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CAPELLI IDENTITIES FOR LIE SUPERALGEBRAS

BY MAXIM NAZAROV

ABSTRACT. – We study a distinguished basis in the centre of the enveloping algebra of the Lie superalgebra $\mathfrak{q}_N$. The irreducible polynomial representations of $\mathfrak{q}_N$ are labelled by decreasing sequences of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell \leq N$. The elements of our basis are labelled by the same sequences. The basic element C_λ of the centre vanishes in the irreducible representation corresponding to a sequence μ if and only if $\lambda_i > \mu_i$ for some i. We obtain an explicit formula for the element C_λ.

RÉSUMÉ. – Nous étudions une base distinguée du centre de l'algèbre enveloppante de la super-algèbre de Lie $\mathfrak{q}_N$. Les représentations polynomiales irréductibles de $\mathfrak{q}_N$ sont indexées par des suites décroissantes d'entiers positifs $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ où $\ell \leq N$. Les éléments de notre base sont indexés par les mêmes suites. L'élément C_λ de cette base s'annule dans la représentation correspondante à une suite μ si et seulement si $\lambda_i > \mu_i$ pour quelque i. Nous obtenons une formule explicite pour l'élément C_λ.

1. Introduction

The Capelli identity [1] is one of the best exploited results of the classical invariant theory. It provides a set of distinguished generators $C_1, \ldots, C_N$ for the centre of the enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ of the general linear Lie algebra. For any non-negative integer M consider the natural action of the Lie algebra $\mathfrak{gl}_N$ in the vector space $\mathbb{C}^N \otimes \mathbb{C}^M$. Extend it to the action of the algebra $\mathrm{U}(\mathfrak{gl}_N)$ in the space of polynomial functions on $\mathbb{C}^N \otimes \mathbb{C}^M$. The resulting representation of $\mathrm{U}(\mathfrak{gl}_N)$ by differential operators on $\mathbb{C}^N \otimes \mathbb{C}^M$ with polynomial coefficients is faithful when $M \geqslant N$.

The image of the centre $\mathrm{Z}(\mathfrak{gl}_N)$ of the algebra $\mathrm{U}(\mathfrak{gl}_N)$ under this representation coincides with the ring $\mathcal{J}$ of $\mathfrak{gl}_N \times \mathfrak{gl}_M$-invariant differential operators on $\mathbb{C}^N \otimes \mathbb{C}^M$ with polynomial coefficients. The latter ring has a distinguished set of generators $\Omega_1, \ldots, \Omega_L$ with $L = \min(M, N)$ which are called the *Cayley operators* [2]. If x_{ib} with $i = 1, \ldots, N$ and $b = 1, \ldots, M$ are the standard coordinates on $\mathbb{C}^N \otimes \mathbb{C}^M$ and ∂_{ib} are the corresponding partial derivations, then Ω_n equals

$$(1.1) \qquad \sum_{g \in S_n} \sum_{i_1, \ldots, i_n} \sum_{b_1, \ldots, b_n} \mathrm{sgn}(g) \cdot x_{i_1 b_1} \ldots x_{i_n b_n} \partial_{i_{g(1)} b_1} \ldots \partial_{i_{g(n)} b_n} .$$

Here $\mathrm{sgn}(g)$ stands for the sign of the element g of the symmetric group S_n. The Capelli identity gives explicit formula for a preimage in $\mathrm{Z}(\mathfrak{gl}_N)$ of the operator I_n. Let E_{ij} be the standard generators of the enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ so that in the above representation

$$E_{ij} \mapsto \sum_b x_{ib}\,\partial_{jb}\,.$$

The Cayley operator Ω_n is then the image of the element $C_n \in \mathrm{Z}(\mathfrak{gl}_N)$ equal to

$$\sum_{g \in S_n} \sum_{i_1,\ldots,i_n} \operatorname{sgn}(g) \cdot \prod_s^{\rightarrow} \left(E_{i_s i_{g(s)}} + (s-1)\cdot \delta_{i_s i_{g(s)}}\right) \tag{1.2}$$

where the index s runs through $1,\ldots,n$ and factors in the ordered product are arranged from the left to right while s increases. Here δ_{ij} is the Kronecker delta.

Let χ be the irreducible character of the symmetric group S_n corresponding to any partition of n into not more than M, N parts. Let us normalize χ so that $\chi(1) = 1$. When we replace the character sgn in (1.1) by χ we obtain an element of a distinguished basis in the vector space $\mathcal{J}$; see [3]. Explicit formula for a preimage in $\mathrm{Z}(\mathfrak{gl}_N)$ of this basic element was given in [4, 5]. This result generalizes classical formula (1.2) and may be called the *higher Capelli identity* [4]. In the present article we extend this result to the queer Lie superalgebra $\mathfrak{q}_N$ [6]. In particular, we obtain an analogue for $\mathfrak{q}_N$ of the classical Capelli identity. Such an analogue has been so far unknown. See, however [7, 8] for related results.

The symmetric group S_n appears in the formulas (1.1) and (1.2) because its permutational action in the tensor product $(\mathbb{C}^N)^{\otimes n}$ generates the commutant of the action of the Lie algebra $\mathfrak{gl}_N$. For the n-th tensor power of the defining representation of the Lie superalgebra $\mathfrak{q}_N$ the role of S_n is played [9] by the semidirect product $S_n \ltimes A_n$ where A_n is the Clifford algebra with anticommuting generators $a_1,\ldots,a_n$. We will denote this product by H_n and call it the Sergeev algebra. Note that to obtain the higher Capelli identity one replaces sgn in (1.2) by a diagonal matrix element relative to the Young orthogonal basis in the irreducible representation with character χ. In Section 2 we give an analogue of this matrix element for any irreducible representation of H_n.

The irreducible H_n-modules are parametrized by the strict partitions λ of n. Note that the algebra H_n has a natural $\mathbb{Z}_2$-gradation, and we use the notion of a $\mathbb{Z}_2$-graded irreducibility. Let ℓ_λ be the number of parts in λ. For each λ we construct a certain element $\Psi_\lambda \in H_n$ such that under the left regular action of H_n the space $H_n \cdot \Psi_\lambda$ splits into a direct sum of $2^{[\ell_\lambda/2]}$ copies of the irreducible module corresponding to λ; see Theorem 3.4 and the subsequent remark. Moreover, Ψ_λ is invariant with respect to the natural involutive antiautomorphism of the algebra H_n; see Lemma 2.3. If $\lambda = (n)$ then

$$\Psi_\lambda = \prod_{1\leqslant r<n}^{\rightarrow} \left(\prod_{r<s\leqslant n}^{\rightarrow} \left(1 - \frac{(r\,s)}{u_r - u_s} + \frac{(r\,s)\cdot a_r\,a_s}{u_r + u_s}\right)\right) \tag{1.3}$$

where $u_s = \sqrt{s(s-1)}$ while $(r\,s) \in S_n$ is the transposition of r and s. To give an explicit formula for Ψ_λ with a general λ we use the fusion procedure [10]; see Theorem 2.2 here. More detailed exposition of this construction appeared in [11].

In Section 3 we introduce our main technical tool – the Jucys-Murphy elements of the algebra H_n. These are the pairwise commuting elements $x_1, \ldots, x_n$ defined by (3.1); see also Lemma 3.2. With respect to the left regular action of H_n the element Ψ_λ is a joint eigenvector of $x_1, \ldots, x_n$ and the corresponding eigenvalues are easy to describe; see Proposition 3.3. Our proof of the Capelli identity for $\mathfrak{q}_N$ will be based on Proposition 3.6; cf. [12,13]. Namely, we will use Corollary 3.7.

Section 4 contains the main result of this article. We define actions of the Lie superalgebras $\mathfrak{q}_N$ and $\mathfrak{q}_M$ in the free supercommutative algebra $\mathcal{P}$ with the even generators x_{ib} and odd generators $x_{-i,b}$ where $i = 1, \ldots, N$ and $b = 1, \ldots, M$; see Proposition 4.1. Let $\mathcal{PD}$ be the algebra generated by the operators of left multiplication in $\mathcal{P}$ by $x_{\pm i,b}$ and by the corresponding left derivations $\partial_{\pm i,b}$. So we get a representation $\gamma : \mathrm{U}(\mathfrak{q}_N) \to \mathcal{PD}$ for the enveloping algebra of $\mathfrak{q}_N$. The image of the centre $\mathrm{Z}(\mathfrak{q}_N) \subset \mathrm{U}(\mathfrak{q}_N)$ with respect to γ coincides with the ring $\mathcal{I}$ of $\mathfrak{q}_N \times \mathfrak{q}_M$-invariants in $\mathcal{PD}$. We introduce a distinguished basis in the vector space $\mathcal{I}$; see Proposition 4.3. The elements of this basis are parametrized by the strict partitions λ of $n = 0, 1, 2, \ldots$ with $\ell_\lambda \leqslant M, N$ and are determined by (4.7). Our main result is the explicit formula (4.6) for a preimage $C_\lambda \in \mathrm{Z}(\mathfrak{q}_N)$ of the basic element $I_\lambda \in \mathcal{I}$ corresponding to λ; see also (4.5). The equality $I_\lambda = \gamma(C_\lambda)$ with $\lambda = (n)$ may be regarded as an analogue for $\mathfrak{q}_N$ of the classical Capelli identity.

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2. Fusion procedure for the Sergeev algebra

We start with recalling several known facts about irreducible modules over the *Sergeev algebra* H_n. By definition, H_n is the semidirect product of the symmetric group S_n and the Clifford algebra A_n with n generators over the complex field $\mathbb{C}$. These generators are denoted by $a_1, \ldots, a_n$ and subjected to the relations

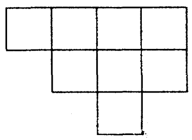
$$a_i^2 = -1\,; \qquad a_i\, a_j = -a_j\, a_i\,, \quad i \neq j.$$

The symmetric group S_n acts on the algebra A_n by permutations of these n generators. Denote by the superscript $*$ the involutive antiautomorphism of the algebra H_n defined by the assignments $a_i \mapsto a_i^{-1}$ and $g \mapsto g^{-1}$ for any $g \in S_n$.

For any $\mathbb{Z}_2$-graded algebra A a representation $\mathrm{A} \to \mathrm{End}(\mathbb{C}^{K|K})$ will be called *irreducible* if the even part of its supercommutant equals $\mathbb{C}$. If the supercommutant coincides with $\mathbb{C}$ this representation is called *absolutely irreducible*. We will equip the algebra H_n with a $\mathbb{Z}_2$-gradation so that $\deg a_i = 1$ and $\deg g = 0$ for any element $g \in S_n$. The irreducible modules over the $\mathbb{Z}_2$-graded algebra H_n are parametrized by partitions λ of n with pairwise distinct parts. Such a partition is called *strict*. The H_n-module U_λ corresponding to λ is absolutely irreducible if and only if the number ℓ_λ of non-zero parts in λ is even [9, Lemma 6].

Consider the left regular representation of the algebra H_n. In this section for any strict partition λ we will construct a certain element $\Psi_\lambda \in H_n$ such that the left ideal $H_n \cdot \Psi_\lambda$ is a direct sum of $2^{[\ell_\lambda/2]}$ copies of the irreducible H_n-module corresponding to λ. Moreover, we will have the equality $\Psi^*_\lambda = \Psi_\lambda$. Our construction is motivated by the results of [10]; see [11, Section 3] for more detail.

Strict partitions are usually depicted as shifted Young diagrams. For instance, here is the diagram corresponding to the partition $\lambda = (4, 3, 1)$:



We will denote by Λ the shifted *column tableau* of the shape λ. It is obtained by filling the boxes of λ with the numbers $1, \ldots, n$ by columns from the left to the right, downwards in every column. For each $i = 1, \ldots, n$ we put $c_i = p - q$ if the number i appears in the p-th column and q-th row of the tableau Λ. The difference $p - q$ is then called the *content* of the box of the diagram λ occupied by the number i. For example, here on the left we show the shifted column tableau of the shape $\lambda = (4, 3, 1)$:

1	2	4	7
	3	5	8
		6	

0	1	2	3
	0	1	2
		0	

On the right we indicated the contents of the boxes of the shifted Young diagram.

