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ON THE UNIVERSAL COVERINGS OF ALGEBRAIC SURFACES

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ABSTRACT. – In this paper we use some recent developments in Nonabelian Hodge theory to study the existence of holomorphic functions on the universal coverings of algebraic surfaces. In particular we prove that if the fundamental group of an algebraic surface is reductive then its universal covering is holomorphically convex. This is a partial verification of the Shafarevich conjecture claiming that the universal covering of a smooth projective variety is holomorphically convex. © Elsevier, Paris

RÉSUMÉ. – Dans cet article, nous utilisons certains développements récents de la théorie de Hodge non-abélienne pour étudier l'existence de fonctions holomorphes sur le revêtement universel de surfaces algébriques. En particulier, nous montrons que si le groupe fondamental d'une surface algébrique est réductif alors son revêtement universel est holomorphiquement convexe. Nous vérifions ainsi particulièrement la conjecture de Shafarevich selon laquelle le revêtement universel d'une variété projective lisse est holomorphiquement convexe. © Elsevier, Paris

1. Introduction

The question of characterizing the universal covering of a compact Kähler manifold X is a difficult question. Central to the subject is the Shafarevich conjecture asking if the universal covering $\widetilde{X}$ of compact Kähler manifold X is holomorphically convex. A complex manifold M is holomorphically convex if for every infinite discrete subset S of M there exists a holomorphic function on M that is unbounded on S. An easier question to try to study is the existence of nonconstant holomorphic functions on the universal covering of X. Clearly we need to restrict ourselves to the case when the fundamental group of X is infinite. It is clear that if we can show that some intermediate covering X'

$$\widetilde{X} \longrightarrow X' \longrightarrow X$$

is holomorphically convex then there will be nonconstant holomorphic functions on the universal covering of X. We would like to point out that that there are intermediate coverings of abelian varieties with infinite covering group which do not admit any nonconstant holomorphic functions (see e.g. [GR]).

It is known that a noncompact holomorphically convex manifold M admits real analytic plurisubharmonic exhaustion functions. Unfortunately the converse fails if

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$\dim_{\mathbf{C}}(M) > 1$ (see e.g. [GR]). Nevertheless the holomorphically convex manifolds enjoy some functoriality properties which we recall for future reference.

1) Let $f : X \to Y$ be a proper holomorphic map between complex manifolds. Then the holomorphic convexity of Y implies the holomorphic convexity of X.

2) The Cartan-Remmert theorem [Ca] says that if $\widetilde{X}$ is holomorphically convex and noncompact then there exists a proper map with connected fibers $Sh : \widetilde{X} \to \widetilde{Sh(X)}$ onto a Stein space $Sh(\widetilde{X})$. This map is called the Cartan-Remmert reduction of $\widetilde{X}$. Following Kollár [K1], [K2] we denote this map by $Sh : \widetilde{X} \to \widetilde{Sh(X)}$.

3) T. Napier has studied the question when (unramified, infinite) coverings of a complex surface are holomorphically convex and has shown [N1] that one of the basic obstructions to the holomorphic convexity is the existence of connected noncompact analytic sets all of whose irreducible components are compact. Throughout the paper we will call such an analytic set an infinite chain.

We will restrict ourselves to the case of complex surfaces where more tools for testing holomorphic convexity are developed at this stage. In particular we will make a frequent use of the following result of T. Napier and the second author.

THEOREM 1.1 (Theorem 4.8, [NR]). – *Let X be a compact Kähler surface. Then any intermediate regular covering $\widetilde{X} \longrightarrow X' \longrightarrow X$, which does not have two ends and admits a real analytic plurisubharmonic exhaustion function is holomorphically convex.*

The main result of this paper can be seen as a nonabelian version of the well known elementary fact that any intermediate abelian covering of a compact Kähler manifold which does not have two ends is holomorphically convex. In order to state our result we first need the following definition.

DEFINITION 1.1. – *We call a finitely generated group Γ reductive if it admits an almost faithful Zariski dense representation in a reductive complex Lie group G.*

Now we have

THEOREM 1.2 ([Main theorem). – *Let $X' \longrightarrow X$ be a Galois covering of a compact Kähler surface with a reductive Galois group Γ which does not have two ends. Then X' is holomorphically convex.*

The proof of this theorem builds up on the following two important achievements in Kähler geometry:

1) The work of Hitchin [HI], Corlette [C] and Simpson [S1] on nonabelian Hodge theory.

