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SYMPLECTIC RIGIDITY OF GEODESIC FLOWS ON TWO-STEP NILMANIFOLDS

BY CAROLYN S. GORDON, YIPING MAO AND DOROTHEE SCHUETH

ABSTRACT. – We show that if two 2-step Riemannian nilmanifolds have symplectically conjugate geodesic flows, then they must be isometric. By 2-step Riemannian nilmanifold, we mean a Riemannian manifold of the form $(\Gamma\backslash N, g)$, where N is a 2-step nilpotent Lie group, Γ is a cocompact discrete subgroup of N, and g is a metric whose pullback to N is left invariant.

RÉSUMÉ. – On montre que si les flots géodésiques de deux nilvariétés riemanniennes de rang deux sont conjugués par un symplectomorphisme, alors les deux variétés sont nécessairement isométriques. Une nilvariété riemannienne de rang deux est une variété riemannienne de la forme $(\Gamma\backslash N, g)$ où N est un groupe de Lie nilpotent de rang deux, Γ est un sous-groupe discret et cocompact de N, et g est une métrique induite par une métrique invariante à gauche sur N.

Introduction

Two compact Riemannian manifolds M_1 and M_2 are said to have *symplectically conjugate geodesic flows* if there exists a symplectomorphism $F : T^*M_1\backslash\{0\} \to T^*M_2\backslash\{0\}$ which intertwines the geodesic flows G_1^t and G_2^t of M_1 and M_2, i.e., such that $F \circ G_1^t = G_2^t \circ F$ for all t.

In this article, we consider compact 2-step Riemannian nilmanifolds. A *Riemannian nilmanifold* is a quotient $\Gamma\backslash N$ of a simply-connected nilpotent Lie group N by a discrete subgroup Γ together with a Riemannian metric whose lift to N is left-invariant. We say $\Gamma\backslash N$ is a *k-step nilmanifold* if N is k-step nilpotent.

THEOREM. – *If M_1 and M_2 are compact 2-step Riemannian nilmanifolds with symplectically conjugate geodesic flows, then M_1 and M_2 are isometric.*

Our interest in this question is motivated in part by a consideration of the relationship between the geodesic flow of a Riemannian manifold (the classical dynamics) and the Laplacian of the manifold (the quantum dynamics). Two compact Riemannian manifolds

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are said to be *isospectral* if the associated Laplacians have the same eigenvalue spectrum. Many compact Riemannian nilmanifolds (including many 2-step nilmanifolds) admit continuous families of isospectral, non-isometric Riemannian metrics ([GW]). The Main Theorem implies that in the 2-step case, these metrics never have symplectically conjugate geodesic flows. On the other hand, R. Kuwabara [Ku] showed that the geodesic flows of these isospectral nilmanifolds, when restricted to suitable open dense subsets of the cotangent bundles, are symplectically conjugate. F. Marhuenda [Ma] studied these isospectral deformations microlocally, examining the operators intertwining the Laplacians. We briefly describe his results here. Given a pair of Riemannian manifolds M_1 and M_2 and the associated symplectic structures ω_1, ω_2 on their cotangent bundles, define a symplectic structure on $T^*M_1\backslash\{0\}\times T^*M_2\backslash\{0\}$ by $\omega = \omega_1 - \omega_2$. A *canonical relation* is a Lagrangian submanifold C of $T^*M_1\backslash\{0\}\times T^*M_2\backslash\{0\}$. Note that if C is actually the graph of a function $F : T^*M_1\backslash\{0\} \to T^*M_2\backslash\{0\}$, then F must be a symplectomorphism. Hörmander's theory associates canonical relations with Fourier integral operators. Marhuenda showed for some of the above deformations $\{M_t\}_t$ that the Laplacians of the isospectral manifolds M_0 and M_t are intertwined by a type of singular Fourier integral operator for each t. These operators are associated with canonical relations on $T^*M_0\backslash\{0\} \times T^*M_t\backslash\{0\}$ which, off a hypersurface in each cotangent bundle, are graphs of symplectic maps. These maps must again intertwine the geodesic flows.

The Main Theorem is not true for arbitrary Riemannian manifolds. A. Weinstein [We] exhibited Zoll surfaces of non-constant curvature whose geodesic flows are symplectically conjugate to that of the round sphere.