For any distinct $i, j = 1, \ldots, n$ let (ij) be the transposition in the symmetric group S_n. Consider the rational function of two complex variables u, v valued in the algebra H_n

$$\varphi_{ij}(u, v) = 1 - \frac{(ij)}{u - v} + \frac{(ij) \cdot a_i a_j}{u + v}.$$

As a direct calculation shows, this rational function satisfies the equations

$$\varphi_{ij}(u, v)\, \varphi_{ik}(u, w)\, \varphi_{jk}(v, w) = \varphi_{jk}(v, w)\, \varphi_{ik}(u, w)\, \varphi_{ij}(u, v) \tag{2.1}$$

for all pairwise distinct i, j, k. Evidently, for all pairwise distinct i, j, k, l we have

$$\varphi_{ij}(u, v)\, \varphi_{kl}(z, w) = \varphi_{kl}(z, w)\, \varphi_{ij}(u, v) \tag{2.2}$$

We will also make use of the relations for all distinct i, j

$$\varphi_{ij}(u, v)\, \varphi_{ji}(v, u) = 1 - \frac{1}{(u - v)^2} - \frac{1}{(u + v)^2}\,; \tag{2.3}$$

$$a_i\,\varphi_{ij}(u,v)\,a_i^{-1}=\varphi_{ij}(-u,v), \tag{2.4}$$

$$a_j\,\varphi_{ij}(u,v)\,a_j^{-1}=\varphi_{ij}(u,-v). \tag{2.5}$$

Note that due to (2.3) the element $\varphi_{ij}(u,v)\in H_n$ is not invertible if and only if

$$\frac{1}{(u-v)^2}+\frac{1}{(u+v)^2}=1. \tag{2.6}$$

Observe also that in the latter case the element $\varphi_{ij}(u,v)/2\in H_n$ is an idempotent.

Consider the rational function of u,v,w appearing at either side of (2.1). Denote by $\varphi_{ijk}(u,v,w)$ this function. The factor $\varphi_{ik}(u,w)$ in (2.1) has a pole at $u=\pm w$. Still we have the following lemma. It will be the basis for constructing $\Psi_\lambda\in H_n$.

LEMMA 2.1. – *Restriction of $\varphi_{ijk}(u,v,w)$ to the set of (u,v,w) such that the pair (u,v) satisfies the condition* (2.6), *is continuous at $u=\pm w$.*

Proof. – Assume that the the condition (2.6) is satisfied. Then due to (2.3) and (2.5)

$$\varphi_{ij}(u,v)\,\varphi_{ji}(v,u)=0\,,\qquad \varphi_{ij}(u,v)\,a_i\,\varphi_{ji}(v,-u)=0\,.$$

Hence the product $\varphi_{ijk}(u,v,w)$ can be rewritten as

$$\varphi_{ij}(u,v)\,\varphi_{jk}(v,w)-\varphi_{ij}(u,v)\,\frac{(ik)}{u-w}\,\big(\varphi_{jk}(v,w)-\varphi_{jk}(v,u)\big)+$$
$$\varphi_{ij}(u,v)\,\frac{(ik)\,a_i\,a_k}{u+w}\,\big(\varphi_{jk}(v,w)-\varphi_{jk}(v,-u)\big).$$

Under the condition (2.6) the latter function is continuous at $u=\pm w$ □

For any two real non-negative variables s,t let us substitute in the equation (2.6)

$$u=\sqrt{s(s+1)},\quad v=\sqrt{t(t+1)}.$$

Observe that (2.6) will be then satisfied if $s-t=\pm1$. We will denote for short

$$\psi_{ij}(s,t)=\varphi_{ij}\big(\sqrt{s(s+1)},\sqrt{t(t+1)}\,\big),$$

$$\psi_{ijk}(s,t,r)=\varphi_{ijk}\big(\sqrt{s(s+1)},\sqrt{t(t+1)},\sqrt{r(r+1)}\,\big).$$

Now introduce a real non-negative parameter t_i for each $i=1,\dots,n$. Equip the set of all pairs (i,j) where $1\leqslant i<j\leqslant n$ with the lexicographical ordering. Introduce the ordered product over this set

$$\prod_{(i,j)}^{\rightarrow}\psi_{ij}(c_i+t_i\,,c_j+t_j). \tag{2.7}$$

Consider this product as a function of the parameters $t_1,\dots,t_n$ valued in the algebra H_n. Let us denote by $\Psi_\lambda(t_1,\dots,t_n)$ this function. It may have singularities when $c_i + t_i = c_j + t_j$ for some $i \neq j$. Consider the set $\mathcal{T}$ of all tuples $(t_1, \ldots, t_n)$ such that $t_i = t_j$ whenever the numbers i and j appear in the same row of the tableau Λ. So $\mathcal{T} = \mathbb{R}^{\ell_\lambda}_{\geqslant 0}$. Next theorem goes back to [10] and [11, Theorem 5.6].

THEOREM 2.2. – *Restriction of* $\Psi_\lambda(t_1, \ldots, t_n)$ *to* $\mathcal{T}$ *is continuous at* $t_1 = \ldots = t_n$.

Proof. – We shall provide an expression for the restriction of the function (2.7) to $\mathcal{T}$ which is manifestly continuous at $t_1 = \ldots = t_n$. Let us reorder the pairs (i, j) in the product (2.7) as follows. This reordering will not affect the value of the product due to the relations (2.1) and (2.2). Let $\mathcal{C}$ be the sequence of numbers obtained by reading the tableau Λ in the usual way, that is by rows from the top to the bottom, eastwards in every row. For each $j = 1, \ldots, n$ denote by $\mathcal{A}_j$ and $\mathcal{B}_j$ the subsequences of $\mathcal{C}$ consisting of all numbers $i < j$ which appear respectively after and before j in that sequence. Now set $(i, j) \prec (k, l)$ if one of the following conditions is satisfied:

– the number i appears in $\mathcal{B}_j$ while k appears in $\mathcal{A}_l$;

– the numbers i and k appear respectively in $\mathcal{B}_j$ and $\mathcal{B}_l$ where $j < l$;

– the numbers i and k appear respectively in $\mathcal{A}_j$ and $\mathcal{A}_l$ where $j > l$;

– we have the equality $j = l$ and i appears before k in $\mathcal{B}_j$ or $\mathcal{A}_j$.

From now on we assume that the factors in (2.7) corresponding to the pairs (i, j) are arranged with respect to this new ordering. The factor $\psi_{ij}(c_i + t_i, c_j + t_j)$ has a singularity at $t_i = t_j$ if and only if i and j stand on the same diagonal of the tableau Λ. We will then call the pair (i, j) *singular.* Observe that the number i occurs in the subsequence $\mathcal{A}_j$ exactly when i stands to the left and below of j in the tableau Λ. In this case $c_j - c_i > 1$ and the pair (i, j) cannot be singular.

Let a singular pair (i, j) be fixed. Suppose that the number i appears in the p-th column and the q-th row of the tableau Λ. In our new ordering the next pair after (i, j) is (h, j) where the number h appears in the $(p+1)$-th column and the q-th row of Λ. In particular, we have $c_i = c_j = c_h - 1$. Moreover, $(i, h) \prec (i, j)$. Due to the relations (2.1), (2.2) the product

$$\prod_{(k,l) \prec (i,j)}^{\longrightarrow} \psi_{kl}(c_k + t_k, c_l + t_l)$$

is divisible on the right by $\psi_{ih}(c_i + t_i, c_h + t_h)$; cf. [11, p. 222]. Note that each value of the restriction of $\psi_{ih}(c_i + t_i, c_h + t_h)/2$ to $t_i = t_h$ is an idempotent in H_n.

Now for each singular pair (i, j) let us replace the two adjacent factors in (2.7)

$$\psi_{ij}(c_i + t_i, c_j + t_j)\, \psi_{hj}(c_h + t_h, c_j + t_j)$$

by

$$\begin{aligned} (2.8) \qquad & \psi_{ih}(c_i + t_i, c_h + t_h)\, \psi_{ij}(c_i + t_i, c_j + t_j)\, \psi_{hj}(c_h + t_h, c_j + t_j)/2 \\ & = \psi_{ihj}(c_i + t_i, c_h + t_h, c_h + t_h)/2. \end{aligned}$$

This replacement does not affect the value of restriction to $\mathcal{T}$ of the function (2.7). But the restriction to $t_i = t_h$ of (2.8) is continuous at $t_i = t_j$ by Lemma 2.1 □

The process of continuation of the function $\Psi_\lambda(t_1,\ldots,t_n)$ along the set $\mathcal{T}$ is called the *fusion procedure*. In the proof of Theorem 2.2 we established the decomposition

$$\Psi_\lambda(t_1,\ldots,t_n) = \Upsilon_\lambda(t_1,\ldots,t_n)\cdot\Theta_\lambda(t_1,\ldots,t_n)$$

where $\Upsilon_\lambda(t_1,\ldots,t_n)$ and $\Theta_\lambda(t_1,\ldots,t_n)$ are products of the factors in (2.7) which correspond to the pairs (i,j) with i appearing in $\mathcal{B}_j$ and $\mathcal{A}_j$ respectively. The function $\Theta_\lambda(t_1,\ldots,t_n)$ is continuous at $t_1 = \ldots = t_n$. Moreover, any value of this function at $t_1 = \ldots = t_n$ is invertible. Let us denote by Θ_λ the value $\Theta_\lambda(0,\ldots,0)$. Restriction to $\mathcal{T}$ of the function $\Upsilon_\lambda(t_1,\ldots,t_n)$ is continuous at $t_1 = \ldots = t_n$ as well as the restriction of the function $\Psi_\lambda(t_1,\ldots,t_n)$. Denote respectively by Υ_λ and Ψ_λ the values of these restrictions at $t_1 = \ldots = t_n = 0$. Then $\Psi_\lambda = \Upsilon_\lambda\Theta_\lambda$. Let α be the linear map $H_n \to A_n$ identical on A_n such that $\alpha(ga) = 0$ for $g \neq 1$ in S_n and any element $a \in A_n$.

LEMMA 2.3. – *We have* $\Psi_\lambda^* = \Psi_\lambda$ *and* $\alpha(\Psi_\lambda) = 1$.

Proof. – By the definition of the antiautomorphism $*$ we have $\varphi_{ij}(u,v)^* = \varphi_{ij}(u,v)$ for any distinct indices i and j. Therefore due to the relations (2.1) and (2.2) the product (2.7) is invariant with respect to this antiautomorphism. So is the value Ψ_λ of its restriction to $\mathcal{T}$. Further, we have the equality $\alpha\big(\Psi_\lambda(t_1,\ldots,t_n)\big) = 1$ by the definition (2.7). Hence $\alpha(\Psi_\lambda) = 1$ □

Throughout this article we will denote $z_i = \sqrt{c_i(c_i+1)}$ for $i = 1,\ldots,n$.

PROPOSITION 2.4. – *Let the numbers* $k < l$ *stand next to each other in one row of the tableau* Λ. *Then the element* $\Upsilon_\lambda \in H_n$ *is divisible on the right by* $\varphi_{kl}(z_k,z_l)$.

Proof. – Due to the relations (2.1) and (2.2) the restriction of $\Upsilon_\lambda(t_1,\ldots,t_n)$ to $\mathcal{T}$ is divisible on the right by $\psi_{kl}(c_k+t_k, c_l+t_l)$. Here $t_k = t_l$ and the element $\psi_{kl}(c_k+t_k, c_l+t_l)/2 \in H_n$ is an idempotent. Restriction of $\Upsilon_\lambda(t_1,\ldots,t_n)$ to $\mathcal{T}$ is continuous at $t_1 = \ldots = t_n = 0$. So Υ_λ is divisible on right by $\psi_{kl}(c_k,c_l)/2$ □

COROLLARY 2.5. – *Let the numbers* $k < l$ *stand next to each other in the first row of the tableau* Λ. *Then the element* $\Psi_\lambda \in H_n$ *is divisible on the right by*

$$\varphi_{kl}(z_k,z_l)\cdot\prod_{k<m<l}^{\longrightarrow}\varphi_{ml}(z_m,z_l). \tag{2.9}$$

The element Ψ_λ *is then also divisible on the left by*

$$\prod_{k<m<l}^{\longleftarrow}\varphi_{ml}(z_m,z_l)\cdot\varphi_{kl}(z_k,z_l). \tag{2.10}$$

Proof. – By Proposition 2.4 the element Υ_λ is divisible on the right by $\psi_{kl}(c_k,c_l)$. But due to the relations (2.1) and (2.2) the product $\psi_{kl}(c_k,c_l)\cdot\Theta_\lambda$ is divisible on the right by (2.9). Since $\Psi_\lambda = \Upsilon_\lambda\Theta_\lambda$ we obtain the first statement of Corollary 2.5. Note that the image of (2.9) with respect to the antiautomorphism $*$ is (2.10). Since by Lemma 2.3 the element Ψ_λ is invariant with respect to this antiautomorphism we get the second statement of Corollary 2.5 $\square$

Consider again the rational function $\varphi_{ijk}(u,v,w)$ appearing at either side of (2.1). The values at $u=w$ of its restriction to (u,v) subjected to (2.6) need not be divisible on the right by $\varphi_{jk}(v,u)=\varphi_{jk}(v,w)$. Yet we have the following lemma.