2) The work of Gromov and Schoen [GS] on equivariant harmonic maps to buildings.

One of the main outputs of these theories is an existence theorem for equivariant pluriharmonic maps. According to this theorem, given any reductive representation $\rho : \pi_1(X) \to G$ one can construct an equivariant pluriharmonic map $U : \widetilde{X} \to N(\rho)$ to an appropriate non-positively curved space $N(\rho)$ that is intrinsically attached to the representation ρ. The space $N(\rho)$ can be the symmetric space G/K if G is defined over $\mathbf{R}$ or $\mathbf{C}$, the affine Bruhat-Tits building of G if G is defined over an archimedian local field, or a suitably chosen product of symmetric spaces and buildings if G happens to be defined over $\overline{\mathbf{Q}}$. If we fix a point $o \in N(\rho)$ the distance function $\operatorname{dist}(\bullet, o)$ will be convex on $N(\rho)$ and hence $\varphi = U \circ (\operatorname{dist}(\bullet, o)^2)$ will be plurisubharmonic on $\widetilde{X}$. If in addition φ is an exhaustion function we may try to apply Theorem 1.1. There are several serious obstructions one has to overcome in order to carry out this idea.

1) To claim that the function φ is an exhaustion function it is necessary that the action of $\pi_1(X)$ on $N(\rho)$ is properly discontinuous. To assure that we deform our representation to a representation defined over $\overline{\mathbf{Q}}$ and then apply a construction of adelic type. We use the information from all representations defined over $\overline{\mathbf{Q}}$ in a neighborhood of the original representation.

2) We also need to analyze the image of the pluriharmonic maps $U : \widetilde{X} \to N(\rho)$.

DEFINITION 1.2. – *The rank of a pluriharmonic map is the codimension of the holomorphic foliation it induces.*

We use the following theorem proved in [FT] and independently by Jost and Zuo in [JZ].

THEOREM 1.3. – *Let $\mathcal{K}$ be a non-archimedian local field and let $\rho : \pi_1(X) \longrightarrow G$ be a Zariski dense representation to some simple Lie group G over $\mathcal{K}$. Then either the image of ρ is in a maximal compact subgroup of G or, there exists a diagram*

$$\begin{array}{ccc} & \widehat{X} & \\ {}^{\varepsilon}\swarrow & & \searrow^{h} \\ X & & Y \end{array}$$

where

(i) *Y is a smooth projective variety of general type with* $\dim_{\mathbf{C}} Y = \operatorname{rank}(U) \leq \operatorname{rank}_{\mathcal{K}}(G)$*;*

(ii) *$\varepsilon : \widehat{X} \to X$decomposes in a sequence of blow-ups with smooth centers and a finite non-ramified covering;*

(iii) *$h : \widehat{X} \longrightarrow Y$ is a morphism with connected fibers such that $\rho : \pi_1(\widehat{X}) \longrightarrow G$ factors through a representation of $\pi_1(Y)$.*

The same statement holds also for a representation $\rho : \pi_1(X) \longrightarrow G$ into a group G defined over an archimedean local field provided that ρ is not rigid, see e.g. [ZUO].

3) Observe that a deformation of the representation can change the rank of the corresponding harmonic maps. We control this by applying the above theorem. By deforming the representation to one defined over $\overline{\mathbf{Q}}$ we can also create a big kernel and so the only thing we can show this way is that some intermediate covering is holomorphically convex. But as we have mentioned already the work of T. Napier [N1] indicates that in case of complex surfaces essentially the only obstruction to the holomorphic convexity will be the existence of infinite chains. Nonabelian Hodge theory was used by Lasell and the second author in [LR] (Theorem 1.1) to rule out this possibility. Before we formulate this theorem let us give some explanations. Let X be an algebraic surface and D be an effective divisor in X that is a connected reducible curve having only rational curves as

irreducible components. We can produce a counterexample to the Shafarevich conjecture if we can arrange that $\mathrm{im}[\pi_1(Y) \to \pi_1(X))]$ is infinite. Deligne [D] has shown that it is impossible to achieve this on the level of homology since we have

$$\begin{aligned}\mathrm{im}[H_1(D, \mathbf{Q}) \to H_1(X, \mathbf{Q})] &= \mathrm{im}[H_1(D', \mathbf{Q}) \to H_1(X, \mathbf{Q})] \\ &= \mathrm{im}[\cup H_1(\mathbf{P}^1, \mathbf{Q}) \to H_1(X, \mathbf{Q})] = 0.\end{aligned}$$

