

A CORRECTOR FOR A WAVE PROBLEM WITH PERIODIC COEFFICIENTS IN A 1D BOUNDED DOMAIN

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Abstract. We consider a wave problem posed in a bounded open interval of \mathbb{R} , where the coefficients, the initial conditions and the right-hand side are highly oscillating, periodic in the space variable and almost periodic in the time one. Our purpose is to find not only the corresponding limit equation but a corrector, *i.e.* a strong approximation in the H^1 topology, which for the wave equation is known to be non-local. In a previous paper we have studied this problem in the whole \mathbb{R}^N , here we consider the case of a bounded domain in dimension one. Thus the novelty in this paper is the analysis of the boundary conditions.

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1. INTRODUCTION

The homogenization of a wave problem with oscillating coefficients in a bounded open set $\Omega \subset \mathbb{R}^N$ such as

$$\begin{cases} \partial_t(\rho_\varepsilon(x)\partial_t u_\varepsilon) - \operatorname{div}_x(A_\varepsilon(x)\nabla_x u_\varepsilon) = f_\varepsilon & \text{in } (0, T) \times \Omega \\ u_\varepsilon = 0 & \text{on } (0, T) \times \partial\Omega \\ u_\varepsilon|_{t=0} = u_\varepsilon^0, \partial_t u_\varepsilon|_{t=0} = \vartheta_\varepsilon & \text{in } \Omega, \end{cases} \quad (1.1)$$

has been studied in several papers ([5, 9, 11]). Assuming the coefficients $\rho_\varepsilon, A_\varepsilon$ uniformly elliptic and bounded, the right-hand side converging strongly in $L^1(0, T; L^2(\Omega))$ to some function f and the initial data $u_\varepsilon^0, \vartheta_\varepsilon$ converging weakly in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively to some functions u^0, ϑ , it is known that the solution u_ε of (1.1) converges in $L^\infty(0, T; H_0^1(\Omega))$ weak-* to the solution u of

$$\begin{cases} \partial_t(\rho(x)\partial_t u) - \operatorname{div}_x(A(x)\nabla_x u) = f & \text{in } (0, T) \times \Omega \\ u = 0 & \text{on } (0, T) \times \partial\Omega \\ u|_{t=0} = u^0, \partial_t u|_{t=0} = \vartheta & \text{in } \Omega. \end{cases}$$

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Here the coefficient ρ is the weak-* limit of ρ_ε in $L^\infty(\Omega)$ and the coefficient matrix A is the limit of A_ε in the sense of the elliptic homogenization ([13, 17, 18]). This homogenization result provides a weak approximation of the derivatives of u_ε . It is also interesting to get an approximation of these derivatives in the strong topology of $L^2((0, T) \times \Omega)$. This is called a corrector result in homogenization. Since A is the limit of A_ε in the sense of the elliptic homogenization one could expect that the elliptic corrector also provides a corrector for the wave problem. However it has been proved in [5] that this only holds true if the initial data are “well posed”.

A corrector result for problem (1.1) in the case of periodic coefficients and $\Omega = \mathbb{R}^N$ has been obtained in [6] and [10]. For the elliptic or parabolic problems the corrector in every point is just obtained from the value of the derivative of the limit function in such point. In particular initial and boundary conditions do not affect to the corrector. However for the wave equation the corrector is non-local. In general, its value in a point depends on the value of the limit function and the right-hand side in the whole domain (or more exactly in a certain cone of dependence) and of the initial conditions. This is due to the dispersion of the waves in an heterogeneous domain. Namely, it has been considered in [10] the wave problem

$$\begin{cases} \partial_t(\rho_\varepsilon(t, x)\partial_t u_\varepsilon) - \operatorname{div}_x(A_\varepsilon(t, x)\nabla_x u_\varepsilon) + B_\varepsilon(t, x) \cdot \nabla_{t,x} u_\varepsilon = f_\varepsilon & \text{in } (0, T) \times \mathbb{R}^N \\ u_\varepsilon|_{t=0} = u_\varepsilon^0, \quad \partial_t u_\varepsilon|_{t=0} = \vartheta_\varepsilon & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where the functions $\rho_\varepsilon, A_\varepsilon, B_\varepsilon, f_\varepsilon, u_\varepsilon^0$ and ϑ_ε have the following structure

$$\begin{aligned} \rho_\varepsilon(t, x) &= \rho^0\left(\frac{x}{\varepsilon}\right) + \varepsilon\rho^1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), & A_\varepsilon(t, x) &= A^0\left(\frac{x}{\varepsilon}\right) + \varepsilon A^1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), & B_\varepsilon(t, x) &= B\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \\ f_\varepsilon(t, x) &= f\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), & u_\varepsilon^0(x) &= u^0(x) + \varepsilon u^1\left(x, \frac{x}{\varepsilon}\right), & \vartheta_\varepsilon(x) &= \vartheta\left(x, \frac{x}{\varepsilon}\right), \end{aligned}$$

and they are periodic in $y = x/\varepsilon$ of period the unitary cube Y in \mathbb{R}^N and almost-periodic in $s = t/\varepsilon$. Then it has been proved the corrector result

$$u_\varepsilon(t, x) - u_0(t, x) - \varepsilon u_1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \rightarrow 0 \quad \text{in } H^1((0, T) \times \mathbb{R}^N), \tag{1.3}$$

where u_0, u_1 (and some u_2) solve

$$\begin{cases} \rho^0 \partial_{ss}^2 u_1 - \operatorname{div}_y(A^0(\nabla_x u_0 + \nabla_y u_1)) = 0 \\ \rho^0 \partial_{tt}^2 u_0 - \operatorname{div}_x(A^0 \nabla_x u_0) + 2\rho^0 \partial_{st}^2 u_1 - \operatorname{div}_x(A^0 \nabla_y u_1) - \operatorname{div}_y(A^0 \nabla_x u_1) \\ + \partial_s(\rho^1(\partial_t u_0 + \partial_s u_1)) - \operatorname{div}_y(A^1(\nabla_x u_0 + \nabla_y u_1)) \\ + B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) + \rho^0 \partial_{ss}^2 u_2 - \operatorname{div}_y(A^0 \nabla_y u_2) = f \\ u_1(t, x, s, y), \quad u_2(t, x, s, y) \text{ periodic of period } Y \text{ in } y \text{ and almost periodic in } s \\ u_0|_{t=0} = u^0, \quad (\partial_t u_0 + \partial_s u_1)|_{t=s=0} = \vartheta, \quad \partial_y u_1|_{t=s=0} = \partial_y u^1. \end{cases} \tag{1.4}$$

Here the second equation is just formal. The corresponding variational formulation consider test functions $\psi = \psi(t, x, s, y)$ satisfying the wave equation

$$\rho^0 \partial_{ss}^2 \psi - \operatorname{div}_y(A^0 \nabla_y \psi) = 0,$$

and then the terms containing u_2 disappear. Therefore, in this formulation we get a system of two equations for the two functions u_0, u_1 which appear in (1.3).

We observe that the second equation in (1.4) contains derivatives of u_1 not only in the microscopic variables (s, y) but also in the macroscopic ones (t, x) . This is completely different to the elliptic and parabolic cases and

as we mentioned above shows that the corrector is non-local for the wave equation. The behavior of u_1 in a point (t, x, s, y) does not only depend on the value of u_0 in (t, x) . As a consequence of this non-local behavior, it is proved in [10] that the presence of the first order term $B_\varepsilon(t, x) \cdot \nabla u_\varepsilon$ in (1.2) provides a non-local problem for the limit u_0 of u_ε . On the other hand, we observe that due to the presence of derivatives in t for u_1 in the second equation of (1.4) we have needed to introduce initial conditions for u_1 in $t = 0$.

An interesting question is if a result like (1.3) holds also true for a bounded domain Ω . In such case a formal calculus shows that u_0, u_1 must also satisfy (1.4) where now it would be necessary to introduce some boundary conditions for u_1 on $\partial\Omega$. These boundary conditions must depend on the boundary conditions imposed for u_ε in (1.2) and probably on the geometry of Ω . For this reason the problem in a bounded domain seems to be much more difficult than the problem in \mathbb{R}^N . In the present paper we analyze this question in the simplest case where Ω is a bounded one-dimensional interval (α, β) . Our formulation considers Dirichlet, Neumann and mixed boundary conditions. We show that (1.3) holds still true, but just for a subsequence because the behavior of the derivatives of u_ε depends on the relative position of the extremes of the interval (α, β) with respect to the periodicity cell. The main difficulty is to find the good boundary conditions for u_1 . They describe the shocks of the wave with the walls. Namely, we show that u_1 can be decomposed as

$$u_1(t, x, s, y) = \hat{u}_1(t, x, y) + \sum_{j \in J_1} \tilde{u}_j(t, x) \Phi_j(y) e^{i\lambda_j s} + \sum_{j \in J_2} (\tilde{u}_{j1}(t, x) \Phi_{j1}(y) + \tilde{u}_{j2}(t, x) \Phi_{j2}(y)) e^{i\lambda_j s},$$

where \hat{u}_1 is the classical elliptic corrector, the numbers λ_j are the (negative and positive) square roots of the positive eigenvalues μ_j of the following problem, with a^0 the above function A^0 which is now denoted by a lowercase letter to emphasize that it is a scalar function.

$$\begin{cases} -\frac{d}{dy} \left(a^0 \frac{d\Phi}{dy} \right) = \mu_j \rho^0 \Phi & \text{in } \mathbb{R} \\ \Phi, a^0 \frac{d\Phi}{dy} & \text{periodic of period } Y. \end{cases} \tag{1.5}$$

The sets J_1 and J_2 correspond to the index j such that μ_j has multiplicity one or two respectively and the functions $\Phi_j, \Phi_{j1}, \Phi_{j2}$ are a basis of the corresponding eigenfunctions spaces. The boundary conditions for u_1 are in fact given for the functions $\tilde{u}_{j1}, \tilde{u}_{j2}$ associated to the eigenvalues of multiplicity two while boundary conditions for the functions \tilde{u}_j are not needed. This different behavior corresponds to the fact that the perturbations associated to simple eigenvalues do not propagate in space while the corresponding to eigenvalues of multiplicity two travel in space.

We observe that (1.5) corresponds to the eigenvalue problem (2) in [6] for $l = 0$, where a corrector for problem (1.1) with $\Omega = \mathbb{R}^N$ and coefficients periodic of period ε is obtained by using the Bloch theory. This is not surprising because we are studying very related problems but in our case this is the only eigenvalue problem we need, while in [6] it is necessary to introduce the second parameter $l \in \mathbb{Z}^N$.

2. NOTATION

The functions will be assumed valued in the complex field \mathbb{C} , with i the imaginary unit. The conjugate of a vector $u \in \mathbb{C}^k$ is denoted by \bar{u} . The real part of a complex number z is denoted by $\Re e(z)$.

In order to write shorter expressions, we will only write the arguments of the functions when it is essential.

For $\alpha, \beta \in \mathbb{R}, \alpha < \beta$, we denote by I the interval $I = (\alpha, \beta) \subset \mathbb{R}$.

For $T > 0$, we denote by Q_T the open set $(0, T) \times I$.

We denote by O_ε an arbitrary sequence which converges to zero when ε tends to zero and which can change from line to line.

We denote by Y the unitary interval $(0, 1)$.

For functions defined in Y , we use the index \sharp to note periodicity with respect to Y (thus the functions are in fact defined in the whole of \mathbb{R}). For example $L^1_\sharp(Y)$ denotes the space of functions in $L^1_{loc}(\mathbb{R})$ which are periodic of period Y . The integral in Y of a function u in $L^1_\sharp(Y)$ will be denoted by $M_y(u)$ (mean value).

For functions defined in \mathbb{R} , the index \sharp denotes almost-periodicity. Namely, we will use the following spaces of almost-periodic functions.

1. We denote by $C_\sharp(\mathbb{R})$ the space of almost-periodic functions in the Bohr sense, *i.e.* the closure of the trigonometric polynomials u of the form

$$u(s) = \sum_{j=1}^m c_{p_j} e^{ip_j s}, \quad \forall s \in \mathbb{R}, \quad \text{with } m \in \mathbb{N}, p_j \in \mathbb{R}, c_{p_j} \in \mathbb{C}, 1 \leq j \leq m, \tag{2.1}$$

with respect to the uniform convergence topology.