One can also consider weaker notions of geodesic conjugacy by requiring that the geodesic conjugacy F be only a C^k-diffeomorphism as opposed to a symplectomorphism. For example, any closed surface whose geodesic flow is C^0-conjugate to that of a negatively curved surface must be isometric to that surface ([CFF]).

This article is a companion to the paper [GM2] in which it is shown that for some large classes of 2-step compact Riemannian nilmanifolds M, any Riemannian nilmanifold whose geodesic flow is C^2-conjugate to that of M must be isometric to M. We expect that this actually holds for arbitrary 2-step compact Riemannian nilmanifolds. However, the proof in [GM2] is quite technical and involves a careful analysis of the behavior of the geodesics. By assuming symplectic conjugacy, we not only remove the extra hypotheses but also obtain a more elegant proof.

We note that the 1-step nilmanifolds are precisely the flat tori. In this case it is well-known that the analog of the Main Theorem holds even if the geodesic flows are only assumed to be C^0-conjugate.

This paper is organized as follows: In §1 (Background), we describe a family of automorphisms, called *almost inner automorphisms*, of nilpotent Lie groups which were first used in [GW] to construct isospectral nilmanifolds. We then recall a theorem of P. Eberlein [Eb] stating that if a pair of 2-step nilmanifolds have the same marked length spectrum (this will always be the case if their geodesic flows are conjugate), then they must be of the form $(\Gamma\backslash N, g)$ and $(\Phi(\Gamma)\backslash N, g)$ for some almost inner automorphism Φ of N. The manifolds will be isometric if the automorphism is actually inner. Finally we review a result of [GM2] concerning geodesic conjugacies between 2-step nilmanifolds.

In §2, we begin our study of 2-step nilmanifolds with symplectically conjugate flows and show that the associated almost inner automorphism Φ must satisfy a certain additional condition. Thus the Main Theorem is reduced to an algebraic problem: to show that the only almost inner automorphisms satisfying this additonal condition are inner.

In §3 we resolve this question by a Lie algebra cohomology argument.

1. Background

A *Riemannian nilmanifold* is a quotient $M = \Gamma\backslash N$ of a nilpotent Lie group N by a discrete subgroup Γ, together with a Riemannian metric g whose lift to N, also denoted by g, is left invariant. We say that $\Gamma\backslash N$ is a *k-step nilmanifold* if N is k-step nilpotent.

1.1. NOTATION AND REMARKS.

(i) Let N be a simply-connected 2-step nilpotent Lie group with a left invariant metric g. The metric g defines an inner product $\langle .,. \rangle$ on the Lie algebra $\mathcal{N}$ of N. We will denote by $\mathcal{Z}$ the derived algebra $\mathcal{Z} = [\mathcal{N}, \mathcal{N}]$ and denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$ relative to $\langle .,. \rangle$. Since N is nilpotent, the Lie group exponential map $\exp : \mathcal{N} \to N$ is a diffeomorphism; its inverse is denoted by $\log$.

(ii) For left-invariant vector fields X, Y, U the Levi-Cività connection satisfies

$$2\langle \nabla_X Y, U \rangle = \langle [X,Y], U \rangle + \langle [U,X], Y \rangle + \langle [U,Y], X \rangle.$$

(The remaining three terms in the usual expression for ∇ vanish since the metric is left invariant.) In particular, $\langle \nabla_X X, U \rangle = \langle [U,X], X \rangle$, so the integral curves $n \exp(tX)$ of X are geodesics if and only if $[X, \mathcal{N}] \perp X$. Observe that this condition always holds when $X \in \mathcal{V}$ or $X \in \mathcal{Z}$.

1.2. COCOMPACT DISCRETE SUBGROUPS. – A nilpotent Lie group N admits a cocompact discrete subgroup Γ if and only if the Lie algebra $\mathcal{N}$ admits a basis relative to which the constants of structure are rational. If Γ is a cocompact discrete subgroup, then one can choose such a basis which consists of elements of $\log \Gamma$. Let $\mathcal{N}_{\mathbf{Q}} = \operatorname{span}_{\mathbf{Q}}(\log \Gamma)$. Elements of $\mathcal{N}_{\mathbf{Q}}$ are said to be *rational* elements of $\mathcal{N}$, and a Lie subalgebra $\mathcal{H}$ of $\mathcal{N}$ is called *rational* if it is spanned by rational elements. Note that the notion of rationality depends on Γ. If $H = \exp \mathcal{H}$ is a connected Lie subgroup of N with rational Lie algebra $\mathcal{H}$, then $\Gamma \cap H$ is a cocompact discrete subgroup of H. The derived algebra Z is always rational. Moreover, the image of $\log \Gamma$ under the projection from $\mathcal{N}$ to $\mathcal{N}/\mathcal{Z}$ generates a lattice of full rank in $\mathcal{N}/\mathcal{Z}$; in the 2-step case, this image is actually a lattice itself.

