2}}, \text{P-a.s.}$$

i.e., $0 \geq Y_t^{1,u^*} \geq \frac{\theta}{2C^*}(1 - \mathrm{e}^{-C^*\alpha}) - C\alpha^{\frac{3}{2}}$, P-a.s. Therefore,

$$0 \geq \frac{\theta}{2C^*}\frac{1 - \mathrm{e}^{-C^*\alpha}}{\alpha} - C\alpha^{\frac{1}{2}}.$$

Letting $\alpha \to 0+$, we get $0 \geq \frac{\theta}{2}$, which contradicts our assumption that $\theta > 0$. Therefore, we have $\sup_{u \in U} F(t, x, 0, 0, u) \leq 0$, which implies by the definition of F that

$$\frac{\partial \varphi}{\partial t}(t, x) + H\left(t, x, W, D\varphi, D^2\varphi\right) \leq 0, \quad \text{if } x \in D.$$

We now consider the case $x \in \partial D$. We must prove that

$$\min\left\{\frac{\partial \varphi}{\partial t}(t,x) + H\left(t,x,\varphi,D\varphi,D^2\varphi\right),\ \ \frac{\partial \varphi}{\partial n}(t,x) + g(t,x,\varphi)\right\} \le 0$$

If this is not true, then there exists some $\theta > 0$ such that

$$\sup_{u\in U} F(t,x,0,0,u) \ge \theta > 0, \quad G(t,x,0) \ge \theta > 0, \tag{4.40}$$

therefore, there exists $u^* \in U$ such that $F(t,x,0,0,u^*) \ge \frac{2\theta}{3}$.

Choose $\bar\alpha \in (0, T-t]$ such that

$$F\left(s,y,0,0,u^*\right) \ge \tfrac{1}{2}\theta, \tag{4.41}$$

$$G(s,y,0) \ge \tfrac{1}{2}\theta, \text{ for all } t \le s \le t+\bar\alpha,\ |y-x| \le \bar\alpha. \tag{4.42}$$

Now we fix $\bar\alpha$, and we consider any $\alpha \in (0,\bar\alpha]$. Similarly, we still consider GBSDE (4.31) with $x \in \partial D$. For this $u^* \in \mathcal{U}_{t,t+\alpha}$ we still have (4.32) and define

$$\tau = \inf\left\{s \ge t : \left|X_s^{t,x;u^*} - x\right| \ge \bar\alpha\right\} \wedge (t+\alpha).$$

We observe that, for all $(s,y,z) \in [t,\tau] \times \mathbb{R} \times \mathbb{R}^d$, from (4.41), (4.42) and the definition of τ

$$\begin{aligned} F\left(s, X_s^{t,x;u^*}, y, z, u_s^\varepsilon\right) &\ge -C^*(|y|+|z|) + F\left(s, X_s^{t,x;u^*}, 0, 0, u_s^\varepsilon\right) \\ &\ge -C^*(|y|+|z|) + \frac{1}{2}\theta; \\ G\left(s, X_s^{t,x;u^*}, y\right) \ge -C^*|y| + G\left(s, X_s^{t,x;u^*}, 0\right) &\ge -C^*|y| + \frac{1}{2}\theta. \end{aligned}$$

Consequently, from the comparison result for GBSDEs (Lem. 2.7, or Rem. 1.5 in [20]) we have that $Y_s^{1,u^*} \ge \widehat{Y}_s^4,\ s \in [t,\tau],$ P-a.s., where $\widehat{Y}^4$ is defined by the following BSDE:

$$\begin{cases} -\mathrm{d}\widehat{Y}_s^4 = \left(-C^*\left(\left|\widehat{Y}_s^4\right| + \left|\widehat{Z}_s^4\right|\right) + \frac{1}{2}\theta\right)\mathrm{d}s + \left(-C^*\left|\widehat{Y}_s^4\right| + \frac{1}{2}\theta\right)\mathrm{d}K_s^{t,x;u^*} - \widehat{Z}_s^4 \mathrm{d}B_s, \\ \widehat{Y}_\tau^4 = Y_\tau^{1;u^*}. \end{cases} \tag{4.43}$$

On the other hand, we also have to introduce the following BSDE:

$$\begin{cases} -\mathrm{d}\widehat{Y}_s^5 = \left(-C^*\left(\left|\widehat{Y}_s^5\right| + \left|\widehat{Z}_s^5\right|\right) + \frac{1}{2}\theta\right)\mathrm{d}s + \left(-C^*\left|\widehat{Y}_s^5\right| + \frac{1}{2}\theta\right)\mathrm{d}K_s^{t,x;u^*} - \widehat{Z}_s^5 \mathrm{d}B_s, \\ \widehat{Y}_{t+\alpha}^5 = 0. \end{cases} \tag{4.44}$$

Notice that $-C^*|\widehat{Y}_s^2| + \frac{1}{2}\theta > 0$, therefore $\widehat{Y}_s^5 \ge \widehat{Y}_s^2,\ s \in [t,t+\alpha],$ P-a.s., from Lemma 2.4.

From Lemma 2.3 we have

$$\begin{aligned} \left|\widehat{Y}_t^4 - \widehat{Y}_t^5\right| &\le C\left(E\left[\left|\widehat{Y}_\tau^4 - \widehat{Y}_\tau^5\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \le C\left(E\left[\left|\widehat{Y}_\tau^4\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} + C\left(E\left[\left|\widehat{Y}_\tau^5\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \\ &\le C\left(E\left[\left|Y_\tau^{1;u^*}\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} + C\left(E\left[\left|\widehat{Y}_\tau^2\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} + C\left(E\left[\left|\widehat{Y}_\tau^5 - \widehat{Y}_\tau^2\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \\ &\le C\alpha^{\frac{3}{2}} + C\left(E\left[\left|\widehat{Y}_\tau^5 - \widehat{Y}_\tau^2\right|^2 \middle| \mathcal{F}_t\right]\right)^{\frac{1}{2}} \text{(from the proof of (4.39))}, \end{aligned} \tag{4.45}$$

for any $\alpha \in (0,\bar\alpha]$.