2. We denote by $L^p_\sharp(\mathbb{R})$, $1 \leq p < +\infty$ the space of Besicovitch defined as the closure of the trigonometric polynomials as above with respect to the norm

$$\|u\|_{L^p_\sharp(\mathbb{R})} = \left(\lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R |u(s)|^p ds \right)^{\frac{1}{p}}.$$

Indeed, $\|\cdot\|_{L^p_\sharp(\mathbb{R})}$ is not a norm but a seminorm. In order to have a structure of normed space in $L^p_\sharp(\mathbb{R})$ it is necessary to work with a quotient space, *i.e.* to identify functions $u, v \in L^p_\sharp(\mathbb{R})$ such that $\|u - v\|_{L^p_\sharp(\mathbb{R})} = 0$. It is well known that every function u in the space $L^1_\sharp(\mathbb{R})$ has a mean value $M_s(u) \in \mathbb{C}$, which is defined by

$$\lim_{\varepsilon \rightarrow 0} \int_E u\left(\frac{s}{\varepsilon}\right) ds = M_s(u)|E|, \quad \forall E \subset \mathbb{R}, \text{ bounded and measurable.}$$

3. We denote by $H^k_\sharp(\mathbb{R})$, $k \in \mathbb{R}$ (in the paper we just use $k = -1, 0, 1$) by

$$H^k_\sharp(\mathbb{R}) = \left\{ \sum_{p \in \mathbb{R}} c_p e^{ips} : c_p \in \mathbb{C}, \forall p \in \mathbb{R}, \sum_{p \in \mathbb{R} \setminus \{0\}} |c_p|^2 |p|^{2k} < \infty \right\}.$$

It is a Hilbert space endowed with the norm

$$\left\| \sum_{p \in \mathbb{R}} c_p e^{ips} \right\|_{H^k_\sharp(\mathbb{R})} = \left(|c_0|^2 + \sum_{p \in \mathbb{R} \setminus \{0\}} |c_p|^2 |p|^{2k} \right)^{\frac{1}{2}}.$$

We remark that $H^0_\sharp(\mathbb{R}) = L^2_\sharp(\mathbb{R})$. Moreover, we have $H^k_\sharp(\mathbb{R})' = H^{-k}_\sharp(\mathbb{R})$.

In $H^k_\sharp(\mathbb{R})$ we define the derivative operator $\frac{d}{ds} : H^k_\sharp(\mathbb{R}) \rightarrow H^{k-1}_\sharp(\mathbb{R})$ just by

$$\frac{d}{ds} \left(\sum_{p \in \mathbb{R}} c_p e^{ips} \right) = \sum_{p \in \mathbb{R}} ip c_p e^{ips}.$$

We will also use the index \sharp for functions defined in $\mathbb{R} \times Y$ to denote periodicity with respect to Y and almost periodicity with respect to \mathbb{R} . Namely, we will work with the spaces $C_\sharp(\mathbb{R} \times Y)$, $L^p_\sharp(\mathbb{R} \times Y)$, $H^k_\sharp(\mathbb{R} \times Y)$, which are constructed as above but starting from functions of the form

$$u(s, y) = \sum_{j=1}^m c_{p_j n_j} e^{i(p_j s + 2\pi n_j y)}, \quad \forall (s, y) \in \mathbb{R} \times Y,$$

$$m \in \mathbb{N}, p_j \in \mathbb{R}, n_j \in \mathbb{Z}, c_{p_j n_j} \in \mathbb{C}, \quad 1 \leq j \leq m.$$

The mean value of a function u in $L^1_\sharp(\mathbb{R} \times Y)$ will be denoted by $M_{s,y}(u) = M_s(M_y(u)) = M_y(M_s(u))$.

Along the paper we will consider two fixed periodic real functions $\rho^0 \in L^\infty_\#(Y)$ and $a^0 \in L^\infty_\#(Y)$, such that

$$\inf_{y \in Y} \rho^0(y) > 0, \quad \inf_{y \in Y} a^0(y) > 0. \tag{2.2}$$

Then, we introduce the spaces \mathcal{W}^k , $k \in \mathbb{R}$ by

$$\mathcal{W}^k = \left\{ \sum_{p \in \mathbb{R}} c_p(y) e^{ips} \in H^k_\#(\mathbb{R} \times Y) : c_0 \equiv 0, \quad \rho^0 \partial_{ss}^2 w - \partial_y(a^0 \partial_y w) = 0 \right\}. \tag{2.3}$$

A spectral decomposition for the elements of \mathcal{W}^k can be obtained in the following way: we introduce $\mu_0 = 0 < \mu_1 < \mu_2 < \dots$ as the numbers (eigenvalues) $\mu_j \in [0, +\infty)$ satisfying that the space

$$W_j = \left\{ \Phi \in H^1_\#(Y) : -\frac{d}{dy} \left(a^0 \frac{d\Phi}{dy} \right) = \mu_j \rho^0 \Phi \text{ in } \mathbb{R} \right\} \tag{2.4}$$

does not reduce to the null space. For an eigenvalue μ_j , we denote

$$W_{-j} = W_j, \quad \lambda_j = \sqrt{\mu_j}, \quad \lambda_{-j} = -\sqrt{\mu_j}. \tag{2.5}$$

Then,

$$\mathcal{W}^k = \left\{ \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} c_j(y) e^{i\lambda_j s} : c_j \in W_j, \forall j \in \mathbb{Z}, \quad \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \|c_j\|_{L^2_\#(Y)}^2 |\lambda_j|^{2k} < \infty \right\}. \tag{2.6}$$

We remark that with our definitions, the space $H^1_\#(\mathbb{R} \times Y)$ is not contained in $L^2_\#(\mathbb{R} \times Y)$ since

$$\sum_{p \in \mathbb{R} \setminus \{0\}} \|c_p\|_{L^2_\#(Y)}^2 |p|^2 < \infty \not\equiv \sum_{p \in \mathbb{R} \setminus \{0\}} \|c_p\|_{L^2_\#(Y)}^2 < \infty,$$

but \mathcal{W}^1 is contained in \mathcal{W}^0 because

$$\sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \|c_j\|_{L^2_\#(Y)}^2 \leq \frac{1}{\lambda_1^2} \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \|c_j\|_{L^2_\#(Y)}^2 |\lambda_j|^2.$$

We also recall the following well known result: If Φ^1, Φ^2 are two solutions of the differential equation

$$-\frac{d}{dy} \left(a^0 \frac{d\Phi}{dy} \right) = \mu_j \rho^0 \Phi \text{ in } \mathbb{R} \tag{2.7}$$

then

$$a^0 \left(\Phi^1 \frac{d\Phi^2}{dy} - \Phi^2 \frac{d\Phi^1}{dy} \right) \text{ is constant in } \mathbb{R}, \tag{2.8}$$

where the constant is zero if and only if Φ^1, Φ^2 are linearly dependent.

Taking into account that the spaces W_j can be of dimension one or two, we denote

$$J_1 = \{j \in \mathbb{Z} \setminus \{0\} : \dim(W_j) = 1\}, \quad J_2 = \{j \in \mathbb{Z} \setminus \{0\} : \dim(W_j) = 2\},$$

and

$$J_1^+ = J_1 \cap \mathbb{Z}^+, \quad J_2^+ = J_2 \cap \mathbb{Z}^+.$$

For a function

$$u = \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} c_j(y)e^{i\lambda_j s} \in \mathcal{W}^0,$$

we define P_1u, P_2u by

$$P_1u = \sum_{j \in J_1} c_j(y)e^{i\lambda_j s}, \quad P_2u = \sum_{j \in J_2} c_j(y)e^{i\lambda_j s}.$$

Besides of the functions ρ^0, a^0 , we will also consider functions $\rho^1 \in L^\infty(Q_T; C_{\sharp}(\mathbb{R} \times Y)), a^1 \in L^\infty(Q_T; C_{\sharp}(\mathbb{R} \times Y)), \rho^1, a^1$ real, such that

$$\partial_t \rho^1, \partial_s \rho^1, \partial_t a^1, \partial_s a^1 \in L^\infty(Q_T; C_{\sharp}(\mathbb{R} \times Y)), \tag{2.9}$$

and a function $B \in L^\infty(Q_T; C_{\sharp}(\mathbb{R} \times Y))^2$.

With these functions, and $\varepsilon > 0$, we will define $\rho_\varepsilon, a_\varepsilon \in W^{1,\infty}(0, T; L^\infty(I)), B_\varepsilon \in L^\infty(Q_T)^2$ by

$$a_\varepsilon(t, x) = a^0\left(\frac{x}{\varepsilon}\right) + \varepsilon a^1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \tag{2.10}$$

$$\rho_\varepsilon(t, x) = \rho^0\left(\frac{x}{\varepsilon}\right) + \varepsilon \rho^1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \tag{2.11}$$

$$B_\varepsilon(t, x) = B\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \tag{2.12}$$

a.e. $(t, x) \in Q_T$.

We denote by $c_\alpha, d_\alpha, c_\beta, d_\beta$ complex constants such that

$$|c_\alpha| + |d_\alpha| > 0, \quad |c_\beta| + |d_\beta| > 0.$$

With these constants, we define the space V by

$$V = \{v \in H^1(I) : v(\alpha) = 0 \text{ if } c_\alpha = 0 \text{ and } v(\beta) = 0 \text{ if } c_\beta = 0\}.$$

The space V is endowed with the $H^1(I)$ norm.

3. MAIN RESULTS

For $u_\varepsilon^0, \vartheta_\varepsilon$ and f_ε bounded in $V, L^2(I)$ and $L^2(Q_T)$ respectively, we consider the wave problem

$$\begin{cases} \partial_t(\rho_\varepsilon \partial_t u_\varepsilon) - \partial_x(a_\varepsilon \partial_x u_\varepsilon) + B_\varepsilon \cdot \nabla_{t,x} u_\varepsilon = f_\varepsilon & \text{in } Q_T \\ (-c_\alpha a_\varepsilon \partial_x u_\varepsilon + d_\alpha u_\varepsilon)|_{x=\alpha} = 0, (c_\beta a_\varepsilon \partial_x u_\varepsilon + d_\beta u_\varepsilon)|_{x=\beta} = 0 & \text{in } (0, T) \\ u_\varepsilon|_{t=0} = u_\varepsilon^0, \partial_t u_\varepsilon|_{t=0} = \vartheta_\varepsilon & \text{in } I \\ u_\varepsilon \in L^\infty(0, T; V), \partial_t u_\varepsilon \in L^\infty(0, T; L^2(I)). \end{cases} \tag{3.1}$$

Our purpose in the present paper is to study the asymptotic behavior of u_ε when ε tends to zero. First we remark that since for $\varepsilon > 0$ small enough, $\rho_\varepsilon, a_\varepsilon$, are uniformly elliptic and bounded in $W^{1,\infty}(0, T; L^\infty(I)), B_\varepsilon$ is bounded in $L^\infty(Q_T)^2$, and $u_\varepsilon^0, \vartheta_\varepsilon, f_\varepsilon$ bounded in $V, L^2(I)$ and $L^2(Q_T)$ respectively, we immediately have

Theorem 3.1. *For $\varepsilon > 0$ small enough, problem (3.1) has a unique solution. Moreover, there exists $C > 0$ such that*

$$\|u_\varepsilon\|_{L^\infty(0, T; V)} + \|\partial_t u_\varepsilon\|_{L^\infty(0, T; L^2(I))} \leq C. \tag{3.2}$$

Theorem 3.1 implies that, at least for a subsequence, there exists the limit u_0 of u_ε in the weak-* topology of $L^\infty(0, T; V)$. In order to characterize this limit and to obtain a corrector result for problem (3.1), let us use the two-scale convergence theory.