1.3. DEFINITION.

(i) Let Γ be a uniform discrete subgroup of a simply-connected nilpotent Lie group N. An automorphism Φ of N is said to be Γ-*almost inner* if $\Phi(\gamma)$ is conjugate to γ for all $\gamma \in \Gamma$. The automorphism is said to be *almost inner* if $\Phi(x)$ is conjugate to x for all $x \in N$.

(ii) A derivation φ of the Lie algebra $\mathcal{N}$ is said to be Γ-*almost inner*, respectively *almost inner*, if $\varphi(X) \in [\mathcal{N}, X]$ for all $X \in \log \Gamma$, respectively, for all $X \in \mathcal{N}$.

1.4. *Remark* (See [GW], [Go].)

(i) The Γ-almost inner automorphisms and the almost inner automorphisms form connected Lie subgroups of $\mathrm{Aut}(N)$. In many cases, these groups properly contain the group $\mathrm{Inn}(N)$ of inner automorphisms. The spaces of Γ-almost inner (respectively, almost inner) derivations of $\mathcal{N}$ are the Lie algebras of these groups of automorphisms. In particular, if φ is a (Γ-)almost inner derivation, then there exists a one-parameter family Φ_t of (Γ-) almost inner automorphisms of N such that $\Phi_{t*} = e^{t\varphi}$. Conversely, if Φ is a (Γ-)almost inner automorphism of N, then $\Phi_* = e^{\varphi}$ for some (Γ-)almost inner derivation of $\mathcal{N}$.

(ii) Note that a Γ-almost inner derivation φ satisfies $\varphi(\mathcal{N}) \subseteq [\mathcal{N}, \mathcal{N}]$ and $\varphi(Z) = 0$ if Z is central. In particular, if $\mathcal{N}$ is 2-step nilpotent, then (letting $\mathcal{Z} = [\mathcal{N}, \mathcal{N}]$ as before), we have $\varphi(\mathcal{N}) \subseteq \mathcal{Z}$ and $\varphi(\mathcal{Z}) = \{0\}$, so $\varphi^2 = 0$. Thus $e^{t\varphi} = \mathrm{Id} + t\varphi$.

The notion of almost inner automorphisms first arose in the construction of continuous families of isospectral nilmanifolds [GW]. If $(\Gamma\backslash N, g)$ is a compact Riemannian nilmanifold and Φ is a Γ-almost inner automorphism of N, then $(\Phi(\Gamma)\backslash N, g)$ is isospectral to $(\Gamma\backslash N, g)$. Conversely (see [OP]), if N is 2-step nilpotent and if $\{\Gamma_t\}_{t\geq 0}$ is a continuous family of discrete subgroups of N such that the family of manifolds $(\Gamma_t\backslash N, g)$ are all isospectral, then there exists a family $\{\Phi_t\}_{t\geq 0}$ of Γ_0-almost inner automorphisms of N such that $\Gamma_t = \Phi_t(\Gamma_0)$ for all t.

1.5. Remark. – If Φ is an inner automorphism of N, say Φ is conjugation by $a \in N$, then $(\Phi(\Gamma)\backslash N, g)$ is isometric to $(\Gamma\backslash N, g)$. The isometry is induced from the isometry L_a of (N, g) given by left translation.

We now consider compact Riemannian nilmanifolds with conjugate geodesic flows.