Similar to (4.25) and (4.26), $P\{\tau < t+\alpha \,|\mathcal{F}_t\} \le \frac{C}{\alpha^8}\alpha^4$; and

$$E\left[\left|\widehat{Y}^5_\tau - \widehat{Y}^2_\tau\right|^2 \Big|\mathcal{F}_t\right] \le CE\left[\int_\tau^{t+\alpha} \mathrm{e}^{2K_s^{t,x;u^*}}\left(C^*\left|\widehat{Y}^2_s\right| - \frac{1}{2}\theta\right)^2 \mathrm{d}K_s^{t,x;u^*}\Big|\mathcal{F}_t\right]$$
$$= CE\left[\int_\tau^{t+\alpha} \mathrm{e}^{2K_s^{t,x;u^*}}\frac{\theta^2}{4}\mathrm{e}^{2C^*(s-(t+\alpha))}\mathrm{d}K_s^{t,x;u^*}\Big|\mathcal{F}_t\right] \le C\theta^2\alpha^{\frac{5}{2}}. \tag{4.46}$$

Therefore,

$$\left|\widehat{Y}^4_t - \widehat{Y}^5_t\right| \le C\alpha^{\frac{3}{2}} + C\theta\alpha^{\frac{5}{4}}. \tag{4.47}$$

Now we obtain

$$0 \ge Y_t^{1,u^*} \ge \widehat{Y}^4_t \ge \widehat{Y}^5_t - \left|\widehat{Y}^4_t - \widehat{Y}^5_t\right| \ge \widehat{Y}^2_t - C\alpha^{\frac{3}{2}} - C\theta\alpha^{\frac{5}{4}}, \text{ P-a.s.}$$

i.e., $0 \ge Y_t^{1,u^*} \ge \frac{\theta}{2C^*}(1-\mathrm{e}^{-C^*\alpha}) - C\alpha^{\frac{3}{2}} - C\theta\alpha^{\frac{5}{4}}$, P-a.s. Therefore,

$$0 \ge \frac{\theta}{2C^*}\frac{1-\mathrm{e}^{-C^*\alpha}}{\alpha} - C\alpha^{\frac{1}{2}} - C\theta\alpha^{\frac{1}{4}},$$

and by taking the limit as $\alpha \downarrow 0$, we get $0 \ge \frac{\theta}{2}$ which contradicts our assumption that $\theta > 0$. Therefore, it must hold

$$\min\left\{\frac{\partial\varphi}{\partial t}(t,x) + H\left(t,x,W,D\varphi,D^2\varphi\right),\ \frac{\partial\varphi}{\partial n}(t,x) + g(t,x,W)\right\} \le 0. \qquad \square$$

Hence, we have

Theorem 4.10. *Under assumptions* (H4.1) *and* (H4.2)*, the value function* W *is the unique viscosity solution to* (4.1)*.*

Remark 4.11. From Propositions 4.2 and 4.7, it remains to show the uniqueness assertion, which can be referred to Barles ([1], Sect. 3), Bourgoing ([6], Sect. 3), and Crandall *et al.* ([8], Sect. 7B).

Appendix. A

A.1. Forward–Backward SDES (FBSDEs)

In this section we give some results on GBSDEs associated with forward reflected SDEs (for short: FSDEs). Consider the following assumption.

The functions $b : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable and satisfy the following conditions:

(HA.1)

(i) The processes $b(\cdot,0)$ and $\sigma(\cdot,0)$ are $\mathbb{F}$-adapted, and there is a constant $C>0$ such that, for all $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|), \text{ a.s.};$$

(ii) b and σ are Lipschitz in x, *i.e.*, there is a constant $C>0$ such that, for all $t\in[0,T]$, and $x,\ x'\in\mathbb{R}^d$,

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| \le C|x-x'|, \text{ a.s.}$$

Under the assumption (HA.1), it follows from the results in Lions and Sznitman [15] that for each initial condition $(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \bar{D})$ there exists a unique pair of progressively measurable continuous processes $\{(X^{t,\zeta}, K^{t,\zeta})\}$, with values in $\bar{D} \times \mathbb{R}_+$, such that

$$\begin{cases} X_s^{t,\zeta} = \zeta + \int_t^s b\left(r, X_r^{t,\zeta}\right) \mathrm{d}r + \int_t^s \sigma\left(r, X_r^{t,\zeta}\right) \mathrm{d}B_r + \int_t^s \nabla\phi\left(X_r^{t,\zeta}\right) \mathrm{d}K_r^{t,\zeta},\ s \in [t,T], \\ K_s^{t,\zeta} = \int_t^s I_{\left\{X_r^{t,\zeta} \in \partial D\right\}} \mathrm{d}K_r^{t,\zeta},\ K^{t,\zeta} \text{ is increasing.} \end{cases} \tag{A.1}$$

Proposition A.1. *For each $T \geq 0$, there exists a constant C_T such that, for all $\zeta,\ \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$,*

$$E\left(\sup_{t \leq s \leq T} \left|X_s^{t,\zeta} - X_s^{t,\zeta'}\right|^4 \Big| \mathcal{F}_t\right) \leq C_T \left|\zeta - \zeta'\right|^4, \tag{A.2}$$

and

$$E\left(\sup_{t \leq s \leq T} \left|K_s^{t,\zeta} - K_s^{t,\zeta'}\right|^4 \Big| \mathcal{F}_t\right) \leq C_T \left|\zeta - \zeta'\right|^4. \tag{A.3}$$

Moreover, for each $\mu > 0, s \in [t,T]$, there exists $C(\mu, s)$ such that for all $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$,

$$E\left(\mathrm{e}^{\mu K_s^{t,\zeta}} \Big| \mathcal{F}_t\right) \leq C(\mu, s). \tag{A.4}$$

The proof is similar to that of Propositions 3.1 and 3.2 in Pardoux and Zhang [20].

We assume that the three functions f, g and Φ satisfy the following conditions:

(HA.2)

(i) $\Phi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is an $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$-measurable random variable and $f : [0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a measurable process such that $f(\cdot, x, y, z)$ is $\mathbb{F}$-adapted, for all $(x,y,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$; $g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $g(\cdot) \in C^{1,2,2}([0,T] \times \mathbb{R}^d \times \mathbb{R})$;

(ii) There exists a constant $C > 0$ such that for all $t \in [0,T],\ x, x', z, z' \in \mathbb{R}^d,\ y, y' \in \mathbb{R}$,

$$|f(t,x,y,z) - f(t,x',y',z')| + |g(t,x,y) - g(t,x',y')| + |\Phi(x) - \Phi(x')| \\ \leq C(|x - x'| + |y - y'| + |z - z'|), \quad \text{a.s.};$$

(iii) f and Φ satisfy a linear growth condition, *i.e.*, there exists some $C > 0$ such that, for all $x \in \mathbb{R}^d$, $\quad |f(t,x,0,0)| + |\Phi(x)| \leq C(1 + |x|)$, a.s.