Definition 3.2. We say that a sequence $v_\varepsilon \in L^2(I)$ two-scale converges to a function $v \in L^2(I; L^2_\#(Y))$ and we write $v_\varepsilon \xrightarrow{2\varepsilon} v$, if for every $\psi \in C_c(I; C_\#(Y))$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_I v_\varepsilon \psi \left(x, \frac{x}{\varepsilon} \right) dx = \int_I M_y(v\psi) dx. \tag{3.3}$$

Analogously, we say that a sequence $v_\varepsilon \in L^2(Q_T)$ two-scale converges to a function $v \in L^2(Q_T; L^2_\#(\mathbb{R} \times Y))$ and we write $v_\varepsilon \xrightarrow{2\varepsilon} v$, if for every $\psi \in C_c(Q_T; C_\#(\mathbb{R} \times Y))$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_{Q_T} v_\varepsilon \psi \left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) dt dx = \int_{Q_T} M_{s,y}(v\psi) dt dx. \tag{3.4}$$

The interest of the two-scale convergence follows from the classical compactness results which establishes the existence of a two-scale limit for bounded sequences in L^2 (see [1, 7, 8, 14–16]). As a consequence of these results and $u_\varepsilon^0, \vartheta_\varepsilon, f_\varepsilon$ bounded in $V, L^2(I)$ and $L^2(Q_T)$, there exist $u^0 \in V, u^1 \in L^2(I; H^1_\#(Y)), \vartheta \in L^2(I; L^2_\#(Y))$ and $f \in L^2(Q_T; L^2_\#(\mathbb{R} \times Y))$ such that, up to a subsequence, we have

$$u_\varepsilon^0 \rightharpoonup u^0 \text{ in } V, \quad \partial_x u_\varepsilon^0 \xrightarrow{2\varepsilon} \partial_x u^0 + \partial_y u^1, \quad \vartheta_\varepsilon \xrightarrow{2\varepsilon} \vartheta, \quad f_\varepsilon \xrightarrow{2\varepsilon} f. \tag{3.5}$$

The main result of the paper is given by Theorem 3.4 below which provides the limit function u_0 of the solution u_ε of (3.1) and the two-scale limit of the sequence $\nabla_{t,x} u_\varepsilon$. Related results have been obtained in [2–6, 9, 11, 12]. Here the novelty is the analysis of the boundary conditions which (because the waves travel and shock with the walls) influence the behavior of u_ε at the interior of Ω .

Although the problem is periodic, we will prove that the two-scale limit for the whole sequence $\nabla_{t,x} u_\varepsilon$ does not exist, instead we will need to consider a subsequence of ε such that there exist $\alpha^*, \beta^* \in \mathbb{R}/\mathbb{Z}$ satisfying

$$\frac{\alpha}{\varepsilon} \rightarrow \alpha^*, \quad \frac{\beta}{\varepsilon} \rightarrow \beta^* \text{ in } \mathbb{R}/\mathbb{Z}. \tag{3.6}$$

Associated to this subsequence, we define the following spaces which will appear in Theorem 3.4.

Definition 3.3. For a subsequence of ε such that (3.6) holds, we define the spaces \mathcal{V}_1 and \mathcal{V}_2 by

$$\mathcal{V}_1 = H^1_0(0, T; L^2(I; P_1(\mathcal{W}^1))) \tag{3.7}$$

$$\mathcal{V}_2 = \left\{ \psi \in H^1(Q_T; P_2(\mathcal{W}^1)) : \begin{array}{l} \psi|_{t=0} = \psi|_{t=T} = 0, \\ (a^0 \partial_y \psi)|_{y=\alpha^*} = 0 \text{ if } c_\alpha \neq 0, \\ \psi|_{x=\alpha, y=\alpha^*} = 0 \text{ if } c_\alpha = 0 \\ (a^0 \partial_y \psi)|_{y=\beta^*} = 0 \text{ if } c_\beta \neq 0 \\ \psi|_{x=\beta, y=\beta^*} = 0 \text{ if } c_\beta = 0. \end{array} \right\}. \tag{3.8}$$

Theorem 3.4. We consider a subsequence of ε , still denoted by ε such that (3.5) and (3.6) hold. Then, the sequence u_ε satisfies

$$u_\varepsilon \xrightarrow{*} u_0 \text{ in } L^\infty(0, T; V) \tag{3.9}$$

$$\nabla_{t,x} u_\varepsilon \xrightarrow{2\varepsilon} \nabla_{t,x} u_0 + \nabla_{s,y} u_1, \tag{3.10}$$

where u_0, u_1 are the unique solutions of the variational system

$$u_0 \in L^\infty(0, T; V), \quad \partial_t u_0 \in L^\infty(0, T; L^2(I)) \tag{3.11}$$

$$u_1 \in L^\infty(0, T; L^2(I; H_\#^1(\mathbb{R} \times Y))), \quad M_{s,y}(u_1) = 0 \text{ a.e. in } Q_T \tag{3.12}$$

$$u_0|_{t=0} = u^0, \quad (\partial_t u_0 + \partial_s u_1)|_{t=s=0} = \vartheta, \quad \partial_y u_1|_{t=s=0} = \partial_y u^1 \tag{3.13}$$

$$\left\{ \begin{aligned} & \int_{Q_T} M_{s,y} (-\rho^0(\partial_t u_0 \partial_t v_0 + \partial_s u_1 \partial_s v_1) + a^0(\partial_x u_0 + \partial_y u_1)(\partial_x v_0 + \partial_y v_1)) dt dx \\ & + \frac{d_\alpha}{c_\alpha} \int_{\{x=\alpha\}} u_0 v_0 dt + \frac{d_\beta}{c_\beta} \int_{\{x=\beta\}} u_0 v_0 dt \\ & + \int_{Q_T} M_{s,y} (B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) v_0) dt dx = \int_{Q_T} M_{s,y} (f v_0) dt dx \\ & \forall v_0 \in W_0^{1,1}(0, T; L^2(I)) \cap L^1(0, T; V), \quad \forall v_1 \in L^2(Q_T; H_\#^1(\mathbb{R} \times Y)) \end{aligned} \right. \tag{3.14}$$

$$\left\{ \begin{aligned} & \int_{Q_T} M_{s,y} (-2\rho^0 \partial_s u_1 \partial_t \psi - \rho^1(\partial_t u_0 + \partial_s u_1) \partial_s \psi + a^1(\partial_x u_0 + \partial_y u_1) \partial_y \psi) dt dx \\ & + \int_{Q_T} M_{s,y} (B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) \psi) dt dx = \int_{Q_T} M_{s,y} (f \psi) dt dx, \quad \forall \psi \in \mathcal{V}_1 \end{aligned} \right. \tag{3.15}$$

$$\left\{ \begin{aligned} & \int_{Q_T} M_{s,y} (-2\rho^0 \partial_s u_1 \partial_t \psi + a^0 \partial_y u_1 \partial_x \psi - a^0 \partial_{x,y}^2 \psi u_1) dt dx \\ & + \int_{Q_T} M_{s,y} (-\rho^1(\partial_t u_0 + \partial_s u_1) \partial_s \psi + a^1(\partial_x u_0 + \partial_y u_1) \partial_y \psi) dt dx \\ & + \int_{Q_T} M_{s,y} (B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) \psi) dt dx = \int_{Q_T} M_{s,y} (f \psi) dt dx, \quad \forall \psi \in \mathcal{V}_2. \end{aligned} \right. \tag{3.16}$$

Remark 3.5. In the case where $c_\alpha = 0$, the term

$$\frac{d_\alpha}{c_\alpha} \int_{\{x=\alpha\}} u_0 v_0 dt$$

which appears in (3.14) must be understood as zero. Observe that in this case, the functions u_0 and v_0 vanish at $x = \alpha$. Analogously, if $c_\beta = 0$, the term

$$\frac{d_\beta}{c_\beta} \int_{\{x=\beta\}} u_0 v_0 dt$$

is defined as zero.

Remark 3.6. In order to have uniqueness for u_1 we have taken it satisfying $M_{s,y}(u_1) = 0$. Clearly, (3.10) still holds adding to u_1 any function independent of the variables (s, y) .

Remark 3.7. Observe that the two-scale limit of the solution u_ε of (3.1) depends on the position of the extremes of I with respect to the periodic cell, which correspond to the points α^* and β^* given by (3.6).

Remark 3.8. In equations (3.15) and (3.16) the function ψ applies Q_T into the spaces $P_1(\mathcal{W}^1)$ and $P_2(\mathcal{W}^1)$ respectively. This shows that the behavior of elementary waves associated to simple or multiple eigenvalues of (2.4) is different.

Remark 3.9. From (3.14), with $v_0 = 0$, we deduce the equation

$$\rho^0 \partial_{ss}^2 u_1 - \partial_y(a^0(\partial_x u_0 + \partial_y u_1)) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}, \quad \text{a.e. in } Q_T. \tag{3.17}$$

A solution of this problem independent of s is given by the elliptic corrector \hat{u}_1 (see e.g. [1, 4, 14]) solution of

$$\begin{cases} \hat{u}_1 \in L^\infty(0, T; L^2(I; H_{\#}^1(Y))), & M_y(\hat{u}_1) = 0 \quad \text{a.e. } (t, x) \in Q_T \\ -\partial_y(a^0(\partial_x u_0 + \partial_y \hat{u}_1)) = 0 \quad \text{in } \mathbb{R}, & \text{a.e. in } Q_T, \end{cases} \tag{3.18}$$

or equivalently

$$\partial_y \hat{u}_1(t, x, y) = a_h^0 \left(\frac{1}{a^0(y)} - \frac{1}{a_h^0} \right) \partial_x u_0(t, x), \quad \text{a.e. } (t, x, y) \in Q_T, \times \mathbb{R}, \quad M_y(\hat{u}_1) = 0, \tag{3.19}$$

with a_h^0 the homogenized coefficient corresponding to a^0 , i.e.

$$a_h^0 = \frac{1}{M_y\left(\frac{1}{a^0}\right)}. \tag{3.20}$$

From (3.17) and (3.18), we deduce that u_1 can be decomposed as

$$u_1 = \hat{u}_1 + \tilde{u}_1, \tag{3.21}$$

with

$$\tilde{u}_1 \in L^\infty(0, T; L^2(I; \mathcal{W}^1)). \tag{3.22}$$

Remark 3.10. Taking $v_1 = 0$ in (3.14) and using (3.21), (3.19) and $M_s(\tilde{u}_1) = 0$, which is a consequence of (3.22), we deduce that u_0 satisfies

$$\begin{cases} M_y(\rho^0 \partial_{tt}^2 u_0 - a_h^0 \partial_{xx}^2 u_0 + B_h \cdot \nabla_{t,x} u_0 + M_{s,y}(B \cdot \nabla_{s,y} \tilde{u}_1)) = M_{s,y}(f) \quad \text{in } Q_T \\ (-c_\alpha a_h^0 \partial_x u_0 + d_\alpha u_0)|_{x=\alpha} = 0, \quad (c_\beta a_h^0 \partial_x u_0 + d_\beta u_0)|_{x=\beta} = 0 \quad \text{in } (0, T) \\ u_0|_{t=0} = u^0, \end{cases} \tag{3.23}$$

where, denoting by b_t, b_x the two components of B , the function B_h is given by

$$B_h = (b_{h,t}, b_{h,x}) \quad \text{with } b_{h,t} = M_{s,y}(b_t), \quad b_{h,x} = \frac{M_{s,y}\left(\frac{b_x}{a^0}\right)}{M_y\left(\frac{1}{a^0}\right)}. \tag{3.24}$$

If we assume that B satisfies

$$M_{s,y}(B \cdot \nabla_{s,y} v) = 0, \quad \forall v \in \mathcal{W}^1, \quad \text{a.e. in } Q_T, \tag{3.25}$$

then the fourth term in the PDE in (3.23) vanishes providing an equation for u_0 (the limit equation). This holds, for example, if B does not depend on s . In general (3.25) is not satisfied and then, since \tilde{u}_1 does not depend locally on u_0 , we get a non-local equation for u_0 . An example of this situation is given in [10], where $Q_T = (0, T) \times \mathbb{R}^N$.

Remark 3.11. Equation (3.23) for u_0 must be completed with initial conditions for u_0 and $\partial_t u_0$. The first one is contained in (3.13). The second one can be obtained as follows: Using (3.21), we can write the second equation in (3.13) as

$$(\partial_t u_0 + \partial_s \tilde{u}_1)|_{t=s=0} = \vartheta, \tag{3.26}$$

where $\partial_s \tilde{u}_1$ is a combination of eigenfunctions of problem (2.7) corresponding to non-vanishing eigenvalues. Thus $M_y(\rho^0 \partial_s \tilde{u}_1) = 0$ and therefore, multiplying (3.26) by ρ^0 and taking the mean value in y we get

$$\partial_t u_0|_{t=0} = \frac{M_y(\rho^0 \vartheta)}{M_y(\rho^0)}. \tag{3.27}$$

Equation (3.16) implicitly provides boundary conditions for the function \tilde{u}_1 given by (3.21). This is given by the following Proposition which shows that $P_2 \tilde{u}_1$ satisfies the boundary conditions imposed to ψ in (3.16).

