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LUIS GUIJARRO
PETER PETERSEN

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RIGIDITY IN NON-NEGATIVE CURVATURE

BY LUIS GUIJARRO AND PETER PETERSEN

ABSTRACT. – In this paper we will show that any complete manifold of non-negative curvature has a flat soul provided it has curvature going to zero at infinity. We also show some similar results about manifolds with bounded curvature at infinity. To establish these theorems we will prove some rigidity results for Riemannian submersions, eg., any Riemannian submersion with complete flat total space and compact base in fact must have a flat base space.

RÉSUMÉ. – Dans cet article nous montrons que toute variété complète de courbure non-négative a une âme plate, lorsque sa courbure tend vers zéro à l'infini. Pour ces théorèmes nous démontrons quelques résultats de rigidité pour des submersions riemanniennes ; par exemple, chaque submersion riemannienne ayant un espace total complet et plat et ayant une base compacte doit avoir une base plate.

1. Introduction

In this paper we establish some rigidity theorems for souls of complete non-compact manifolds with non-negative curvature. More precisely we show that the soul is flat provided that there are some constraints on the geometry at infinity of the manifold. Our first result is:

THEOREM 1.1. – *Let M be a complete Riemannian manifold of non-negative curvature. If the curvature goes to zero at infinity then the soul is flat.*

This result was first mentioned by Marenich in [12]. The proof there has been acknowledged to be incorrect (*see* also [11]). It was also independently considered in [4], where the authors proved it when the soul has codimension ≤ 3. Our method for proving the above theorem uses completely different ideas from those used in [12] and [4]. It furthermore yields some new and interesting perturbation results. Our proof also rests on some rigidity theorems of independent interest, one of which goes back to an unpublished paper by the second author (*see* [17].) These results, which can be found in section 3, are concerned with Riemannian submersions from a complete space to a compact base

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space. The idea is to find conditions that make the base space a flat manifold. The simplest such condition is to assume that the total space is flat (*see* [17].) This is the rigidity phenomenon behind the above theorem. Our other rigidity results imply the following two extensions of the above theorem:

THEOREM 1.2. – *Let M be a complete Riemannian n-manifold of non-negative curvature. Given $D > 0$ there is an $\varepsilon(n, D) > 0$ such that if M has an end of diameter growth $\leq D$ and curvature $\leq \varepsilon$ at infinity then the soul is flat.*

THEOREM 1.3. – *Let M be a complete Riemannian n-manifold of non-negative curvature. Given $i, D > 0$ there is an $\varepsilon(n, i, D) > 0$ such that if the soul of M has diameter $\leq D$ and injectivity radius $\geq i$ and furthermore the curvature of M is $\leq \varepsilon$ at infinity then the soul is flat.*

It is possible that the last theorem is true without an assumption on the injectivity radius.

The idea of the proof of all of the above results is to choose an appropriate sequence $\{p_i\}$ of points on M, which goes to infinity. We then use convergence techniques to get a limit space (X, p) from the sequence $\{M, p_i\}$. If S is a soul of M then we have a Riemannian submersion $sh : M \to S$ (*see* [19], [14]). This gives rise to a map $sh : X \to S$ which is a Riemannian submersion provided X is sufficiently smooth (*see* [1]). It is now clear that whenever we have a result for Riemannian submersions which says that the base must be flat, then we can hope to apply it to the above situation and get a result which claims that the soul should be flat.

2. Preliminaries

2.1. Non-negative Curvature

Let M be a complete non-compact manifold of non-negative sectional curvature. The *soul* S of M is a compact totally convex submanifold which contains all the topology of M in the sense that M is diffeomorphic to the normal bundle of S (*see* [3] and [18]). In [19] Sharafutdinov constructed a distance non-increasing map $sh : M \to S$. In a rather amazing development Perel'man showed in [14] that this is a $C^{1,1}$ Riemannian submersion (*see* also [1], [9]), whose fiber at $s \in S$ consists exactly of the geodesics that emanate from s and are normal to S.

