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A PURITY THEOREM FOR THE WITT GROUP

BY MANUEL OJANGUREN AND IVAN PANIN

ABSTRACT. – Let $A$ be a regular local ring and $K$ its field of fractions. We denote by $W$ the Witt group functor that classifies quadratic spaces. We say that purity holds for $A$ if $W(A)$ is the intersection of all $W(A_p) \subset W(K)$, as $p$ runs over the height-one prime ideals of $A$. We prove purity for every regular local ring containing a field of characteristic $\neq 2$. The question of purity and of the injectivity of $W(A)$ into $W(K)$ for arbitrary regular local rings is still open. © Elsevier, Paris

RÉSUMÉ. – Soit $A$ un anneau local régulier et $K$ son corps des fractions. Soit $W$ le foncteur groupe de Witt qui classifie les espaces quadratiques. On dit que le théorème de pureté vaut pour $A$ si $W(A)$ est l’intersection de tous les $W(A_p) \subset W(K)$, où $p$ parcourt les idéaux premiers de hauteur égale à 1 de $A$. Nous démontrons le théorème de pureté pour tout anneau local régulier qui contient un corps de caractéristique $\neq 2$. La question de la pureté et de l’injectivité de $W(A)$ dans $W(K)$ pour un anneau local régulier arbitraire est encore ouverte. © Elsevier, Paris

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ANNALES SCIENTIFIQUES DE L’ÉCOLE NORMALE SUPÉRIEURE. – 0012-9593/99/01© Elsevier, Paris
1. Introduction

We briefly review the definitions of quadratic spaces and Witt groups. A very detailed exposition of these topics may be found in [9] and in [10].

Let \( X \) be a scheme such that 2 is invertible in \( \Gamma(\mathcal{O}_X) \). A quadratic space over \( X \) is a pair \( \mathbf{q} = (\mathcal{E}, q) \) consisting of a locally free coherent sheaf (we also say "vector bundle") \( \mathcal{E} \) and a symmetric isomorphism \( q : \mathcal{E} \to \mathcal{E}^* = \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \): this means that, after identifying \( \mathcal{E} \) with \( \mathcal{E}^* \) in the usual way, it satisfies \( q = q^* \).

An isometry \( \varphi : \mathbf{q} \to \mathbf{q}' \) is an isomorphism \( \varphi : \mathcal{E} \to \mathcal{E}' \) such that the square

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\varphi} & \mathcal{E}' \\
\downarrow q & & \downarrow q' \\
\mathcal{E}^* & \xleftarrow{\varphi^*} & \mathcal{E}'^*
\end{array}
\]

commutes.

The orthogonal sum of \( \mathbf{q} \) and \( \mathbf{q}' \) is the space \( \mathbf{q} \perp \mathbf{q}' = (\mathcal{E} \oplus \mathcal{E}', q \oplus q') \).

Let \( \mathbf{q} = (\mathcal{E}, q) \) be a quadratic space over \( X \) and \( \mathcal{F} \) a subsheaf of \( \mathcal{E} \). The orthogonal \( \mathcal{F}^\perp \) of \( \mathcal{F} \) is the kernel of \( i^* \circ q \), where \( i \) denotes the inclusion of \( \mathcal{F} \) into \( \mathcal{E} \).

A subbundle \( \mathcal{L} \) of \( \mathcal{E} \) is a sublagrangian of \( \mathbf{q} \) if \( \mathcal{L} \subseteq \mathcal{L}^\perp \), and it is a lagrangian if \( \mathcal{L} = \mathcal{L}^\perp \). Note that lagrangians and sublagrangians are subbundles, i.e. locally direct factors, not just subsheaves. A space \( \mathbf{q} = (\mathcal{E}, q) \) is said to be metabolic if it has a lagrangian.

Let \( \text{GW}(X) \) denote the Grothendieck group of quadratic spaces over \( X \) with respect to the orthogonal sum. Let \( M \) be the subgroup of \( \text{GW}(X) \) generated by metabolic spaces. The Witt group of \( X \) is the quotient \( \text{W}(X) = \text{GW}(X)/M \). If \( f : X \to Y \) is a map of schemes and \( \mathbf{q} = (\mathcal{E}, q) \) is space over \( Y \), the pair \( f^* \mathbf{q} = (f^* \mathcal{E}, f^* q) \) is a quadratic space over \( X \). It is easily seen that \( f^* \) respects orthogonal sums and maps metabolic spaces to metabolic spaces; thus \( f \) induces a group homomorphism \( \text{W}(f) : \text{W}(Y) \to \text{W}(X) \) and \( \text{W} \) turns out to be a contravariant functor from the category of schemes to the category of abelian groups.

If \( X = \text{Spec}(A) \) is affine, a quadratic space over \( X \) is the same as a pair \( (P, q) \) consisting of a finitely projective \( A \)-module \( P \) and an \( A \)-linear isomorphism \( q : P \to P^* \) such that \( q = q^* \). In this case a space \( (P, q) \) is metabolic if and only if it is isometric to a space of the form \( (L \oplus L^*, (0^1_1 0)) \).

For an affine scheme \( X = \text{Spec}(A) \) we denote \( \text{W}(X) \) by \( \text{W}(A) \).

Let now \( X \) be an integral scheme and \( K = k(X) \) its field of rational functions. By the functoriality of \( \text{W} \) there is a canonical map \( \text{W}(X) \to \text{W}(K) \) and, for every point \( x \in X \), a canonical map \( \text{W}(\mathcal{O}_{X,x}) \to \text{W}(K) \). We say that an element \( \xi \in \text{W}(K) \) is defined at \( x \) if \( \xi \) is in the image of \( \text{W}(\mathcal{O}_{X,x}) \). We say that an element \( \xi \in \text{W}(K) \) is unramified (over \( X \)) if it is defined at every height-one point \( x \in X \). We say that purity holds for \( X \) if every unramified element of \( \text{W}(K) \) belongs to the image of \( \text{W}(X) \) in \( \text{W}(K) \).

Purity is known to hold for every regular integral noetherian scheme of dimension at most two [3] and for every regular integral noetherian affine scheme of dimension 3 [15].

The main result of this paper is the following purity theorem (§7).

**Theorem A.** Purity holds for any regular local ring containing a field of characteristic different from 2.
Theorem A will be deduced from the same statement for essentially smooth local algebras over a field, using a well-known result of Dorin Popescu. Further, using essentially the same methods, we prove (§8)

**Theorem B.** Let $A$ be a regular local ring containing a field of characteristic $\neq 2$ and $K$ the field of fractions of $A$. Let $f$ be a regular parameter of $A$. The natural homomorphism $W(A) \rightarrow W(K)$ is injective.

From this, using a result of Piotr Jaworski for 2-dimensional regular rings, we deduce (§9)

**Theorem C.** Let $A$ be a regular local ring containing a field of characteristic $\neq 2$ and $f$ a regular parameter of $A$. There is a short exact sequence

$$0 \rightarrow W(A) \rightarrow W(A_f) \rightarrow W(A/\mathfrak{p}) \rightarrow 0,$$

where $\delta$ is induced by the second residue homomorphism $\partial_f$ at the height-one prime $\mathfrak{p} = Af$.