1.6. Notation and Remarks.

(i) The left invariant vector fields on N induce global vector fields on $\Gamma\backslash N$. Thus the tangent bundles of both N and $\Gamma\backslash N$ are completely parallelizable, and we will make the identifications

$$TN = N \times \mathcal{N},$$
$$T(\Gamma\backslash N) = \Gamma\backslash N \times \mathcal{N}.$$

(ii) For convenience, we are viewing the geodesic flows on the tangent bundles rather than the cotangent bundles as in the introduction. Suppose that $(\Gamma\backslash N, g)$ and $(\Gamma'\backslash N', g')$ are two compact 2-step Riemannian nilmanifolds and that there is a homeomorphism $F : T(\Gamma\backslash N)\backslash\{0\} \to T(\Gamma'\backslash N')\backslash\{0\}$ which intertwines their geodesic flows. By (i), we can write

$$F : \Gamma\backslash N \times (\mathcal{N}\backslash\{0\}) \to \Gamma'\backslash N' \times (\mathcal{N}'\backslash\{0\}).$$

Consider the universal coverings $\pi : N \times (\mathcal{N}\backslash\{0\}) \to \Gamma\backslash N \times (\mathcal{N}\backslash\{0\})$, $\pi' : N' \times (\mathcal{N}'\backslash\{0\}) \to \Gamma'\backslash N' \times (\mathcal{N}'\backslash\{0\})$. Choose an arbitrary lift $\tilde{F} : N \times (\mathcal{N}\backslash\{0\}) \to N' \times (\mathcal{N}'\backslash\{0\})$ of F, *i.e.*, a map which satisfies $\pi' \circ \tilde{F} = F \circ \pi$. Note that the group of deck transformations of π consists of the left translations $dL_\gamma : (n, U) \mapsto (\gamma n, U)$ with $\gamma \in \Gamma$, and similarly for π'. Thus to every $\gamma \in \Gamma$ belongs a unique $\gamma' \in \Gamma'$ such that $\tilde{F} \circ dL_\gamma = dL_{\gamma'} \circ \tilde{F}$, and $\gamma \mapsto \gamma'$ is an isomorphism which we denote by $F_* : \Gamma \to \Gamma'$.

Note that $\tilde{F}$ intertwines the geodesic flows of (N, g) and (N', g') since F does so on the quotients. This, together with $\tilde{F} \circ dL_\gamma = dL_{F_*(\gamma)} \circ \tilde{F}$, implies that the isomorphism F_* induces a marking of the length spectra of $(\Gamma\backslash N, g)$ and $(\Gamma'\backslash N', g')$, that is, the collection of lengths of closed geodesics in $(\Gamma\backslash N, g)$ which belong to the free homotopy class $[\gamma]_\Gamma$ is the same as the collection of lengths of closed geodesics in $(\Gamma'\backslash N', g')$ belonging to $[F_*(\gamma)]_{\Gamma'}$.

1.7. Proposition ([Eb]). – *Suppose $(\Gamma\backslash N, g)$ and $(\Gamma'\backslash N', g')$ are compact 2-step Riemannian nilmanifolds, and $F_* : \Gamma \to \Gamma'$ is an isomorphism which induces a marking between their length spectra. Then there exists a Γ-almost inner automorphism Φ of N and an isometric isomorphism $\Psi : (N, g) \to (N', g')$ with $\Psi(\Phi(\Gamma)) = \Gamma'$ such that $F_* = \Psi|_{\Phi(\Gamma)} \circ \Phi|_\Gamma$.*

1.8. Notation and Remarks.

(i) Let $\tilde{F}$ and F_* be as in 1.6(ii), and let Φ and Ψ be as in 1.7 with $F_* = \Psi|_{\Phi(\Gamma)} \circ \Phi|_\Gamma$. Then Ψ induces an isometry, also denoted Ψ, from $(\Phi(\Gamma)\backslash N, g)$ to $(\Gamma'\backslash N', g')$, so we may replace $(\Gamma'\backslash N', g')$ by $(\Phi(\Gamma)\backslash N, g)$. Moreover, we replace F by $\Psi_*^{-1} \circ F : T(\Gamma\backslash N)\backslash\{0\} \to T(\Phi(\Gamma)\backslash N)\backslash\{0\}$ and $\tilde{F}$ by $\Psi_*^{-1} \circ \tilde{F} : TN\backslash\{0\} \to TN\backslash\{0\}$. The new F is a geodesic conjugacy from $(\Gamma\backslash N, g)$ to $(\Phi(\Gamma)\backslash N, g)$, and the new

$$\tilde{F} : TN\backslash\{0\} \to TN\backslash\{0\}$$

is a lift of F which satisfies

$$\tilde{F} \circ dL_\gamma = dL_{\Phi(\gamma)} \circ \tilde{F} \tag{1}$$

for all $\gamma \in \Gamma$. Also, denoting by G^t the geodesic flow of (N, g), we have

$$\tilde{F} \circ G^t = G^t \circ \tilde{F}. \tag{2}$$

(ii) By Remark 1.4, there exists a Γ-almost inner derivation of $\mathcal{N}$, which we denote by φ, such that the differential $\Phi_* : \mathcal{N} \to \mathcal{N}$ is given by $\Phi_* = \mathrm{Id} + \varphi$. We have $\varphi(\mathcal{Z}) = \{0\}$ and $\varphi(\mathcal{V}) \subseteq \mathcal{Z}$.