Proposition 3.12. *In the conditions of Theorem 3.4, and defining \tilde{u}_1 by (3.21), we have*

$$\begin{aligned} (a^0 \partial_y P_2 \tilde{u}_1)|_{x=\alpha, y=\alpha^*} &= 0 \quad \text{if } c_\alpha \neq 0, & P_2 \tilde{u}_1|_{x=\alpha, y=\alpha^*} &= 0 \quad \text{if } c_\alpha = 0, \\ (a^0 \partial_y P_2 \tilde{u}_1)|_{x=\beta, y=\beta^*} &= 0 \quad \text{if } c_\beta \neq 0, & P_2 \tilde{u}_1|_{x=\beta, y=\beta^*} &= 0 \quad \text{if } c_\beta = 0. \end{aligned} \tag{3.28}$$

From Theorem 3.4 it is also possible to obtain a corrector result for problem (3.1). For this purpose, it is necessary to assume that in (3.5) the two-scale convergence holds in the following strong sense:

$$\begin{cases} \lim_{\varepsilon \rightarrow 0} \int_I \partial_x u_\varepsilon^0 \psi_\varepsilon \, dx = \int_I M_y((\partial_x u^0 + \partial_y u^1) \psi) \, dx \\ \lim_{\varepsilon \rightarrow 0} \int_I \vartheta_\varepsilon \psi_\varepsilon \, dx = \int_I M_y(\vartheta \psi) \, dx \\ \forall \psi_\varepsilon \text{ bounded in } L^2(I)^N, \psi \in L^2(I; L^2_\#(Y)), \Psi_\varepsilon \stackrel{2e}{\rightharpoonup} \psi, \end{cases} \tag{3.29}$$

which is possible to prove (see e.g. [1]) that it is equivalent to assume that the second and third assertions in (3.5) hold and

$$\lim_{\varepsilon \rightarrow 0} \int_I |\partial_x u_\varepsilon^0|^2 \, dx = \int_I M_y(|\partial_x u^0|^2 + |\partial_y u^1|^2) \, dx, \quad \lim_{\varepsilon \rightarrow 0} \int_I |\vartheta_\varepsilon|^2 \, dx = \int_I M_y(|\vartheta|^2) \, dx.$$

The corrector result is given by Theorem 3.13 below. Its proof is very similar to the one of Theorem 3.7 in [10] and then it will not be given here. The main idea consists in using Theorem 3.4 to pass to the limit in the energy identity for problem (3.1).

Theorem 3.13. *We assume that in Theorem 3.4, the sequences u_ε^0 and ϑ_ε satisfy (3.29), then, for every sequence $\Gamma_\varepsilon \in L^2(Q_T)^2$ which two-scale converges strongly to $\nabla_{s,y} u_1$, we have*

$$\lim_{\varepsilon \rightarrow 0} \int_{Q_T} |\nabla_{t,x}(u_\varepsilon - u_0) - \Gamma_\varepsilon|^2 \, dx dt = 0. \tag{3.30}$$

Remark 3.14. If the function $\nabla_{s,y} u_1$ is in $L^2(Q_T; C^0_\#(\mathbb{R} \times Y)^2)$, then we can take in (3.30)

$$\Gamma_\varepsilon = \nabla_{s,y} u_1 \left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right).$$

In this case Theorem 3.13 asserts

$$\nabla_{t,x} u_\varepsilon - \nabla_{t,x} u_0 - \nabla_{s,y} u_1 \left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \rightarrow 0 \quad \text{in } L^2(Q_T)^2,$$

which assuming further regularity in u_1 reads as

$$u_\varepsilon - u_0 - \varepsilon u_1 \left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \rightarrow 0 \text{ in } H^1(Q_T).$$

We finish this section with a simplified model of (3.1) where we can explicitly obtain the function u_1 which appears in Theorem 3.4 from u_0 , the initial data and the right-hand side. For this purpose, let us consider an orthonormal basis of the spaces W_j composed by real functions. Namely:

If $j \in J_1$, we consider a real eigenfunction $\Phi_j \in W_j$ such that

$$M_y(\rho^0 |\Phi_j|^2) = 1. \tag{3.31}$$

If $j \in J_2$, we consider two real eigenfunctions $\Phi_{j1}, \Phi_{j2} \in W_j$ such that

$$M_y(\rho^0 \Phi_{jk} \Phi_{jl}) = \delta_{kl}, \quad k, l \in \{1, 2\}. \tag{3.32}$$

Since $W_j = W_{-j}$ for every $j \in \mathbb{Z} \setminus \{0\}$, we can also assume

$$\Phi_j = \Phi_{-j}, \quad \forall j \in J_1, \quad \Phi_{j1} = \Phi_{-j1}, \quad \Phi_{j2} = \Phi_{-j2}, \quad \forall j \in J_2. \tag{3.33}$$

Moreover, we remark that these functions $\Phi_j, \Phi_{j1}, \Phi_{j2}$ with $j \in \mathbb{Z}^+$ are a basis of $L^2_{\#}(Y)/\mathbb{R}$ and $H^1_{\#}(Y)/\mathbb{R}$.

Taking into account that f belongs to $L^2(Q_T; C_{\#}(\mathbb{R} \times Y))$, and then to the space $L^2(Q_T; L^2_{\#}(\mathbb{R} \times Y))$, that u^1 belongs to $L^2(I; H^1_{\#}(Y))$, and can be chosen with zero mean value in y , and that ϑ belongs to $L^2(I; L^2_{\#}(Y))$, we can decompose these functions as

$$\left\{ \begin{aligned} \frac{f(t, x, s, y)}{\rho^0(y)} &= \sum_{j \in J_1} f_j(t, x) \Phi_j(y) e^{i\lambda_j s} \\ &+ \sum_{j \in J_2} (f_{j1}(t, x) \Phi_{j1}(y) + f_{j2}(t, x) \Phi_{j2}(y)) e^{i\lambda_j s} + f^*(t, x, s, y) \end{aligned} \right. \tag{3.34}$$

$$u^1(x, y) = \sum_{j \in J_1^+} u_j^1(x) \Phi_j(y) + \sum_{j \in J_2^+} (u_{j1}^1(x) \Phi_{j1}(y) + u_{j2}^1(x) \Phi_{j2}(y)) \tag{3.35}$$

$$\vartheta(x, y) = \sum_{j \in J_1^+} \vartheta_j(x) \Phi_j(y) + \sum_{j \in J_2^+} (\vartheta_{j1}(x) \Phi_{j1}(y) + \vartheta_{j2}(x) \Phi_{j2}(y)) + M_y(\vartheta), \tag{3.36}$$

with

$$M_{s,y}(\rho^0 f^* v) = 0 \text{ a.e. in } Q_T, \quad \forall v \in \mathcal{W}^0 \tag{3.37}$$

$$\sum_{j \in J_1} \|f_j\|_{L^2(Q_T)}^2 + \sum_{j \in J_2} (\|f_{j1}\|_{L^2(Q_T)}^2 + \|f_{j2}\|_{L^2(Q_T)}^2) < +\infty \tag{3.38}$$

$$\sum_{j \in J_1^+} \lambda_j^2 \|u_j^1\|_{L^2(I)}^2 + \sum_{j \in J_2^+} \lambda_j^2 (\|u_{j1}^1\|_{L^2(I)}^2 + \|u_{j2}^1\|_{L^2(I)}^2) < +\infty \tag{3.39}$$

$$\sum_{j \in J_1^+} \|\vartheta_j\|_{L^2(I)}^2 + \sum_{j \in J_2^+} (\|\vartheta_{j1}\|_{L^2(I)}^2 + \|\vartheta_{j2}\|_{L^2(I)}^2) < +\infty. \tag{3.40}$$

We also denote (recall that by (2.8) the definition of κ_j below does not depend on y)

$$m_j = M_y(a^0 \partial_y \Phi_j), \quad j \in J_1, \quad m_{jl} = M_y(a^0 \partial_y \Phi_{jl}), \quad j \in J_2, \quad l \in \{1, 2\} \tag{3.41}$$

$$\kappa_j = a^0(y) \left(\Phi_{j1}(y) \frac{d\Phi_{j2}}{dy}(y) - \Phi_{j2}(y) \frac{d\Phi_{j1}}{dy}(y) \right), \quad \text{a.e. } y \in \mathbb{R} \tag{3.42}$$

$$\mu_{jl}^{\alpha^*} = \Phi_{jl}(\alpha^*), \quad \mu_{jl}^{\beta^*} = \Phi_{jl}(\beta^*), \quad j \in J_2, \quad l \in \{1, 2\} \tag{3.43}$$

$$\nu_{jl}^{\alpha^*} = \left(a_0 \frac{d\Phi_{jl}}{dy} \right) (\alpha^*), \quad \nu_{jl}^{\beta^*} = \left(a_0 \frac{d\Phi_{jl}}{dy} \right) (\beta^*), \quad j \in J_2, \quad l \in \{1, 2\}. \tag{3.44}$$

Theorem 3.15. *We assume that the functions ρ^1, a^1, B in Theorem 3.4 vanish. Then the function u_1 is given by (3.21), where the functions \hat{u}_1 and \tilde{u}_1 can be written as*

$$\hat{u}_1(t, x, y) = \sum_{j \in J_1^+} \hat{u}_j(t, x) \Phi_j(y) + \sum_{j \in J_2^+} (\hat{u}_{j1}(t, x) \Phi_{j1}(y) + \hat{u}_{j2}(t, x) \Phi_{j2}(y)) \tag{3.45}$$

$$\tilde{u}_1(t, x, s, y) = \sum_{j \in J_1} \tilde{u}_j(t, x) \Phi_j(y) e^{i\lambda_j s} + \sum_{j \in J_2} (\tilde{u}_{j1}(t, x) \Phi_{j1}(y) + \tilde{u}_{j2}(t, x) \Phi_{j2}(y)) e^{i\lambda_j s}, \tag{3.46}$$

with

$$\sup_{t \in (0, T)} \left(\sum_{j \in J_1^+} \lambda_j^2 \|\hat{u}_j(t, x)\|_{L^2(I)}^2 + \sum_{j \in J_2^+} \lambda_j^2 (\|\hat{u}_{j1}(t, x)\|_{L^2(I)}^2 + \|\hat{u}_{j2}(t, x)\|_{L^2(I)}^2) \right) < \infty \tag{3.47}$$

$$\sup_{t \in (0, T)} \left(\sum_{j \in J_1} \lambda_j^2 \|\tilde{u}_j(t, x)\|_{L^2(I)}^2 + \sum_{j \in J_2} \lambda_j^2 (\|\tilde{u}_{j1}(t, x)\|_{L^2(I)}^2 + \|\tilde{u}_{j2}(t, x)\|_{L^2(I)}^2) \right) < \infty. \tag{3.48}$$

The coefficients \hat{u}_j and \hat{u}_{jl} are given by

$$m_j \partial_x u_0 + \lambda_j^2 \hat{u}_j = 0, \quad \text{a.e. in } I, \quad \forall t \in [0, T], \quad \forall j \in J_1^+ \tag{3.49}$$

$$m_{jl} \partial_x u_0 + \lambda_j^2 \hat{u}_{jl} = 0, \quad \text{a.e. in } I, \quad \forall t \in [0, T], \quad \forall j \in J_2^+, \quad \forall l \in \{1, 2\}. \tag{3.50}$$

For $j \in J_1^+$, the coefficients $\tilde{u}_j, \tilde{u}_{-j}$ are given by

$$\tilde{u}_j(t, x) = \frac{1}{2} \left(u_j^1(x) + \frac{m_j}{\lambda_j^2} \partial_x u_0(0, x) - \frac{\vartheta_j(x)}{\lambda_j} i \right) - \frac{i}{2\lambda_j} \int_0^t f_j(\tau, x) d\tau, \quad \text{a.e. in } Q_T \tag{3.51}$$

$$\tilde{u}_{-j}(t, x) = \frac{1}{2} \left(u_j^1(x) + \frac{m_j}{\lambda_j^2} \partial_x u_0(0, x) + \frac{\vartheta_j(x)}{\lambda_j} i \right) - \frac{i}{2\lambda_j} \int_0^t f_j(\tau, x) d\tau, \quad \text{a.e. in } Q_T. \tag{3.52}$$