(Here we denote by D' the desingularization of D.) The theorem of Lasell and the second author [LR] is an nonabelian analog of the above result of Deligne's. In case of surfaces it says that:

THEOREM 1.4. – *If we have an algebraic surface X a reductive Zariski dense representation to a complex reductive Lie group $\rho : \pi_1(X) \to G$ and $Y = \cup D_i \hookrightarrow X$ a divisor in X such that the restricted representations $\rho : \pi_1(D_i) \longrightarrow G$ are trivial then $\rho : \pi_1(Y) \longrightarrow G$ factors through a finite group δ_n the order of which depends only on $n = \mathrm{rank}_{\mathbf{C}}(G)$.*

This theorem shows that if we have an algebraic surface X a reductive Zariski dense almost faithful representation to a complex reductive Lie group $\rho : \pi_1(X) \to G$ and $Y = \cup D_i \hookrightarrow X$ then there are no infinite chains in $\widetilde{X}$.

In [NIL] the first author has shown that every smooth projective variety with a nilpotent fundamental group has a holomorphically convex universal covering. Combining this result with our main theorem we get:

COROLLARY 1.1. – *Let X be a smooth projective surface and $\rho : \pi_1(X) \to L$ be a linear representation with an infinite image. Then the universal covering of X admits nontrivial holomorphic functions.*

2. Reduction to the case of discrete action

Start with a smooth projective surface X and a Zariski dense representation $\rho : \pi_1(X) \to G$ to a complex reductive group. To get an exhaustion function we need a discrete action on a nonpositively curved space. So we need to adjust both the target group G and the representation ρ in order to get a representation with discrete image. First we reduce to the case of a representation to a semisimple group.

Up to isogeny any reductive complex group G is a direct product a semisimple part S and a torus C. After taking a finite nonramified covering of X we may assume that $G = S \times C$ and that $\mathrm{im}(\rho)$ has no torsion. This does not affect holomorphic convexity.

Consider the quotient representation $\rho_C : \pi_1(X) \to G \to C = G/S$.

LEMMA 2.1. – *Every abelian representation $\rho_C : \pi_1(X) \to C$ can be replaced with a another representation $\pi_1(X) \to C$ which is defined over $\overline{\mathbf{Q}}$ and whose kernel is commensurable to* $\ker(\rho_C)$.

Proof. – Since ρ_C is Zariski dense and since every affine complex torus C is a product of finitely many copies of $\mathbf{C}^\times$ it suffices to prove the lemma when $\rho_C : \pi_1(X) \to \mathbf{C}^\times$ is a non-torsion character. In this case the inclusion $\pi_1(X)/\ker(\rho_C) \cong \mathbf{Z} \hookrightarrow \mathbf{C}^\times$ is determined by a number $\lambda \in \mathbf{C}^\times$ which is not a root of unity and hence we can just replace λ by any number in $\overline{\mathbf{Q}}^\times$ which is not a root of unity. □

We need an analogue of the above lemma for the semisimple representation $\rho_S : \pi_1(X) \to S$. We deform $\rho_S : \pi_1(X) \to S$ to a representation that is defined over $\overline{\mathbf{Q}}$.

LEMMA 2.2. – *A sufficiently small neighborhood of $\rho_S : \pi_1(X) \to S$ consists of Zariski dense representations. In particular, every neighborhood of ρ_S contains a Zariski dense representation $\pi_1(X)$ defined over $\overline{\mathbf{Q}}$.*

A different proof of the above statement was given by N. A'Campo and M. Burger in [BC] (Proposition 8.2).