Under the above assumptions the coefficients $f(s, X_s^{t,\zeta}, y, z)$ and $g(s, X_s^{t,\zeta}, y)$ satisfy (H2.1) and $\xi = \Phi(X_T^{t,\zeta}) \in L^2(\Omega, \mathcal{F}_T, P)$. Therefore, the following GBSDE possesses a unique solution:

$$\begin{cases} -\mathrm{d}Y_s^{t,\zeta} = f\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}\right) \mathrm{d}s + g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right) \mathrm{d}K_s^{t,\zeta} - Z_s^{t,\zeta} \mathrm{d}B_s,\ s \in [t,T], \\ \quad Y_T^{t,\zeta} = \Phi\left(X_T^{t,\zeta}\right). \end{cases} \tag{A.5}$$

Proposition A.2. *Let assumptions* (HA.1) *and* (HA.2) *hold. Then, for any $0 \leq t \leq T$ and $\zeta,\ \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$,*

(i) $E\left[\sup_{t \leq s \leq T} \left|Y_s^{t,\zeta}\right|^2 + \int_t^T \left|Z_s^{t,\zeta}\right|^2 \mathrm{d}s \Big| \mathcal{F}_t\right] \leq C\left(1 + |\zeta|^2\right)$, *a.s.; and in particular,* $\left|Y_t^{t,\zeta}\right| \leq C(1 + |\zeta|)$, *a.s.*;

(ii) $\left|Y_t^{t,\zeta} - Y_t^{t,\zeta'}\right| \leq C\left|\zeta - \zeta'\right| + C\left|\zeta - \zeta'\right|^{\frac{1}{2}}$, *a.s.*,

where the constant $C > 0$ depends only on the Lipschitz and the growth constants of b, σ, f, g and Φ.

Remark A.3. Since D is bounded, we have

$$E\left[\sup_{t\le s\le T}\left|Y_s^{t,\zeta}\right|^2+\int_t^T\left|Z_s^{t,\zeta}\right|^2\mathrm{d}s\middle|\mathcal{F}_t\right]\le C,\ \text{a.s.},\tag{A.6}$$

where C is independent of ζ.

Proof. From Lemma 2.2 and Proposition A.1, we have assertion (i). Now we prove assertion (ii). First notice that from (i) we have $|Y_t^{t,\zeta}|\le C(1+|\zeta|)$, a.s., therefore we can get from the uniqueness of the solution of equations (A.1) and (A.5) that

$$\left|Y_s^{t,\zeta}\right|=\left|Y_s^{s,X_s^{t,\zeta}}\right|\le C\left(1+\left|X_s^{t,\zeta}\right|\right)\le C,\ \text{a.s.},\tag{A.7}$$

since D is bounded. From Burkholder–Davis–Gundy inequality and (A.5), as well as from the boundedness of the processes $X^{t,\zeta}$, $Y^{t,\zeta}$,

$$\begin{aligned}E\left[\left(\int_s^T\left|Z_r^{t,\zeta}\right|^2\mathrm{d}r\right)^2\middle|\mathcal{F}_t\right]&\le CE\left[\sup_{r\in[s,T]}\left|\int_s^r Z_v^{t,\zeta}\mathrm{d}B_v\right|^4\middle|\mathcal{F}_t\right]\\&\le C+C_0(T-s)^2E\left[\left(\int_s^T\left|Z_r^{t,\zeta}\right|^2\mathrm{d}r\right)^2\middle|\mathcal{F}_t\right]+CE\left[\left(K_T^{t,\zeta}\right)^4\middle|\mathcal{F}_t\right]\\&\le C+C_0(T-s)^2E\left[\left(\int_s^T\left|Z_r^{t,\zeta}\right|^2\mathrm{d}r\right)^2\middle|\mathcal{F}_t\right].\end{aligned}$$

Consequently, for $T-s\le(\frac{1}{2C_0})^{1/2}$, $E[(\int_s^T|Z_r^{t,\zeta}|^2\mathrm{d}r)^2|\mathcal{F}_t]\le C$. This argument allows to choose a partition $t=t_0<t_1<\ldots<t_N=T$ of the interval $[t,T]$ such that $E[(\int_{t_{i-1}}^{t_i}|Z_r^{t,\zeta}|^2\mathrm{d}r)^2|\mathcal{F}_t]\le C$, $1\le i\le N$. Therefore, we have

$$E\left[\left(\int_t^T\left|Z_r^{t,\zeta}\right|^2\mathrm{d}r\right)^2\middle|\mathcal{F}_t\right]\le C.\tag{A.8}$$

For any $\lambda>0$, applying Itô's formula to $\mathrm{e}^{\lambda K_s^{t,\zeta'}}|Y_s^{t,\zeta}-Y_s^{t,\zeta'}|^2$, we have

$$\begin{aligned}&\left|Y_s^{t,\zeta}-Y_s^{t,\zeta'}\right|^2+\lambda\int_s^T\mathrm{e}^{\lambda K_r^{t,\zeta'}}\left|Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right|^2\mathrm{d}K_r^{t,\zeta'}+\int_s^T\mathrm{e}^{\lambda K_r^{t,\zeta'}}\left|Z_r^{t,\zeta}-Z_r^{t,\zeta'}\right|^2\mathrm{d}r\\&=\mathrm{e}^{\lambda K_T^{t,\zeta'}}\left|Y_T^{t,\zeta}-Y_T^{t,\zeta'}\right|^2-2\int_s^T\mathrm{e}^{\lambda K_r^{t,\zeta'}}\left(Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right)\left\langle Z_r^{t,\zeta}-Z_r^{t,\zeta'},\mathrm{d}B_r\right\rangle\\&+2\int_s^T\mathrm{e}^{\lambda K_r^{t,\zeta'}}\left(Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right)\left(f\left(r,X_r^{t,\zeta},Y_r^{t,\zeta},Z_r^{t,\zeta}\right)-f\left(r,X_r^{t,\zeta'},Y_r^{t,\zeta'},Z_r^{t,\zeta'}\right)\right)\mathrm{d}r\\&+2\int_s^T\mathrm{e}^{\lambda K_r^{t,\zeta'}}\left(Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right)\left(g\left(r,X_r^{t,\zeta},Y_r^{t,\zeta}\right)-g\left(r,X_r^{t,\zeta'},Y_r^{t,\zeta'}\right)\right)\mathrm{d}K_r^{t,\zeta'}\\&+2\int_s^T\mathrm{e}^{\lambda K_r^{t,\zeta'}}\left(Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right)g\left(r,X_r^{t,\zeta},Y_r^{t,\zeta}\right)\mathrm{d}\left(K_r^{t,\zeta}-K_r^{t,\zeta'}\right).\end{aligned}\tag{A.9}$$