The coefficients \tilde{u}_{jl} are the solutions of the system

$$\begin{cases} 2i\lambda_j \partial_t \tilde{u}_{j1} - \kappa_j \partial_x \tilde{u}_{j2} = f_{j1} \\ 2i\lambda_j \partial_t \tilde{u}_{j2} + \kappa_j \partial_x \tilde{u}_{j1} = f_{j2}, \end{cases} \tag{3.53}$$

with the boundary conditions

$$\begin{cases} \left(\nu_{j1}^{\alpha*} \tilde{u}_{j1} + \nu_{j2}^{\alpha*} \tilde{u}_{j2} \right)_{|x=\alpha} = 0 \text{ if } c_\alpha \neq 0, & \left(\mu_{j1}^{\alpha*} \tilde{u}_{j1} + \mu_{j2}^{\alpha*} \tilde{u}_{j2} \right)_{|x=\alpha} = 0 \text{ if } c_\alpha = 0 \\ \left(\nu_{j1}^{\beta*} \tilde{u}_{j1} + \nu_{j2}^{\beta*} \tilde{u}_{j2} \right)_{|x=\beta} = 0 \text{ if } c_\beta \neq 0, & \left(\mu_{j1}^{\beta*} \tilde{u}_{j1} + \mu_{j2}^{\beta*} \tilde{u}_{j2} \right)_{|x=\beta} = 0 \text{ if } c_\beta = 0, \end{cases} \quad (3.54)$$

and the initial conditions

$$\tilde{u}_{jl}|_{t=0} = \frac{1}{2} \left(u_{jl}^1 + \frac{m_{jl}}{\lambda_j^2} \partial_x u_0|_{t=0} - \frac{\vartheta_{jl}}{\lambda_j} i \right), \quad a.e. \text{ in } I, \forall j \in J_2^+, l = 1, 2 \quad (3.55)$$

$$\tilde{u}_{-j}|_{t=0} = \frac{1}{2} \left(u_{jl}^1 + \frac{m_{jl}}{\lambda_j^2} \partial_x u_0|_{t=0} + \frac{\vartheta_{jl}}{\lambda_j} i \right), \quad a.e. \text{ in } I, \forall j \in J_2^+, l = 1, 2. \quad (3.56)$$

Remark 3.16. The existence and uniqueness of solution for system (3.53) with the boundary conditions given by (3.54) and the initial conditions given by (3.55) and (3.56), follows from Theorem 3.4. It can also be proved directly. For this purpose we remark that it is easy to obtain an *a priori* estimate for this problem. Namely, assume to simplify $c_\alpha = c_\beta = 0$ (Dirichlet conditions), the other cases are similar. Then, multiplying the first equation in (3.53) by \tilde{u}_{j1} , the second equation by \tilde{u}_{j2} , integrating in (α, β) , dividing by $2i\lambda_j$ and adding the two equations, we get

$$\int_\alpha^\beta (\partial_t \tilde{u}_{j1} \bar{\tilde{u}}_{j1} + \partial_t \tilde{u}_{j2} \bar{\tilde{u}}_{j2}) dx + \frac{\kappa_{j1} i}{2\lambda_j} \int_\alpha^\beta (\partial_x \tilde{u}_{j2} \bar{\tilde{u}}_{j1} - \partial_x \tilde{u}_{j1} \bar{\tilde{u}}_{j2}) dx = -\frac{i}{2\lambda_j} \int_\alpha^\beta (f_{j1} \bar{\tilde{u}}_{j1} + f_{j2} \bar{\tilde{u}}_{j2}) dx. \quad (3.57)$$

In the second term an integration by parts shows

$$\int_\alpha^\beta (\partial_x \tilde{u}_{j2} \bar{\tilde{u}}_{j1} - \partial_x \tilde{u}_{j1} \bar{\tilde{u}}_{j2}) dx = 2 \int_\alpha^\beta \mathcal{R}e(\partial_x \tilde{u}_{j2} \bar{\tilde{u}}_{j1}) dx - \tilde{u}_{j1}(t, \beta) \bar{\tilde{u}}_{j2}(t, \beta) + \tilde{u}_{j1}(t, \alpha) \bar{\tilde{u}}_{j2}(t, \alpha). \quad (3.58)$$

Now, we observe that if $\mu_{j1}^{\alpha*} = 0$ then $\tilde{u}_{j2}(t, \alpha) = 0$ by (3.54) and therefore

$$\tilde{u}_{j1}(t, \alpha) \bar{\tilde{u}}_{j2}(t, \alpha) = 0 \in \mathbb{R}.$$

If $\mu_{j1}^{\alpha*} \neq 0$, then, multiplying the first equation in (3.54) by $\tilde{u}_{j2}(t, \alpha)$ we have

$$\tilde{u}_{j1}(t, \alpha) \bar{\tilde{u}}_{j2}(t, \alpha) = -\frac{\mu_{j2}^{\alpha*}}{\mu_{j1}^{\alpha*}} |\tilde{u}_{j2}(t, \alpha)|^2 \in \mathbb{R}.$$

So, we always have that the third term on the right-hand side of (3.58) is real. Analogously, we can also show that the second term is real and therefore (3.58) shows that the second term in (3.57) is purely imaginary. Thus, taking the real part in (3.57) we get

$$\frac{1}{2} \frac{d}{dt} \int_\alpha^\beta (|\tilde{u}_{j1}|^2 + |\tilde{u}_{j2}|^2) dx \leq \frac{1}{2\lambda_j} \int_\alpha^\beta (|f_{j1}|^2 + |f_{j2}|^2)^{\frac{1}{2}} (|\tilde{u}_{j1}|^2 + |\tilde{u}_{j2}|^2)^{\frac{1}{2}} dx.$$

This estimate allows us to use Gronwall's inequality to deduce the *a priori* estimate

$$\sum_{k=1}^2 \|\tilde{u}_{jk}\|_{L^\infty(0,T;L^2(\alpha,\beta))} \leq \frac{C}{\lambda_j} \sum_{k=1}^2 \|f_{jk}\|_{L^2(Q_T)} + C \|u_j^1\|_{L^2(\alpha,\beta)} + \frac{C}{\lambda_j} \|\theta_j^1\|_{L^2(\alpha,\beta)} + \frac{C}{\lambda_j^2} \|\partial_x u_0|_{t=0}\|_{L^2(\alpha,\beta)}.$$

Then, thanks to this *a priori* estimate and using for example a Galerkin approximation, is now easy to prove existence and uniqueness of solution for (3.53)–(3.56).

Remark 3.17. From (3.53), we can deduce that the functions $\tilde{u}_{j1}, \tilde{u}_{j2}$ are the solutions of the wave equations

$$4\lambda_j^2 \partial_{tt}^2 \tilde{u}_{j1} - \kappa_j^2 \partial_{xx}^2 \tilde{u}_{j1} = -2i\lambda_j \partial_t f_{j1} - \kappa_j \partial_x f_{j2} \tag{3.59}$$

$$4\lambda_j^2 \partial_{tt}^2 \tilde{u}_{j2} - \kappa_j^2 \partial_{xx}^2 \tilde{u}_{j2} = \kappa_j \partial_x f_{j1} - 2i\lambda_j \partial_t f_{j2}, \tag{3.60}$$

combined with some boundary and initial conditions which can also be deduced from (3.54) and (3.55), (3.56). The right-hand side of (3.59) and (3.60) has a lack of smoothness (something similar happens for the initial and the boundary conditions), the corresponding solution can be defined by transposition.

Remark 3.18. In Theorem 3.15 we can observe how the physical behavior of the elementary waves corresponding to frequencies λ_j is completely different in the case of simple or multiple eigenvalues. Namely, if λ_j belongs to J_1 the corresponding coefficient \tilde{u}_j of \tilde{u} satisfies the equation (see 3.51)

$$2\lambda_j i \partial_t \tilde{u}_j = f_j.$$

It corresponds to a perturbation which does not moves in space, such as it can be observed in (3.51) which provides a value for \tilde{u}_j in a point (t, x) depending only on the values of the data for the same spatial point x . Thus, the behavior of these wave does not depend on the boundary conditions.

When λ_j is an eigenvalue of multiplicity two, the equations corresponding to the coefficients $\tilde{u}_{j1}, \tilde{u}_{j2}$ are given by (3.59) and (3.60). They correspond to waves moving in the space at a velocity $\kappa_j/(2\lambda_j)$ through the characteristic

$$x = \pm \frac{\kappa_j}{2\lambda_j} t.$$

Thus, perturbations originated in a certain point of (α, β) arrive to the boundary in a finite time and there, they are reflected depending on the chosen boundary conditions.

4. PROOF OF THE RESULTS

This section is devoted to show the different results stated in the previous one.

In order to prove Theorem 3.4, we need the following lemma which is a particular case of Lemma 4.1 in [10] and then it is given without proof.

Lemma 4.1. *We consider a bounded sequence ζ_ε in $H^1(0, T; L^2(I))$ such that there exist $\zeta_0 \in H^1(0, T; L^2(I; L^2_\#(Y)))$ and $\zeta_1 \in L^2(0, T; L^2(I; H^1_\#(\mathbb{R}; L^2_\#(Y))))$ satisfying*

$$\zeta_\varepsilon \xrightarrow{2\varepsilon} \zeta_0, \quad \partial_t \zeta_\varepsilon \xrightarrow{2\varepsilon} \partial_t \zeta_0 + \partial_s \zeta_1. \tag{4.1}$$

Then,

$$\zeta_\varepsilon(t, \cdot) \xrightarrow{2\varepsilon} \zeta_0(t, \cdot, \cdot), \quad \forall t \in [0, T]. \tag{4.2}$$

Besides, for every $\varphi \in C^1(\overline{Q_T})$, $\varphi|_{t=0} = \varphi|_{t=T} = 0$, $h \in L^2_\#(Y)$ and $g \in W^{1,\infty}(\mathbb{R})$, periodic with period S , for some $S > 0$, and zero mean value, we have

$$\exists \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{Q_T} \zeta_\varepsilon(t, x) \varphi(t, x) g\left(\frac{t}{\varepsilon}\right) h\left(\frac{x}{\varepsilon}\right) dt dx = \int_{Q_T} M_{s,y}(\zeta_1(t, x, s, y) \varphi(t, x) g(s) h(y)) dt dx. \tag{4.3}$$

Proof of Theorem 3.4. We split the proof in several steps.