Proof. – We need to show that Zariski denseness is an open condition in the moduli space of representations. In other words we need to show that the subset in the moduli space $M = \mathrm{Hom}(\pi_1(X), S)//S$ consisting of representations with images that are not Zariski dense subgroups in S is a Zariski closed subset.

Let S act faithfully and irreducibly on a vector space V. Consider the induced action of S on the exterior algebra of V, $\Lambda^\bullet V$. Let $F_1, \ldots, F_k$ denote all the irreducible representations of S that appear in $\Lambda^\bullet V$ with a non-zero multiplicity.

CLAIM 2.1. – *Every proper algebraic subgroup J of S fixes some line D in one of the S-modules $F_1, \ldots, F_k$.*

Proof. – Assume that J is a maximal proper algebraic subgroup of S which is not reductive. In view of Morozov's theorem the connected component of the identity in J must be a maximal parabolic but since every parabolic in a simple algebraic group is self normalized it follows that J is connected, i.e. J a maximal parabolic subgroup. Thus J fixes a proper vector subspace in V or equivalently fixes a line in some exterior power of V and hence fixes a line in one of $F_1, \ldots, F_k$.

To deal with the reductive subgroups we invoke a result of M. Larsen [LAR, Proposition 3.7] according to which every reductive subgroup J of S has an invariant line in one of $F_1, \ldots, F_k$. □

Now consider the set of all representations

$$R_{F_i} = \{\rho \in M \mid a_i \circ \rho \text{ fixes } D \in F_i\}.$$

Here $a_i : S \to \mathrm{Aut}(F_i)$ is the natural action and D is a line in F_i. Clearly $R_{F_i} \subset M$ is a closed subvariety. Thus the union of all the R_{F_i}'s is also closed and due to Proposition 2.1 all the representations in the complement of $\cup R_{F_i}$ are Zariski dense in S. This proves the lemma. □

In general $\ker(\rho_S)$ can change drastically under deformations. We give the following example.

Example. – Consider the subgroup $\Gamma \subset SL(2, \mathbf{C}[t, t^{-1}])$ generated by the following two elements

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

For every $\lambda \in \mathbf{C}^\times$ one has the specialization homomorphism $s_\lambda : SL(2, \mathbf{C}[t, t^{-1}]) \to SL(2, \mathbf{C})$ that sends $F(t) \in SL(2, \mathbf{C}[t, t^{-1}])$ to $F(\lambda) \in SL(2, \mathbf{C})$. By restricting the s_λ's to Γ we obtain a family of representations $\rho^\lambda : \Gamma \to SL(2, \mathbf{C})$ parameterized by $\lambda \in \mathbf{C}^\times$. Notice that for a transcendental numbe rλ the specialization morphism s_λ will be injective and hence ρ^λ will be faithful. On the other hand if $\lambda \in \overline{\mathbf{Q}}^\times$ the representation ρ^λ will have an infinite kernel. Namely the sequence of elements $A, BAB^{-1}, B^2AB^{-2}, B^3AB^{-3}, \dots$ in Γ is an infinite sequence of pairwise commuting elements of infinite order. Thus Γ contains $\mathbf{Z}^\infty$. However a group Γ that contains $\mathbf{Z}^\infty$ can not have almost faithful representations in $SL(2, \mathbf{Q}(\lambda))$, $\lambda \in \overline{\mathbf{Q}}^\times$. Indeed, according to [LZ] all finitely generated subgroups of $SL(2, F)$ (F-a number field) have virtually finite cohomological dimension and $\mathbf{Z}^\infty$ certainly does not. There is a different way to see that the representation ρ^λ will have an infinite kernel if $\lambda \in \overline{\mathbf{Q}}^\times$. The matrices $A, BAB^{-1}, B^2AB^{-2}, B^3AB^{-3}, \dots$ are all uppertriangular and since $\lambda \in \overline{\mathbf{Q}}^\times$ we cannot get infinitely many algebraically independent powers of λ.