Let $A((t)) = A[[t]]_t$ be the ring of formal Laurent series over $A$. As a special case of Theorem C we can formulate (§9):

**Theorem D.** Let $A$ be a regular local ring containing a field of characteristic $\neq 2$. There exists a split short exact sequence

$$0 \rightarrow W(A) \rightarrow W(A((t))) \rightarrow W(A) \rightarrow 0.$$

**Remark.** The method used for proving purity for an essentially smooth local $k$-algebra $A$ also yields a new proof of the injectivity of $W(A)$ into the Witt group $W(K)$ of its field of fractions. Since this result is well-known and not very difficult (see for instance [14]), we use it whenever it is convenient, without proving it again.

Our proof has been inspired by Vladimir Voevodsky's work [20] and makes essential use of a non-degenerate trace form for finite extensions of smooth algebras, which was discovered by Leonhard Euler in a special case. We recall its definition and main properties in §§ 2 and 3.

### 2. The Euler trace

Let $k$ be any field and $A \hookrightarrow B$ a finite extension of smooth, purely $d$-dimensional $k$-algebras. Let $\Omega_A$ and $\Omega_B$ be the modules of Kähler differentials of $A$ and $B$ over $k$ and let $\Omega_{B/A}$ be the module of relative differentials of $B$ over $A$. Let $\omega_A = \bigwedge^d \Omega_A$, $\omega_B = \bigwedge^d \Omega_B$.

**Proposition 2.1.** There exists an isomorphism of $B$-modules

$$\omega_B \cong \text{Hom}_A(B, \omega_A).$$

**Proof.** Let $R$ be the polynomial algebra $A[X_1, \ldots, X_n]$ and $\rho : R \rightarrow B$ a surjective homomorphism of $A$-algebras. Let $I = \ker(\rho)$. Since $B$ is a local complete intersection over $A$, by Lemma 4.4 of [18] there exists an isomorphism of $B$-modules

$$(*) \quad \text{Hom}_A(B, A) \simeq \bigwedge^n (\text{Hom}_B(I/I^2, B)).$$
On the other hand, from the canonical exact sequence of projective $B$-modules (see [1], VII, Theorem 5.8)

$$0 \rightarrow I/I^2 \rightarrow B \otimes_R \Omega_R \rightarrow \Omega_B \rightarrow 0,$$

we deduce, taking maximal exterior powers, that

$$(\dagger) \quad \omega_B \otimes_B \bigwedge^n (I/I^2) \simeq B \otimes_A \omega_A .$$

From $(\dagger)$ we get, using the fact that $I/I^2$ is a finitely generated projective $B$-module,

$$\omega_B \simeq (B \otimes_A \omega_A) \otimes_B \text{Hom}_B \left( \bigwedge^n (I/I^2), B \right) \simeq (B \otimes_A \omega_A) \otimes_B \bigwedge^n \text{Hom}_B(I/I^2, B)$$

and then, from $(\ast)$,

$$(B \otimes_A \omega_A) \otimes_B \bigwedge^n \text{Hom}_B(I/I^2, B) \simeq \omega_A \otimes_A \text{Hom}_A(B, A) \simeq \text{Hom}_A(B, \omega_A) . \quad \square$$

**Corollary 2.2.** If $\omega_A$ and $\omega_B$ are trivial, there exists an isomorphism of $B$-modules

$$\lambda : B \sim \rightarrow \text{Hom}_A(B, A) . \quad \square$$

The isomorphism $\lambda$ induces an $A$-linear map

$$\epsilon : B \rightarrow A$$

defined by $\epsilon(x) = \lambda(1)(x)$. We call it an *Euler trace* because Euler discovered a special case of it (see [5] and also [17], Chap. III). Conversely, from $\epsilon$ we get back $\lambda$ as $\lambda(x)(y) = \epsilon(xy)$.

In the next proposition we record, without proof, a few obvious properties of $\epsilon$ and $\lambda$.

**Proposition 2.3.** Let $B$ be a finite locally free $A$-algebra and $\epsilon : B \rightarrow A$ an $A$-linear map such that the bilinear map $\lambda : B \rightarrow \text{Hom}_A(B, A)$ given by $\lambda(x)(y) = \epsilon(xy)$ is an isomorphism.

Then, for every $A \rightarrow A'$, we have an $A'$-linear map $\epsilon' = \epsilon \otimes_A A' : B' = B \otimes_A A' \rightarrow A'$ such that the associated $\lambda' : B' \rightarrow \text{Hom}_{A'}(B', A')$ is an isomorphism of $B'$-modules. If $B = B_1 \times B_2$, $\lambda$ decomposes as $\lambda_1 \times \lambda_2$, where $\lambda_i : B_i \rightarrow \text{Hom}_A(B_i, A)$ is the map associated to $\epsilon|_{B_i}$. In particular, if $B = B_1 \times A$, the map $\lambda_2 : A \rightarrow A$ is multiplication by a unit of $A$.

### 3. Traces and quadratic spaces

Let $A \hookrightarrow B$ be a finite flat extension of commutative rings. Let $\epsilon : B \rightarrow A$ be an $A$-linear map such that the associated $\lambda : B \rightarrow \text{Hom}_A(B, A)$ is an isomorphism. To every quadratic space $q = (P, q)$ over $B$ we associate the bilinear form $\text{Tr}^\epsilon(q) = (P_A, \epsilon \circ q)$, where $P_A$ denotes $P$ considered as an $A$-module. This bilinear form is in fact a quadratic space and it is easy to check (see [10], I, §7) that $\text{Tr}$ has the following properties:
(1) $\text{Tr}^e(q \perp q') = \text{Tr}^e(q) \perp \text{Tr}^e(q').$

(2) If $q$ is hyperbolic, $\text{Tr}^e(q)$ is hyperbolic.

(3) For any homomorphism of commutative rings $A \to A'$ we have

$$\text{Tr}^e(q \otimes_A A') = \text{Tr}^e(q) \otimes_A A',$$

where $e' = e \otimes_A A'$.

(4) If, as at the end of §2, $B = B_1 \times B_2$ and $e_i = e|_{B_i}$,

$$\text{Tr}^e(q) = \text{Tr}^{e_1}(q_1) \perp \text{Tr}^{e_2}(q_2),$$

where $q_i = q \otimes_B B_i$.

(5) If, as in (4), $B = B_1 \times B_2$ but $B_2 = A$, then $e_2$ is multiplication by a unit $u \in A^*$

and thus, for any quadratic space $q$,

$$\text{Tr}^{e_2}(q_2) = u \cdot q_2.$$

If $f : A \to A'$ is a ring homomorphism and $B' = B \otimes_A A'$, clearly $B' = B_1' \times B_2'$ with $B_2' = A'$, and $e_2'$ is multiplication by $f(u)$.