We also need some well known rigidity results:

THEOREM 2.1. – *Let M be a compact Riemannian manifold of non-negative sectional curvature. Then the universal cover of M is isometric to $\mathbb{R}^k \times C$, where C is compact. In particular if the universal cover is contractible then M is flat.*

This result is established in [3] and only depends on the splitting theorem. It can therefore also be generalized to compact manifolds with non-negative Ricci curvature, and to Alexandrov spaces which have non-negative curvature and no boundary.

THEOREM 2.2 (See [6]). – *Let M be a compact n-manifold with $|\sec(M)| \leq 1$ and diam$(M) \leq \varepsilon(n)$, then M is an infra-nil manifold and in particular has contractible universal cover.*

COROLLARY 2.3. – *Let M be a compact n-manifold with* $0 \leq \sec(M) \leq 1, \operatorname{diam}(M) \leq \varepsilon(n)$. *Then M is flat.*

The construction of the Sharafutdinov map was also used to get the following bound for the injectivity radius:

THEOREM 2.4. – *Suppose M has curvature bounded by K. Then:*

$$\operatorname{inj}(M) \geq \min\left\{\operatorname{inj}(S), \frac{\pi}{\sqrt{K}}\right\}$$

We get in particular from this result that M must have a lower bound for the injectivity radius as long as no sectional curvatures become large at infinity. We can therefore apply the standard techniques of convergence theory to sequences of the form (M, p_i) where M is complete, has non-negative curvature and bounded curvature, and p_i is any sequence of points on M (*see* [2], [5], [15], [16].)

2.2. Submetries

Let X and Y be metric spaces. An isometry between these spaces is by definition a map which preserves distances. Berestovskii has found a very natural generalization of this concept. Namely he considers so called *submetries* which by definition are maps that send metric balls to metric balls of the same radius. More precisely $f : X \to Y$ is called a submetry if for all $x \in X$ and $r \in [0, r(x)]$ we have that $f(B(x,r)) = B(f(x), r)$, where $B(p,r)$ denotes the open metric ball centered at p of radius r and $r(x)$ is some positive continuous function.

It is by now classical that a map between Riemannian manifolds is an isometry iff it is a Riemannian isometry, *i.e.*, it is smooth and preserves the metric tensor. In [1] this was generalized to submetries. It is of course obvious that Riemannian submersions are submetries, the converse is contained in:

THEOREM 2.5. – *A submetry $f : X \to Y$ between smooth Riemannian manifolds is a C^1 Riemannian submersion.*

Outline of Proof. – Any distance function $g(\cdot) = d(p, \cdot)$ is C^1 on $B(p,i) - \{p\}$ (where i is the injectivity radius at p) and a Riemannian submersion onto $(0, i)$. To prove the result, it suffices to check it for $g \circ f$ where g varies over a suitable number of distance functions in Y. Since these are also submetries, we can just consider the case where Y is 1-dimensional. Let $x \in X$. By passing to a small convex neighborhood of x, we can assume that the fibers of f are closed and that any two points in the domain are joined by a unique geodesic. We now wish to show that f has a continuous unit gradient field ∇f. We know that the integral curves for ∇f should be exactly the unit speed geodesics which are mapped to unit speed geodesics by f. Since f is distance non-increasing it is clear that any piecewise smooth unit speed curve which is mapped to a unit speed geodesic must be a smooth unit speed geodesic. Thus these integral curves are unique and vary continuously to the extent that they exist. To establish the existence of these curves we use the submetry property. First fix $p \in X$ and let $c(t)$ be the unit speed segment in Y with $c(0) = f(p)$. Denote by F_t the fiber of f above $c(t)$. Now let $\gamma(t) : [0, a] \to X$ be a unit speed segment

with $\gamma(0) = p$ and $\gamma(a) \in F_a$, this is possible since $f(B(p,a)) = B(c(0),a)$. It is now easy to check, again using the submetry property, that $c(t) = f \circ \gamma(t)$, as desired. ■

For the proof we clearly only used that one has C^1 distance functions on Y and that geodesics in X are locally unique and vary continuously. These conditions are certainly satisfied if e.g., X has bounded curvature in the sense of Alexandrov, and Y has a lower bound for the injectivity radius and a lower bound for the curvature in the sense of Alexandrov.