REMMARK 2.1. – *We should point out that the group Γ considered in the above example is not finitely presented. An interesting question suggested by the referee is if there exists an example as above but with Γ - a finitely presented group. We do not know the answer of this question.*

We use the following construction of adelic type. Since $\pi_1(X)$ is a finitely generated group, $\rho'_S : \pi_1(X) \to S$ is defined over a finite extension E of $\mathbf{Q}$. Let O be the ring of integers in E. Put $O_\mathfrak{p}$ for the localization of O at some prime ideal $\mathfrak{p}$ in O and $\widehat{O}_\mathfrak{p}$ for the completion of $O_\mathfrak{p}$ with respect to the natural discrete valuation. Denote by $E_\mathfrak{p}$ the field of fractions of $\widehat{O}_\mathfrak{p}$ and let $S(E_\mathfrak{p})$ be the group of $E_\mathfrak{p}$-points of S. Since $E_\mathfrak{p}$ is a local field we can use the Bruhat-Tits theory to attach to $S(E_\mathfrak{p})$ an affine building $B_\mathfrak{p}$. Similarly we consider the symmetric space $\mathrm{Symm}_\nu = S(E_\nu)/(\text{maximal compact subgroup})$ at an archimedean place ν. From the fact that $E_\mathfrak{p}$ is discrete in $\prod_\mathfrak{p} E_\mathfrak{p} \times \prod_\nu E_\nu$ we conclude that the image of ρ'_S acts discretely on $\prod_\mathfrak{p} B_\mathfrak{p} \times \prod_\nu \mathrm{Symm}_\nu$ via the diagonal action (see [BS]).

By the results of Corlette [C] (Corollary 3.5) and Gromov and Schoen [GS] (Theorem 7.3) there exist $\pi_1(X)$-equivariant pluriharmonic maps $U_\nu : \widetilde{X} \to \mathrm{Symm}_\nu$ and $U_\mathfrak{p} : \widetilde{X} \to B_\mathfrak{p}$ respectively. Clearly only finitely many among the maps $U_\mathfrak{p}$ and U_ν are nontrivial since $\rho'_S(\pi_1(X))$ is finitely generated and hence only finitely many elements of O occur in the denominators of the traces of its generators. It is easy to see that at least one of these maps is not constant. Otherwise we will get that the image of $\pi_1(X)$ is contained in a maximal compact subgroup for each of the representations $\pi_1(X) \xrightarrow{\rho'_S} S(E) \to S(E_\mathfrak{p})$ and $\pi_1(X) \xrightarrow{\rho'_S} S(E) \to S(E_\nu)$. Therefore $\rho'_S(\pi_1(X))$ will be a compact discrete group - so it is finite and this contradicts the Zariski density of our representation.

In the same way we handle the representation $\rho_C : \pi_1(X) \to C$ modified according to Lemma 2.1. The image of $\rho_C : \pi_1(X) \to C$ is defined over some number field E and repeating the above procedure we again get a discrete representation in some torus.

Finally in order to make the whole reductive representation $\rho' : \pi_1(X) \to G$ act discretely on a nonpositively curved space we need to take the diagonal action of C and of S on the product N of $\mathbf{C}^n$ and $\prod_\mathfrak{p} B_\mathfrak{p} \times \prod_\nu \mathrm{Symm}_\nu$. To summarize, for every $\rho' : \pi_1(X) \to G$, defined over $\bar{\mathbf{Q}}$ we obtain a nonpositively curved space $N(\rho')$ equipped with a properly discontinuous action of $\rho'(\pi_1(X)$ and a pluriharmonic map $U : \widetilde{X} \to N(\rho')$ which is $\pi_1(X)$-equivariant.

3. The proof of the main theorem

In this section we prove the Main theorem. To simplify the exposition we will assume that $X' = \widetilde{X}$ is the universal covering of X. The Hodge theorem implies that $\widetilde{X}$ does not have two ends. The proof for Galois covering $X' \longrightarrow X$ of a compact Kähler surface X with a reductive Galois group Γ, which does not have two ends, is the same.

We will need the following technical lemma which is well known to the experts. It is an easy consequence of the work of Narasimhan [N].