(6) Suppose that the map $f : A \to A'$ considered in (5) has a section $s : A' \to A$ and

that $B \otimes_A A' = B' = B_1' \times B_2'$ with $B_2' = A'$. Then, by (5), $e_2'$ is the multiplication by a unit $u'$ of $A'$. Replacing $e$ by $s(u')^{-1} e$, we get a new Euler map $e : B \to A$ for which

$$e_2' = \text{id}_{A'}$$

and, for any ring homomorphism $A' \to A''$, we have $B'' = B_1'' \times B_2''$ with $B_2'' = A''$ and $e_2'' = \text{id}_{A''}$. Thus, for any quadratic space $q''$ over $B''$,

$$\text{Tr}^{e_2''}(q_2'') = q_2''.$$

(7) The linear map $e : B \to A$ induces a homomorphism of Witt groups

$$\text{Tr}^e : W(B) \to W(A).$$

(8) If $B$ is of the form $A[t]/(f) = A[\tau]$, where $f$ is a monic polynomial of odd degree

and $\tau$ the class of $t$, we can define an Euler map by

$$e(\tau^i) = \begin{cases} 0 & \text{if } i < n - 1, \\ 1 & \text{if } i = n - 1. \end{cases}$$

In this case, a direct computation shows that the composite homomorphism

$$W(A) \to W(B) \to W(A)$$

is the identity of $W(A)$. 

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4. Reduction of purity to infinite base fields

Let $F$ be a finite field of odd characteristic $p$ and $A$ a local, essentially smooth $F$-algebra with maximal ideal $m$. Suppose that purity holds for essentially smooth local algebras over any infinite field $k$. Let $K$ be the field of fractions of $A$ and $\xi$ an unramified element of $W(K)$. Let $p^m$ be the cardinality of the algebraic closure of $F$ in $A/m$ and $s$ an odd integer greater than 2 and prime to $m$. For any $i$ let $k_i$ be the field (in some fixed algebraic closure of $F$) of degree $s^i$ over $F$. Let $K$ be the union of all $k_i$. Since $k \otimes_F (A/m)$ is still a field, $B = k \otimes_F A$ is a local, essentially smooth algebra over the infinite field $k$. Let $L = k \otimes_F K$ be its field of fractions. The image $\xi_L$ of $\xi$ in $W(L)$ is unramified. In fact, let $q$ be a height-one prime of $B$ and $p = A \cap q$. By assumption $\xi \in W(A_p)$ and since $A_p \to L$ factors through $B_q$, the class $\xi_L$ is in $W(B_q)$ for every $q$. Since purity holds for $B$, $\xi_L$ is in the image of $W(B)$. We can now find a finite subfield $F'$ of $k$ and, for $A' = F' \otimes_F A$, a $\xi' \in W(A')$ which maps to $\xi_L$. Let $K'$ be the field of fractions of $A'$. Further enlarging $F'$, we may assume that the images of $\xi$ and $\xi'$ in $W(K')$ coincide. Consider now the diagram

\[
\begin{align*}
W(A) & \xrightarrow{\pi} W(A') & W(A) \\
\downarrow & & \downarrow \alpha \\
W(K) & \xrightarrow{\pi} W(K') & W(K)
\end{align*}
\]

where $\epsilon$ has been chosen as in §3 (8). Since the composition of the horizontal maps is the identity, we have $\alpha \circ \text{Tr}^\epsilon(\xi') = \xi$ in $W(K)$. Thus $\xi$ is indeed in the image of $W(A)$.

5. The geometric presentation lemma

We state and prove a lemma that will play a crucial role in the sequel. In geometrical disguise it sounds like this:

**Lemma 5.1.** Let $A$ be a local ring of a smooth variety over an infinite field $k$. Let $U = \text{Spec}(A)$ and let $u$ be the closed point of $U$. Let $p : \mathcal{X} \to U$ be an affine $U$-scheme, essentially smooth over $k$. Let $f$ be an element of $k[\mathcal{X}]$ such that $k[\mathcal{X}]/(f)$ is finite over $A$. We denote by $\mathcal{X}_f$ the principal open set defined by $f \neq 0$. Assume that there exists a finite surjective morphism $\mathcal{X} \to U \times \mathbb{A}^1$ of $U$-schemes and that there exists a section $\Delta : U \to \mathcal{X}$ of $p$ such that $p$ is smooth along $\Delta(U)$.

Then there exists a finite surjective morphism $\pi : \mathcal{X} \to U \times \mathbb{A}^1$ of $U$-schemes with the following properties:

(a) $\pi^{-1}(U \times \{1\})$ is in $\mathcal{X}_f$.
(b) $\pi^{-1}(U \times \{0\}) = \Delta(U)$ if $\mathcal{D}$, where $\mathcal{D} \subset \mathcal{X}_f$.

Clearly the statement above is equivalent to the following, purely algebraic one.

**Lemma 5.2.** Let $A$ be a local essentially smooth algebra over an infinite field $k$, $m$ its maximal ideal and $R$ an essentially smooth $k$-algebra, which is finite over the polynomial algebra $A[t]$. Suppose that $\epsilon : R \to A$ is an $A$-augmentation and let $I = \ker(\epsilon)$. Assume that $R$ is smooth over $A$ at every prime containing $I$. Given $f \in R$ such that $R/Rf$ is finite over $A$ we can find an $s \in R$ such that

1. $R$ is finite over $A[s]$.
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(2) \( R/Rs = R/I \times R/J \) for some ideal \( J \) of \( R \).

(3) \( J + Rf = R \).

(4) \( R(s - 1) + Rf = R \).

Proof. - Replacing \( t \) by \( t - \epsilon(t) \) we may assume that \( t \in I \). We denote by “bar” the reduction modulo \( m \). By the assumptions made on \( R \) the quotient \( \bar{R} \) is smooth over \( \bar{A} \) at its maximal ideal \( \bar{I} \). Choose an \( \alpha \in R \) such that \( \bar{\alpha} \) is a local parameter of the localization \( \bar{R}_I \) of \( \bar{R} \) at \( \bar{I} \). By the chinese remainders’ theorem we may assume that \( \bar{\alpha} \) does not vanish at the zeros of \( \bar{f} \) different from \( \bar{I} \). Without changing \( \bar{\alpha} \) we may replace \( \bar{\alpha} \) by \( \bar{\alpha} - \bar{\epsilon} \bar{\alpha} \) and assume that \( \alpha \in I \). Since \( R \) is integral over \( A[t] \), there exists a relation of integral dependence

\[
\alpha^n + p_1(t)\alpha^{n-1} + \ldots + p_n(t) = 0.
\]

For any \( r \in k^* \) and any \( N \) larger than the degree of each \( p_i(t) \), putting \( s = \alpha - rtN \) we see from the equation above that \( t \) is integral over \( A[s] \). Hence \( R \), which is integral over \( A[t] \), is integral over \( A[s] \). Clearly \( s \in I \). To insure that \( \bar{s} \) is also a local parameter of \( \bar{R}_I \) it suffices to take \( N \geq 2 \). By assumption \( R \) and \( A[s] \) are both regular and since \( R \) is finite over \( A[s] \), \( R \) is locally free over \( A[s] \) (see for instance Corollary 18.17 of [4]) and hence \( R/Rs \) is free over \( A \). By the next sublemma \( R/Rs \) splits as in (2).