LEMMA 2.6. – *Restriction of the function* $\varphi_{ijk}(u,v,w)\,\varphi_{kj}(w,v)$ *to those* (u,v) *which satisfy* (2.6), *takes at* $u=w$ *the value*

$$(ik)\,\varphi_{kj}(w,v)\cdot\left(\frac{2}{(v+w)^3}-\frac{2}{(v-w)^3}\right).$$

Proof. – It consists of a direct calculation. Namely, by our definition

$$\varphi_{ijk}(u,v,w)\,\varphi_{kj}(w,v)=\varphi_{ij}(u,v)\,\varphi_{ik}(u,w)\cdot\varphi_{jk}(v,w)\,\varphi_{kj}(w,v).$$

Under the condition (2.6) we have by (2.3) the equality

$$\varphi_{jk}(v,w)\,\varphi_{kj}(w,v)=\frac{1}{(u-v)^2}+\frac{1}{(u+v)^2}-\frac{1}{(v-w)^2}-\frac{1}{(v+w)^2}. \tag{2.11}$$

Dividing the right hand side of (2.11) by u^2-w^2 and then setting $u=w$ we get

$$-\frac{1}{w}\cdot\left(\frac{1}{(v+w)^3}-\frac{1}{(v-w)^3}\right).$$

On the other hand, by setting $u=w$ in the product

$$\varphi_{ij}(u,v)\,\varphi_{ik}(u,w)\cdot(u^2-w^2)=$$
$$\varphi_{ij}(u,v)\cdot\big((u^2-w^2)-(ik)\,(u+w)+(ik)\,a_ia_k\,(u-w)\big)$$

we obtain

$$-\,2\,w\cdot\varphi_{ij}(u,v)\,(ik)=-\,2\,w\cdot(ik)\,\varphi_{kj}(w,v)\quad\square$$

COROLLARY 2.7. – *Restriction of the function* $\varphi_{ijk}(u,v,w)\,\varphi_{kj}(w,v)$ *to those* (u,v) *which satisfy* (2.6), *vanishes at* $u=w=0$.

The next proposition makes the central part of the present section; cf. [14].

PROPOSITION 2.8. – *Let the numbers* k *and* $k+1$ *stand in the same column of the tableau* Λ. *Then the element* $\Upsilon_\lambda\in H_n$ *is divisible on the left by* $\varphi_{k,k+1}(z_k,z_{k+1})$.

Proof. – Observe first that Proposition 2.8 follows from its particular case $k+1=n$. Indeed, let ν be the shape of the tableau obtained from Λ by removing each of the numbers $k+2,\dots,n$. Then

$$\Upsilon_\lambda(t_1,\dots,t_n)=\Upsilon_\nu(t_1,\dots,t_{k+1})\cdot\prod_{(i,j)}^{\rightarrow}\psi_{ij}(c_i+t_i\,,c_j+t_j)$$

where $j = k+2, \dots, n$ and i runs through the sequence $\mathcal{B}_j$. Consider the value Υ_ν at $t_1 = \dots = t_{k+1} = 0$ of the restriction of $\Upsilon_\nu(t_1, \dots, t_{k+1})$ to $\mathcal{T}$. According to our proof of Theorem 2.2 then $\Upsilon_\lambda = \Upsilon_\nu \Upsilon$ for a certain element $\Upsilon \in H_n$.

From now on we will assume that $k+1 = n$. Since the element Θ_λ is invertible, it suffices to prove that $\Upsilon_\lambda \Theta_\lambda = \Psi_\lambda$ is divisible by $\psi_{n-1,n}(c_{n-1}, c_n)$ on the left. But $\Psi_\lambda^* = \Psi_\lambda$ by Lemma 2.3. So we will prove that Ψ_λ is divisible by $\psi_{n-1,n}(c_{n-1}, c_n)$ on the right. Since $\psi_{n-1,n}(c_{n-1}, c_n) + \psi_{n,n-1}(c_n, c_{n-1}) = 2$, that is to prove

$$\Psi_\lambda \cdot \psi_{n,n-1}(c_n, c_{n-1}) = 0. \tag{2.12}$$

Suppose that the number n appears in the p-th column and the q-th row of the tableau Λ. Then by our assumption the number $n-1$ appears in the same column and $(q-1)$-th row of Λ. Let $i_1 < \dots < i_r$ be all the numbers in the q-th row. So we have $i_r = n$. Then due to the relations (2.1) and (2.2) we have for a certain element $\Theta \in H_n$ the equality

$$\Theta_\lambda \cdot \psi_{n,n-1}(c_n, c_{n-1}) = \prod_{s<r}^{\rightarrow} \psi_{i_s,n-1}(c_{i_s}, c_{n-1}) \cdot \psi_{n,n-1}(c_n, c_{n-1}) \cdot \Theta.$$

Therefore to get (2.12) we have to prove that

$$\Upsilon_\lambda \cdot \prod_{s<r}^{\rightarrow} \psi_{i_s,n-1}(c_{i_s}, c_{n-1}) \cdot \psi_{n,n-1}(c_n, c_{n-1}) = 0. \tag{2.13}$$

We will now prove (2.13) by induction on r. Suppose that $r = 1$. Let m be the number appearing in the $(p-1)$-th column and $(q-1)$-th row of Λ. Then according to our proof of Theorem 2.2 the function $\Upsilon_\lambda(t_1, \dots, t_n)$ has the form

$$\Upsilon(t_1, \dots, t_n) \cdot \psi_{mn}(c_m + t_m, c_n + t_n)\, \psi_{n-1,n}(c_{n-1} + t_{n-1}, c_n + t_n)$$

where the restriction of $\Upsilon(t_1, \dots, t_n)$ to $\mathcal{T}$ is continuous at $t_1 = \dots = t_n = 0$. Moreover, this restriction is divisible on the right by $\psi_{m,n-1}(c_m + t_m, c_{n-1} + t_{n-1})$ where $t_m = t_{n-1}$. Since $c_m = c_n = 0$ and $c_{n-1} = 1$, restriction to $t_m = t_{n-1}$ of

$$\psi_{m,n-1,n}(c_m + t_m, c_{n-1} + t_{n-1}, c_n + t_n) \cdot \psi_{n,n-1}(c_n + t_n, c_{n-1} + t_{n-1})$$

vanishes at $t_m = t_n = 0$ by Corollary 2.7. This proves (2.13) for $r = 1$.

Now suppose that $r > 1$. We have to prove that the restriction to $\mathcal{T}$ of

$$\Upsilon_\lambda(t_1, \dots, t_n) \cdot \prod_{s \leqslant r}^{\rightarrow} \psi_{i_s,n-1}(c_{i_s} + t_{i_s}, c_{n-1} + t_{n-1}) \tag{2.14}$$

vanishes at $t_1 = \dots = t_n = 0$. Now denote $i_{r-1} = m$. The number $m-1$ appears in the $(p-1)$-th column and the $(q-1)$-th row of Λ. So we have $c_{m-1} = c_n$. Let μ be the shape of the tableau obtained from Λ by removing each of the numbers $m+1,\ldots,n$. Then the function $\Upsilon_\lambda(t_1,\ldots,t_n)$ has the form

$$\Upsilon_\mu(t_1,\ldots,t_m)\cdot\Psi(t_1,\ldots,t_{n-1})\cdot\psi_{m-1,n-1}(c_{m-1}+t_{m-1},c_{n-1}+t_{n-1})\times$$
$$\Phi(z_1,\ldots,z_n)\cdot\psi_{m-1,n}(c_{m-1}+t_{m-1},c_n+t_n)\,\psi_{n-1,n}(c_{n-1}+t_{n-1},c_n+t_n)\times$$
$$\prod_{s<r}^{\rightarrow}\psi_{i_sn}(c_{i_s}+t_{i_s},c_n+t_n).$$

Here we have denoted by $\Psi(t_1,\ldots,t_{n-1})$ the product

$$\prod_{(i,j)}^{\rightarrow}\psi_{ij}(c_i+t_i,c_j+t_j);\qquad j=m+1,\ldots,n-1 \tag{2.15}$$

where i runs through $\mathcal{B}_j$ but $(i,j)\neq(m-1,n-1)$. Further, we have denoted

$$\Phi(t_1,\ldots,t_n)=\prod_{(i,n)}^{\rightarrow}\psi_{in}(c_i+t_i,c_n+t_n) \tag{2.16}$$

where i runs through the sequence $\mathcal{B}_n$ but $i\neq m-1,n-1,\ldots,m$. In particular, any factor in the product (2.16) commutes with

$$\psi_{m-1,n-1}(c_{m-1}+t_{m-1},c_{n-1}+t_{n-1})$$

due to (2.2). Therefore the product (2.14) takes the form

$$\begin{aligned}(2.17)\qquad &\Upsilon_\mu(t_1,\ldots,t_m)\cdot\Psi(t_1,\ldots,t_{n-1})\cdot\Phi(t_1,\ldots,t_n)\times\\ &\psi_{m-1,n-1,n}(c_{m-1}+t_{m-1},c_{n-1}+t_{n-1},c_n+t_n)\times\\ &\prod_{s<r}^{\rightarrow}\psi_{i_sn}(c_{i_s}+t_{i_s},c_n+t_n)\cdot\prod_{s<r}^{\rightarrow}\psi_{i_s,n-1}(c_{i_s}+t_{i_s},c_{n-1}+t_{n-1})\times\\ &\psi_{n,n-1}(c_n+t_n,c_{n-1}+t_{n-1})=\\ &\Upsilon_\mu(t_1,\ldots,t_m)\cdot\Psi(t_1,\ldots,t_{n-1})\cdot\Phi(t_1,\ldots,t_n)\times\\ &\psi_{m-1,n-1,n}(c_{m-1}+t_{m-1},c_{n-1}+t_{n-1},c_n+t_n)\;\psi_{n,n-1}(c_n+t_n,c_{n-1}+t_{n-1})\\ &\prod_{s<r}^{\rightarrow}\psi_{i_s,n-1}(c_{i_s}+t_{i_s},c_{n-1}+t_{n-1})\cdot\prod_{s<r}^{\rightarrow}\psi_{i_sn}(c_{i_s}+t_{i_s},c_n+t_n).\end{aligned}$$

To get the latter equality we used the relations (2.1) and (2.2). Restriction to $\mathcal{T}$ of the product of factors in the first line of (2.17) is continuous at $t_1=\ldots=t_n=0$ according to our proof of Theorem 2.2. Each of the factors in the last line is also continuous at