Then from (HA.1), (HA.2), (A.4), (A.7) and (A.8), taking a suitable $\lambda > 0$, we get

$$\begin{aligned}\left|Y_s^{t,\zeta} - Y_s^{t,\zeta'}\right|^2 \le{}& C\left|\zeta - \zeta'\right|^2 + CE\left[\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left|Y_r^{t,\zeta} - Y_r^{t,\zeta'}\right|^2 \mathrm{d}r\middle|\mathcal{F}_s\right] \\ &+ CE\left[\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left|X_r^{t,\zeta} - X_r^{t,\zeta'}\right|^2 \mathrm{d}r\middle|\mathcal{F}_s\right] + CE\left[\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left|X_r^{t,\zeta} - X_r^{t,\zeta'}\right|^2 \mathrm{d}K_r^{t,\zeta'}\middle|\mathcal{F}_s\right] \\ &+ E\left[2\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left(Y_r^{t,\zeta} - Y_r^{t,\zeta'}\right) g\left(r, X_r^{t,\zeta}, Y_r^{t,\zeta}\right)\mathrm{d}\left(K_r^{t,\zeta} - K_r^{t,\zeta'}\right)\middle|\mathcal{F}_s\right]. \end{aligned} \tag{A.10}$$

Furthermore, from Proposition A.1, we have

$$\begin{aligned}\left|Y_s^{t,\zeta} - Y_s^{t,\zeta'}\right|^2 \le{}& C\left|\zeta - \zeta'\right|^2 + CE\left[\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left|Y_r^{t,\zeta} - Y_r^{t,\zeta'}\right|^2 \mathrm{d}r\middle|\mathcal{F}_s\right] \\ &+ E\left[2\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left(Y_r^{t,\zeta} - Y_r^{t,\zeta'}\right) g\left(r, X_r^{t,\zeta}, Y_r^{t,\zeta}\right)\mathrm{d}\left(K_r^{t,\zeta} - K_r^{t,\zeta'}\right)\middle|\mathcal{F}_s\right]. \end{aligned} \tag{A.11}$$

On the other hand, applying Itô's formula to $\mathrm{e}^{\lambda K_s^{t,\zeta'}}(Y_s^{t,\zeta} - Y_s^{t,\zeta'})g(s, X_s^{t,\zeta}, Y_s^{t,\zeta})(K_s^{t,\zeta} - K_s^{t,\zeta'})$, we have

$$\begin{aligned}&E\left[\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left(Y_r^{t,\zeta} - Y_r^{t,\zeta'}\right) g\left(r, X_r^{t,\zeta}, Y_r^{t,\zeta}\right)\mathrm{d}\left(K_r^{t,\zeta} - K_r^{t,\zeta'}\right)\middle|\mathcal{F}_s\right] \\ &= E\left[\mathrm{e}^{\lambda K_T^{t,\zeta'}}\left(Y_T^{t,\zeta} - Y_T^{t,\zeta'}\right) g\left(T, X_T^{t,\zeta}, Y_T^{t,\zeta}\right)\left(K_T^{t,\zeta} - K_T^{t,\zeta'}\right)\middle|\mathcal{F}_s\right] + E\left[\int_s^T f_1(r)\left(K_r^{t,\zeta'} - K_r^{t,\zeta}\right)\mathrm{d}r\middle|\mathcal{F}_s\right] \\ &+ E\left[\int_s^T f_2(r)\left(K_r^{t,\zeta'} - K_r^{t,\zeta}\right)\mathrm{d}K_r^{t,\zeta'}\middle|\mathcal{F}_s\right] + E\left[\int_s^T f_3(r)\left(K_r^{t,\zeta'} - K_r^{t,\zeta}\right)\mathrm{d}K_r^{t,\zeta}\middle|\mathcal{F}_s\right], \end{aligned} \tag{A.12}$$

where

$$\begin{aligned}f_1(s) ={}& -\mathrm{e}^{\lambda K_s^{t,\zeta'}}\left(f\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}\right) - f\left(s, X_s^{t,\zeta'}, Y_s^{t,\zeta'}, Z_s^{t,\zeta'}\right)\right) g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right) \\ &+ \mathrm{e}^{\lambda K_s^{t,\zeta'}}\left(Y_s^{t,\zeta} - Y_s^{t,\zeta'}\right)\left\{\frac{\partial}{\partial s} g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right) + \nabla_x g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right) b\left(s, X_s^{t,\zeta}\right)\right. \\ &- \nabla_y g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right) f\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}\right) + \frac{1}{2}\mathrm{tr}\left(D_x^2 g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right)\sigma\sigma^T\left(s, X_s^{t,\zeta}\right)\right) \\ &\left.+ \frac{1}{2} D_y^2 g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right)\left|Z_s^{t,\zeta}\right|^2 + \frac{1}{2} tr\left\langle D_{xy} g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right)\sigma\left(s, X_s^{t,\zeta}\right), Z_s^{t,\zeta}\right\rangle\right\} \\ &+ \mathrm{e}^{\lambda K_s^{t,\zeta'}}\left(Z_s^{t,\zeta} - Z_s^{t,\zeta'}\right)\left\{\nabla_x g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right)\sigma\left(s, X_s^{t,\zeta}\right) + \nabla_y g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right) Z_s^{t,\zeta}\right\}; \\ f_2(s) ={}& \left\{\lambda \mathrm{e}^{\lambda K_s^{t,\zeta'}}\left(Y_s^{t,\zeta} - Y_s^{t,\zeta'}\right) + \mathrm{e}^{\lambda K_s^{t,\zeta'}} g\left(s, X_s^{t,\zeta'}, Y_s^{t,\zeta'}\right)\right\} g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right); \\ f_3(s) ={}& \mathrm{e}^{\lambda K_s^{t,\zeta'}}\left(Y_s^{t,\zeta} - Y_s^{t,\zeta'}\right)\left\{\nabla_x g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right)\nabla\phi\left(s, X_s^{t,\zeta}\right) - \nabla_y g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right) g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right)\right\} \\ &- \mathrm{e}^{\lambda K_s^{t,\zeta'}}\left|g\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}\right)\right|^2. \end{aligned}$$

From assertion (i), Propositions A.1, (A.7) and (A.8), we have

$$E\left[\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left(Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right) g\left(r, X_r^{t,\zeta}, Y_r^{t,\zeta}\right) \mathrm{d}\left(K_r^{t,\zeta}-K_r^{t,\zeta'}\right)\Big|\mathcal{F}_s\right] \le C\left|\zeta-\zeta'\right|^2 + C\left|\zeta-\zeta'\right|.$$

Furthermore, from (A.11) and (A.4) we have

$$\begin{aligned} E\left[\left|Y_s^{t,\zeta}-Y_s^{t,\zeta'}\right|^4\Big|\mathcal{F}_t\right] &\le C\left|\zeta-\zeta'\right|^4 + C\left|\zeta-\zeta'\right|^2 + CE\left[\left(\int_s^T \mathrm{e}^{\lambda K_r^{t,\zeta'}}\left|Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right|^2 \mathrm{d}r\right)^2\Big|\mathcal{F}_t\right] \\ &\le C\left|\zeta-\zeta'\right|^4 + C\left|\zeta-\zeta'\right|^2 + CE\left[\mathrm{e}^{2\lambda K_T^{t,\zeta'}}\Big|\mathcal{F}_t\right] E\left[\int_s^T \left|Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right|^4 \mathrm{d}r\Big|\mathcal{F}_t\right] \\ &\le C\left|\zeta-\zeta'\right|^4 + C\left|\zeta-\zeta'\right|^2 + CE\left[\int_s^T \left|Y_r^{t,\zeta}-Y_r^{t,\zeta'}\right|^4 \mathrm{d}r\Big|\mathcal{F}_t\right], \quad s\in[t,T], \end{aligned}$$

then from Gronwall's Lemma, we get $E[|Y_s^{t,\zeta}-Y_s^{t,\zeta'}|^4|\mathcal{F}_t] \le C|\zeta-\zeta'|^4 + C|\zeta-\zeta'|^2$, a.s, $s\in[t,T]$, which means (ii) for $s=t$. □