SUBLEMMA 5.3. - Let \( B \) be a commutative ring, \( \gamma : B \to C \) a finite commutative \( B \)-algebra and \( \lambda : C \to B \) an augmentation with augmentation ideal \( I \). Let \( h \in C \) be such that

(a) \( C_h \) is étale over \( B \).

(b) \( \lambda(h) \) is invertible in \( B \).

Then \( C \) splits as \( C/I \times C/J \) for some ideal \( J \) of \( C \).

Proof. - Since \( B \to C_h \) is étale and the composite map

\[
B \xrightarrow{\gamma} C_h \xrightarrow{\lambda} B
\]

is the identity of \( B \), by Proposition 4.7 of [1], \( C_h \to B \) is étale. But \( C \to C_h \) is étale, hence \( \lambda : C \to B \) is étale and in particular it induces an open morphism \( \lambda^* : \text{Spec}(B) \to \text{Spec}(C) \). Its image \( \lambda^*(\text{Spec}(B)) = \text{Spec}(C/I) \) is therefore open and since it is also closed, \( C \) splits as claimed.

To complete the proof of Lemma 5.2 we still have to choose \( r \in k^* \) so that conditions (3) and (4) are satisfied. Since \( R/Rf \) is semilocal, there are only finitely many maximal ideals of \( R \) containing \( f \). We denote by \( m_1, \ldots, m_p \) those which, in case \( f \in I + mR \), are different from \( I + mR \). Recalling that \( \alpha \) was chosen outside \( m_1 \cup \ldots \cup m_p \), we have \( s \notin m_1 \cup \ldots \cup m_p \) for almost any choice of \( r \in k^* \). To see that condition (3) is satisfied it suffices to show that \( J \not\subseteq m_i \) for \( 1 \leq i \leq p \) and that \( J \not\subseteq mR + I \). The first assertion is clear because \( s \in J \setminus m_i \) for \( 1 \leq i \leq p \). For the second one note that, since \( R/Rs = R/I \times R/J \), we have \( I + J = R \) and therefore \( J \not\subseteq mR + I \). It remains to satisfy (4). Since \( R/Rf \) is semilocal there exists a \( \lambda \in k \) such that \( s - \lambda \) is invertible in \( R/Rf \). Without perturbing conditions (1), (2) and (3) we may replace \( s \) by \( \frac{1}{\lambda} s \) and thus satisfy (4) as well.
6. A commutative diagram for relative curves

**Lemma 6.1.** With the notation and the hypotheses of Lemma 5.2, let \( U = \text{Spec}(A) \) and \( X = \text{Spec}(R) \). Let \( p : X \to U \) be the structural morphism and \( \Delta : U \to X \) the morphism corresponding to the augmentation \( \epsilon : R \to A \). Let \( Z \subset X \) be a closed set of codimension at least 2, contained in the vanishing locus of \( f \). Suppose that \( \omega_{X/k} \) is trivial. Then there exists a homomorphism \( \psi : W(X \setminus Z) \to W(U) \) such that, for any \( g \in A \) with \( X_g \subseteq X \setminus Z \), the diagram

\[
\begin{array}{ccc}
W(X \setminus Z) & \xrightarrow{\psi} & W(U) \\
w(i)\downarrow & & \downarrow w(i) \\
W(X_g \setminus Z_g) & \xrightarrow{\psi} & W(U_g)
\end{array}
\]

commutes, where \( i : U_g \to U \) and \( j : X_g \to X \setminus Z \) are the inclusions.

**Proof.** By Lemma 5.2 there exists an element \( s \in R \) satisfying the conditions (1) to (4). The \( A \)-algebra homomorphism \( A[t] \to R \) sending \( t \) to \( s \) defines a finite surjective morphism \( \pi : X \to U \times A^1 \) of \( U \)-schemes such that, putting \( \pi^{-1}(U \times \{0\}) = \Delta(U) \cap D_0 \) and \( \pi^{-1}(U \times \{1\}) = D_1 \), we have \( D_0 \cup D_1 \subset X_f \). Since \( \omega_{X/k} \) is obviously trivial and \( \omega_{X/k} \) is trivial by assumption, we can use Corollary 2.2 to find an Euler trace \( \epsilon : R \to A[t] \) such that the associated map \( \lambda : R \to \text{Hom}_{A[t]}(R, A[t]) \) is an isomorphism. We can then choose a trace map \( \text{Tr} : W(X) \to W(U \times A^1) \) as in \( \S \)3. Restricting \( \text{Tr} \) to \( W(\pi^{-1}(U \times \{0\})) \) yields a homomorphism \( W(\pi^{-1}(U \times \{0\})) \to W(U \times \{0\}) \). Since the evaluation at \( t = 0 \) has as retraction the natural embedding \( A \hookrightarrow A[t] \), by (6) of \( \S \)3 we may choose the Euler trace \( \epsilon : R \to A[t] \) such that \( \text{Tr}|_{W(\pi^{-1}(U \times \{0\}))} = W(\epsilon) \) (see (6) of \( \S \)3). Further restricting \( \epsilon \) to \( D_i, i = 0,1 \), we get trace maps \( \text{Tr}_i : W(D_i) \to W(U) \). Let \( \varphi_i : D_i \to X \setminus Z \) be the inclusion. We put

\[
\psi = \text{Tr}_1 \circ W(\varphi_1) - \text{Tr}_0 \circ W(\varphi_0) .
\]

Since \( Z \) is of codimension \( \geq 2 \) in \( X \) and \( \pi : X \to U \times A^1 \) is finite, the image of \( Z \) in \( U \) under the structural map is contained in the vanishing locus of some non zero \( g \in A \). Making now the base change of \( \epsilon \) by means of the inclusion \( i : U_g \to U \) we get \( \epsilon_g \) and \( \text{Tr}_g \) such that we still have \( \text{Tr}_g|_{W(\Delta(U_g))} = W(\Delta_g) \) (see (6) of \( \S \)3). Further restricting \( \epsilon_g \) to \( D_{ig}, i = 0,1 \), we get trace maps \( \text{Tr}_{ig} : W(D_{ig}) \to W(U_g) \). Let \( \varphi_{ig} : D_{ig} \to X_g \setminus Z_g = X_g \), \( i = 0,1 \), be the inclusions. We put

\[
\psi_g = \text{Tr}_{1g} \circ W(\varphi_{1g}) - \text{Tr}_{0g} \circ W(\varphi_{0g}) .
\]

Clearly property (3) of \( \S \)3 implies the relation \( W(i) \circ \psi = \psi_g \circ W(j) \). Thus, to complete the proof of the lemma, it suffices to check the relation \( \psi_g = W(\Delta_g) \). For this take any \( \xi \) in \( W(X_g) \) and, using property (4) of \( \S \)3, write a chain of relations

\[
\text{Tr}_g(\xi)|_{U_g \times \{1\}} = \text{Tr}_g(\xi)|_{U_g \times \{0\}}
\]

\[
= \text{Tr}_{1g}(\xi)|_{D_{1g}} - \text{Tr}_{0g}(\xi)|_{D_{0g}} - \text{Tr}_g(\xi|_{\Delta(U_g)}) = \psi_g(\xi) - W(\Delta_g)(\xi) .
\]

A well-known theorem of Max Karoubi (see [10], VII, \( \S \)4) asserts that for any affine \( k \)-scheme \( S \) the canonical homomorphism \( W(S) \to W(S \times A^1) \) is an isomorphism, and therefore, the left hand side of the relation above is zero. This proves the relation \( \psi_g = W(\Delta_g) \), whence the commutativity of the diagram.
7. Purity

**Theorem 7.1.** Let $A$ be a local, essentially smooth algebra over an infinite field $k$ and let $K$ be its field of fractions. Every unramified element of $W(K)$ belongs to $W(A)$.