$t_1 = \ldots = t_n = 0$. Therefore by Lemma 2.6 the restriction of (2.17) to $\mathcal{T}$ has at $t_1 = \ldots = t_n = 0$ the same value as restriction to $\mathcal{T}$ of the product

$$\begin{aligned} (2.18) \qquad & \Upsilon_\mu(t_1,\ldots,t_m)\cdot\Psi(t_1,\ldots,t_{n-1})\cdot\Phi(t_1,\ldots,t_n)\times \\ & (m-1,n)\cdot\varphi_{n,n-1}(c_n+t_n,c_{n-1}+t_{n-1})\cdot f\ \times \\ & \overrightarrow{\prod_{s<r}}\varphi_{i_s,n-1}(c_{i_s}+t_{i_s},c_{n-1}+t_{n-1})\cdot\overrightarrow{\prod_{s<r}}\varphi_{i_s,n}(c_{i_s}+t_{i_s},c_n+t_n)= \\ & \Upsilon_\mu(t_1,\ldots,t_m)\cdot\Psi(t_1,\ldots,t_{n-1})\cdot\Phi(t_1,\ldots,t_n)\times \\ & \overrightarrow{\prod_{s<r}}\psi_{i_s,m-1}(c_{i_s}+t_{i_s},c_n+t_n)\cdot\overrightarrow{\prod_{s<r}}\psi_{i_s,n-1}(c_{i_s}+t_{i_s},c_{n-1}+t_{n-1})\times \\ & (m-1,n)\cdot\psi_{n,n-1}(c_n+t_n,c_{n-1}+t_{n-1})\cdot f \end{aligned}$$

for a certain number $f\in\mathbb{R}$. Here each of the factors $\psi_{i_s,m-1}(c_{i_s}+t_{i_s},c_n+t_n)$ commutes with $\Phi(t_1,\ldots,t_n)$ by the relations (2.2). In each of these factors we can replace c_n+t_n by $c_{m-1}+t_{m-1}$ without affecting the value at $t_1=\ldots=t_n=0$ of the restriction to $\mathcal{T}$ of (2.18). Denote

$$\Gamma(t_1,\ldots,t_m)=\overrightarrow{\prod_{s<r}}\psi_{i_s,m-1}(c_{i_s}+t_{i_s},c_{m-1}+t_{m-1}).$$

According to our proof of Theorem 2.2 it now suffices to demonstrate vanishing at $t_1=\ldots=t_{n-1}$ of the restriction to $\mathcal{T}$ of the product

$$(2.19) \qquad \Upsilon_\mu(t_1,\ldots,t_m)\cdot\Psi(t_1,\ldots,t_{n-1})\cdot\Gamma(t_1,\ldots,t_m).$$

Consider the product (2.15). Here the factors corresponding to the pairs (i,j) are arranged with respect to ordering chosen in the proof of Theorem 2.2. Let us now reorder the pairs (i,j) in (2.15) as follows. For each number $j>m$ appearing in the $(p-1)$-th column of Λ change the sequence

$$(m-1,j),(i_1,j),\ldots,(i_{r-1},j)$$

to

$$(i_1,j),\ldots,(i_{r-1},j),(m-1,j).$$

Denote by $\Psi'(t_1,\ldots,t_{n-1})$ the resulting ordered product. Then by (2.1) and (2.2)

$$(2.20) \qquad \Psi(t_1,\ldots,t_{n-1})\cdot\Gamma(t_1,\ldots,t_m)=\Gamma(t_1,\ldots,t_m)\cdot\Psi'(t_1,\ldots,t_{n-1}).$$

Now let (i,j) be any singular pair in (2.15). Let (h,j) be the pair following (i,j) in the ordering from our proof of Theorem 2.2. Then (h,j) follows (i,j) in our new ordering as well. Furthermore, by the relations (2.1) and (2.2) the product

$$(2.21) \qquad \Upsilon_\mu(t_1,\ldots,t_m)\cdot\Gamma(t_1,\ldots,t_m)$$

is divisible on the right by $\psi_{ih}(c_i+t_i,\,c_h+t_h)$. Therefore

$$\Upsilon_\mu(t_1,\dots,t_m)\cdot\Gamma(t_1,\dots,t_m)\cdot\Psi'(t_1,\dots,t_{n-1})= \Upsilon_\mu(t_1,\dots,t_m)\cdot\Gamma(t_1,\dots,t_m)\cdot\Psi''(t_1,\dots,t_{n-1}) \tag{2.22}$$

where restriction of $\Psi''(t_1,\dots,t_{n-1})$ to $\mathcal{T}$ is continuous at $t_1=\dots=t_{n-1}=0$. But by the inductive assumption the restriction of the product (2.21) to $\mathcal{T}$ vanishes at $t_1=\dots=t_m=0$. Thus due to (2.20) and (2.22) the restriction of (2.19) to $\mathcal{T}$ vanishes at $t_1=\dots=t_{n-1}=0$ as well $\square$

We have shown that the element $\Psi_\lambda=\Upsilon_\lambda\Theta_\lambda$ of H_n is non-zero and $*$-invariant.

COROLLARY 2.9. – *Let the numbers k and $k+1$ stand in the same column of the tableau Λ Then the element $\Psi_\lambda\in H_n$ is divisible on both sides by $\varphi_{k,k+1}(z_k,z_{k+1})$.*

The proof of the next lemma is similar to that of Lemma 2.1 and will be omitted.

LEMMA 2.10. – *Restriction of $\varphi_{ijk}(u,v,w)$ to the set of (u,v,w) such that*

$$\frac{1}{(v-w)^2}+\frac{1}{(v+w)^2}=1.$$

is continuous at $u=\pm w$.

We conclude this section with two brief remarks. First, notice that the proofs of Lemma 2.1 and Theorem 2.2 yield explicit formulas for both elements $\Psi_\lambda,\Upsilon_\lambda\in H_n$; cf. [11, Example 9.7]. The element $\Upsilon_\lambda\in H_n$ should be regarded as an analogue of the classical Young symmetrizer [15] in the group ring $\mathbb{C}\cdot S_n$; see [11, Section 9].

3. Jucys-Murphy elements of the Sergeev algebra

For each $k=1,2,\dots$ we can regard the algebra H_k as a subalgebra in H_{k+1}. Here the symmetric group $S_k\subset S_{k+1}$ acts on $1,\dots,k+1$ preserving the number $k+1$. So we get a chain of subalgebras

$$H_1\subset H_2\subset\dots\subset H_n\subset H_{n+1}\subset\dots.$$

For each $k=1,2,\dots$ introduce the element of the algebra H_k

$$x_k=\sum_{1\leqslant i<k}(i\,k)+(i\,k)\,a_ia_k. \tag{3.1}$$

In particular, $x_1=0$. The following proposition will be used in the next section.

PROPOSITION 3.1. – *We have equality of rational functions of u valued in H_{n+1}*

$$\prod_{1\leqslant i\leqslant n}^{\longrightarrow}\varphi_{n+1,i}(u,z_i)\cdot\Psi_\lambda=\Big(1-\frac{x_{n+1}}{u}\Big)\cdot\Psi_\lambda. \tag{3.2}$$

Proof. – Denote by $\mathrm{X}(u)$ the rational function at the left hand side of (3.2). The value of this function at $u=\infty$ is Ψ_λ. Moreover, the residue of $\mathrm{X}(u)$ at $u=0$ is $-x_{n+1}\Psi_\lambda$.

It remains to prove that $\mathrm{X}(u)$ has a pole only at $u = 0$ and this pole is simple. Let an index $i \in \{2, \ldots, n\}$ be fixed. The factor $\varphi_{n+1,i}(u, z_i)$ in (3.2) has a pole at $u = \pm z_i$. We shall prove that when estimating from above the order of the pole at $u = \pm z_i$ of $\mathrm{X}(u)$, this factor does not count.

Suppose that the number i does not appear in the first row of the tableau Λ. Then the number $i-1$ appears in Λ straight above i. In particular, $c_{i-1} = c_i + 1$. By Corollary 2.8 the element Ψ_λ is divisible on the left by

$$\psi_{i-1,i}(c_{i-1}, c_i) = \varphi_{i-1,i}(z_{i-1}, z_i).$$

But the product

$$\varphi_{n+1,i-1}(u, z_{i-1})\, \varphi_{n+1,i}(u, z_i)\, \varphi_{i-1,i}(z_{i-1}, z_i) = \varphi_{n+1,i-1,i}(u, z_{i-1}, z_i)$$

is regular at $u = c_i$ due to Lemma 2.10.

Now suppose that the number i appears in the first row of Λ. Let k be the number k adjacent to i on the left in the first row of Λ. Then $c_k = c_i - 1$. By Corollary 2.5 the element Ψ_λ is divisible on the left by

$$\prod_{k<j<i}^{\longleftarrow} \varphi_{ji}(z_j, z_i) \cdot \varphi_{ki}(z_k, z_i). \tag{3.3}$$

Consider the product of factors in (3.2)

$$\varphi_{n+1,k}(u, z_k) \cdot \prod_{k<j<i}^{\longrightarrow} \varphi_{n+1,j}(u, z_j) \cdot \varphi_{n+1,i}(u, z_i). \tag{3.4}$$

Multiplying the product (3.4) on the right by (3.3) and using (2.1), (2.2) we get

$$\prod_{k<j<i}^{\longleftarrow} \varphi_{ji}(z_j, z_i) \cdot \varphi_{n+1,k,i}(u, z_k, z_i) \cdot \prod_{k<j<i}^{\longrightarrow} \varphi_{1,j+1}(u, c_j).$$

The latter product is regular at $u = z_i$ by Lemma 2.10. The proof is complete □

The elements $x_1, \ldots, x_n$ of the algebra H_n will play an important role in this article. They will be called the *Jucys-Murphy elements* of the algebra H_n; cf. [16].

LEMMA 3.2. – *a) The elements $x_1, \ldots, x_n$ of the algebra H_n pairwise commute. b) For any $r = 1, 2, \ldots$ the element $x_1^{2r} + \ldots + x_n^{2r}$ belongs to the centre of H_n.*

Proof. – The first statement of this lemma can be verified by direct calculation. One can also check directly the relations in the algebra H_n

$$a_k\, x_k = -\, x_k\, a_k; \qquad a_k\, x_l = x_l\, a_k, \quad k \neq l; \tag{3.5}$$

$$(k, k+1)\, x_{k+1} - x_k\, (k, k+1) = 1 + a_k a_{k+1}; \tag{3.6}$$

$$x_{k+1}\, (k, k+1) - (k, k+1)\, x_k = 1 - a_k a_{k+1}. \tag{3.7}$$

By making use of (3.6) and (3.7) one can verify

$$\big[\,(k,k+1),x_k\,x_{k+1}\big]=0;\quad \big[\,(k,k+1),x_k^2+x_{k+1}^2\,\big]=0 \tag{3.8}$$

where the square brackets stand for the commutator. By the definition (3.1) the transposition $(k,k+1)\in S_n$ also commutes with x_l if $l\neq k,k+1$. So the commutation relations (3.5), (3.8) imply the second statement of Lemma 3.2 □

According to the following theorem, the element $\Psi_\lambda\in H_n$ is a joint eigenvector of $x_1,\ldots,x_n$ with respect to the left regular action of the algebra H_n.

PROPOSITION 3.3. – *For each* $k=1,\ldots,n$ *we have* $x_k\cdot\Psi_\lambda=z_i\cdot\Psi_\lambda$.

Proof. – Note that $x_1=0$ and $z_1=0$ by definition. So we will assume that $k\neq 1$. For any $k=2,\ldots,n$ an argument similar to that already used in the proof of Proposition 3.1 shows the equality of rational functions of u valued in H_n

$$u\cdot\prod_{1\leqslant i<k}^{\longrightarrow}\varphi_{ki}(u,z_i)\cdot\Psi_\lambda=(u-x_k)\cdot\Psi_\lambda. \tag{3.9}$$

Denote by $\Phi_k(u)$ the rational function appearing at the left hand side of (3.9). We shall prove that $\Phi_k(z_k)=0$.

Consider any factor $\varphi_{ki}(u,z_i)$ at the left hand side of (3.9) with $z_i=z_k$. Then this factor has a pole at $u=z_k$. However, according to the proof of Proposition 3.1 this factor does not count when estimating from above the order of the pole at $u=z_k$ of $\Phi_k(u)$, provided $i\neq 1$. But $u\cdot\varphi_{k1}(u,z_1)$ has no pole at $u=0$.

Suppose the number k does not appear in the first row of the tableau Λ. Then the number $k-1$ appears in Λ straight above k. So $c_{k-1}=c_k+1$. Consider the factor $\varphi_{k,k-1}(u,z_{k-1})$ in (3.9). By Corollary 2.8 the element Ψ_λ is divisible on the left by

$$\psi_{k-1,k}(c_{k-1},c_k)=\varphi_{k-1,k}(z_{k-1},z_k).$$

But the product

$$\varphi_{k,k-1}(u,z_{k-1})\cdot\varphi_{k-1,k}(z_{k-1},z_k)$$

vanishes at $u=z_k$ due to (2.3). So the function $\Phi_k(u)$ vanishes at $u=z_k$ as well.