Remark A.4. If g is a bounded random variable, assertion (ii) of (A.6) still holds. Indeed, from Lemma 2.3 in [7] and Proposition A.1, we get

$$\begin{aligned} \left|Y_t^{t,\zeta}-Y_t^{t,\zeta'}\right|^2 &\le CE\left[\left|\zeta-\zeta'+g(\omega)\left(K_T^{t,\zeta}-K_T^{t,\zeta'}\right)\right|^2\Big|\mathcal{F}_t\right] \\ &\quad + CE\left[\int_t^T \left|f\left(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}\right) - f\left(s, X_s^{t,\zeta'}, Y_s^{t,\zeta}, Z_s^{t,\zeta}\right)\right|^2 \mathrm{d}s\Big|\mathcal{F}_t\right] \\ &\le C\left|\zeta-\zeta'\right|^2, \text{ a.s.} \end{aligned}$$

Proposition A.5. *Let assumptions* (HA.1) *and* (HA.2) *hold. Then, for any* $0\le\alpha\le T-t$ *and the associated initial conditions* $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\bar{D})$*, we have the following estimates:*

$$E\left[\left|K_{t+\alpha}^{t,\zeta}\right|^2\Big|\mathcal{F}_t\right] \le C\alpha, \ \text{a.s.}, \tag{A.13}$$

where the constant $C>0$ *depends only on the Lipschitz and the growth constants of* b*,* σ*,* f*,* g *and* Φ*.*

Proof. For $\zeta'\in L^2(\Omega,\mathcal{F}_t,P;\bar{D})$, from Itô's formula we have

$$\begin{aligned} \left|X_s^{t,\zeta}-\zeta'\right|^2 &= \left|\zeta-\zeta'\right|^2 + 2\int_t^s \left(X_r^{t,\zeta}-\zeta'\right) b\left(r, X_r^{t,\zeta}\right)\mathrm{d}r + 2\int_t^s \left(X_r^{t,\zeta}-\zeta'\right)\sigma\left(r, X_r^{t,\zeta}\right)\mathrm{d}B_r \\ &\quad + \int_t^s \left|\sigma\left(r, X_r^{t,\zeta}\right)\right|^2 \mathrm{d}r + 2\int_t^s \left(X_r^{t,\zeta}-\zeta'\right)\nabla\phi\left(X_r^{t,\zeta}\right)\mathrm{d}K_r^{t,\zeta}, \ s\in[t,T]. \end{aligned} \tag{A.14}$$

Since $D\subset\mathbb{R}^d$ is convex, we have

$$\int_t^s \left(X_r^{t,\zeta}-\zeta'\right)\nabla\phi\left(X_r^{t,\zeta}\right)\mathrm{d}K_r^{t,\zeta} \le 0. \tag{A.15}$$

Therefore, we have $E[\sup_{s\in[t,t+\alpha]}|X_s^{t,\zeta}-\zeta'|^2|\mathcal{F}_t]\le C(|\zeta-\zeta'|^2+\alpha)$. Recall that D is an open connected bounded convex subset. In particular, we have,

$$E\left[\sup_{s\in[t,t+\alpha]}\left|X_s^{t,\zeta}-\zeta\right|^2\Big|\mathcal{F}_t\right]\le C\alpha. \tag{A.16}$$

Because $\phi\in C_b^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\phi\left(X_s^{t,\zeta}\right)=\phi(\zeta)+\int_t^s\nabla\phi\left(X_r^{t,\zeta}\right)b\left(r,X_r^{t,\zeta}\right)\mathrm{d}r+\int_t^s\nabla\phi\left(X_r^{t,\zeta}\right)\sigma\left(r,X_r^{t,\zeta}\right)\mathrm{d}B_r\\+\frac{1}{2}\int_t^s\operatorname{tr}\left(D^2\phi\sigma\left(r,X_r^{t,\zeta}\right)\sigma^T\left(r,X_r^{t,\zeta}\right)\right)\mathrm{d}r+\int_t^s\left|\nabla\phi\left(X_r^{t,\zeta}\right)\right|^2\mathrm{d}K_r^{t,\zeta},\ s\in[t,T].\end{aligned}$$

Therefore, we get

$$K_s^{t,\zeta}\le\left|\phi\left(X_s^{t,\zeta}\right)-\phi(\zeta)\right|+C\int_t^s\left(1+\left|X_r^{t,\zeta}\right|^2\right)\mathrm{d}r+\left|\int_t^s\nabla\phi\left(X_r^{t,\zeta}\right)\sigma\left(r,X_r^{t,\zeta}\right)\mathrm{d}B_r\right|,$$

and furthermore, from Burkholder–Davis–Gundy inequality, we have

$$E\left[\left|K_{t+\alpha}^{t,\zeta}\right|^2\Big|\mathcal{F}_t\right]\le CE\left[\sup_{s\in[t,t+\alpha]}\left|X_s^{t,\zeta}-\zeta\right|^2\Big|\mathcal{F}_t\right]+C\alpha.$$

In view of (A.16), the proof is complete. □

Remark A.6. In view of (A.13) and (A.14), using Burkholder–Davis–Gundy inequality, we have

$$E\left[\sup_{s\in[t,t+\alpha]}\left|X_s^{t,\zeta}-\zeta\right|^8\Big|\mathcal{F}_t\right]\le C\alpha^4. \tag{A.17}$$

Let us now define the random field:

$$u(t,x)=Y_s^{t,x}|_{s=t},\ (t,x)\in[0,T]\times\bar{D}, \tag{A.18}$$

where $Y^{t,x}$ is the solution of GBSDE (A.5) with $x\in\bar{D}$ at the place of $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\bar{D})$.