The optimum smoothness one would expect for a submetry is $C^{1,1}$. Berestovski has been able to prove this, but we only need the weaker result for our purposes.

For our applications we are concerned with the stability of submetries under pointed Gromov-Hausdorff convergence. Consider sequences $\{X_i, x_i\}$ and $\{Y_i, y_i\}$ of complete pointed separable metric spaces and assume that $\{X_i, x_i\} \to (X, x)$ and $\{Y_i, y_i\} \to (Y, y)$ in the pointed Gromov-Hausdorff topology. If we have distance non-increasing maps $f_i : (X_i, x_i) \to (Y_i, y_i)$ (or more generally equi-continuous maps) then an immediate generalization of the classical Arzela-Ascoli Theorem [7] tells us that there must be a subsequence of spaces and maps converging to a distance non-increasing map $f : (X, x) \to (Y, y)$. If we know that the maps f_i are in fact isometries then is easy to check that the limit map must also be an isometry. Similarly one can also show:

LEMMA 2.6. – *Let spaces and maps be as above. Then any limit map of a sequence of submetries is again a submetry.*

Proof. – First observe that any submetry is distance non-increasing. So we can always find a limit map. Now suppose that $f : (X, x) \to (Y, y)$ is the limit of a sequence of submetries $f_i : (X_i, x_i) \to (Y_i, y_i)$. Fix $p \in X$ and $r > 0$. Since the limit map is again distance non-increasing it must certainly satisfy: $f(B(p,r)) \subset B(f(p), r)$. For the reverse inclusion choose some point $q \in B(f(p), r)$. Then choose a sequence of points $q_i \in Y_i$ converging to q and $p_i \in X_i$ converging to p. Now choose $\varepsilon > 0$ such that $q \in B(f(p), r - \varepsilon)$. For sufficiently large i we must have that $q_i \in B(f_i(p_i), r - \varepsilon)$. Consequently we can find $x_i \in B(p_i, r - \varepsilon)$ such that $f_i(x_i) = q_i$. Using completeness we can now assume that x_i (sub)converges to a point x which must lie in the closure of $B(p, r - \varepsilon)$ which is clearly contained in $B(p, r)$. Now from the convergence of the maps we get that $f_i(x_i) \to q_i$ and hence $f(x) = q$. Whence we get the other inclusion. ■

We will also need the following result:

THEOREM 2.7 (See [10]). – *Let $f : M \to N$ be a C^1Riemannian submersion between complete Riemannian manifolds, then $f : M \to N$ is a fibration.*

3. Rigidity of Riemannian submersions

We will consider certain Riemannian submersions $f : M \to N$, where M, is complete and N is compact. Our first result was initially proved in [17] using topological arguments. A more geometric approach was soon after found by Walschap in [20]. We shall here use the topological approach as it seems to lead more easily to the kind of generalizations we are interested in.

THEOREM 3.1. – *Let $f : M \to N$ be as above. If M is flat then N is also flat and hence the Riemannian submersion is locally a product.*

Proof. – We will prove that under considerably weaker conditions the universal covering of N is contractible. Thus N must be flat if it has non-negative Ricci curvature.

Suppose $f : V \to W$ is a submersion and a fibration between manifolds. We claim that if V has contractible universal cover and W has finitely generated homotopy groups, then W also has contractible universal cover.