Now suppose that the number k appears in the first row of Λ. Note that $k\neq 1$ by our assumption. Let i be the number adjacent to k on the left in the first row of Λ. Then $c_i=c_k-1$. Consider the product of the factors in (3.9)

$$\varphi_{ki}(u,z_i)\cdot\prod_{i<j<k}^{\longrightarrow}\varphi_{kj}(u,z_j) \tag{3.10}$$

None of them has a pole at $u=z_k$. By Corollary 2.5 the element Ψ_λ is divisible on the left by

$$\prod_{i<j<k}^{\longleftarrow}\varphi_{jk}(z_j,z_k)\cdot\varphi_{ik}(z_i,z_k).$$

Due to (2.3) when we multiply the latter product on the left by (3.10) and set $u = z_k$ we get

$$\prod_{i \leqslant j < k} \left(1 - \frac{1}{(z_j - z_k)^2} - \frac{1}{(z_j + z_k)^2}\right)$$

which is equal to zero because the factor corresponding to $j = i$ vanishes □

Let μ run through the set $\mathcal{D}_\lambda$ of all strict partitions of $n-1$ which can be obtained by decreasing one of the parts of λ. Put $m_{\lambda\mu} = 1$ if the number ℓ_μ is odd while ℓ_λ is even; otherwise put $m_{\lambda\mu} = 2$. Restriction of the H_n-module U_λ to H_{n-1} splits into a direct sum of the modules U_μ where each U_μ appears $m_{\lambda\mu}$ times; see [9, Lemma 6]. This branching property determines the irreducible module U_λ over $\mathbb{Z}_2$-graded algebra H_n uniquely. We set $H_0 = \mathbb{C}$.

THEOREM 3.4. – *Under the left regular action of H_n the space $H_n \cdot \Psi_\lambda$ splits into a direct sum of copies of the irreducible module U_λ.*

Proof. – Let us prove by induction on n the following statement: for any $r = 1, 2, \ldots$ the central element $x_1^{2r} + \ldots + x_n^{2r} \in H_n$ acts as

$$z_1^{2r} + \ldots + z_n^{2r} = c_1^r (c_1 + 1)^r + \ldots + c_n^r (c_n + 1)^r \tag{3.11}$$

in the irreducible module U_λ. Due to Proposition 3.3 the latter statement implies Theorem 3.4. Indeed, the collection of the eigenvalues (3.11) determines the shifted Young diagram λ uniquely. Therefore if an irreducible H_n-module U contains an eigenvector of $x_1^2, \ldots, x_n^2$ with the respective eigenvalues $z_1^2, \ldots, z_n^2$ then $U = U_\lambda$.

If $n = 1$ then $x_1 = 0$ and $c_1 = 0$ so the statement to prove is trivial.

Now assume that $n > 1$. Suppose there exist two distinct diagrams $\mu, \nu \in \mathcal{D}_\lambda$. Each of them can be obtained by adding a box to the same shifted diagram. Let c and d be the contents of these two boxes. Here $c \neq d$. Consider the irreducible components U_μ and U_ν in the restriction of U_λ to H_{n-1}. The element $x_n \in H_n$ commutes with the subalgebra H_{n-1} and acts in U_μ, U_ν by certain numbers $z, w \in \mathbb{C}$ respectively. By comparing the actions of the central element $x_1^{2r} + \ldots + x_n^{2r} \in H_n$ in U_μ, U_ν and by applying the inductive assumption to these H_{n-1}-modules we get the equality

$$c^r (c+1)^r + z^{2r} = d^r (d+1)^r + w^{2r}.$$

These equalities for $r = 1, 2$ imply that $z^2 = d(d+1)$ and $w^2 = c(c+1)$. So we obtain the required statement.

Assume that $\mathcal{D}_\lambda = \{\mu\}$. Take a joint eigenvector $\xi \in U_\lambda$ of $x_n, x_{n-1} \in H_n$. Let $u, v \in \mathbb{C}$ be the respective eigenvalues. Observe that $u \neq \pm v$. If $u = v$ then by applying (3.6), (3.7) with $k = n-1$ to the vector ξ we obtain that $(x_n - x_{n-1}) \cdot (n-1, n)\, \xi = 2\,\xi$ while $(x_n - x_{n-1}) \cdot \xi = 0$. This contradicts the property $(x_n - x_{n-1})^* = x_n - x_{n-1}$. Thus $u \neq v$. Similarly, by taking the vector $a_{n-1} \cdot \xi$ instead of ξ we prove that $u \neq -v$. Now consider the element

$$h_n = (n-1, n) \cdot (x_{n-1}^2 - x_n^2) + (x_{n-1} + x_n) - a_{n-1}\, a_n\, (x_{n-1} - x_n) \in H_n.$$

One can derive that $x_n h_n = h_n x_{n-1}$ from (3.7). So $x_n \cdot h_n \xi = v \cdot h_n \xi$. But $x_n c_n = -c_n x_n$ while $d_{\lambda\mu} \leqslant 2$. So any eigenvalue of x_n in U_λ is either u or $-u$. Hence $h_n \cdot \xi = 0$, that is

$$(3.12) \qquad (n-1,n) \cdot \xi = \Big(\frac{1}{u-v} + \frac{a_{n-1}\, a_n}{u+v} \Big) \cdot \xi.$$

In particular, the pair (u,v) satisfies the condition (2.6) since $(n-1,n)^2 \cdot \xi = \xi$.

Let us now take any H_{n-1}-irreducible component $U_\mu \subset U_\lambda$. We can choose $\xi \in U_\mu$. We will show that $u^2 = z_n^2$. Then we will get the required statement by the inductive assumption. If $\lambda = (2)$ then $v = 0$, so $u^2 = 2 = z_2^2$ by (2.6). We will assume that $n > 2$. Let us choose $\xi \in U_\mu$ so that $x_i \cdot \xi = z_i \cdot \xi$ for $i = n-1, n-2$. Here we use Proposition 3.3 and the inductive assumption. So we have $v = z_{n-1}$. We will also write $w = z_{n-2}$. Consider the following five cases.

(i) Suppose that $\ell_\lambda = 2$ while both parts of λ are greater than 1. Then the number $n-1$ stands straight above n in Λ while $n-2$ stands next to the left of n. Now

$$x_{n-1} \cdot h_{n-1}\, \xi = h_{n-1}\, x_{n-2} \cdot \xi = w \cdot h_{n-1}\, \xi,$$
$$h_{n-1}\, \xi = (w^2 - v^2) \Big((n-2,n-1) - \frac{1}{v-w} - \frac{a_{n-2}\, a_{n-1}}{v+w} \Big) \cdot \xi \neq 0.$$

So the pair (u,w) satisfies the condition (2.6) as well the pair (u,v). Note that here $w^2 = (c_n - 1)\, c_n$ and $v^2 = (c_n+1)(c_n+2)$. Therefore $u^2 = c_n\,(c_n+1) = z_n^2$.

In the remaining four cases the number $n-1$ will stand next to $n-2$ in the tableau Λ. Due to Corollaries 2.5 and 2.9 we can now assume that

$$(3.13) \qquad (n-2,n-1) \cdot \xi = \Big(\frac{1}{v-w} + \frac{a_{n-2}\, a_{n-1}}{v+w} \Big) \cdot \xi.$$

(ii) Suppose that $\ell_\lambda > 2$. Then the numbers $n-1, n-2$ stand straight above n in the tableau Λ. So $v^2 = (c_n+1)(c_n+2)$ and $w^2 = (c_n+2)(c_n+3)$. In particular, here we have $w \neq 0$. The condition (2.6) implies that either $u^2 = c_n\,(c_n+1)$ or $u^2 = (c_n+2)(c_n+3) = w^2$. But if $u = \pm\, w$ then by substituting (3.12), (3.13) in

$$(n-1,n)\,(n-2,n-1)\,(n-1,n) \cdot \xi = (n-2,n-1)\,(n-1,n)\,(n-2,n-1) \cdot \xi$$

we obtain that $w = 0$. This contradiction demonstrates that $u^2 = c_n\,(c_n+1) = z_n^2$.

(iii) If $\lambda = (2{,}1)$ then $m_{\lambda\mu} = 1$, so $u = -\,u$. Thus $u^2 = 0 = z_3^2$. Note that here

$$(3.14) \qquad (12) \cdot \xi = \frac{1}{\sqrt{2}}\,(a_1 a_2 + 1) \cdot \xi, \qquad (23) \cdot \xi = \frac{1}{\sqrt{2}}\,(a_2\, a_3 - 1) \cdot \xi.$$

(iv) Suppose that $\ell_\lambda = 1$ and $n > 3$. Then $n-1, n-2$ stand straight to the left of n in the tableau Λ. So $v^2 = (c_n - 1)\, c_n$ and $w^2 = (c_n-2)(c_n-1)$. Now (2.6) implies that either $u^2 = c_n\,(c_n+1)$ or $u^2 = (c_n-2)(c_n-1) = w^2$. But $w \neq 0$. As in (ii) one proves that only the case $u^2 = c_n\,(c_n+1) = z_n^2$ is possible.

(v) Finally, suppose that $\lambda = (3)$. Here $v^2 = 2$ and (2.6) implies that either $u^2 = 6$ or $u^2 = 0$. But in the latter case the transpositions (12),(23) $\in H_3$ would act in the irreducible module $U_{(3)}$ by the same formulas (3.14) as in the module $U_{(2,1)}$. Since the H_3-modules $U_{(3)}$ and $U_{(2,1)}$ are non-equivalent, we get $u^2 = 6 = z_3^2$ $\square$

More detailed analysis shows that the left ideal $H_n \cdot \Psi_\lambda$ splits into a direct sum of $2^{[\ell_\lambda/2]}$ copies of the irreducible module U_λ [11, Theorem 8.3]. However, we will not use the latter fact in the present article. We will need the following corollary to Theorem 3.4. Let h run through the set of basic elements of H_n

$$g\, a_1^{l_1} \dots a_n^{l_n}\,; \qquad g \in S_n, \quad l_1, \dots, l_n = 0, 1. \tag{3.15}$$

We denote by $\chi_\lambda(h)$ the trace of the element $h \in H_n$ in the module U_λ normalized so that $\chi_\lambda(1) = 1$. Consider the corresponding central element of H_n

$$\mathrm{X}_\lambda = \sum_h \chi_\lambda(h) \cdot h^{-1}. \tag{3.16}$$

COROLLARY 3.5. – *We have the equality*

$$\mathrm{X}_\lambda \cdot 2^n\, n\,! = \sum_h h\ \Psi_\lambda\, h^{-1}.$$

Now fix any non-negative integer m and consider H_n as a subalgebra in H_{n+m}. We will write $\tilde{\jmath} = j + n$ for every $j = 1, \dots, m$. For each $k = 1, \dots, n$ denote

$$y_k = \sum_{1 \leqslant j \leqslant m} (\,k\,\tilde{\jmath}\,) - (\,k\,\tilde{\jmath}\,)\, a_k\, a_{\tilde{\jmath}}. \tag{3.17}$$

The next auxiliary statement easily follows from Proposition 3.3; cf. [12, 13].