Proposition A.2 yields that, for all $t\in[0,T]$, P-a.s.,

$$\begin{aligned}&\text{(i) } |u(t,x)-u(t,y)|\le C|x-y|+C|x-y|^{\frac{1}{2}},\ \text{for all } x,y\in\bar{D};\\&\text{(ii) } |u(t,x)|\le C(1+|x|),\ \text{for all } x\in\bar{D}.\end{aligned} \tag{A.19}$$

Theorem A.7. *Under the assumptions* (H3.1) *and* (H3.2), *for any* $t\in[0,T]$ *and* $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\bar{D})$, *we have*

$$u(t,\zeta)=Y_t^{t,\zeta},\ P\text{-}a.s. \tag{A.20}$$

The proof of Theorem A.7 is similar to that of Theorem 3.1 in Peng [21] or Theorem A.2 in Buckdahn and Li [7]. Therefore it is omitted here.

A.2. Proofs of Proposition 3.1 and Theorem 3.1

Proof of Proposition 3.1. Let H be the Cameron–Martin space of all absolutely continuous elements $h \in \Omega$ whose derivative $\dot{h}$ is in $L^2([0,T],\mathbb{R}^d)$.

For any $h \in H$, we define $\tau_h\omega := \omega + h$, $\omega \in \Omega$. Obviously, $\tau_h : \Omega \to \Omega$ is a bijection with the inverse τ_h^{-1}. The law is given by

$$P \circ \left[\tau_h^{-1}\right] = \exp\left\{\int_0^T \dot{h}_s \mathrm{d}B_s - \frac{1}{2}\int_0^T \left|\dot{h}_s\right|^2 \mathrm{d}s\right\} P.$$

Fix any $(t,x) \in [0,T] \times \bar{D}$, and define $H_t := \{h \in H | h(\cdot) = h(\cdot \wedge t)\}$. The rest of the proof is divided into the following three steps:

Step 1. For any $u \in \mathcal{U}_{t,T}$ and $h \in H_t$, $J(t,x;u)\ (\tau_h) = J(t,x;u(\tau_h))$, P-a.s.

Indeed, the τ_h-shifted reflected SDE (3.1) (with $\zeta = x$) is the same reflected SDE (3.1) with u being substituted into the τ_h-shifted control process $u(\tau_h)$. From the uniqueness of the solution of the reflected SDE (3.1), we get $X_s^{t,x;u}(\tau_h) = X_s^{t,x;u(\tau_h)}$ and $K_s^{t,x;u}(\tau_h) = K_s^{t,x;u(\tau_h)}$ for $s \in [t,T]$ P-a.s. Furthermore, by a similar shift argument and the associated Girsanov transformation, we get from the uniqueness of the solution of GBSDE (3.3) that

$$Y_s^{t,x;u}(\tau_h) = Y_s^{t,x;u(\tau_h)} \text{ for any } s \in [t,T],\ P\text{-a.s.},$$

$$Z_s^{t,x;u}(\tau_h) = Z_s^{t,x;u(\tau_h)},\ \mathrm{d}s\mathrm{d}P\text{-a.e. on } [t,T] \times \Omega.$$

It means

$$J(t,x;u)\,(\tau_h) = J\left(t,x;u\left(\tau_h\right)\right),\ P\text{-a.s.}$$

Step 2. For all $h \in H_t$ we have

$$\left\{\operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u)\right\}(\tau_h) = \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} \left\{J(t,x;u)\,(\tau_h)\right\},\ P\text{-a.s.}$$

Indeed, define

$$W(t,x) := \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u).$$

we have $W(t,x) \geq J(t,x;u)$, and thus $W(t,x)(\tau_h) \geq J(t,x;u)(\tau_h)$, P-a.s., for all $u \in \mathcal{U}_{t,T}$. On the other hand, for any random variable ζ satisfying $\zeta \geq J(t,x;u)(\tau_h)$, and hence also $\zeta(\tau_{-h}) \geq J(t,x;u)$, P-a.s., for all $u \in \mathcal{U}_{t,T}$, we have $\zeta(\tau_{-h}) \geq W(t,x)$, P-a.s., *i.e.*, $\zeta \geq W(t,x)(\tau_h)$, P-a.s. Consequently,

$$W(t,x)\,(\tau_h) = \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} \left\{J(t,x;u)\,(\tau_h)\right\},\ P\text{-a.s.}$$

Step 3. $W(t,x)$ is invariant with respect to the shift τ_h, *i.e.*,

$$W(t,x)\,(\tau_h) = W(t,x),\ P\text{-a.s., for any } h \in H.$$

Indeed, from Step 1 to Step 2, we have, for any $h \in H_t$,

$$\begin{aligned} W(t,x)\,(\tau_h) &= \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} \left\{J(t,x;u)\,(\tau_h)\right\} = \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J\left(t,x;u\left(\tau_h\right)\right) \\ &= \operatorname*{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u) = W(t,x),\ P\text{-a.s.,} \end{aligned}$$

where we have used $\{u(\tau_h) | u(\cdot) \in \mathcal{U}_{t,T}\} = \mathcal{U}_{t,T}$ so as to obtain the 3rd equality. Therefore, $W(t,x)(\tau_h) = W(t,x)$, P-a.s. for any $h \in H_t$. Since $W(t,x)$ is $\mathcal{F}_t$-measurable, it holds for all $h \in H$. Indeed, since $\Omega = C_0([0,T];\mathbb{R}^d)$, by the definition of the filtration, the $\mathcal{F}_t$-measurable random variable $W(t,x)(\omega)$, $\omega \in \Omega$, only depends on the restriction of ω to the time interval $[0,t]$.

The result of Step 3, combined with the following Lemma A.8 (refer to Buckdahn and Li ([7], Lem. 3.4) completes the proof. □

Lemma A.8. *Let ζ be a random variable defined on the Wiener space $(\Omega, \mathcal{F}_T, P)$ such that $\zeta(\tau_h) = \zeta$ P-a.s. for any $h \in H$. Then $\zeta = E\zeta$ P-a.s.*

Proof of Theorem 3.1. To simplify our exposition, define

$$I_\delta(t,x,u) := G^{t,x;u}_{t,t+\delta}\left[W\left(t+\delta, X^{t,x;u}_{t+\delta}\right)\right]$$

and

$$W_\delta(t,x) := \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} I_\delta(t,x,u) = \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u}_{t,t+\delta}\left[W\left(t+\delta, X^{t,x;u}_{t+\delta}\right)\right].$$

The proof of Theorem 3.5 is reduced to the following three lemmas. Similar to the proof of Proposition 3.2, we first have

Lemma A.9. *$W_\delta(t,x)$ is deterministic for any $0 \le t < t+\delta \le T$, $x \in \bar{D}$.*

Lemma A.10. $W_\delta(t,x) \le W(t,x)$, $0 \le t < t+\delta \le T$, $x \in \bar{D}$. □