We can immediately construct another submersion/fibration $\widetilde{f} : \widetilde{V} \to \widetilde{W}$ between the universal covers. Let F be the homotopy fiber of $\widetilde{f} : \widetilde{V} \to \widetilde{W}$. Since $\widetilde{f}$ is a submersion we know that F is a finite dimensional manifold. Since the homotopy groups of $\widetilde{W}$ (and $\widetilde{V}$) are finitely generated their homology groups must also be finitely generated. We can then conclude that the same must be true of F. Now let $p \leq \dim F$ be the largest number such that $H_p(F, \mathbb{Z}) \neq 0$ and $q \leq \dim \widetilde{W}$ the largest number so that $H_q(\widetilde{W}, \mathbb{Z}) \neq 0$. Then the spectral sequence for the homology of the fibration can be applied and says that:

$$H_{p+q}(\widetilde{V}, L) = H_q(\widetilde{W}, H_p(F, L)) = H_q(\widetilde{W}, L) \otimes_L H_p(F, L), \text{ where } L \text{ is any field}$$

However, $H_{p+q}(\widetilde{V}, L) = 0$, unless $p = q = 0$. So we have arrived at a contradiction unless $p = q = 0$. Whence $\widetilde{W}$ is contractible.

To see how this implies the original statement of the theorem observe that flat manifolds have contractible universal coverings and that Riemannian submersions are curvature increasing so that N must have non-negative sectional curvature. Whence N must be flat. ■

We need to extend this theorem to a slightly more general situation where N is merely a $C^{0,1}$ Riemannian manifold with curvature ≥ 0 in the sense of Alexandrov and f is a submetry from the flat manifold M. In this case it still follows from Berestovskii's work that f is a C^1 Riemannian submersion. Furthermore Theorem 2.1. is also valid for such N (*see* [8].) Thus the universal covering must be flat.

THEOREM 3.2. – *Given an integer $n \geq 2$, and numbers $D > 0, i > 0$ there is an $\varepsilon(n, D, i) > 0$, such that any Riemannian submersion as above, where $n = \dim M$, $\operatorname{inj}(N) \geq i$, $\operatorname{diam}(N) \leq D$, and $-\varepsilon \leq \delta \leq \sec(M) \leq \varepsilon$, must have the property that N is diffeomorphic to a flat manifold, and therefore N is flat if $\delta = 0$.*

Proof. – We argue by contradiction. So suppose we have a sequence $f_k : M_k \to N_k$ of Riemannian submersions where N_k has $\operatorname{inj}(N_k) \geq i$ and $\operatorname{diam}(N_k) \leq D$, while $|\sec(M_k)| \leq 1/k$. Fix $p_k \in M_k$ and consider the exponential map $g_k = \exp : B\left(0, \sqrt{k}\right) \subset T_{p_k} M_k \to M_k$. If we use the pull-back metric on $B\left(0, \sqrt{k}\right)$ then we get a Riemannian submersion $f_k \circ g_k : B\left(0, \sqrt{k}\right) \to N_k$. As $k \to \infty$ the curvatures on $B\left(0, \sqrt{k}\right)$ converge to zero and there is no collapse so the limit space will be $\mathbb{R}^n$, while the limit N of N_k will be a space with a compact Riemannian space of type $C^{0,1}$ and $\operatorname{inj} \geq i$. Thus the results from the preceding section yields a Riemannian submersion $\mathbb{R}^n \to N$. This implies from above that N is a flat manifold and hence N_k is diffeomorphic to a flat manifold for large k. This contradicts our assumptions. ■

It is possible that this theorem is true without any assumptions on the injectivity radius. Another variant of the above result is:

THEOREM 3.3. – *Given an integer $n \geq 2$, and a number $D > 0$ there is an $\varepsilon(n, D) > 0$, such that any Riemannian submersion as above, where $n = \dim M$, $\operatorname{diam}(M) \leq D$, and $-\varepsilon \leq \delta \leq \sec(M) \leq \varepsilon$, must have the property that N has contractible universal cover, and therefore N is flat if $\delta = 0$.*

Proof. – Simply observe that [6] implies M has contractible universal covering if ε is sufficiently small. ■