PROPOSITION 3.6. – *The product* $\Psi_\lambda \cdot (y_1 - z_1) \dots (y_n - z_n) \in H_{n+m}$ *is equal to*

$$\Psi_\lambda \cdot \sum_{j_1 \cdots j_n} (\,1\,\tilde{\jmath}_1\,) \dots (\,n\,\tilde{\jmath}_n\,) \cdot (1 - a_1 a_{\tilde{\jmath}_1}) \dots (1 - a_n a_{\tilde{\jmath}_n}) \tag{3.18}$$

where all the indices $j_1, \dots, j_n \in \{1, \dots, m\}$ *are pairwise distinct.*

Proof. – We will use the induction on n. Note that $z_1 = 0$ so for $n = 1$ we get the required statement by the definition (3.17). Let us now suppose that $n > 1$. Then by the inductive assumption and by the definition (3.17) we have the equality

$$\begin{gathered}\Psi_\lambda \cdot (y_1 - z_1) \dots (y_n - z_n) = \\ \Psi_\lambda \cdot \sum_{j_1 \cdots j_{n-1}} (\,1\,\tilde{\jmath}_1\,) \dots (\,n-1, \tilde{\jmath}_{n-1}\,) \big(1 - a_1 a_{\tilde{\jmath}_1}\big) \dots \big(1 - a_{n-1} a_{\tilde{\jmath}_{n-1}}\big) \times \\ \times \sum_j (\,n\,\tilde{\jmath}\,) \big(1 - a_n\, a_{\tilde{\jmath}}\big) \ - \\ \Psi_\lambda \cdot \sum_{j_1 \cdots j_{n-1}} z_n\, (\,1\,\tilde{\jmath}_1\,) \dots (\,n-1, \tilde{\jmath}_{n-1}\,) \big(1 - a_1 a_{\tilde{\jmath}_1}\big) \dots \big(1 - a_{n-1} a_{\tilde{\jmath}_{n-1}}\big)\end{gathered} \tag{3.19}$$

where $j_1, \ldots, j_{n-1}, j \in \{1, \ldots, m\}$ and the indices $j_1, \ldots, j_{n-1}$ are pairwise distinct. Due to Lemma 2.3 and Proposition 3.3 by the definition (3.1) the sum in the last line of this equality can be replaced by the sum

$$\sum_i \sum_{j_1 \ldots j_{n-1}} (n\, i)\,(1 - a_n a_i) \cdot (1\, \tilde{j}_1) \ldots (n-1, \tilde{j}_{n-1})\,(1 - a_1 a_{\tilde{j}_1}) \ldots (1 - a_{n-1} a_{\tilde{j}_{n-1}})$$

where the index i runs through $1, \ldots, n-1$. The latter sum can be rewritten as

$$\sum_{j_1 \ldots j_{n-1}} \sum_j (1\, \tilde{j}_1) \ldots (n-1, \tilde{j}_{n-1}) \cdot (n\, \tilde{j})\,(1 - a_n a_{\tilde{j}}) \cdot (1 - a_1 a_{\tilde{j}_1}) \ldots (1 - a_{n-1} a_{\tilde{j}_{n-1}})$$

where j runs through $j_1, \ldots, j_{n-1}$. Now Proposition 3.6 follows from (3.19) □

Let μ be any strict partition of m. We will identify the partitions λ and μ with their shifted Young diagrams. Take the embedding of the algebra H_m into H_{n+m} such that the symmetric group $S_m \subset S_{n+m}$ acts by permutations of the numbers $n+1, \ldots, n+m$. Denote by $\Psi_\mu^\sim$ the image of the element $\Psi_\mu \in H_m$ with respect to this embedding.

COROLLARY 3.7. – *We have the equality* $\Psi_\lambda \cdot (y_1 - z_1) \ldots (y_n - z_n) \cdot \Psi_\mu^\sim = 0$ *if the diagram* λ *is not contained in* μ.

Proof. – If $m < n$ there is no summand in (3.18) and $\Psi_\lambda \cdot (y_1 - z_1) \ldots (y_n - z_n) = 0$. Suppose that $m \geqslant n$ but the diagram λ is not contained in μ. Consider H_n as a subalgebra in H_m with respect to the standard embedding. The restriction of the H_m-module U_μ to H_n does not contain any irreducible component isomorphic to U_λ. So by Lemma 2.3 and Theorem 3.4 we get $\Psi_\lambda \cdot \Psi_\mu = 0$ in H_m. Moreover, $\Psi_\lambda \cdot a_{j_1} \ldots a_{j_s} \cdot \Psi_\mu = 0$ for any $j_1, \ldots, j_s \in \{1, \ldots, m\}$. Now the required equality follows from Proposition 3.6 □

In the next section we interpret Corollary 3.7 in terms of classical invariant theory.

4. Capelli identity for the queer Lie superalgebra

In this section we will let the indices i, j run through $\pm 1, \ldots, \pm N$. We will write $\bar{\imath} = 0$ if $i > 0$ and $\bar{\imath} = 1$ if $i < 0$. Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N|N}$. Let $e_i \in \mathbb{C}^{N|N}$ be an element of the standard basis. The $\mathbb{Z}_2$-gradation on $\mathbb{C}^{N|N}$ is defined so that $\deg e_i = \bar{\imath}$. Let $E_{ij} \in \mathrm{End}(\mathbb{C}^{N|N})$ be the standard matrix units. The algebra $\mathrm{End}(\mathbb{C}^{N|N})$ is $\mathbb{Z}_2$-graded so that $\deg E_{ij} = \bar{\imath} + \bar{\jmath}$.

We will also regard E_{ij} as generators of the complex Lie superalgebra $\mathfrak{gl}_{N|N}$. The *queer* Lie superalgebra $\mathfrak{q}_N$ is the subalgebra in $\mathfrak{gl}_{N|N}$ spanned by the elements $F_{ij} = E_{ij} + E_{-i,-j}$. Thus $F_{-i,-j} = F_{ij}$ by definition. Note that the image of the defining representation $\mathfrak{q}_N \to \mathrm{End}(\mathbb{C}^{N|N})$ coincides with the supercommutant of the element

$$J = \sum_j E_{j,-j} \cdot (-1)^{\bar{\jmath}} \in \mathrm{End}(\mathbb{C}^{N|N}).$$

In this section we will use the following convention. Let A and B be any two associative complex $\mathbb{Z}_2$-graded algebras. Their tensor product $\mathrm{A} \otimes \mathrm{B}$ will be a $\mathbb{Z}_2$-graded algebra such that for any homogeneous elements $X, X' \in \mathrm{A}$ and $Y, Y' \in \mathrm{B}$

$$(X \otimes Y)(X' \otimes Y') = X X' \otimes Y Y' \cdot (-1)^{\deg X' \deg Y},$$
$$\deg (X \otimes Y) = \deg X + \deg Y .$$

If the algebra A is unital denote by ι_s its embedding into the tensor product $\mathrm{A}^{\otimes n}$ as the s-th tensor factor:

$$\iota_s(X) = 1^{\otimes (s-1)} \otimes X \otimes 1^{\otimes (n-s)}, \qquad 1 \leqslant s \leqslant n.$$

We will also use various embeddings of the algebra $\mathrm{A}^{\otimes m}$ into $\mathrm{A}^{\otimes n}$ for any $m \leqslant n$. For any choice of pairwise distinct indices $s_1, \ldots, s_m \in \{1, \ldots, n\}$ and an element $X \in \mathrm{A}^{\otimes m}$ of the form $X = X^{(1)} \otimes \ldots \otimes X^{(m)}$ we will denote

$$X_{s_1 \ldots s_m} = \iota_{s_1}\big(X^{(1)}\big) \, \ldots \, \iota_{s_m}\big(X^{(m)}\big) \in \mathrm{A}^{\otimes n}.$$

Denote

$$P = \sum_{i,j} E_{ij} \otimes E_{ji} \cdot (-1)^{\bar{\jmath}} \in \mathrm{End}\,(\mathbb{C}^{N|N})^{\otimes 2}.$$

For any positive integer n a representation $H_n \to \mathrm{End}\,(\mathbb{C}^{N|N})^{\otimes n}$ of the Sergeev algebra can be determined by the assignments

$$a_k \mapsto J_k \quad \text{and} \quad (k\,l) \mapsto P_{kl}\,. \tag{4.1}$$

Let us now consider the enveloping algebra $\mathrm{U}(\mathfrak{q}_N)$ of the Lie superalgebra $\mathfrak{q}_N$. The algebra $\mathrm{U}(\mathfrak{q}_N)$ is a Hopf superalgebra: the comultiplication, counit and the antipodal map are defined for $F \in \mathfrak{q}_N$ respectively by

$$\Delta(F) = F \otimes 1 + 1 \otimes F, \qquad \varepsilon(F) = 0, \qquad S(F) = -F.$$

Take the n-th tensor power of the defining representation $\mathrm{U}(\mathfrak{q}_N) \to \mathrm{End}\,(\mathbb{C}^{N|N})$. Its image coincides [9, Theorem 3] with supercommutant of the image of Sergeev algebra H_n relative to (4.1). Let λ be any strict partition of n. Take the irreducible module U_λ over the $\mathbb{Z}_2$-graded algebra H_n. Denote by V_λ the subspace in $\mathrm{Hom}\big(U_\lambda, \mathrm{End}\,(\mathbb{C}^{N|N})^{\otimes n}\big)$ consisting of all the elements which supercommute with the action of H_n. We have an irreducible representation of the $\mathbb{Z}_2$-graded algebra $\mathrm{U}(\mathfrak{q}_N)$ in the space V_λ. We denote it by π_λ. Here $V_\lambda \neq \{0\}$ if and only if $\ell_\lambda \leqslant N$ [9, Theorem 4]. From now on we will assume that this is the case.

Now let two positive integers N and M be fixed. Let the indices a, b run through $\pm 1, \ldots, \pm M$ while the indices i, j keep running through $\pm 1, \ldots, \pm N$. We will also write $\bar{a} = 0$ if $a > 0$ and $\bar{a} = 1$ if $a < 0$. We will use the generators $F_{ij} \in \mathfrak{q}_N$ and $F_{ab} \in \mathfrak{q}_M$.

Introduce a supercommutative algebra $\mathcal{P}$ with the free generators x_{ia} where $a > 0$ and the $\mathbb{Z}_2$-gradation is defined by $\deg x_{ia} = \bar{\imath}$. It will be convenient to set

$$x_{i,-a} = \sqrt{-1} \cdot x_{-i,a}\,; \quad a > 0.$$

Let ∂_{ia} be the left derivations in the supercommutative algebra $\mathcal{P}$ corresponding to the generators x_{ia}. Here we allow both indices i and a to be negative, so that

$$\partial_{i,-a} = -\sqrt{-1}\cdot\partial_{-i,a}\,; \quad a>0.$$

The algebra generated by all the left derivatives ∂_{ia} in $\mathcal{P}$ along with the operators of left multiplication by x_{jb} will be denoted by $\mathcal{PD}$. Note that for the arbitrary indices $i=\pm 1,\ldots,\pm N$ and $a=\pm 1,\ldots,\pm M$ we have $\deg x_{ia}=\bar\imath+\bar a$ in $\mathcal{P}$.

Proposition 4.1. – *The assignments for* $i,j=\pm 1,\ldots,\pm N$ *and* $a,b=\pm 1,\ldots,\pm M$

$$F_{ij}\mapsto\sum_b x_{ib}\,\partial_{jb} \quad \text{and} \quad F_{ab}\mapsto\sum_j x_{ja}\partial_{jb}\cdot(-1)^{\bar\jmath(\bar a+\bar b)} \tag{4.2}$$

define representations in $\mathcal{PD}$ *of the Lie superalgebras* $\mathfrak{q}_N$ *and* $\mathfrak{q}_M$. *The images of* $\mathrm{U}(\mathfrak{q}_N)$ *and* $\mathrm{U}(\mathfrak{q}_M)$ *in these representations are the supercommutants of each other.*

Proof. – Let $e_i\in\mathbb{C}^{N|N}$ and $e_a\in\mathbb{C}^{M|M}$ be elements of the standard bases. Identify the tensor product $\mathrm{End}(\mathbb{C}^{N|N})\otimes\mathrm{End}(\mathbb{C}^{M|M})$ with the algebra $\mathrm{End}(\mathbb{C}^{N|N}\otimes\mathbb{C}^{M|M})$ so that

$$E_{ij}\otimes E_{ab}\cdot e_k\otimes e_c = e_i\otimes e_a\cdot\delta_{jk}\,\delta_{bc}\,(-1)^{\bar\jmath(\bar a+\bar b)}.$$