Step 1. From (3.2), we know that, at least for a subsequence, there exists $u_0 \in L^\infty(0, T; V)$, with $\partial_t u_0 \in L^\infty(0, T; L^2(I))$ such that

$$u_\varepsilon \xrightarrow{*} u_0 \text{ in } L^\infty(0, T; V), \quad \partial_t u_\varepsilon \xrightarrow{*} \partial_t u_0 \text{ in } L^\infty(0, T; L^2(I)). \tag{4.4}$$

Moreover, the compactness property of the two-scale convergence (see [1, 7, 8, 14–16]) implies the existence of $u_1 \in L^\infty(0, T; L^2(I; H^1_\#(\mathbb{R} \times Y)))$, $u_1 = u_1(t, x, s, y)$, such that

$$\nabla_{t,x} u_\varepsilon \xrightarrow{2e} \nabla_{t,x} u_0 + \nabla_{s,y} u_1. \tag{4.5}$$

The problem is to characterize these functions. For this purpose, as it is usual in the two-scale convergence method we take as test function in (3.1) the sequence

$$v_\varepsilon(t, x) = v_0(t, x) + \varepsilon v_1\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

with $v_0 \in C_c^\infty(0, T; V)$, $v_1 \in C_c^\infty(Q_T; C^\infty_\#(\mathbb{R} \times Y))$. Passing to the limit when ε tends to zero we easily deduce that u_0, u_1 satisfy (3.14). However this is not enough to characterize the function u_1 and then we need to use another type of test functions. For this purpose, we reason as in [10]. Namely, for $j \in \mathbb{Z} \setminus \{0\}$, $\Phi \in W_j$ and $\varphi \in C^\infty([0, T]; C^\infty(I))$, $\varphi|_{t=T} = 0$ we take as test function in (3.1) the sequence

$$v_\varepsilon(t, x) = \varphi(t, x) e^{i\lambda_j \frac{t}{\varepsilon}} \Phi\left(\frac{x}{\varepsilon}\right), \quad \text{a.e. } (t, x) \in Q_T.$$

Then, the same calculus which appears in Step 3 of the proof of Theorem 3.5 in [10] shows

$$\begin{aligned} & - \int_I M_y(\vartheta \varphi(0, x) \Phi) dx - 2 \int_{Q_T} M_{s,y}(\rho^0 \partial_s u_1 \partial_t \varphi e^{i\lambda_j s} \Phi) dt dx \\ & + \frac{i}{\lambda_j} \int_I M_y\left(a^0 \partial_y u_1 \frac{d\Phi}{dy} \varphi(0, x)\right) dx - i\lambda_j \int_{Q_T} M_{s,y}(\rho^1(\partial_t u_0 + \partial_s u_1) \varphi e^{i\lambda_j s} \Phi) dt dx \\ & + \int_{Q_T} M_{s,y}(a^0 \partial_y u_1 \partial_x \varphi e^{i\lambda_j s} \Phi) dt dx - \int_{Q_T} M_{s,y}\left(a^0 \frac{d\Phi}{dy} \partial_x \varphi u_1 e^{i\lambda_j s}\right) dt dx \\ & - \frac{i}{\lambda_j} \int_I M_y\left(a^0 \frac{d\Phi}{dy}\right) \partial_x \varphi(0, x) u_0(0, x) dx + \int_{Q_T} M_{s,y}\left(a^1(\partial_x u_0 + \partial_y u_1) \frac{d\Phi}{dy} \varphi e^{i\lambda_j s}\right) dt dx \\ & + \int_{Q_T} M_{s,y}(B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) \varphi e^{i\lambda_j s} \Phi) dt dx = \int_{Q_T} M_{s,y}(f \varphi e^{i\lambda_j s} \Phi) dt dx, \end{aligned} \tag{4.6}$$

which combined to (3.14) allows us to prove (see [10] for more details) that u_0, u_1 satisfy the variational equation

$$\left\{ \begin{aligned} & \int_{Q_T} M_{s,y}(-2\rho^0 \partial_s u_1 \partial_t \psi + a^0 \partial_y u_1 \partial_x \psi - a^0 \partial_{xy}^2 \psi u_1) dt dx \\ & + \int_{Q_T} M_{s,y}(-\rho^1(\partial_t u_0 + \partial_s u_1) \partial_s \psi + a^1(\partial_x u_0 + \partial_y u_1) \partial_y \psi) dt dx \\ & + \int_{Q_T} M_{s,y}(B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) \psi) dt dx = \int_{Q_T} M_{s,y}(f \psi) dt dx, \quad \forall \psi \in H_0^1(Q_T; \mathcal{W}^1) \end{aligned} \right. \tag{4.7}$$

and the initial conditions (3.13).

Contrarily to [10] where the wave equation was considered in the whole space \mathbb{R}^N , problem (3.1) is stated in a bounded interval with boundary conditions. Thus, besides of equations (3.14), (4.7) and the initial conditions (3.13) we need to obtain some boundary conditions for u_1 in order to determine the functions u_0, u_1 . This is the main novelty with respect to Theorem 3.5 in [10] and it is carried out in the next step.

Step 2. Let us extend (4.7) to more general functions ψ which do not necessarily vanish in $x = \alpha, x = \beta$. Since we consider functions ψ from Q_T into \mathcal{W}^1 , they are a linear combination of functions of the type $g_j(t, x, y) e^{i\lambda_j s}$,

with $g_j(t, x, \cdot) \in W_j$. By linearity it is enough to assume that ψ is just of the form $\psi = g_j(t, x, y)e^{i\lambda_j s}$ for a particular j . We distinguish the cases $j \in J_1$ or $j \in J_2$ and we take special functions g_j in every case.

• $j \in J_1$: By (3.14) (see Rem. 3.9), we can decompose u_1 as $\hat{u}_1 + \tilde{u}_1$, where \hat{u}_1 does not depend on s and $\tilde{u}_1 \in L^\infty(0, T; L^2(I; \mathcal{W}^1))$ can be written as

$$\tilde{u}_1(t, x, s, y) = \sum_{l \in \mathbb{Z} \setminus \{0\}} h_l(t, x, y) e^{i\lambda_l s}, \quad (4.8)$$

for some functions $h_l \in L^\infty(0, T; L^2(I; W_l))$ satisfying

$$\sup_{t \in (0, T)} \sum_{l \in \mathbb{Z} \setminus \{0\}} \lambda_l^2 \int_{I \times Y} |h_l|^2 dx dy < +\infty.$$

Then, taking in (4.7) $\psi = \psi(t, x, s, y) = \varphi(t, x)\Phi(y)e^{-i\lambda_j s}$, with $\varphi \in H_0^1(Q_T)$, $\Phi \in W_j$, $\Phi \not\equiv 0$ and using (4.8), and $M_s(e^{ips}) = 0$ if $p \neq 0$, $M_s(e^{ips}) = 1$ if $p = 0$, we have

$$\begin{aligned} \int_{Q_T} M_{s,y} (a^0 \partial_y u_1 \partial_x \psi - \partial_x (a^0 \partial_y \psi) u_1) dt dx &= \int_{Q_T} \partial_x \varphi M_y (a^0 \partial_y \hat{u}_1 \Phi - a^0 \partial_y \Phi \hat{u}_1) M_s(e^{i\lambda_j s}) dt dx \\ &+ \sum_{l \in \mathbb{Z} \setminus \{0\}} \int_{Q_T} \partial_x \varphi M_y (a^0 \partial_y h_l \Phi - a^0 \partial_y \Phi h_l) M_s(e^{i(\lambda_l - \lambda_j)s}) dt dx \\ &= \int_{Q_T} \partial_x \varphi M_y \left(a^0 \left(\partial_y h_j \Phi - h_j \frac{d\Phi}{dy} \right) \right) dt dx, \end{aligned}$$

but since $\dim(W_j) = 1$ the function h_j satisfies $h_j(t, x, y) = c_j(t, x)\Phi(y)$, with $c_j \in L^\infty(0, T; L^2(I))$ and thus

$$\int_{Q_T} \partial_x \varphi M_y \left(a^0 \left(\partial_y h_j \Phi - h_j \frac{d\Phi}{dy} \right) \right) dt dx = 0.$$

So, for such ψ , equation (4.7) reduces to

$$\begin{cases} \int_{Q_T} M_{s,y} (-2\rho^0 \partial_s u_1 \partial_t \psi - \rho^1 (\partial_t u_0 + \partial_s u_1) \partial_s \psi + a^1 (\partial_x u_0 + \partial_y u_1) \partial_y \psi) dt dx \\ + \int_{Q_T} M_{s,y} (B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) \psi) dt dx = \int_{Q_T} M_{s,y} (f\psi) dt dx. \end{cases}$$

By the density of $H_0^1(I)$ into $L^2(I)$, this equation holds true for $\psi(t, x, s, y) = \varphi(t, x)\Phi(y)e^{-i\lambda_j s}$, with $\varphi \in H_0^1(0, T; L^2(I))$ and not only in $H_0^1(Q_T)$. Reasoning then by linearity and density, we deduce (3.15).

• $j \in J_2$: Since $\dim(W_j) = 2$, we can consider a basis $\{\Phi, \Psi\}$ of W_j , such that

$$\begin{cases} a^0(\alpha^*) \frac{d\Phi}{dy}(\alpha^*) = 0, \quad a^0(\alpha^*) \frac{d\Psi}{dy}(\alpha^*) \neq 0 & \text{if } c_\alpha \neq 0 \\ \Phi(\alpha^*) = 0, \quad \Psi(\alpha^*) \neq 0 & \text{if } c_\alpha = 0. \end{cases} \quad (4.9)$$

For $\varphi \in H^1(Q_T)$ such that $\varphi|_{t=0} = \varphi|_{t=T} = \varphi|_{x=\beta} = 0$ we define

$$v_\varepsilon(t, x) = \varphi(t, x) \left(\Phi\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \Psi\left(\frac{x}{\varepsilon}\right) \right) e^{i\lambda_j \frac{t}{\varepsilon}},$$

with

$$\mu_\varepsilon = \begin{cases} \frac{d\Phi}{dy}\left(\frac{\alpha}{\varepsilon}\right) / \frac{d\Psi}{dy}\left(\frac{\alpha}{\varepsilon}\right) & \text{if } c_\alpha \neq 0 \\ \Phi\left(\frac{\alpha}{\varepsilon}\right) / \Psi\left(\frac{\alpha}{\varepsilon}\right) & \text{if } c_\alpha = 0. \end{cases}$$

Observe that thanks to $\Phi, \Psi, a^0 \frac{d\Phi}{dy}, a^0 \frac{d\Psi}{dy}$ continuous and (3.6), we have that μ_ε is well defined for ε small enough and satisfies

$$\mu_\varepsilon \rightarrow 0. \tag{4.10}$$

Taking v_ε as test function in (3.1), we get

$$\begin{aligned} & - \int_{Q_T} \rho_\varepsilon \partial_t u_\varepsilon \partial_t \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \Phi\left(\frac{x}{\varepsilon}\right) dt dx - \frac{i\lambda_j}{\varepsilon} \int_{Q_T} \rho_\varepsilon \partial_t u_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \left(\Phi\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \Psi\left(\frac{x}{\varepsilon}\right)\right) dt dx \\ & + \frac{d_\alpha}{c_\alpha} \int_{\{x=\alpha\}} u_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \left(\Phi\left(\frac{\alpha}{\varepsilon}\right) - \mu_\varepsilon \Psi\left(\frac{\alpha}{\varepsilon}\right)\right) dt + \int_{Q_T} a_\varepsilon \partial_x u_\varepsilon \partial_x \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \Phi\left(\frac{x}{\varepsilon}\right) dt dx \\ & + \frac{1}{\varepsilon} \int_{Q_T} a_\varepsilon \partial_x u_\varepsilon \left(\frac{d\Phi}{dy}\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \frac{d\Psi}{dy}\left(\frac{x}{\varepsilon}\right)\right) \varphi e^{i\lambda_j \frac{t}{\varepsilon}} dt dx + \int_{Q_T} B_\varepsilon \cdot \nabla_{t,x} u_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \Phi\left(\frac{x}{\varepsilon}\right) dt dx \\ & = \int_{Q_T} f_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \Phi\left(\frac{x}{\varepsilon}\right) dt dx + O_\varepsilon, \end{aligned} \tag{4.11}$$

where O_ε is due to (4.10) and where the third term is defined as zero in the case where $c_\alpha = 0$ because by definition of μ_ε , we have $\Phi(\frac{\alpha}{\varepsilon}) - \mu_\varepsilon \Psi(\frac{\alpha}{\varepsilon}) = 0$ if $c_\alpha = 0$.

Let us pass to the limit in each term of this equality. The calculus is similar to the one in the proof of Theorem 3.5 in [10] but simplified thanks to the absence of terms at $t = 0$. We prefer to explicit the calculus because in [10] the corresponding sequence v_ε has compact support and then there is not problem in the integrations by parts, while here we must use the special structure of the test functions v_ε defined above.