Proof. For $u_1(\cdot) \in \mathcal{U}_{t,t+\delta}$ and $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, we define $u_1 \oplus u_2 := u_1 \mathbf{1}_{[t,t+\delta]} + u_2 \mathbf{1}_{(t+\delta,T]}$, which lies in $\mathcal{U}_{t,T}$. Note that there exists a sequence $\{u_i^1,\ i \ge 1\} \subset \mathcal{U}_{t,t+\delta}$ such that

$$W_\delta(t,x) = \operatorname*{esssup}_{u_1\in\mathcal{U}_{t,t+\delta}} I_\delta\left(t,x,u_1\right) = \sup_{i\ge1} I_\delta\left(t,x,u_i^1\right), \quad P\text{-a.s.}$$

For any $\varepsilon > 0$, we define $\widetilde{\Gamma}_i := \{W_\delta(t,x) \le I_\delta(t,x,u_i^1) + \varepsilon\} \in \mathcal{F}_t$, $i \ge 1$. Then the following mutually disjoint events $\Gamma_1 := \widetilde{\Gamma}_1$, $\Gamma_i := \widetilde{\Gamma}_i \backslash (\cup_{l=1}^{i-1} \widetilde{\Gamma}_l) \in \mathcal{F}_t$, $i \ge 2$, form a $(\Omega, \mathcal{F}_t)$-partition. It is obvious that $u_1^\varepsilon := \sum_{i\ge1} \mathbf{1}_{\Gamma_i} u_i^1 \in \mathcal{U}_{t,t+\delta}$. Moreover, from the uniqueness of the solution of the forward-backward SDE, we have $I_\delta(t,x,u_1^\varepsilon) = \sum_{i\ge1} \mathbf{1}_{\Gamma_i} I_\delta(t,x,u_i^1)$, P-a.s. Hence,

$$\begin{aligned} W_\delta(t,x) &\le \sum_{i\ge1} \mathbf{1}_{\Gamma_i} I_\delta\left(t,x,u_i^1\right) + \varepsilon = I_\delta\left(t,x,u_1^\varepsilon\right) + \varepsilon \\ &= G^{t,x;u_1^\varepsilon}_{t,t+\delta}\left[W\left(t+\delta, X^{t,x;u_1^\varepsilon}_{t+\delta}\right)\right] + \varepsilon,\ P\text{-a.s.} \end{aligned} \tag{A.21}$$

On the other hand, from the definition of $W(t+\delta, y)$ we have, for any $y \in \bar{D}$,

$$W(t+\delta,y) = \operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J\left(t+\delta, y; u_2\right), \quad P\text{-a.s.}$$

Finally, since there exists a constant $C \in \mathbb{R}$ such that for any $y, y' \in \bar{D}$, $u_2 \in \mathcal{U}_{t+\delta,T}$,

$$\begin{aligned} &\text{(i)}\ \ |W(t+\delta,y) - W\left(t+\delta,y'\right)| \le C\left(|y-y'| + |y-y'|^{\frac{1}{2}}\right); \\ &\text{(ii)}\ |J\left(t+\delta,y,u_2\right) - J\left(t+\delta,y',u_2\right)| \le C\left(|y-y'| + |y-y'|^{\frac{1}{2}}\right), \quad P\text{-a.s.}, \end{aligned} \tag{A.22}$$

(see Lem. 3.3(i) and (3.4)(i)) we can prove by approximating $X^{t,x;u_1^\varepsilon}_{t+\delta}$ that

$$W\left(t+\delta, X^{t,x;u_1^\varepsilon}_{t+\delta}\right) \le \operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J\left(t+\delta, X^{t,x;u_1^\varepsilon}_{t+\delta}; u_2\right),\ P\text{-a.s.}$$

To estimate the right side of the above inequality we notice that there exists some sequence $\{u_j^2, j \ge 1\} \subset \mathcal{U}_{t+\delta,T}$ such that

$$\operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J\left(t+\delta, X^{t,x;u_1^\varepsilon}_{t+\delta}; u_2\right) = \sup_{j\ge1} J\left(t+\delta, X^{t,x;u_1^\varepsilon}_{t+\delta}; u_j^2\right),\ P\text{-a.s.}$$

Then, putting $\widetilde{\Delta}_j := \{\operatorname{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon}; u_2) \le J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon}; u_j^2)+\varepsilon\} \in \mathcal{F}_{t+\delta},\ j\ge 1$; we have with $\Delta_1 := \widetilde{\Delta}_1,\ \Delta_j := \widetilde{\Delta}_j\backslash(\cup_{l=1}^{j-1}\widetilde{\Delta}_l) \in \mathcal{F}_{t+\delta},\ j\ge 2$, an $(\Omega,\mathcal{F}_{t+\delta})$-partition and $u_2^\varepsilon := \sum_{j\ge1}\mathbf{1}_{\Delta_j}u_j^2 \in \mathcal{U}_{t+\delta,T}$. Therefore, from the uniqueness of the solution of our reflected SDE and GBSDE, we have

$$J\left(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon}; u_2^\varepsilon\right) = Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon}; u_2^\varepsilon} \qquad \text{(see (3.6))}$$

$$= \sum_{j\ge1}\mathbf{1}_{\Delta_j} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon}; u_j^2} = \sum_{j\ge1}\mathbf{1}_{\Delta_j} J\left(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon}; u_j^2\right),\ P\text{-a.s.}$$

Thus,

$$W\left(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon}\right) \le \operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J\left(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon}; u_2\right)$$

$$\le \sum_{j\ge1}\mathbf{1}_{\Delta_j} Y_{t+\delta}^{t,x;u_1^\varepsilon\oplus u_j^2} + \varepsilon = Y_{t+\delta}^{t,x;u_1^\varepsilon\oplus u_2^\varepsilon} + \varepsilon = Y_{t+\delta}^{t,x;u^\varepsilon} + \varepsilon,\ P\text{-a.s.}, \tag{A.23}$$

where $u^\varepsilon := u_1^\varepsilon \oplus u_2^\varepsilon \in \mathcal{U}_{t,T}$. From (A.21) and (A.23) and Lemmas 2.4 and 2.3, we get

$$W_\delta(t,x) \le G_{t,t+\delta}^{t,x;u_1^\varepsilon}\left[Y_{t+\delta}^{t,x;u^\varepsilon} + \varepsilon\right] + \varepsilon \le G_{t,t+\delta}^{t,x;u_1^\varepsilon}\left[Y_{t+\delta}^{t,x;u^\varepsilon}\right] + (C+1)\varepsilon$$

$$= G_{t,t+\delta}^{t,x;u^\varepsilon}\left[Y_{t+\delta}^{t,x;u^\varepsilon}\right] + (C+1)\varepsilon = Y_t^{t,x;u^\varepsilon} + (C+1)\varepsilon$$

$$\le \operatorname*{esssup}_{u\in\mathcal{U}_{t,T}} Y_t^{t,x;u} + (C+1)\varepsilon,\ P\text{-a.s.} \tag{A.24}$$

That is,

$$W_\delta(t,x) \le W(t,x) + (C+1)\varepsilon. \tag{A.25}$$

Finally, letting $\varepsilon\downarrow 0$, we get $W_\delta(t,x) \le W(t,x)$. □

Lemma A.11. $W(t,x) \le W_\delta(t,x),\ 0\le t < t+\delta \le T,\ x\in\bar{D}$.