Now the standard embeddings $X\mapsto X\otimes 1$ and $Y\mapsto 1\otimes Y$ of $\mathrm{End}(\mathbb{C}^{N|N})$ and $\mathrm{End}(\mathbb{C}^{M|M})$ into $\mathrm{End}(\mathbb{C}^{N|N})\otimes\mathrm{End}(\mathbb{C}^{M|M})$ respectively define actions of the Lie superalgebras $\mathfrak{q}_N$ and $\mathfrak{q}_M$ in the space $\mathbb{C}^{N|N}\otimes\mathbb{C}^{M|M}$. These actions preserve the subspace W in $\mathbb{C}^{N|N}\otimes\mathbb{C}^{M|M}$ with the basis consisting of the vectors

$$e_i\otimes e_a-\sqrt{-1}\cdot e_{-i}\otimes e_{-a}\,; \quad a>0.$$

The supersymmetric algebra $\mathrm{S}(W)$ coincides with $\mathcal{P}$, the above basic vector of W being identified with the generator x_{ia}. The action of $\mathfrak{q}_N$ and $\mathfrak{q}_M$ in the $\mathrm{S}(W)$ is then given by (4.2). So the images of the enveloping algebras $\mathrm{U}(\mathfrak{q}_N)$ and $\mathrm{U}(\mathfrak{q}_M)$ in $\mathcal{PD}$ supercommute. Moreover, due to [9, Theorem 3] the spectrum of the $\mathrm{U}(\mathfrak{q}_N)\otimes\mathrm{U}(\mathfrak{q}_M)$-module $\mathrm{S}(W)$ is simple. Hence the images of $\mathrm{U}(\mathfrak{q}_N)$ and $\mathrm{U}(\mathfrak{q}_M)$ in $\mathcal{PD}$ are the supercommutants of each other □

Thus we have defined representations $\mathrm{U}(\mathfrak{q}_N)\to\mathcal{PD}$ and $\mathrm{U}(\mathfrak{q}_M)\to\mathcal{PD}$, we will denote them by γ and γ' respectively. In this section we will consider the subspace

$$\mathcal{I}=\gamma\big(\mathrm{U}(\mathfrak{q}_N)\big)\cap\gamma'\big(\mathrm{U}(\mathfrak{q}_M)\big)\subset\mathcal{PD}.$$

Next corollary to Proposition 4.1 gives another description of the subspace $\mathcal{I}\subset\mathcal{PD}$.

Corollary 4.2. – *The subspace* $\mathcal{I}\subset\mathcal{PD}$ *consists of those operators which supercommute with the images of* γ *and* γ'.

Let us also equip the algebra $\mathcal{P}$ with the $\mathbb{Z}$-gradation such that $\deg x_{ia}=1$. Denote by $\mathcal{P}_n$ the subspace in $\mathcal{P}$ consisting of the elements of degree n. According to [9, Theorem 3] the irreducible components of the $\mathrm{U}(\mathfrak{q}_N)\otimes\mathrm{U}(\mathfrak{q}_M)$-module $\mathcal{P}_n$ are parametrized by the strict partitions λ of n with not more than M,N parts. Denote by W_λ the irreducible component of $\mathcal{P}_n$ corresponding to λ. If the number ℓ_λ is even then $W_\lambda = V_\lambda \otimes V'_\lambda$ where the second tensor factor is the irreducible $\mathrm{U}(\mathfrak{q}_M)$-module corresponding to the partition λ. If ℓ_λ is odd then the tensor product $V_\lambda \otimes V'_\lambda$ splits into direct sum of two copies of the irreducible module W_λ.

Let μ run through the set of strict partitions of $m = 0, 1, 2, \ldots$ with not more than M, N parts. The next proposition gives a distinguished decomposition of the vector space $\mathcal{I}$ into a direct sum of one-dimensional subspaces; cf. [3, Theorem 1]. Consider the $\mathbb{Z}$-filtration on the algebra $\mathcal{PD}$ by degree of the differential operator.

PROPOSITION 4.3. – *There is a unique one-dimensional subspace $\mathcal{I}_\lambda \in \mathcal{I}$ of degree n such that $\mathcal{I}_\lambda \cdot W_\mu = \{0\}$ when $m < n$ or $m = n$ but $\mu \neq \lambda$. We have a decomposition*

$$\mathcal{I} = \bigoplus_\lambda \mathcal{I}_\lambda \tag{4.3}$$

where λ runs through the set of all strict partitions with not more than M, N parts.

Proof. – Consider the actions $\mathrm{ad} \circ \gamma$ and $\mathrm{ad} \circ \gamma'$ in the space $\mathcal{PD}$ of the Lie superalgebras $\mathfrak{q}_N$ and $\mathfrak{q}_M$ respectively. By Corollary 4.2 the subspace $\mathcal{I} \subset \mathcal{PD}$ consists of all the invariants of both actions. Denote by $\mathcal{D}$ the subalgebra in $\mathcal{PD}$ generated by all the left derivatives ∂_{ia}. The actions of $\mathfrak{q}_N$ and $\mathfrak{q}_M$ in $\mathcal{PD}$ preserve the subspace $\mathcal{D}$. Obviously, we have $\mathcal{PD} = \mathcal{P} \cdot \mathcal{D}$. Moreover, as a $\mathrm{U}(\mathfrak{q}_N) \otimes \mathrm{U}(\mathfrak{q}_M)$-module the space $\mathcal{PD}$ now decomposes into the tensor product $\mathcal{P} \otimes \mathcal{D}$.

Let us equip the algebra $\mathcal{D}$ with the $\mathbb{Z}$-gradation such that $\deg \partial_{ia} = 1$. Denote by $\mathcal{D}_n$ the subspace in $\mathcal{D}$ consisting of the elements of degree n. As a module over $\mathrm{U}(\mathfrak{q}_N) \otimes \mathrm{U}(\mathfrak{q}_M)$ the subspace $\mathcal{D}_n$ splits into direct sum of irreducible modules W^*_λ such that the tensor product $W_\lambda \otimes W^*_\mu$ contains an invariant subspace if and only if $\lambda = \mu$. Let $\mathcal{I}_\lambda$ be the invariant subspace in $W_\lambda \otimes W^*_\lambda \subset \mathcal{PD}$, it is one-dimensional.

Choose any non-zero element $I_\lambda \in \mathcal{I}_\lambda$. Since $W_\mu \subset \mathcal{P}_m$ we have $I_\lambda \cdot W_\mu = 0$ for $m < n$. Suppose that $m = n$. Consider the linear map $\mathcal{P}_n \to \mathcal{P}_n : X \to I_\lambda \cdot X$. It commutes with the actions of $\mathfrak{q}_N$ and $\mathfrak{q}_M$ in $\mathcal{P}_n$. But the image of this map is contained in W_λ by the definition of the space $\mathcal{I}_\lambda$. So the restriction of this map to $W_\mu \subset \mathcal{P}_n$ with $\mu \neq \lambda$ is zero. Thus we obtain the decomposition (4.3). It is unique since the spectrum of the $\mathrm{U}(\mathfrak{q}_N) \otimes \mathrm{U}(\mathfrak{q}_M)$-module $\mathcal{P}$ is simple □

Let $\mathrm{Z}(\mathfrak{q}_N)$ be the centre of the enveloping algebra $\mathrm{U}(\mathfrak{q}_N)$. By definition, an element of $\mathrm{U}(\mathfrak{q}_N)$ is central if it supercommutes with any element of $\mathrm{U}(\mathfrak{q}_N)$. However, the centre $\mathrm{Z}(\mathfrak{q}_N)$ consists of even elements only [17, Theorem 1]. Note that the representation $\gamma : \mathrm{U}(\mathfrak{q}_N) \to \mathcal{PD}$ is faithful when $M \geqslant N$. Then we get the equality $\mathcal{I} = \gamma\big(\mathrm{Z}(\mathfrak{q}_N)\big)$ by Proposition 4.1; cf. [18, Section 3]. In this section for any M, N we will give an explicit formula for a non-zero element in $\mathcal{I}_\lambda$. We will also construct a non-zero element in $\gamma^{-1}(\mathcal{I}_\lambda) \cap \mathrm{Z}(\mathfrak{q}_N)$. An element of $(\gamma')^{-1}(\mathcal{I}_\lambda) \cap \mathrm{Z}(\mathfrak{q}_\mathrm{M})$ can be constructed in a similar way. Thus for any M, N we will get an evidential proof of the equality

$$\mathcal{I} = \gamma\big(\mathrm{Z}(\mathfrak{q}_N)\big) = \gamma'\big(\mathrm{Z}(\mathfrak{q}_\mathrm{M})\big).$$

Now let λ be any strict partition of n with $\ell_\lambda \leqslant N$. Let $R_\lambda \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes n}$ correspond to $\Psi_\lambda \in H_n$ with respect to (4.1). Let $L_\lambda \subset (\mathbb{C}^{N|N})^{\otimes n}$ be the image of the endomorphism R_λ. The representation of $\mathrm{U}(\mathfrak{q}_N)$ in the space L_λ is a direct sum of $2^{[\ell_\lambda/2]}$ copies of the irreducible representation in V_λ; see Theorem 3.4. Denote by ω_λ the respective homomorphism $\mathrm{U}(\mathfrak{q}_N) \to \mathrm{End}(L_\lambda)$. We will identify the algebra $\mathrm{End}(L_\lambda)$ with the subalgebra in $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes n}$ consisting of all the elements which have the form $X\,R_\lambda = R_\lambda Y$ for some $X, Y \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes n}$.

Denote

$$F = \sum_{i,j} E_{ij} \otimes F_{ji} \cdot (-1)^{\bar{j}} \in \mathrm{End}(\mathbb{C}^{N|N}) \otimes \mathrm{U}(\mathfrak{q}_N).$$

Note that for any element $X \in \mathfrak{q}_N$ we have the equality

$$[X \otimes 1 + 1 \otimes X,\, F] = 0 \tag{4.4}$$

where the square brackets stand for the supercommutator. For $s = 1, \ldots, n$ we will write

$$F_s = \iota_s \otimes \mathrm{id}\ (F) \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes n} \otimes \mathrm{U}(\mathfrak{q}_N).$$

As well as in the previous section put $z_s = \sqrt{c_s(c_s+1)}$ where c_s is the content of the box with number s in the shifted column tableau Λ. Consider the element

$$F_\lambda = R_\lambda \otimes 1 \cdot (F_1 - z_1) \ldots (F_n - z_n) \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes n} \otimes \mathrm{U}(\mathfrak{q}_N). \tag{4.5}$$

Now let μ by any strict partition of m with not more than N parts. The next proposition makes the central part of this section; cf. [19, Section I.2].

PROPOSITION 4.4. – *We have* $\mathrm{id} \otimes \pi_\mu(F_\lambda) = 0$ *if the diagram* λ *is not contained in* μ.

Proof. – We can replace in Proposition 4.4 the representation π_μ by the direct sum ω_μ of its copies. With respect to the defining representation $\mathrm{U}(\mathfrak{q}_N) \to \mathrm{End}(\mathbb{C}^{N|N})$

$$\mathrm{End}(\mathbb{C}^{N|N}) \otimes \mathrm{U}(\mathfrak{q}_N) \to \mathrm{End}(\mathbb{C}^{N|N})^{\otimes 2} : \ F \mapsto P\,(1 - J_1 J_2).$$

So the element $\mathrm{id} \otimes \omega_\mu(F_\lambda)$ of $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes n} \otimes \mathrm{End}(L_\mu) \subset \mathrm{End}(\mathbb{C}^{N|N})^{\otimes (n+m)}$ equals

$$R_\lambda \otimes 1 \cdot \prod_{1 \leqslant s \leqslant n}^{\rightarrow} \Big(\sum_{1 \leqslant r \leqslant m} P_{s,n+r}(1 - J_s\, J_{n+r}) - z_s \Big) \cdot 1 \otimes R_\mu.$$

This product is the image in $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes (n+m)}$ of the element from the Sergeev algebra H_{n+m}

$$\Psi_\lambda \cdot (y_1 - z_1) \ldots (y_n - z_n) \cdot \Psi_\mu^{\sim}\ ;$$

see the end of the previous section. By Corollary 3.7 the latter product vanishes if the shifted diagram λ is not contained in the shifted diagram μ □

Consider the linear functional $\mathrm{str} : \mathrm{End}(\mathbb{C}^{N|N})^{\otimes n} \to \mathbb{C}$ called the *supertrace*. By definition, we have

$$\mathrm{str} : E_{i_1 j_1} \otimes \ldots \otimes E_{i_n j_n} \mapsto \delta_{i_1 j_1} \ldots \delta_{i_n j_n} \cdot (-1)^{\bar{\imath}_1 + \ldots + \bar{\imath}_n}.$$

This functional is invariant with respect to the adjoint action ad of the Lie superalgebra $\mathfrak{gl}_{N|N}$ in $\mathrm{End}(\mathbb{C}^{N|N})^{\otimes n}$. Let us now introduce the *Capelli element*

$$C_\lambda = \mathrm{str} \otimes \mathrm{id}\,(F_\lambda) \in \mathrm{U}(\mathfrak{q}_N), \tag{4.6}$$

see (4.5). This definition of the element C_λ is motivated by the results of [4,5].