First term. Using the expression (2.11) of ρ_ε , (4.5) and $M_s(e^{i\lambda_j s}) = 0$, we get

$$- \int_{Q_T} \rho_\varepsilon \partial_t u_\varepsilon \partial_t \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \Phi\left(\frac{x}{\varepsilon}\right) dt dx = - \int_{Q_T} M_{s,y}(\rho^0 \partial_s u_1 \partial_t \varphi e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon. \tag{4.12}$$

Second term. Using the expression (2.11) of ρ_ε , integrating by parts with respect to t the term containing ρ^0 , and taking into account $\varphi|_{t=0} = 0$, (4.10) and (4.5), we have

$$\begin{aligned} - \frac{i\lambda_j}{\varepsilon} \int_{Q_T} \rho_\varepsilon \partial_t u_\varepsilon e^{i\lambda_j \frac{t}{\varepsilon}} \varphi \left(\Phi\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \Psi\left(\frac{x}{\varepsilon}\right)\right) dt dx &= \frac{i\lambda_j}{\varepsilon} \int_{Q_T} \rho^0\left(\frac{x}{\varepsilon}\right) u_\varepsilon \partial_t \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \left(\Phi\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \Psi\left(\frac{x}{\varepsilon}\right)\right) dt dx \\ & - \frac{\lambda_j^2}{\varepsilon^2} \int_{Q_T} \rho^0\left(\frac{x}{\varepsilon}\right) u_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \left(\Phi\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \Psi\left(\frac{x}{\varepsilon}\right)\right) dt dx \\ & - i\lambda_j \int_{Q_T} M_{s,y}(\rho^1(\partial_t u_0 + \partial_s u_1) \varphi e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon. \end{aligned}$$

In the first term on the right-hand side of this equality we use (4.3) with $\zeta_\varepsilon = u_\varepsilon$, which gives

$$\begin{aligned} \frac{i\lambda_j}{\varepsilon} \int_{Q_T} \rho^0\left(\frac{x}{\varepsilon}\right) u_\varepsilon \partial_t \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \left(\Phi\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \Psi\left(\frac{x}{\varepsilon}\right)\right) dt dx &= i\lambda_j \int_{Q_T} M_{s,y}(\rho^0 u_1 \partial_t \varphi e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon \\ &= \int_{Q_T} M_{s,y}(\rho^0 u_1 \partial_t \varphi \partial_s e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon = - \int_{Q_T} M_{s,y}(\rho^0 \partial_s u_1 \partial_t \varphi e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon. \end{aligned}$$

Thus, we have proved

$$\begin{aligned}
 -\frac{i\lambda_j}{\varepsilon} \int_{Q_T} \rho_\varepsilon \partial_t u_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \left(\Phi \left(\frac{x}{\varepsilon} \right) - \mu_\varepsilon \Psi \left(\frac{x}{\varepsilon} \right) \right) dt dx &= - \int_{Q_T} M_{s,y} (\rho^0 \partial_s u_1 \partial_t \varphi e^{i\lambda_j s} \Phi) dt dx \\
 &\quad - \frac{\lambda_j^2}{\varepsilon^2} \int_{Q_T} \rho^0 \left(\frac{x}{\varepsilon} \right) u_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \left(\Phi \left(\frac{x}{\varepsilon} \right) - \mu_\varepsilon \Psi \left(\frac{x}{\varepsilon} \right) \right) dt dx \\
 &\quad - i\lambda_j \int_{Q_T} M_{s,y} (\rho^1 (\partial_t u_0 + \partial_s u_1) \varphi e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon.
 \end{aligned} \tag{4.13}$$

Third term. Taking into account that thanks to the compact imbedding of the trace operator from $H^1(Q_T)$ into $L^2(\{x = \alpha\})$, u_ε converges strongly to u_0 in $L^2((0, T) \times \{\alpha\})$ and that $M_s(e^{i\lambda_j s}) = 0$, we get

$$\frac{d_\alpha}{c_\alpha} \int_{\{x=\alpha\}} u_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \left(\Phi \left(\frac{\alpha}{\varepsilon} \right) - \mu_\varepsilon \Psi \left(\frac{\alpha}{\varepsilon} \right) \right) dt = \frac{d_\alpha}{c_\alpha} \int_{\{x=\alpha\}} u_0 \varphi M_{y,s} (e^{i\lambda_j s}) \Phi(\alpha^*) dt + O_\varepsilon = O_\varepsilon. \tag{4.14}$$

Fourth term. From (4.5) and $M_s(e^{i\lambda_j s}) = 0$, we get

$$\int_{Q_T} a_\varepsilon \partial_x u_\varepsilon \partial_x \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \Phi \left(\frac{x}{\varepsilon} \right) dt dx = \int_{Q_T} M_{s,y} (a^0 \partial_y u_1 \partial_x \varphi e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon. \tag{4.15}$$

Fifth term. We reason similarly to the second term, *i.e.* we use expression (2.10) of a_ε , then (4.10), then an integration by parts with respect to x in the term containing a^0 and then (4.5). We get

$$\begin{aligned}
 &\frac{1}{\varepsilon} \int_{Q_T} a_\varepsilon \partial_x u_\varepsilon \left(\frac{d\Phi}{dy} \left(\frac{x}{\varepsilon} \right) - \mu_\varepsilon \frac{d\Psi}{dy} \left(\frac{x}{\varepsilon} \right) \right) \varphi e^{i\lambda_j \frac{t}{\varepsilon}} dt dx \\
 &= -\frac{1}{\varepsilon} \int_{\{x=\alpha\}} a^0 \left(\frac{\alpha}{\varepsilon} \right) u_\varepsilon \left(\frac{d\Phi}{dy} \left(\frac{\alpha}{\varepsilon} \right) - \mu_\varepsilon \frac{d\Psi}{dy} \left(\frac{\alpha}{\varepsilon} \right) \right) \varphi e^{i\lambda_j \frac{t}{\varepsilon}} dt \\
 &\quad - \frac{1}{\varepsilon} \int_{Q_T} a^0 \left(\frac{x}{\varepsilon} \right) u_\varepsilon \left(\frac{d\Phi}{dy} \left(\frac{x}{\varepsilon} \right) - \mu_\varepsilon \frac{d\Psi}{dy} \left(\frac{x}{\varepsilon} \right) \right) \partial_x \varphi e^{i\lambda_j \frac{t}{\varepsilon}} dt dx \\
 &\quad - \frac{1}{\varepsilon^2} \int_{Q_T} u_\varepsilon \frac{d}{dy} \left(a^0 \left(\frac{x}{\varepsilon} \right) \left(\frac{d\Phi}{dy} \left(\frac{x}{\varepsilon} \right) - \mu_\varepsilon \frac{d\Psi}{dy} \left(\frac{x}{\varepsilon} \right) \right) \right) \varphi e^{i\lambda_j \frac{t}{\varepsilon}} dt dx \\
 &\quad + \int_{Q_T} M_{s,y} \left(a^1 (\partial_x u_0 + \partial_y u_1) \frac{d\Phi}{dy} \varphi e^{i\lambda_j s} \right) dt dx + O_\varepsilon,
 \end{aligned} \tag{4.16}$$

where the first term on the right-hand side vanishes due to

$$u_\varepsilon|_{x=\alpha} = 0 \text{ if } c_\alpha = 0, \quad \frac{d\Phi}{dy} \left(\frac{\alpha}{\varepsilon} \right) - \mu_\varepsilon \frac{d\Psi}{dy} \left(\frac{\alpha}{\varepsilon} \right) = 0 \text{ if } c_\alpha \neq 0,$$

and the second term can be estimated using (4.3) with $\zeta_\varepsilon = u_\varepsilon$ and (4.10), which gives

$$-\frac{1}{\varepsilon} \int_{Q_T} a^0 \left(\frac{x}{\varepsilon} \right) u_\varepsilon \left(\frac{d\Phi}{dy} \left(\frac{x}{\varepsilon} \right) - \mu_\varepsilon \frac{d\Psi}{dy} \left(\frac{x}{\varepsilon} \right) \right) \partial_x \varphi e^{i\lambda_j \frac{t}{\varepsilon}} dt dx = - \int_{Q_T} M_{s,y} \left(a^0 \frac{d\Phi}{dy} \partial_x \varphi u_1 e^{i\lambda_j s} \right) dt dx + O_\varepsilon.$$

Thus, we have proved

$$\begin{aligned}
 &\frac{1}{\varepsilon} \int_{Q_T} a_\varepsilon \partial_x u_\varepsilon \left(\frac{d\Phi}{dy} \left(\frac{x}{\varepsilon} \right) - \mu_\varepsilon \frac{d\Psi}{dy} \left(\frac{x}{\varepsilon} \right) \right) \varphi e^{i\lambda_j \frac{t}{\varepsilon}} dt dx = - \int_{Q_T} M_{s,y} \left(a^0 \frac{d\Phi}{dy} \partial_x \varphi u_1 e^{i\lambda_j s} \right) dt dx \\
 &\quad - \frac{1}{\varepsilon^2} \int_{Q_T} u_\varepsilon \frac{d}{dy} \left(a^0 \left(\frac{x}{\varepsilon} \right) \left(\frac{d\Phi}{dy} \left(\frac{x}{\varepsilon} \right) - \mu_\varepsilon \frac{d\Psi}{dy} \left(\frac{x}{\varepsilon} \right) \right) \right) \varphi e^{i\lambda_j \frac{t}{\varepsilon}} dt dx \\
 &\quad + \int_{Q_T} M_{s,y} \left(a^1 (\partial_x u_0 + \partial_y u_1) \frac{d\Phi}{dy} \varphi e^{i\lambda_j s} \right) dt dx + O_\varepsilon.
 \end{aligned} \tag{4.17}$$

Sixth and seventh terms. Using the expression of B_ε given in (2.12), the two-scale convergence of f_ε to f and $M_s(e^{i\lambda_j s}) = 0$, we have

$$\int_{Q_T} B_\varepsilon \cdot \nabla_{t,x} u_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \Phi\left(\frac{x}{\varepsilon}\right) dt dx = \int_{Q_T} M_{s,y} (B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) \varphi e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon. \tag{4.18}$$

$$\int_{Q_T} f_\varepsilon \varphi e^{i\lambda_j \frac{t}{\varepsilon}} \Phi\left(\frac{x}{\varepsilon}\right) dt dx = \int_{Q_T} M_{s,y} (f \varphi e^{i\lambda_j s} \Phi) dt dx + O_\varepsilon. \tag{4.19}$$

Taking into account the estimates obtained for the different terms of (4.11) and

$$-\lambda_j^2 \rho^0\left(\frac{x}{\varepsilon}\right) \left(\Phi\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \Psi\left(\frac{x}{\varepsilon}\right)\right) - \frac{d}{dy} \left(a^0\left(\frac{x}{\varepsilon}\right) \left(\frac{d\Phi}{dy}\left(\frac{x}{\varepsilon}\right) - \mu_\varepsilon \frac{d\Psi}{dy}\left(\frac{x}{\varepsilon}\right)\right)\right) = 0 \text{ a.e. in } Q_T,$$

(which follows from Φ and Ψ in W_j) we have then proved that (4.7) holds for $\psi(t, x, s, y) = \varphi(t, x) e^{i\lambda_j s} \Phi(y)$ with $\varphi \in H^1(Q_T)$, $\varphi|_{t=0} = \varphi|_{t=T} = \varphi|_{x=\beta} = 0$, and $\Phi \in W_j$, satisfying

$$a^0(\alpha^*) \frac{d\Phi}{dy}(\alpha^*) = 0 \text{ if } c_\alpha \neq 0, \quad \Phi(\alpha^*) = 0 \text{ if } c_\alpha = 0.$$

A similar argument proves that (4.7) also holds for $\psi(t, x, s, y) = \varphi(t, x) e^{i\lambda_j s} \Phi(y)$ with $\Phi \in W_j$, such that

$$a^0(\beta^*) \frac{d\Phi}{dy}(\beta^*) = 0 \text{ if } c_\beta \neq 0, \quad \Phi(\beta^*) = 0 \text{ if } c_\beta = 0.$$

and $\varphi \in H^1(Q_T)$ such that $\varphi|_{t=0} = \varphi|_{t=T} = \varphi|_{x=\alpha} = 0$.

Reasoning by linearity and density, we have then proved (3.16).