**Proof.** Let $U = \text{Spec}(A)$ and let $\xi$ be an unramified element of $W(K)$. By assumption there exist a smooth $d$-dimensional $k$-algebra $R = k[t_1, \ldots, t_n]$ and a prime ideal $p$ of $R$ such that $A = R_p$. We first reduce the proof to the case in which $p$ is maximal. To do this, choose a maximal ideal $m$ containing $p$. Since $k$ is infinite, by a standard general position argument we can find $d$ algebraically independent elements $X_1, \ldots, X_d$ such that $R$ is finite over $k[X_1, \ldots, X_d]$ and étale at $m$. After a linear change of coordinates we may assume that $R/p$ is finite over $B = k[X_1, \ldots, X_m]$, where $m$ is the dimension of $R/p$. Clearly $R$ is smooth over $B$ at $m$ and thus, for some $h \in R \setminus m$, the localization $R_h$ is smooth over $B$. Let $S$ be the set of nonzero elements of $B$, $k' = S^{-1}B$ the field of fractions of $B$ and $R' = S^{-1}R_h$. The prime ideal $p' = S^{-1}p_h$ is maximal in $R'$, the $k'$-algebra $R'$ is smooth and $A = R'_{p'}$.

From now on and till the end of the proof of Theorem 7.1 we assume that $A = O_{X,x}$ is the local ring of a closed point $x$ of a smooth $d$-dimensional irreducible affine variety $X$ over $k$.

Replacing $X$ by a sufficiently small affine neighbourhood of $x$ we may assume that $\omega_{X/k}$ is trivial. By Proposition 2.4 of [3] we may assume that $\xi$ is defined on the complement of a closed set $Z$ of codimension at least 2 in $X$. Let $f \neq 0$ be a regular function on $X$ which vanishes on a closed set $Y$ containing $Z$. By Quillen’s trick (see [16], Lemma 5.12) we can find a morphism $q : X \to \mathbb{A}^{d-1}$ with the following properties:

1. $q$ is smooth at $x$.
2. $q|_Y : Y \to \mathbb{A}^{d-1}$ is finite.
3. $q$ factors as

$$
\begin{array}{ccc}
X & \xrightarrow{q_1} & \mathbb{A}^d \\
\downarrow{q} & & \downarrow{p^r} \\
\mathbb{A}^{d-1} & & \\
\end{array}
$$

with $q_1$ finite and surjective.

Consider the cartesian square

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p_X} & X \\
\downarrow{p} & & \downarrow{q} \\
U & \xrightarrow{\Delta} & \mathbb{A}^{d-1} \\
\end{array}
$$

where $U = \text{Spec}(O_{X,x})$, $r = q|_U$, $\mathcal{X} = U \times_{\mathbb{A}^{d-1}} X$, $p$ is the first projection and $\Delta : U \to \mathcal{X}$ the diagonal. Denote again by $f$ the composition of $f$ with $p_X$.

Since $r$ is essentially smooth and $X$ is smooth over $k$, $\mathcal{X}$ is essentially smooth. By base change, condition (3) implies that $\mathcal{X}$ is an affine relative curve over $U$. Since $U$ is local and $q$ is smooth at $x$, $p$ is smooth along $\Delta(U)$. From (3), by base change of $q$ via $r : U \to \mathbb{A}^{d-1}$, we get a commutative triangle

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p_1} & U \times \mathbb{A}^1 \\
\downarrow{p} & & \downarrow{p} \\
U & & U \\
\end{array}
$$
with $p_1$ finite. Again by the same base change we see that $k[X]/(f)$ is finite over $A$. Thus all the hypotheses of Lemma 5.1 are satisfied and we can find a $U$-morphism $\pi : X \to U \times A^1$ satisfying conditions (a) and (b).

We further claim that $\omega_X$ is trivial. To see this observe that

$$\omega_{X/k} \cong p_X^*(\omega_X/\kappa) \otimes_{\mathcal{O}_X} \omega_{X/k}$$

(cf. [7], Proposition 17.2.3) and that $\omega_{X/k} \cong p^*\omega_{U/A^{d-1}}$. Since $U$ is essentially smooth over $A^{d-1}$, $\omega_{U/A^{d-1}}$ is locally free of rank-one, hence trivial because $U$ is local. Thus $p^*\omega_{U/A^{d-1}}$ is trivial and, since $\omega_{X/k}$ is trivial by assumption, we conclude that $\omega_{X/k}$ is trivial.

We can now apply Lemma 6.1 with $Z = U \times A_{d-1}$. We define $\pi = W(p_X)(\xi)$ and claim that $\pi$ is an extension of $\xi$ to $U$. In fact, choosing $g \in A$ as in 6.1 and denoting by $i : U_g \to U$, $i' : U_g \to X \setminus Z$ and $j : X_g \to X \setminus Z$ the inclusions, we have

$$W(i)\pi = W(i) \circ \psi \circ W(p_X)\xi = W(\Delta_g) \circ W(j) \circ W(p_X)\xi = W(p_X \circ j \circ \Delta_g)\xi = W(i')\xi.$$ 

This completes the proof of Theorem 7.1. \hfill \Box

To prove Theorem A we now recall a celebrated result of Dorin Popescu (see [11], [12] and [13] or [2] or, for a self-contained proof, [19]).

Let $k$ be a field and $R$ a local $k$-algebra. We say that $R$ is geometrically regular if $k' \otimes_k R$ is regular for any finite extension $k'$ of $k$. A ring homomorphism $A \to R$ is called geometrically regular if it is flat and if for each prime ideal $q$ of $R$ lying over $p$, $R_q/pR_q = k(p) \otimes_A R_q$ is geometrically regular over $k(p) = A/p$.

Observe that any regular local ring containing a field $k$ is geometrically regular over the prime field of $k$.

**Popescu's Theorem.** A homomorphism $A \to R$ of noetherian rings is geometrically regular if and only if $R$ is a filtered direct limit of smooth $A$-algebras.