LEMMA 4.5. – *We have* $C_\lambda \in \mathrm{Z}(\mathfrak{q}_N)$.

Proof. – By the equality (4.4) and by the definition of F_λ we have for any $X \in \mathfrak{q}_N$

$$\big[X, \mathrm{str} \otimes \mathrm{id}\,(F_\lambda)\big] = -\,(\mathrm{str} \circ \mathrm{ad} X) \otimes \mathrm{id}\,\big(F_\lambda\big) = 0 \quad \square$$

Note that $\gamma\,(\mathrm{Z}(\mathfrak{q}_N)) \subset \mathcal{I}$. So we get the following corollary to Propositions 4.3, 4.4.

COROLLARY 4.6. – *We have* $\gamma\,(C_\lambda) \in \mathcal{I}_\lambda$.

In the remainder of this section we will give an explicit formula for the differential operator $\gamma\,(C_\lambda) \in \mathcal{PD}$. In particular, we will see that $\gamma\,(C_\lambda) \neq 0$ for $\ell_\lambda \leqslant M, N$.

Consider the collection (3.15) of the basic elements $h = g\, a_1^{l_1} \ldots a_n^{l_n}$ of the Sergeev algebra H_n. Here g runs through the symmetric group S_n while each of the indices $l_1, \ldots, l_n$ runs through $0, 1$. Let the indices $i_1, \ldots, i_n$ and $b_1, \ldots, b_n$ run through $\pm 1, \ldots, \pm N$ and $\pm 1, \ldots, \pm M$ respectively. Put

$$I_\lambda = \sum_{g} \sum_{l_1 \ldots l_n} \sum_{i_1 \ldots i_n} \sum_{b_1 \ldots b_n} \chi_\lambda(h) \cdot x_{j_n b_n} \ldots x_{j_1 b_1}\, \partial_{i_1 b_1} \ldots \partial_{i_n b_n} \cdot (-1)^e \tag{4.7}$$

where we write $j_s = i_{g(s)} \cdot (-1)^{l_s}$ for each $s = 1, \ldots, n$ and denote

$$e = \sum_{r<s} (\bar{\imath}_r \bar{b}_s + \bar{\jmath}_r \bar{b}_s + \bar{\jmath}_r \bar{\jmath}_s + \bar{\jmath}_r\, l_s) + \sum_{\substack{r<s \\ g^{-1}(r)<g^{-1}(s)}} \bar{\imath}_r\, \bar{\imath}_s \; + \; \sum_{s} (\bar{\jmath}_s + 1)\, l_s.$$

THEOREM 4.7. – *We have* $\gamma\,(C_\lambda) = I_\lambda$. *Here* $I_\lambda \neq 0$ *if* $\ell_\lambda \leqslant M, N$.

Proof. – Consider the $\mathbb{Z}$-gradation on the vector space $\mathcal{PD} = \mathcal{P} \cdot \mathcal{D}$ by the degree of the differential operator. By definition any element of the subspace $\mathcal{I}_\lambda \subset \mathcal{PD}$ is homogeneous of the degree n. Therefore by (4.6) due to Corollary 4.6 the element $\gamma\,(C_\lambda)$ coincides with the leading term of the element

$$\mathrm{str} \otimes \gamma\,\big(R_\lambda \otimes 1 \cdot F_1 \ldots F_n\big) \in \mathcal{PD}. \tag{4.8}$$

Let $Q_\lambda \in \mathrm{End}(\mathbb{C}^{N|N})^{\otimes n}$ be the image of the element $\mathrm{X}_\lambda \in H_n$ under (4.1); see (3.16). First let us show that the leading term of (4.8) coincides with that of

$$\mathrm{str} \otimes \gamma\,\big(Q_\lambda \otimes 1 \cdot F_1 \ldots F_n\big) \in \mathcal{PD}. \tag{4.9}$$

Observe that by the definition of the element $F \in \mathrm{End}(\mathbb{C}^{N|N}) \otimes \mathrm{U}(\mathfrak{q}_N)$ we have

$$J \otimes 1 \cdot F \cdot J^{-1} \otimes 1 = -\,F.$$

Therefore for any $s=1,\dots,n$ we get the equalities in $\mathcal{PD}$

$$\begin{aligned}&\operatorname{str}\otimes\gamma\big(J_s\,R_\lambda\,J_s^{-1}\otimes 1\cdot F_1\dots F_n\big)=\\&-\operatorname{str}\otimes\gamma\big(J_s\,R_\lambda\otimes 1\cdot F_1\dots F_n\cdot J_s^{-1}\otimes 1\big)=\operatorname{str}\otimes\gamma\big(R_\lambda\otimes 1\cdot F_1\dots F_n\big).\end{aligned}$$

Furthermore, for any $s=1,\dots,n-1$ we have

$$\begin{aligned}&\operatorname{str}\otimes\gamma\big(P_{s,s+1}\,R_\lambda\,P_{s,s+1}\otimes 1\cdot F_1\dots F_s\,F_{s+1}\dots F_n\big)=\\&\operatorname{str}\otimes\gamma\big(P_{s,s+1}\,R_\lambda\otimes 1\cdot F_1\dots F_{s+1}\,F_s\dots F_n\cdot P_{s,s+1}\otimes 1\big)=\\&\operatorname{str}\otimes\gamma\big(R_\lambda\otimes 1\cdot F_1\dots F_{s+1}\,F_s\dots F_n\big).\end{aligned}\tag{4.10}$$

But the elements (4.8) and (4.10) of $\mathcal{PD}$ have the same leading terms. So the equlaity of the leading terms in (4.8) and (4.9) follows from Corollary 3.5.

But the leading term of (4.9) is easy to write down. Under (4.1) for any $g\in S_n$

$$g^{-1}\;\mapsto\;\sum_{i_1\dots i_n}E_{i_{g(1)}i_1}\otimes\dots\otimes E_{i_{g(n)}i_n}\cdot(-1)^{k}$$

where

$$k\;=\;\sum_{r<s}\bar\imath_r\left(\bar\imath_s+\bar\imath_{g(s)}\right)\;+\;\sum_{\substack{r<s\\ g^{-1}(r)>g^{-1}(s)}}\bar\imath_r\,\bar\imath_s.$$

Further,

$$\left(a_1^{l_1}\dots a_n^{l_n}\right)^{-1}\;\mapsto\;\sum_{j_1\dots j_n}E_{j_1,\,\varepsilon_1\cdot j_1}\otimes\dots\otimes E_{j_n,\,\varepsilon_n\cdot j_n}\cdot(-1)^{\,l}$$

where each of the indices $j_1,\dots,j_n$ runs through $\pm 1,\dots,\pm N$; we write $\varepsilon_s=(-1)^{l_s}$ for each $s=1,\dots,n$ and denote

$$l\;=\;\sum_{r<s}l_r\,l_s\;+\;\sum_{s}l_s\,(\,\bar\jmath_s+1\,).$$

Now the definitions (3.16) and (4.2) imply that the leading term of (4.9) equals I_λ.

In particular, we have $I_\lambda\in\mathcal{I}_\lambda$ by Corollary 4.6. Suppose that $\ell_\lambda\leqslant M,N$. Let us show that $I_\lambda\neq 0$. Consider the element

$$I\;=\;\sum_{i_1\dots i_n}\;\sum_{b_1\dots b_n}x_{i_nb_n}\dots x_{i_1b_1}\,\partial_{i_1b_1}\dots\partial_{i_nb_n}\;\in\;\mathcal{PD}$$

This element is $\mathrm{U}(\mathfrak{q}_N)\otimes\mathrm{U}(\mathfrak{q}_M)$-invariant. Due to [9, Theorem 3] the element $I_\lambda\cdot\dim U_\lambda\,/\,(2^n n!)$ is the projection of $I\in\mathcal{I}$ to the direct summand $\mathcal{I}_\lambda$ in (4.3). This projection cannot be zero since the elements $x_{i_1b_1}\dots x_{i_nb_n}\in\mathcal{P}$ span $\mathcal{P}_n$ □

Consider the canonical $\mathbb{Z}$-filtration on the algebra $\mathrm{U}(\mathfrak{q}_N)$. It is defined by assigning the degree 1 to every generator $F_{ij}\in\mathfrak{q}_N$. The corresponding $\mathbb{Z}$-graded algebra is the supersymmetric algebra $\mathrm{S}\,(\mathfrak{q}_N)$. The subalgebra $\mathrm{I}\,(\mathfrak{q}_N)\subset\mathrm{S}\,(\mathfrak{q}_N)$ of invariants with respect to the adjoint action of $\mathfrak{q}_N$ corresponds to the centre $\mathrm{Z}(\mathfrak{q}_N)\subset\mathrm{U}(\mathfrak{q}_N)$.

Take the quotient of the algebra $\mathrm{S}(\mathfrak{q}_N)$ by the ideal generated by all the elements F_{ij} with $i \neq j$. For each $i = 1, \ldots, N$ denote by t_i the image in the quotient of the element $F_{ii} \in \mathrm{S}(\mathfrak{q}_N)$. Then the quotient algebra is $\mathbb{C}[t_1, \ldots, t_N]$. Due to [17, Theorem 1] the image in $\mathbb{C}[t_1, \ldots, t_N]$ of the subalgebra $\mathrm{I}(\mathfrak{q}_N) \subset \mathrm{S}(\mathfrak{q}_N)$ is generated by all the power sums of odd degree $t_1^{2r+1} + \ldots + t_N^{2r+1}$, $r \geqslant 0$. We will desribe the image $T_\lambda \in \mathbb{C}[t_1, \ldots, t_N]$ of the element from $\mathrm{I}(\mathfrak{q}_N) \subset \mathrm{S}(\mathfrak{q}_N)$ corresponding to the central element $C_\lambda \in \mathrm{U}(\mathfrak{q}_N)$. In particular, we will see that $C_\lambda \neq 0$. For description of the eigenvalue of $C_\lambda \in \mathrm{U}(\mathfrak{q}_N)$ in the irreducible module V_μ where the diagram μ does contain λ, see [7, Section 1].

A *shifted Young tableau* of shape λ is any bijective filling of the boxes of diagram λ with the numbers $1, \ldots, n$ such that in every row and column the numbers increase from the left to the right and from the top to the bottom respectively. Let n_λ stand for the total number of shifted Young tableaux of the shape λ. Let $Q_\lambda(t_1, \ldots, t_N; -1)$ be the Schur Q-polynomial symmetric in $t_1, \ldots, t_N$; see [20].

PROPOSITION 4.8. – *We have* $T_\lambda(t_1, \ldots, t_N) = Q_\lambda(t_1, \ldots, t_N; -1) \cdot n!/n_\lambda$.

Proof. – Regard $t_1, \ldots, t_N$ as complex variables. Put $t_{-i} = t_i$ for each $i = 1, \ldots, N$. Denote

$$T = \sum_i t_i\, E_{ii} \cdot (-1)^{\bar{\imath}} \in \mathrm{End}(\mathbb{C}^{N|N})$$

where the index i runs through $\pm 1, \ldots, \pm N$. Then by the definition (4.6) we have $T_\lambda = \mathrm{str}(Q_\lambda \cdot T^{\otimes n})$; see also the proof of Theorem 4.7. Now Proposition 4.8 can be obtained from [9, Section 2.2] and [20, Example III.7.8] □

Due to Proposition 4.8 the elements C_λ where the diagram λ has only one row, generate the centre of enveloping algebra $\mathrm{U}(\mathfrak{q}_N)$. So the equality $\gamma(C_\lambda) = I_\lambda$ for $\lambda = (n)$ may be regarded as an analogue for $\mathfrak{q}_N$ of the classical Capelli identity [1]. Note that for $\lambda = (n)$ the element $\Psi_\lambda \in H_n$ is determined by the formula (1.3).

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