LEMMA 3.1. – *Let M be a complex manifold and let $\chi : M \to \mathbf{R}$ be a continuous plurisubharmonic exhaustion function, which is strictly plurisubharmonic exhaustion function away from a proper complex analytic set $A \subset M$. If all noncompact irreducible components of A are Stein spaces then M is holomorphically convex.*

Proof. – Let A' be the set of all points x in A such that all irreducible components of A passing through x are compact. Observe that A' is also an analytic subset of M. By definition every irreducible component of A' is compact. Moreover every connected component of A' is compact. Indeed if $Y \subset A'$ is a connected component then $\chi(Y) = \text{const}$. But χ is an exhaustion function and hence Y is compact. The fact that A' is an analytic subset implies that $\chi(A')$ is a countable and discrete set in $\mathbf{R}$. Let $\{c_i\}_{i=1}^{\infty} \subset \mathbf{R}$ such that $\{c_i\}_{i=1}^{\infty} \cap \chi(A') = \varnothing$. Define

$$M_{c_i} := \{x \in M \mid \chi(x) < c_i\}.$$

Step 1. – We will show that the sets M_{c_i} are holomorphically convex.

Let $A_1, \ldots, A_k$ be the the Stein irreducible components of A intersecting the closure of M_{c_i}. It follows from a theorem of Siu [S], (Main theorem p. 89) (see also Demailly [DEM] (Theorem 1)), that for every A_s, $s = 1, \ldots, k$ there exists a Stein neighborhood $A_s \subset U_s \subset M$ and thus we have strictly plurisubharmonic functions f_s on every U_s. The set M_{c_i} was chosen so that all irreducible components of A, which intersect its boundary are Stein. By adding a large multiple of χ to f_s multiplied by a cutoff function we get a function ψ_i which is strictly plurisubharmonic outside a compact subset in M_{c_i}. To show that M_{c_i} are holomorphically convex we apply the following.

THEOREM (H. Grauert, [GR]). – *Let U be an open analytic manifold and let $\psi : U \longrightarrow \mathbf{R}$ be a continuous exhaustion function which is strictly plurisubharmonic outside of a compact set. Then U is holomorphically convex.*

Step 2. – We have a sequence of holomorphically convex sets $M_{c_1} \subset M_{c_2} \subset \ldots$. Since the sets M_{c_i} are holomorphically convex they all admit Cartan - Remmert reductions $S(M_{c_i})$.

CLAIM. – *For every i the set $S(M_{c_i})$ is Runge in $S(M_{c_{i+1}})$ and as a consequence M_{c_i} is Runge in $M_{c_{i+1}}$.*

Recall that for two domains $A \subset B$ we say that A is Runge in B if for every holomorphic function f on A and for every compact subset $K \subset A$ there exists a sequence of holomorphic functions on B which approximates f uniformly on K.

Proof. – Observe that the function ψ_{i+1}, defined on $M_{c_{i+1}}$, descends to a function on $S(M_{c_{i+1}})$. Now consider $S(M_{c_{i+1}})$ and the subset $N \subset S(M_{c_{i+1}})$ such that $\psi_{i+1} < c_i$

on N. N is a Stein space and as it follows from [N] (Corollary 1, page 211) it is Runge in $S(M_{c_{i+1}})$. Applying the maximum principle to ψ_{i+1} we get that $N = S(M_{c_i})$. □

Step 3. – Now we finish the proof of the lemma. Let us denote by ST the union of all $S(M_{c_i})$. Due to [N, page 198, (1.3)] ST is a Stein space.

We have a holomorphic map $F : M \to ST$. To show that M is holomorphically convex it is enough to show that this map is proper. If K is a compact subset in ST it is contained in some $S(M_{c_i})$. Therefore the fibers of F over K are compact. □

Now we are ready to prove our main theorem. Let G be a reductive algebraic group and let $\rho : \pi_1(X) \to G(\mathbf{C})$ be a faithful Zariski dense representation in the complex points of G. Lemmas 2.1 and 2.2 imply that all $\overline{\mathbf{Q}}$-points in a neighborhood W of ρ in the moduli space of representations will have Zariski dense images. For every representation ρ' defined over $\overline{\mathbf{Q}}$ we have a nonpositively curved space $N(\rho')$ equipped with a properly discontinuous action of $\rho'(\pi_1(X))$ and a ρ'-equivariant pluriharmonic map $U : \widetilde{X} \to N(\rho')$. (Observe that we have in $N(\rho')$ only finitely many buildings or symmetric spaces with nontrivial pluriharmonic maps to them.)