**Proof of Theorem A.** Let $R$ be a regular local ring containing a field. Let $k$ be the prime field of $R$. By Popescu’s theorem, $R = \lim A_\alpha$, where the $A_\alpha$’s are smooth $k$-algebras. We first observe that we may replace the direct system of the $A_\alpha$’s by a system of essentially smooth local $k$-algebras. In fact, if $m$ is the maximal ideal of $R$, we can replace each $A_\alpha$ by $(A_\alpha)_p$, where $p = m \cap A_\alpha$. Note that in this case the canonical morphisms $\varphi_\alpha : A_\alpha \to R$ are local and that every $A_\alpha$ is a regular local ring, thus in particular a factorial ring.

Let $L$ be the field of fractions of $R$ and, for each $\alpha$, let $K_\alpha$ be the field of fractions of $A_\alpha$. Let $\xi$ be an unramified element of $W(L)$. We may represent $\xi$ by a diagonal matrix $q = \text{diag}(r_1, \ldots, r_n)$ with $r_1, \ldots, r_n$ in $R$. Let $\Sigma$ be the (finite) set of height-one primes of $R$ which divide at least one of the $r_i$. For every $p \in \Sigma$ we can find a matrix $\sigma(p) \in GL_n(L)$ that transforms $q$ into a diagonal form $\text{diag}(u_1(p), \ldots, u_n(p))$ with every $u_i(p) \in R \setminus p$. Clearing denominators we may assume that $\sigma(p) \in M_n(R)$ and that

$$\sigma(p)^T q \sigma(p) = \text{diag}(u_1(p), \ldots, u_n(p))(d(p))^2$$

for some $d(p) \in R$. We can now choose an index $\alpha$ such that, for every $p \in \Sigma$, $A_\alpha$ contains preimages $\check{r}_1, \ldots, \check{r}_n$, $\check{u}_1(p), \ldots, \check{u}_n(p)$, $\check{d}(p)$ and $\check{\sigma}_{ij}(p)$ of the elements
Having chosen these preimages consider the relations
\[(\tilde{\sigma}(p))^T \tilde{\sigma}(p) = \text{diag}(\tilde{u}_1(p), \ldots, \tilde{u}_n(p))(\tilde{d}(p))^2\]
where \(\tilde{\sigma}(p) = \text{diag}(\tilde{r}_1, \ldots, \tilde{r}_n)\) and \(\tilde{\sigma}(p)\) is the matrix \((\tilde{\sigma}_{ij}(p))\). Since they hold over \(R\), we may assume, after replacing \(\alpha\) by some larger index, that they hold over \(A_\alpha\). We claim that the class of \(\tilde{q}\) (which we still denote by \(q\)) is an unramified element of \(W(K_\alpha)\). To show this suppose that \(\tilde{q}\) is ramified at a height-one prime ideal \(pA_\alpha\). Then \(p\) divides some \(\tilde{r}_i\). Any height-one prime \(p\) of \(R\) containing \(pR\) also contains \(r_i\) and thus belongs to \(\Sigma\). Since \(u_i(p) \in R \setminus p\) we have \(\tilde{u}_i(p) \in A_\alpha \setminus pA_\alpha\) and thus the relation (*) shows that \(\tilde{q}\) is unramified at \(pA_\alpha\). By purity for \(A_\alpha\) there exists a \(\xi_\alpha \in W(A_\alpha)\) that coincides with \(q\) in \(W(J_\alpha)\). The ideal \(r = \ker(\phi_\alpha)\) is prime and does not contain any \(\tilde{r}_i\), hence \(q\) is a quadratic space over the essentially smooth local algebra \(B_\alpha = (A_\alpha)_r\). Since \(\tilde{q}\) and \(\xi_\alpha\) coincide in \(W(K_\alpha)\), they already coincide in \(W(B_\alpha)\) because \(W(B_\alpha) \to W(K_\alpha)\) is injective. The commutative diagram of ring homomorphisms

\[
\begin{array}{ccc}
A_\alpha & \xrightarrow{\phi_\alpha} & R \\
\downarrow & & \downarrow \\
B_\alpha & \to & L
\end{array}
\]
shows that \(W(\phi_\alpha)(\xi_\alpha) = q\) in \(W(L)\). This proves that \(q\) is indeed in \(W(R)\).

\section{An injectivity theorem}

If \(A\) is a regular ring of dimension greater than 3 and \(K\) its field of fractions, the canonical homomorphism \(W(A) \to W(K)\) need not be injective. In this section we prove the following injectivity result, from which we shall deduce Theorem C.

\textbf{Theorem 8.1.} Let \(A\) be a local, essentially smooth algebra over an infinite field of characteristic \(\neq 2\). Let \(K\) be the field of fractions of \(A\) and \(f\) a regular parameter of \(A\). The canonical homomorphism \(W(A_f) \to W(K)\) is injective.

The proof of this theorem is similar to that of Theorem 7.1. As we proved there, we can find an infinite field \(k\) and a smooth \(d\)-dimensional irreducible affine variety \(X\) over \(k\) such that \(A\) is the local ring \(O_{X,x}\) of a closed point \(x\) of \(X\). If \(A\) is 1-dimensional \(A_f = K\) and there is nothing to prove, so we assume that \(A\) is at least 2-dimensional. We need the following variant of Quillen’s trick.

\textbf{Lemma 8.2.} Let \(X\) be a smooth \(d\)-dimensional irreducible affine variety over an infinite field \(k\) and \(x\) a closed point of \(X\). Let \(A\) be the local ring of \(x\), \(f \in k[X]\) a regular function on \(X\) which is a regular parameter of \(A\) and \(g \in k[X]\). Denote by \(Y\) the vanishing locus of \(f\) and by \(Z\) the vanishing locus of \(g\). Suppose that \(Y\) is irreducible and not contained in \(Z\). Then there exists a morphism \(q : X \to \mathbb{A}^{d-1}\) with the following properties:

1. \(q\) is smooth at \(x\);
2. \(q|_{Y \cap Z} : Y \cap Z \to \mathbb{A}^{d-1}\) is finite;
We first recall an auxiliary result, which has been proved in slightly different versions by several authors.

**Lemma 8.3.** Under the assumptions of Lemma 8.2 there exists a morphism $q_2 : X \to \mathbb{A}^d$ such that

(i) $q_2$ is finite.
(ii) $q_2$ is étale at $x$.
(iii) $k(x) = k(q_2(x))$.
(iv) $Y \cap q_2^{-1}(q_2(x)) = \{x\}$.

**Proof.** Suppose that $X$ is a closed set of $\mathbb{A}^N \subset \mathbb{P}^N$ and let $\overline{X}$ be its closure in $\mathbb{P}^N$. To prove Lemma 8.3 we will take for $q_2$ the projection from a suitable linear subspace $L$ at infinity. Let $\overline{k}$ be an algebraic closure of $k$ and $\phi : \overline{k} \otimes_k X \to X$ the canonical projection. Then $\phi^{-1}(x)$ is a finite set of closed points $\{x_1, \ldots, x_n\}$ of $\overline{k} \otimes_k X$. Choose a $N - d - 1$-dimensional linear subspace $L$ in $\mathbb{P}^N \setminus \mathbb{A}^N$ with the following properties:

(a) $L$ is defined over $k$;
(b) $L$ does not intersect $\overline{k} \otimes_k \overline{X}$;
(c) $L$ does not intersect the tangent planes of $\overline{k} \otimes_k \overline{X}$ at $x_1, \ldots, x_n$;
(d) For $i \neq j$ we have $q_2(x_i) \neq q_2(x_j)$;
(e) $L$ does not intersect the closures of the cones with vertices $x_1, \ldots, x_n$ and base $\overline{k} \otimes_k Y$.