Proof. Since $W_\delta(t,x) = \operatorname{esssup}_{u_1\in\mathcal{U}_{t,t+\delta}} I_\delta(t,x,u_1)$, we have

$$W_\delta(t,x) \ge I_\delta(t,x,u_1) = G_{t,t+\delta}^{t,x;u_1}\left[W\left(t+\delta, X_{t+\delta}^{t,x;u_1}\right)\right], \tag{A.26}$$

P-a.s., for all $u_1\in\mathcal{U}_{t,t+\delta}$. Moreover, from the definition of $W(t+\delta,y),\ y\in\bar{D}$, we get

$$W(t+\delta,y) = \operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J(t+\delta,y;u_2),\ P\text{-a.s.} \tag{A.27}$$

Let $\{O_i\}_{i\ge1}\subset\mathcal{B}(\mathbb{R}^d)$ be a decomposition of $\bar{D}$ such that $\sum_{i\ge1}O_i = \bar{D}$ and $\operatorname{diam}(O_i)\le\varepsilon,\ i\ge1$. Let y_i be an arbitrarily given element of $O_i,\ i\ge1$. We define $[X_{t+\delta}^{t,x;u_1}] := \sum_{i\ge1} y_i\mathbf{1}_{\{X_{t+\delta}^{t,x;u_1}\in O_i\}}$. Then we have

$$\left|X_{t+\delta}^{t,x;u_1} - \left[X_{t+\delta}^{t,x;u_1}\right]\right| \le \varepsilon,\ \text{everywhere on } \Omega,\ \text{for all } u_1\in\mathcal{U}_{t,t+\delta}. \tag{A.28}$$

Let $u\in\mathcal{U}_{t,T}$ be arbitrarily given and decomposed into $u_1 = u|_{[t,t+\delta]}\in\mathcal{U}_{t,t+\delta}$ and $u_2 = u|_{(t+\delta,T]}\in\mathcal{U}_{t+\delta,T}$. Then, from (A.26), (A.22)(i), (A.28), and Lemmas 2.4 and 2.3, we have

$$W_\delta(t,x) \ge G_{t,t+\delta}^{t,x;u_1}\left[W\left(t+\delta, X_{t+\delta}^{t,x;u_1}\right)\right] \ge G_{t,t+\delta}^{t,x;u_1}\left[W\left(t+\delta,\left[X_{t+\delta}^{t,x;u_1}\right]\right) - C\varepsilon - C\varepsilon^{\frac12}\right] - \varepsilon$$

$$\ge G_{t,t+\delta}^{t,x;u_1}\left[W\left(t+\delta,\left[X_{t+\delta}^{t,x;u_1}\right]\right)\right] - C\varepsilon - C'\varepsilon^{\frac12}$$

$$= G_{t,t+\delta}^{t,x;u_1}\left[\sum_{i\ge1}\mathbf{1}_{\{X_{t+\delta}^{t,x;u_1}\in O_i\}} W(t+\delta,y_i)\right] - C\varepsilon - C'\varepsilon^{\frac12},\quad P\text{-a.s.} \tag{A.29}$$

Furthermore, from (A.27), (A.22)(ii), (A.28), and Lemmas 2.4 and 2.3,

$$\begin{aligned}
W_\delta(t,x) &\geq G^{t,x;u_1}_{t,t+\delta}\left[\sum_{i\geq 1}\mathbf{1}_{\left\{X^{t,x;u_1}_{t+\delta}\in O_i\right\}}J\left(t+\delta,y_i;u_2\right)\right]-C\varepsilon-C'\varepsilon^{\frac12}\\
&= G^{t,x;u_1}_{t,t+\delta}\left[J\left(t+\delta,\left[X^{t,x;u_1}_{t+\delta}\right];u_2\right)\right]-C\varepsilon-C'\varepsilon^{\frac12}\\
&\geq G^{t,x;u_1}_{t,t+\delta}\left[J\left(t+\delta,X^{t,x;u_1}_{t+\delta};u_2\right)-C''\varepsilon-C''\varepsilon^{\frac12}\right]-C\varepsilon-C'\varepsilon^{\frac12}\\
&\geq G^{t,x;u_1}_{t,t+\delta}\left[J\left(t+\delta,X^{t,x;u_1}_{t+\delta};u_2\right)\right]-C\varepsilon-C'\varepsilon^{\frac12}\\
&= G^{t,x;u}_{t,t+\delta}\left[Y^{t,x,u}_{t+\delta}\right]-C\varepsilon-C'\varepsilon^{\frac12}\\
&= Y^{t,x;u}_t-C\varepsilon-C'\varepsilon^{\frac12},\ P\text{-a.s., for any } u\in\mathcal{U}_{t,T},
\end{aligned}\tag{A.30}$$

where the constants C, C', C'' may vary from lines to lines. Consequently,

$$W_\delta(t,x)\geq \operatorname*{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u)-C\varepsilon-C'\varepsilon^{\frac12}=W(t,x)-C\varepsilon-C'\varepsilon^{\frac12},\ P\text{-a.s.}\tag{A.31}$$

Finally, letting $\varepsilon\downarrow 0$ we get $W_\delta(t,x)\geq W(t,x)$. The proof is complete. □

Remark A.12.

(i) For any $u\in\mathcal{U}_{t,t+\delta}$,

$$W(t,x)\left(=W_\delta(t,x)\right)\geq G^{t,x;u}_{t,t+\delta}\left[W\left(t+\delta,X^{t,x;u}_{t+\delta}\right)\right],\quad P\text{-a.s.}\tag{A.32}$$

(ii) From the inequality (A.21), for all $(t,x)\in[0,T]\times\mathbb{R}^n$, $\delta\in(0,T-t]$ and $\varepsilon>0$, the following holds: there exists some $u^\varepsilon(\cdot)\in\mathcal{U}_{t,t+\delta}$ such that

$$W(t,x)\left(=W_\delta(t,x)\right)\leq G^{t,x;u^\varepsilon}_{t,t+\delta}\left[W\left(t+\delta,X^{t,x;u^\varepsilon}_{t+\delta}\right)\right]+C\varepsilon,\ P\text{-a.s.}\tag{A.33}$$

(iii) Recall that the value function W is deterministic. Then, with $\delta=T-t$ and taking the expectation on both sides of (A.32) and (A.33) we can get that

$$W(t,x)=\sup_{u\in\mathcal{U}_{t,T}}E[J(t,x;u)].$$

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