1.9. Proposition ([GM2]). – *Let $(\Gamma\backslash N, g)$ be a compact 2-step Riemannian nilmanifold, and let Φ be a Γ-almost inner automorphism of N. Suppose $\tilde{F} : TN\backslash\{0\} \to TN\backslash\{0\}$ satisfies equations* (1) *and* (2) *from* 1.8 (i) *Then, using the notation from* 1.6 (i):

$$\tilde{F}(N \times \{U\}) = N \times \{U\}$$

for all $U \in \mathcal{V}\backslash\{0\}$ or $U \in \mathcal{Z}\backslash\{0\}$.

We do not repeat the proof here but roughly sketch the main idea. For $U \in \mathcal{Z}$ or $U \in \mathcal{V}$, the integral curves of the left invariant vector field U on N are geodesics (*see* 1.1(ii)); equivalently, for $n \in N$, the curves $(n \exp(tU), U)$ are orbits of the geodesic flow of (N, g). If $U \in \mathcal{Z}$ and U is rational (*see* 1.2), then these orbits are closed in $\Gamma\backslash N$ and are exactly the longest closed orbits in their free homotopy class. Note that Φ, being Γ-almost inner, fixes central homotopy classes, hence by (1) and (2), the aforementioned orbits are carried to orbits of the same form. Since the rational elements are dense in $\mathcal{Z}$, the proposition follows for all $U \in \mathcal{Z}\backslash\{0\}$. A similar, though more complicated, argument is used for the case $U \in \mathcal{V}$.

2. Symplectic conjugacies and reduction of the problem

2.1. SYMPLECTIC STRUCTURE. – For every manifold there is a canonical symplectic structure on the cotangent bundle. A Riemannian metric allows us to view this as a symplectic structure on the tangent bundle. As in 1.6(i), we identify TN with $N \times \mathcal{N}$. Thus for (n, U) in TN, the tangent space $T_{(n,U)}(TN)$ is identified with $\mathcal{N} \times \mathcal{N}$. The canonical symplectic structure ω on TN associated with the Riemannian metric g is then given by

$$\omega_{|(n,U)}((X,S),(Y,T)) = \langle S, Y\rangle - \langle T, X\rangle - \langle U, [X,Y]\rangle.$$

2.2. PROPOSITION. – *We assume $\tilde{F}$ is as in* 1.8(i), *and write*

$$\tilde{F}(n,U) = \big(\exp(A(n,U) + B(n,U))n,\ C(n,U)\big)$$

with $A(n,U) \in \mathcal{Z}$, $B(n,U) \in \mathcal{V}$, and $C(n,U) \in \mathcal{N}$ for all $n \in N$, $U \in \mathcal{N}\backslash\{0\}$. (See the notation 1.1.) *Suppose $\tilde{F}$ is a symplectomorphism. Then:*

(i) *$B(n,V)$ is independent of n for each $V \in \mathcal{V}\backslash\{0\}$.*

(ii) *Writing $B(V) = B(n,V)$ for $V \in \mathcal{V}\backslash\{0\}$, the differential $dB_{|V} : \mathcal{V} \to \mathcal{V}$ is symmetric for all $V \in \mathcal{V}\backslash\{0\}$.*

(iii) *In the notation of* 1.8(ii), *we have $\varphi(V) = [B(V), V]$ for all $V \in \mathcal{V}\backslash\{0\}$.*

(Note that, in particular, φ is almost inner, not only Γ-almost inner.

2.3. REMARK. – In reading the proof below, it is helpful to keep in mind that we are identifying the tangent space at every point of N with the Lie algebra $\mathcal{N}$ of left invariant vector fields on N as in 1.6(i). In particular, for $a, b \in N$, the left translation $dL_a : T_bN \to T_{ab}N$ is identified with the identity map of $\mathcal{N}$.