Dimension considerations show the existence of infinitely many such linear spaces. Condition (a) insures that $q_2$ is defined over $k$. Condition (b) insures that $q_2 : X \to \mathbb{A}^d$ is finite. Condition (c) insures that $q_2$ is étale at $x$. Since the group $\text{Aut}_{\overline{k}}(\overline{k})$ acts transitively on $\{x_1, \ldots, x_n\}$, by condition (d) it acts transitively on $\{q_2(x_1), \ldots, q_2(x_n)\}$ as well. This shows that the separability degree of $k(q_2(x))$ over $k$ is the same as that of $k(x)$. But $q_2$ is étale at $x$, hence the extension $k(x)/k(q_2(x))$, being separable, must be of degree one. Thus condition (iii) is satisfied. Finally, condition (iv) follows from (e).

**Proof of Lemma 8.2.** We choose $q_2$ as in the previous lemma. We put $B = k[\mathbb{A}^d]$ and $C = k[X]$. The map $q_2$ induces an inclusion $\iota_2 : B \hookrightarrow C$ and $C$ is a finite $B$-module. The images of the closed subschemes $Y = \{f = 0\}$ and $Z = \{g = 0\}$ of $X$ are two closed closed subschemes of $\mathbb{A}^d$ defined, respectively, by $f_0 = 0$ and $g_0 = 0$ for some $f_0, g_0 \in k[\mathbb{A}^d]$. The inclusion $\iota$ induces a finite map $B/Bf_0 \to C/Cf$. Let $\mathfrak{m}$ be the maximal ideal of $B$ corresponding to the closed point $q_2(x)$. Since $x$ is the unique closed point of $Y$ lying over $q_2(x)$, the localization $(C/Cf)_\mathfrak{m} = B_\mathfrak{m} \otimes_B (C/Cf)$ is local and finite over $(B/Bf_0)_\mathfrak{m}$. By condition (iii) these two local rings have the same residue field, hence by Nakayama’s lemma they are isomorphic. This shows in particular that $f_0$ is a regular parameter of $B$ at $q_2(x)$. On the other hand, since $C$ is étale over $B$ at $x$, $f_0$ is also a regular parameter of $C$ at $x$.

We now have two polynomials $f_0$ and $g_0$ in $B = k[X_1, \ldots, X_d]$ which we may assume to be monic in $X_1$. The map $\iota_3 : k[Y_1, \ldots, Y_d] \to k[X_1, \ldots, X_d]$ defined by $Y_i \mapsto f_0$ and $Y_i \mapsto X_i$ for $i \geq 1$ induces a finite morphism $q_3 : \mathbb{A}^d \to \mathbb{A}^d$. Composing $q_2$ with $q_3$ we
obtain a finite map \( q_1 = q_3 \circ q_2 : X \to \mathbb{A}^d \). This map is smooth at \( x \) because \( q_2 \) is étale at \( x \) and \( f_0 \) is a regular parameter at \( q_2(x) \). It maps \( Y \) onto the hyperplane \( Y_1 = 0 \). Since \( Y \) is irreducible and not contained in \( Z \), their intersection \( Y \cap Z \) is a proper closed subset of \( Y \). Hence, since \( q_1 \) is a finite morphism, \( q_1(Y \cap Z) \) is a proper closed subset of the hyperplane \( Y_1 = 0 \). We can thus find a nontrivial polynomial \( h_1 \in k[Y_2, \ldots, Y_d] \) identically vanishing on \( q_1(Y \cap Z) \). After a suitable linear change of the coordinates \( Y_2, \ldots, Y_d \) we may assume that \( h_1 \) is monic in \( Y_1 \). The inclusion \( k[Y_1, Y_3, \ldots, Y_d] \subset k[Y_1, Y_3, \ldots, Y_d] \) induces the second projection \( p_{r_2} : \mathbb{A}^d \to \mathbb{A}^{d-1} \). Clearly the restriction of \( p_{r_2} \) to the hypersurface \( H_1 \subset \mathbb{A}^d \) defined by \( h_1 = 0 \) is a finite morphism. Put \( q = p_{r_2} \circ q_1 \) and \( h = (r_2 \circ r_3)(h_1) \in C \). Then the restriction of \( q \) to the hypersurface \( h = 0 \) is a finite morphism and in particular \( q|_{Y \cap Z} : Y \cap Z \to \mathbb{A}^{d-1} \) is also finite. Furthermore, \( q(Y) = \{0\} \times \mathbb{A}^{d-2} \), the hyperplane in \( \mathbb{A}^{d-1} \) defined by \( Y_1 = 0 \). Finally, \( q^{-1}(\{0\} \times \mathbb{A}^{d-2}) \) is a hypersurface in \( X \) defined by the equation \( r_2(f_0) = 0 \). This hypersurface is smooth at \( x \) and therefore contains only one component-namely \( Y \)-that passes through \( x \). This proves the last point (5). 

**Proof of Theorem 8.1.** – Let \( \xi \) be an element in the kernel of \( W(A_f) \to W(K) \). There is a \( g \in A \), which we may suppose prime to \( f \), such that \( \xi \in \ker(W(A_f) \to W(A_{fg})) \). Clearly, making \( X \) sufficiently small, we may assume that \( f \) and \( g \) are regular functions on \( X \) and that \( \xi \in \ker(W(X_f) \to W(X_{fg})) \). Making \( X \) even smaller we may further assume that the vanishing locus \( Y \) of \( f \) is irreducible. Clearly the vanishing locus \( Z \) of \( g \) does not contain \( Y \). In particular the closed set \( W = Y \cap Z \) has codimension at least 2 in \( X \). We may represent \( \xi \) by a quadratic space \( q \) defined over \( X_f \) which becomes hyperbolic over \( X_{fg} \). Patching \( q \) over \( X_f \) with a suitable hyperbolic space over \( X_g \) we get a space over the complement of \( W \). Applying Lemma 8.2 we get a map \( q : X \to \mathbb{A}^{d-1} \) satisfying properties (1) to (5). Let \( h \in k[X] \) be the element given by the proof of Lemma 8.2. It vanishes identically on \( W \) and \( q \) is finite on the closed subscheme defined by \( h = 0 \). As in the proof of Theorem 7.1, but with \( h \) instead of \( f \) and \( W \) instead of \( Z \), we get a commutative square

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p_X} & X \\
p | & & \downarrow \\
U & \to & \mathbb{A}^{d-1}
\end{array}
\]

where \( U = (\mathcal{O}_{X,x}) \), \( r = q|_U \), \( \mathcal{X} = U \times_{\mathbb{A}^{d-1}} X \), \( p \) is the first projection and \( \Delta : U \to \mathcal{X} \) the diagonal. We denote again by \( h \) the composition of \( h \) with \( p_X \) and we put \( W = U \times_{\mathbb{A}^{d-1}} \mathcal{X} \). As in the proof of 7.1, we assume that \( X \) has been so chosen that \( \omega_X \) is trivial.