4. Coming in from Infinity

For this section we will consider a fixed complete Riemannian n-manifold M of non-negative sectional curvature. For this manifold we also select a soul $S \subset M$ and with it the canonical Riemannian submersion $sh : M \to N$. The *upper bound for the curvature at infinity* for M is defined as $K_\infty = \limsup_{r\to\infty}\{\sec(\pi_q) : \pi_q \subset T_qM, d(q, S) \geq r\}$. If $K_\infty < \infty$ then we say that M has *bounded curvature at infinity*, while if $K_\infty = 0$ then we say that the *curvature goes to zero at infinity*. Such manifolds have a particularly nice structure at infinity which relates to the soul via a Riemannian submersion:

THEOREM 4.1. – *Suppose M satisfies $0 \leq K_\infty < \infty$, then for any sequence of points $q_i \to \infty$ we have that the pointed sequence (M, q_i) (sub)converges in the pointed $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ Riemannian manifold (X, q) whose sectional curvatures in the sense of Alexandrov lie in $[0, K_\infty]$. And with this limit space we have a Riemannian submersion $sh : X \to S$. In fact by choosing the sequence judiciously one can ensure that the limit space satisfies: $X = N \times \mathbb{R}^k$, where N is compact.*

Proof. – Since M has bounded curvature and therefore also a lower bound for the injectivity radius we can suppose that the sequence (M, q_i) converges in the pointed $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ Riemannian manifold (X, q) (*see* [6], for instance). For each i we can now select r_i such that $r_i \to \infty$, and the curvatures on $B(q_i, r_i)$ are $\leq K_\infty + 1/i$. Then the pointed metric balls $(B(q_i, r_i), q_i)$ will also converge to (X, q). Since the upper bounds on curvature converge to K_∞ the limit space will inherit this upper curvature bound even if it is zero. This is easily seen using exponential coordinates and using that the sequence already converges in the $C^{1,\alpha}$ topology.

The Riemannian submersions $sh : (M, q_i) \to S$ will obviously converge to a submetry $sh : X \to S$ which will also be a Riemannian submersion by Berestovskii's result.

To prove the last statement of the theorem first observe that the limit space can always be written $N \times \mathbb{R}^l$ where N does not contain any lines. If N is compact then we are done, otherwise N must contain a ray. Now choose a sequence of points $\{q_i\}$ going to infinity along this ray. Then the sequence $(N \times \mathbb{R}^l, q_i)$ will (sub)converge to a space which looks like $N_1 \times \mathbb{R}^{l+1}$. Now for each $q_i \in X = N \times \mathbb{R}^l$ choose $p_i \in M$ close to q_i. Then (M, p_i) will also have $N_1 \times \mathbb{R}^{l+1}$ as a limit space. A simple induction argument now finishes the proof. ■

It might be helpful to have some examples illustrating this theorem.

Example. – Consider a rotationally symmetric surface M of the type: $dr^2 + \varphi^2(r)d\theta^2$, where $\varphi(r) = r$ for r near 0 and $\varphi(r) = a$ for all large r. In this case the soul is a point, and the limit space is always a cylinder where the soul is a circle of length $2\pi a$.

Example. – We will consider 3-dimensional orientable flat manifolds where the soul is a circle of length 2π. These spaces are all gotten by first taking $[0, 2\pi] \times \mathbb{R}^2$ and then gluing the two spaces $\{0\} \times \mathbb{R}^2$, $\{2\pi\} \times \mathbb{R}^2$ together via a rotation. If the angle of the rotation is $2\pi\theta$, then we denote the resulting space as M_θ. If we choose the points $\{p_i\}$ to lie on a ray, then the limit space will clearly split $X = F \times \mathbb{R}$, where F is a 2 dimensional flat manifold. We can immediately eliminate the possibility that F is compact or non-orientable. Thus $F = S \times \mathbb{R}$, where S is either a circle or the real line. We claim that S must be a line if θ is irrational. In case S is a circle it will be a homotopically non-trivial closed geodesic. For large i we can then find loops γ_i based at p_i which converge to S. Since S is homotopically non-trivial we can shorten the γ_i's to become non-trivial geodesic loops c_i based at p_i. These geodesic loops will converge to a geodesic loop in X, but since geodesics there are either closed or infinite, we can actually assume that the c_i's converge to S. It is however a feature of the geometry of M_θ that one can have geodesic loops of a given bounded length arbitrarily far away from the soul only when θ is a rational number.