We consider two cases:

Case 1. – There exists a representation ρ' defined over $\overline{\mathbf{Q}}$ for which $U : \widetilde{X} \to N(\rho')$ has rank equal to two, i.e. the dimension of the leaves of the foliation given by U is generically zero.

Denote by $\widetilde{Y}$ the Galois covering that corresponds to $\rho'(\pi_1(X))$. We have a ρ'-equivariant pluriharmonic map $U : \widetilde{Y} \to N(\rho')$. Let $o \in N(\rho')$ be a point and let $\varphi = V \circ (\mathrm{dist}(\bullet, o)^2)$. Since $N(\rho')$ is nonpositively curved and U is pluriharmonic it follows that φ will be plurisubharmonic on $\widetilde{Y}$. Furthermore the fact that $\rho'(\pi_1(X))$ acts properly discontinuously on $N(\rho')$ implies that φ is an exhaustion function on $\widetilde{Y}$. We apply Lemma 3.1 to $M = \widetilde{Y}$ and $\varphi = \chi$ and get that in this case the Cartan-Remmert reduction of X is two dimensional. Therefore by a theorem of T. Napier [N1, Theorem 6.1] to prove that $\widetilde{X}$ is hollomorphically convex it is sufficient to show that there are no infinite chains of compact curves in $\widetilde{X}$.

Suppose that there exists a chain of curves Z that is compact and has finitely many components in $\widetilde{X}$ and that opens up and becomes an infinite chain of compact curves in $\widetilde{X}$. This is equivalent to saying that $\mathrm{im}[\pi_1(Z) \to \pi_1(X)]$ is infinite and that the image of the fundamental group of every component of Z is finite in $\pi_1(X)$. Theorem 1.4 implies that $\rho(\pi_1(Z))$ is finite and since ρ is faithful we get that $\mathrm{im}[\pi_1(Z) \to \pi_1(X)]$ is finite, which is a contradiction. □

Therefore we assume that we are in:

Case 2. – There exists a neighborhood W of ρ so that for every representation $\rho' \in W$ defined over $\overline{\mathbf{Q}}$ the rank of the pluriharmonic map $U : \widetilde{X} \to N(\rho')$ is one, i.e. the dimension of the generic leaf of the foliation given by U is equal to one.

Suppose that all maps to the building factors in $N(\rho')$ are trivial. Then we have a map $U : \widetilde{X} \to N(\rho')$ just to a product of symmetric spaces and so the pullback of the distance function via U will be a real-analytic plurisubharmonic exhaustion function. Furthermore, the image of ρ' cannot have two ends since ρ' is a Zariski dense representation into a semisimple group whereas groups with two ends always contain an infinite cyclic subgroup of finite index. Thus we are in a position to apply theorem 1.1 which gives us that some intermediate covering X' of $\widetilde{X} \to X' \to X$ is holomorphically convex. To finish the argument we can rule out infinite chains and again use [N1, Theorem 6.1] exactly as in case 1.

Therefore we can assume that there exists a nontrivial map to a building in $N(\rho')$. Using theorem 1.3 we conclude that we have a factorization of the representation ρ'. Thus we have a holomorphic map to an orbicurve $Y_{\rho'}$. Now we consider the map $X \to \Pi_{\rho'} Y_{\rho'}$. Using a theorem of Cartan's [Ca, Main theorem, page 7] we conclude that the image of this map has dimension one or two.

1) If this image is one dimensional we conclude that:

CLAIM. – *Our original representation ρ factors through curve Y_ρ.*

Proof. – Indeed the fact that the image of the map $X \to \Pi_{\rho'} Y_{\rho'}$ is one dimensional implies that there exists a neighborhood W of ρ so that for every representation $\rho' \in W$ defined over $\overline{\mathbf{Q}}$ we have a factorization through the same curve which we denote by Y_ρ.

To prove the claim we need to check that image I of the fundamental group of a generic leave of the foliation $X \to Y_\rho$ belongs to the $ker(\rho)$ or in other words this image is trivial in $\pi_1(X)$. But by construction I is contained in the intersection of the kernels of all representation $\rho' \in W$ defined over $\overline{\mathbf{Q}}$ and therefore is a trivial group. □

The above claim implies that $\widetilde{X}$ maps properly to an infinite covering of $Y_{\rho'}$ and therefore is holomorphically convex.