Applying the geometric presentation lemma we find a map \( \pi : \mathcal{X} \to U \times \mathbb{A}^1 \) of \( U \)-schemes such that \( \pi^{-1}(U \times \{1\}) = D_1 \) is in \( \mathcal{X}_h \) and \( \pi^{-1}(U \times \{0\}) = \Delta(U) \cap D_0 \), where \( D_0 \subset \mathcal{X}_h \). Put \( s = Y_1 \). By condition (5) we have \( W \subset \mathcal{X} \setminus \mathcal{X}_s \) and hence, by Lemma 6.1, there exists a commutative square

\[
\begin{array}{ccc}
W(\mathcal{X} \setminus \mathcal{W}) & \xrightarrow{\psi} & W(U) \\
\downarrow & & \downarrow \\
W(\mathcal{X}_s) & \xrightarrow{W(\Delta_s)} & W(U_s),
\end{array}
\]

where \( i : U_s \to U \) and \( j : \mathcal{X}_s \to \mathcal{X} \setminus \mathcal{W} \) are the inclusions. Repeating the argument of the proof of Theorem 7.1, we define \( \eta = \psi(W(p_X)(\xi)) \in W(A) \) and get \( \eta_s = \xi_s \). By
condition (5), \(A_s = A_f\) and since \(W(A) \to W(K)\) is injective and \(\xi\) vanishes in \(W(K)\) we get \(\eta = 0\). This shows that \(\xi = 0\) as well.

**Proof of Theorem B.** – We first extend Theorem 8.1 to the case of an infinite base field. This is even simpler than for Theorem A: we find a sufficiently large odd degree extension \(F'\) of the finite base field \(F\) such that \(A' = F' \otimes_F A\) is still a local ring and \(\xi_{F'} = 0\) in \(W(A')\). Then, choosing \(e\) as in §3, (8), we see that \(\xi = T^e(\xi_{F'}) = 0\).

We now prove Theorem B. Let \(R\) be a regular local ring containing a field and let \(L\) be the field of fractions of \(R\). Let \(k\) be the prime field of \(R\). As in the proof of Theorem A, \(R = \varinjlim A_\alpha\), where the \(A_\alpha\)'s are essentially smooth local \(k\)-algebras. Let \(f\) be a regular parameter of \(R\) and \(\xi\) an element in the kernel of \(W(R_f) \to W(L)\). There exists a \(g \in R\) such that \(\xi\) vanishes in \(W(R_{fg})\). For a suitable index \(\alpha\) choose lifts \(f_\alpha\) and \(g_\alpha\) of \(f\) and \(g\) in \(A_\alpha\). We may replace the filtered direct system of the \(A_\alpha\) by the subsystem of all \(A_\beta\) with \(\beta \geq \alpha\). Clearly we still have \(R = \varinjlim A_\beta\). We put, for every \(\beta \geq \alpha\), \(f_\beta = \varphi_\beta(f_\alpha)\) and \(g_\beta = \varphi_\beta(g_\alpha)\) where the \(\varphi_\beta : A_\alpha \to A_\beta\) are the transition homomorphisms. It is easy to see that \(\lim_{\beta \geq \alpha} (A_\beta f_\beta) = R_f\) and \(\lim_{\beta \geq \alpha} (A_\beta g_\beta) = R_{fg}\). Since the functor \(W\) commutes with filtered direct limits, we have

\[
\lim_{\beta \geq \alpha} \ker \left(W((A_\beta f_\beta) \to W((A_\beta g_\beta))\right) = \ker \left(W(R_f) \to W(R_{fg})\right).
\]

Since \(\varphi_\beta : A_\beta \to R\) is local, \(f_\beta\) is a regular parameter of \(A_\beta\). Hence the left hand side vanishes and, in particular, \(\xi = 0\). This proves Theorem B.

**9. A short exact sequence**

Let \(B\) be a discrete valuation ring, \(p = Bp\) its maximal ideal and \(L\) its field of fractions. Let \(v : L^* \to \mathbb{Z}\) be the corresponding valuation of \(L\). Recall that there is a homomorphism (which depends on the choice of the local parameter \(p\)) \(\partial_p : W(L) \to W(B/p)\) called *second residue* and defined on rank-one forms \(\langle up^m\rangle\) with \(u \in B^*\) by

\[
\partial_p(\langle up^m\rangle) = \begin{cases} 0 & \text{if } m \text{ is even}, \\ \langle \overline{u} \rangle & \text{if } m \text{ is odd}, \end{cases}
\]

where \(\overline{u}\) is the image of \(u\) in \(B/p\).

This homomorphism fits into the exact sequence

\[0 \to W(B) \to W(L) \to W(B/p) \to 0.\]

**Proof of Theorem C.** – We have a commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
0 & \to & W(A) \\
\downarrow \alpha & & \downarrow \beta \\
0 & \to & W(A_p)
\end{array} & \xleftarrow{t} & \begin{array}{ccc}
W(A_f) & \to & W(A/A_f) \\
\downarrow \beta & & \downarrow \gamma \\
W(K) & \to & W(k(p))
\end{array} & \xrightarrow{\partial_f} & 0
\end{array}
\]
of solid arrows in which the bottom line is exact. We first want to show that
\[ \partial_f \circ \beta(W(A_f)) \subseteq \gamma(W(A/Af)). \]
Since \( \gamma \) is injective this would imply that there is a map \( \delta : W(A_f) \to W(A/Af) \) with \( \partial_f \beta = \gamma \delta \). We then check that the top line is exact.

For the first assertion it suffices to show, by purity, that, for any \( \xi \in W(A_f) \), \( \partial_f \circ \beta(\xi) \) is unramified over \( A/Af \). Let \( q/af \) be a prime of height one of \( A/Af \). We want to show that \( \partial_f \circ \beta(\xi) \) is in the image of \( W(A_q/A_qf) \). For this, after replacing \( A \) by \( A_q \) in the diagram above, we may assume that \( A \) is a local regular ring of dimension 2. But in this case the assertion is precisely Theorem 3 of [8].

Exactness left and right is obvious. Let \( \xi \) be an element of \( \ker(\delta) \). Since \( \beta \) is injective, we may consider \( \xi \) as an element of \( W(K) \). From the exactness of the bottom line we see that \( \xi \) is in the image of \( W(A_p) \). Since it also belongs to \( \beta(W(A_f)) \), it is unramified and by purity it comes from \( W(A) \).

Proof of Theorem D. – Apply Theorem C to the local ring \( A[[t]] \), taking \( t \) as regular parameter and using the fact that \( W(A[[t]]) = W(A) \).

REFERENCES


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