These two examples show that the soul at infinity can be either larger or smaller in dimension and diameter than the original soul. In particular the map $sh : X \to S$ does not necessarily factor through the soul of X. It is therefore important that our rigidity results for Riemannian submersions allow us to have non-compact total space.

We can now prove the theorems mentioned in the introduction.

Proof of Theorem 1.1. – In case $K_\infty = 0$ we have that the limit space X is flat. Hence we have a Riemannian submersion $sh : X \to S$. Which shows that S has to be flat. ■

Proof of Theorem 1.2. – The *diameter growth* with respect to some point o of a manifold M is defined as follows: $\mathrm{diam}(r) = \sup\{d(p,q) : p, q$ lie in the same component of the distance sphere $S(o,r)\}$. Thus the diameter growth is less that D if $\limsup_{r\to\infty}\mathrm{diam}(r) \leq D$. If M has non-negative curvature the splitting theorem implies that either the distance spheres $S(o,r)$ are all connected as $r \to \infty$ or the manifold splits as a product $M = \mathbb{R} \times H$. So if M has bounded diameter growth either the distance spheres $S(o,r)$ have bounded diameter as $r \to \infty$ or the manifold splits as a product $M = \mathbb{R} \times H$, where H is compact and therefore also the soul of M. In the latter situation $\mathrm{diam}(H)$ is obviously the appropriate bound for the diameter function. So if $K_\infty \cdot \mathrm{diam}(H)^2$ is sufficiently small the soul must be flat. In the former case we have that the distance spheres from some fixed point all have uniformly bounded diameter at infinity. Thus the limit space must split as a product: $X = \mathbb{R} \times Y$, where Y is a compact $C^{1,\alpha}$ manifold with $\mathrm{diam}(Y) \leq D$ and the curvatures in the sense of Alexandrov lie in the interval $[0, K_\infty]$. We are therefore done if we can show that Y is flat provided $K_\infty \cdot D^2$ is small. This would definitely be true if Y were a smooth Riemannian manifold, but as the metric is only $C^{1,\alpha}$ we need an extra little argument. The results in [13] show that the metric on Y can be perturbed to a smooth metric which satisfies that $\mathrm{diam} \leq D + \varepsilon$ and the curvatures lie in

$[-\varepsilon - K_\infty, K_\infty + \varepsilon]$, here ε can be chosen arbitrarily. If therefore ε and $K_\infty \cdot D^2$ are small we see that Y is indeed an infra-nilmanifold and in particular has contractible universal cover. Since the original metric on Y had non-negative curvature and the splitting theorem holds for this metric (*see* [8]) we can conclude that Y must be flat. ■

Proof of Theorem 1.3. – We shall proceed as in Theorem 3.2. So suppose we have a sequence of M_k with curvature at infinity $\leq 1/k$ and with souls S_k having $\operatorname{diam} \leq D$ and $\operatorname{inj} \geq i$. We can then select a sequence of points $p_k \in M_k$ such that the curvatures on the metric balls $B(p_k, \sqrt[4]{k})$ are less than $2/k$. We can then again precompose with the exponetial map to get Riemannian submersions $B(p_k, \sqrt[4]{k}) \to S_k$ where as before we use the pull back metric on $B(p_k, \sqrt[4]{k})$. In the limit we then get a Riemannian submersion $\mathbb{R}^n \to S = \lim(S_k)$. Hence N is flat and so S_k is diffeomorphic to a flat manifold for large k. Since S_k has non-negative curvature we can then conclude that it is in fact flat. ■

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L. GUIJARRO
Department of Mathematics,
University of Pennsylvania, Philadelphia
PA 19104
E-mail: guijarro@math. upenn.edu

P. PETERSEN
Departement of Mathematics,
University of California, Los Angeles
PA90095
E-mail: petersen@math. ucla.edu