2) If this image is two dimensional we have that there are two different representation ρ' and ρ'' for which we have a factorizations $X \to Y_{\rho'}$ and $X \to Y_{\rho''}$ through two different Riemann surfaces $Y_{\rho''}$ and $Y_{\rho'}$. Therefore we get a proper map from some intermediate covering X', $\widetilde{X} \to X' \to X$ to a product of two infinite coverings of $Y_{\rho'}$ and $Y_{\rho''}$. Thus $\widetilde{X'}$ is holomorphically convex and we finish the argument as in case 1. □

4. Some examples

All known fundamental groups of smooth projective surfaces have Zariski dense representations in complex reductive groups. Some of these representations are faithful. We would like to mention that if we start with the fundamental group of some smooth projective variety X, $\pi_1(X)$ that has a faithful Zariski dense representation $\pi_1(X) \longrightarrow G$ to some simple Lie group G we can obtain an infinite series of examples to the Main theorem. For this we apply the construction of Catanese, Kollár, Nori, Toledo (see [CKNT]) where they show that one can construct a finite ramified cover $X' \longrightarrow X$ such that $\pi_1(X')$ is an extension of $\pi_1(X)$ by a finite cyclic group:

$$1 \longrightarrow \mathbf{Z}_r \longrightarrow \pi_1(X') \longrightarrow \pi_1(X) \longrightarrow 1.$$

The construction goes as follows. Using a theorem of Goresky, MacPherson (see e.g. [GM], II,1.1) we can and restrict ourselves to the case when X is an algebraic surface. Let L be a very ample line bundle on X which is trivial on $\widetilde{X}$. Let F_1, F_2, F_3, be three sections of L and F_1', F_2', F_3' are their pullbacks on the universal cover of X, $\widetilde{X}$. Define Y to be given in $\widetilde{X} \times \mathbf{C}^3$ by the equations:

$$F_i' = z_i^r, i = 1, 2, 3,$$

where z_1, z_2, z_3, are the coordinates on $\mathbf{C}^3$. The covering $Y \longrightarrow \widetilde{X}$ is a composite of three cyclic coverings with smooth branch loci so it is simply connected. Now it is easy to check that X' is a quotient of Y by $\pi_1(X')$. As a starting point for such an infinite series of examples one can take e.g. a cocompact lattice in $SO^0(2,n)$ (see e.g. [CKNT]).

Interesting examples to which we can apply the Main theorem are Simpson's examples ([SIM]). The construction of these examples goes as follows. Let Z be a smooth projective variety of odd dimension $n+1$. Let L be a sufficiently ample line bundle on Z. Define $\mathbf{P}^n = \mathbf{P}(H^0(Z,L))$ be the projective space of lines. Let D be the locus of singular hyperplane sections in $\mathbf{P}^n$. Choose a general $\mathbf{P}^2 \subset \mathbf{P}^n$. Let $D_1 = D \cap \mathbf{P}^2$, let $U = \mathbf{P}^n - D$ and let $U_1 = \mathbf{P}^2 - D_1$. Define

$$U \longleftarrow X_U \longrightarrow Z$$

to be the incidence variety. Suppose W is a rank one local system on Z. Let $V = R^n f_*(a^*W)$ be the pullback and the following pushforward from Z to U. (Here f is the map $f : X_U \longrightarrow Z$ and a $a : X_U \longrightarrow U$.) According to Simpson [SIM] one can extend V on some compact two sheeted ramified covering $s : S \longrightarrow U_1$ and get a local system on S this way.

The Main theorem also gives the following:

COROLLARY 4.1. – *Let X be a smooth projective surface which has a Zariski dense almost faithful representation ρ of its fundamental group to a reductive complex Lie group G and such that $rank NS(X) = 1$. Then for every subvariety Z in X we have that $im[\pi_1(Z) \longrightarrow \pi_1(X)]$ is infinite.*

The proof is an easy consequence of [K2, Chapter 1, Corollary 1.10].

Finally we would like to mention that for some applications of the Main theorem one might need to relax the smoothness condition. It is easy to see from the proof that the statement of the Main theorem remains true if we require that X is normal.

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