Step 3. We have shown that u_0, u_1 satisfy (3.13), (3.14), (3.15) and (3.16). To finish the proof of Theorem 3.4, it remains to show that this system has a unique solution. Using the decomposition (3.21) of u_1 and taking into account that, thanks to $\tilde{u}_1 \in L^\infty(0, T; L^2(I; \mathcal{W}^1))$, to know $\partial_s u_1, \partial_y u_1$ at $t = 0$ and $s = 0$ is equivalent to know u_1 at $t = 0$, we get that the uniqueness is an immediate consequence of Theorem 4.2 below. The proof is not given here because it is similar to the one of Theorem Appendix B.1 in [10]. \square

Theorem 4.2. For $I = (\alpha, \beta) \subset \mathbb{R}$, four real functions $\rho^0 \in L^\infty_\#(Y)$, $a^0 \in L^\infty_\#(Y)$, $\rho^1 \in L^\infty(Q_T; C_\#(\mathbb{R} \times Y))$, $a^1 \in L^\infty(Q_T; C_\#(\mathbb{R} \times Y))$, satisfying (2.2) and (2.9), and a function $B \in L^\infty(Q_T; C_\#(\mathbb{R} \times Y))^2$. Then, for every $g \in L^1(0, T; L^2(I; L^2_\#(\mathbb{R} \times Y)))$ and every $w^0 \in L^2(I; \mathcal{W}^1)$, there exists a unique solution $w \in L^1(0, T; L^2(I; \mathcal{W}^1))$ of the variational problem

$$\begin{aligned} w|_{t=0} &= w^0 \\ \begin{cases} \int_{Q_T} M_{s,y} (-2\rho^0 \partial_s w \partial_t \psi - \rho^1 \partial_s w \partial_s \psi + a^1 \partial_y w \partial_y \psi + B \cdot \nabla_{s,y} w \psi) dt dx \\ = \int_{Q_T} M_{s,y} (g\psi) dt dx, \quad \forall \psi \in \mathcal{V}_1 \end{cases} \\ \begin{cases} \int_{Q_T} M_{s,y} (-2\rho^0 \partial_s w \partial_t \psi + a^0 \partial_y w \partial_x \psi - a^0 \partial_{xy}^2 \psi w - \rho^1 \partial_s w \partial_s \psi + a^1 \partial_y w \partial_y \psi) dt dx \\ + \int_{Q_T} M_{s,y} (B \cdot \nabla_{s,y} w \psi) dt dx = \int_{Q_T} M_{s,y} (g\psi) dt dx, \quad \forall \psi \in \mathcal{V}_2. \end{cases} \end{aligned}$$

Moreover, this solution w is in $L^\infty(0, T; L^2(I; \mathcal{W}^1))$ and satisfies

$$\|w\|_{L^\infty(0,T;L^2(I;\mathcal{W}^1))} \leq C \left(\|w^0\|_{L^2(I;\mathcal{W}^1)} + \|g\|_{L^1(0,T;L^2(I;L^2_\#(Y \times \mathbb{R}))} \right), \tag{4.20}$$

where the constant C only depends on the ellipticity constant of a^0 and ρ^0 , $\|B\|_{L^\infty(Q_T; C_\#(\mathbb{R} \times Y)^2)}$, $\|\partial_s \rho^1\|_{L^\infty(Q_T; L^\infty_\#(\mathbb{R} \times Y))}$, $\|\partial_s a^1\|_{L^\infty(Q_T; L^\infty_\#(\mathbb{R} \times Y))}$ and T .

Proof of Proposition 3.12. We take $j \in J_2$ and two independent eigenfunctions $\Phi_1, \Phi_2 \in W_j$ such that

$$a^0(\alpha^*) \frac{d\Phi_1}{dy}(\alpha^*) = 0 \text{ if } c_\alpha \neq 0, \quad \Phi_1(\alpha^*) = 0 \text{ if } c_\alpha = 0, \quad a^0 \left(\Phi_1 \frac{d\Phi_2}{dy} - \Phi_2 \frac{d\Phi_1}{dy} \right) = 1 \text{ in } \mathbb{R}.$$

Then, we define $h \in L^\infty(0, T; L^2(I; W_j))$ as the component of \tilde{u}_1 corresponding to $e^{i\lambda_j s}$, when we write it as a Fourier sum with respect to the basis $e^{i\lambda_l s}$, $l \in \mathbb{Z} \setminus \{0\}$, i.e. h is defined by

$$h = M_s(\tilde{u}_1 e^{-i\lambda_j s}), \quad \text{a.e. in } Q_T \times Y. \tag{4.21}$$

Since Φ_1, Φ_2 are a basis of W_j , the function h can be decomposed as

$$h(t, x, y) = h_1(t, x)\Phi_1(y) + h_2(t, x)\Phi_2(y), \quad \text{a.e. } (t, x, y) \in Q_T \times Y. \tag{4.22}$$

Now, for $g \in H^1(Q_T)$, $g|_{t=0} = g|_{t=T} = 0$, $g|_{x=\beta} = 0$, we define ψ by

$$\psi(t, x, s, y) = g(t, x)e^{-i\lambda_j s}\Phi_1(y),$$

and we observe that the decomposition (3.21) of u_1 , with \hat{u}_1 independent of s , and definitions (4.21), (4.22) of h, h_1 and h_2 , imply

$$\begin{aligned} & \int_{Q_T} M_{s,y} (a^0 \partial_y u_1 \partial_x \psi - a^0 \partial_{xy}^2 \psi u_1) \, dt dx \\ &= \int_{Q_T} M_y \left(a^0 \left(h_1 \frac{d\Phi_1}{dy} + h_2 \frac{d\Phi_2}{dy} \right) \partial_x g \Phi_1 - a^0 (h_1(\Phi_1 + h_2\Phi_2) \partial_x g \frac{d\Phi_1}{dy}) \right) \, dt dx = \int_{Q_T} h_2 \partial_x g \, dt dx. \end{aligned}$$

Therefore, using this function ψ as test function in (3.16), we deduce

$$\int_{Q_T} p \partial_t g \, dt dx + \int_{Q_T} h_2 \partial_x g \, dt dx = - \int_{Q_T} q g \, dx dt, \quad \forall g \in H^1(Q_T), \quad g|_{t=0} = g|_{t=T} = 0, \quad g|_{x=\beta} = 0, \tag{4.23}$$

with

$$\begin{aligned} p &= M_{s,y} \left(-2\rho^0 \partial_s u_1 e^{-i\lambda_j s} \Phi_1 \right), \\ q &= M_{s,y} \left(e^{-i\lambda_j s} \left(i\lambda_j \rho^1 (\partial_t u_0 + \partial_s u_1) \Phi_1 + a^1 (\partial_x u_0 + \partial_y u_1) \frac{d\Phi_1}{dy} \right) \right) \\ &\quad + M_{s,y} \left((B \cdot (\nabla_{t,x} u_0 + \nabla_{s,y} u_1) - f) e^{-i\lambda_j s} \Phi_1 \right). \end{aligned}$$

Since in (4.23), we have not imposed any boundary condition to g at $x = \alpha$, this is equivalent to

$$\partial_t p + \partial_x h_2 = q, \quad h_2|_{x=\alpha} = 0.$$

By (4.22), and the properties of Φ_1 , we then deduce

$$(a^0 \partial_y h)|_{x=\alpha, y=\alpha^*} = 0 \text{ if } c_\alpha \neq 0, \quad h|_{x=\alpha, y=\alpha^*} = 0 \text{ if } c_\alpha = 0.$$

Since this holds for every Fourier component of \tilde{u}_1 relative to a Fourier node λ_j , with $j \in J_2$ we get

$$(a^0 \partial_y P_2 \tilde{u}_1)|_{x=\alpha, y=\alpha^*} = 0 \text{ if } c_\alpha \neq 0, \quad P_2 \tilde{u}_1|_{x=\alpha, y=\alpha^*} = 0 \text{ if } c_\alpha = 0.$$

A similar reasoning shows

$$(a^0 \partial_y P_2 \tilde{u}_1)|_{x=\beta, y=\beta^*} = 0 \text{ if } c_\beta \neq 0, \quad P_2 \tilde{u}_1|_{x=\beta, y=\beta^*} = 0 \text{ if } c_\beta = 0. \quad \square$$

Proof of Theorem 3.15. Taking into account the decomposition $u_1 = \tilde{u}_1 + \hat{u}_1$ given in Remark 3.9, where \hat{u}_1 does not depend on s and using that the functions in \mathcal{W}^1 have mean value zero, equations (3.15) and (3.16), with ρ^1 , a^1 and B vanishing, are reduced to

$$\int_{Q_T} M_{s,y}(-2\rho^0 \partial_s \tilde{u}_1 \partial_t \psi) dt dx = \int_{Q_T} M_{s,y}(f\psi) dt dx, \quad \forall \psi \in \mathcal{V}_1 \tag{4.24}$$

$$\int_{Q_T} M_{s,y}(-2\rho^0 \partial_s \tilde{u}_1 \partial_t \psi + a^0 \partial_y \tilde{u}_1 \partial_x \psi - a^0 \partial_{xy}^2 \psi \tilde{u}_1) dt dx = \int_{Q_T} M_{s,y}(f\psi) dt dx, \quad \forall \psi \in \mathcal{V}_2. \tag{4.25}$$

On the other hand, using that \hat{u}_1 belongs to $L^\infty(0, T; L^2(I; H_{\#}^1(Y)))$, and that \tilde{u}_1 belongs to $L^\infty(0, T; L^2(I, \mathcal{W}^1))$ we deduce that \hat{u}_1 and \tilde{u}_1 can be decomposed as in (3.45), (3.46). The problem is now to determine the coefficients in the series defining \hat{u}_1 and \tilde{u}_1 .

Using as test function in (3.18) Φ_j if $j \in J_1^+$, or Φ_{jl} , $l \in \{1, 2\}$, if $j \in J_2^+$ we deduce that the coefficient \hat{u}_j are given by (3.49) and (3.50).

The determination of the coefficients \tilde{u}_j is more involved. We distinguish the cases $j \in J_1$, $j \in J_2$.

- Case $j \in J_1$. We take in (4.24), $\psi(t, x, s, y) = \varphi(t, x)\Phi_j(y)e^{-i\lambda_j s}$, with $\varphi \in C_c^\infty(Q_T)$. This gives

$$-2i\lambda_j \int_{Q_T} \tilde{u}_j \partial_t \varphi dt dx = \int_{Q_T} f_j \varphi dt dx, \quad \forall \varphi \in C_c^\infty(\Omega),$$

and then

$$2i\lambda_j \partial_t \tilde{u}_j = f_j \text{ in } Q_T. \tag{4.26}$$

To determine \tilde{u}_j from this equation we also need an initial condition. This can be obtained from (3.13). Namely, multiplying the second equation in (3.13) by $\rho^0 \Phi_j$, taking the mean value in y , and using (3.36), (3.46) and $M_y(\rho^0 \Phi_j) = 0$, we have

$$i\lambda_j(\tilde{u}_j - \tilde{u}_{-j})|_{t=0} = \vartheta_j, \quad \text{a.e. in } I. \tag{4.27}$$

Analogously, multiplying the third equation in (3.13) by $a^0 \partial_y \Phi_j$, taking the mean value in y and using $u_1 = \hat{u}_1 + \tilde{u}_1$ combined to (3.35), (3.45) and (3.46), we get

$$(\hat{u}_j + \tilde{u}_j + \tilde{u}_{-j})|_{t=0} = u_j^1, \quad \text{a.e. in } I.$$

or taking into account (3.49)

$$(\tilde{u}_j + \tilde{u}_{-j})|_{t=0} = u_j^1 + \frac{m_j}{\lambda_j^2} \partial_x u_0|_{t=0}, \quad \text{a.e. in } I. \tag{4.28}$$

From (4.27) and (4.28) we then get

$$\tilde{u}_j|_{t=0} = \frac{1}{2} \left(u_j^1 + \frac{m_j}{\lambda_j^2} \partial_x u_0|_{t=0} - \frac{\vartheta_j}{\lambda_j} i \right), \quad \text{a.e. in } I, \quad \forall j \in J_1^+. \tag{4.29}$$

$$\tilde{u}_{-j}|_{t=0} = \frac{1}{2} \left(u_j^1 + \frac{m_j}{\lambda_j^2} \partial_x u_0|_{t=0} + \frac{\vartheta_j}{\lambda_j} i \right), \quad \text{a.e. in } I, \quad \forall j \in J_1^+, \tag{4.30}$$

which combined to (4.26) proves (3.51) and (3.52).

- Case $j \in J_2$. As above, taking $\psi(t, x, s, y) = \varphi(t, x)\Phi_{jl}(y)e^{-i\lambda_j s}$, $l = 1, 2$, $\varphi \in C_c^\infty(Q_T)$ as test function in (4.25) we deduce (3.53).

Reasoning as in the case $j \in J_1$ we also deduce that (3.13) implies (3.55) and (3.56). On the other hand, we recall that by Proposition 3.12, (4.25) implies (3.54). □

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