Proof of Proposition 2.2. – We first compute the differential $d\tilde{F}_{|(n,V)}(X,S)$ when $X \in \mathcal{N}$, $S \in \mathcal{V}$, and $V \in \mathcal{V}\backslash\{0\}$. By Proposition 1.9, we have

$$\tilde{F}(n,V) = (f(n,V), V) \tag{3}$$

where $f(n,V) = \exp(A(n,V) + B(n,V))n$. Thus we only need to find $df_{|(n,V)}(X,S)$. Recall (see [Eb], Lemma 1.3) that the differential of the exponential map is given by $d\exp_{|Y}(W) = (dL_{\exp Y})_{|e}(W - \frac{1}{2}[Y,W])$ for $Y, W \in \mathcal{N}$; i.e., with the identifications described in Remark 2.3,

$$d\exp_{|Y}(W) = W - \frac{1}{2}[Y,W]. \tag{4}$$

Next, letting R_n denote right translation by $n \in N$, we have

$$\begin{aligned} df_{|(n,V)}(X,S) = {} & \frac{d}{dt}_{|t=0} \exp(A(n,V) + B(n,V))n\exp(tX) \\ & + (dR_n \circ d\exp)_{|(A(n,V)+B(n,V))}(dA_{|(n,V)}(X,S) + dB_{|(n,V)}(X,S)). \end{aligned} \tag{5}$$

The first term in (5) is just $dL_{f(n,V)}(X) = X$ by Remark 2.3. The second term is more complicated. However, in what follows, we will need only the component of

$df|_{(n,V)}(X,S)$ in $\mathcal{V}$. Now, for any $W \in \mathcal{N}$, we have $dR_n(W) \equiv dL_n(W)$ modulo $\mathcal{Z}$, and thus, modulo $\mathcal{Z}$, $dR_n(W) \equiv W$ by Remark 2.3. Thus by equation (4) and Remark 2.3, the $\mathcal{V}$-component of the second term in (5) is $dB|_{(n,V)}(X,S)$. Thus we have

$$df|_{(n,V)}(X,S) \equiv X + dB|_{(n,V)}(X,S) \bmod \mathcal{Z}. \tag{6}$$

(i) We now prove that $B(n,V)$ does not depend on n for $V \in \mathcal{V}\backslash\{0\}$. Let $X \in \mathcal{N}$ and $S \in \mathcal{V}$. Since F and hence also $\tilde{F}$ is a symplectomorphism we have by 2.1 that

$$\omega|_{\tilde{F}(n,V)}(d\tilde{F}|_{(n,V)}(X,0), d\tilde{F}|_{(n,V)}(0,S)) = \omega|_{(n,V)}((X,0),(0,S)) = -\langle X,S\rangle. \tag{7}$$

By (3) and (6), $d\tilde{F}|_{(n,V)}(X,0) = (W,0)$ where $W \equiv (X + dB|_{(n,V)}(X,0))$ modulo $\mathcal{Z}$, and $d\tilde{F}|_{(n,V)}(0,S) = (U,S)$ for some $U \in \mathcal{N}$. Since V is orthogonal to the derived algebra $\mathcal{Z}$, we thus have

$$\omega|_{\tilde{F}(n,V)}(d\tilde{F}|_{(n,V)}(X,0), d\tilde{F}|_{(n,V)}(0,S)) = -\langle W,S\rangle = -\langle X + dB|_{(n,V)}(X,0), S\rangle.$$

Comparing this with equation (7), we see that $dB|_{(n,V)}(X,0) = 0$ for all $X \in \mathcal{N}$. Thus $B(n,V)$ is independent of n and we can define $B(V) := B(n,V)$.

(ii) To see that $(dB|_V)|_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is symmetric for every $V \in \mathcal{V}\backslash\{0\}$, let $S,T \in \mathcal{V}$. By (3) and (6), $d\tilde{F}|_{(n,V)}(0,S) = (dB|_V(S) + Z, S)$ for some $Z \in \mathcal{Z}$ and similarly for $d\tilde{F}|_{(n,V)}(0,T)$. Again using the fact that V is orthogonal to the derived algebra, we have

$$\begin{aligned} 0 = \omega|_{(n,V)}((0,S),(0,T)) &= \omega|_{\tilde{F}(n,V)}(d\tilde{F}|_{(n,V)}(0,S), d\tilde{F}|_{(n,V)}(0,T)) \\ &= \langle dB|_V(T), S\rangle - \langle dB|_V(S), T\rangle. \end{aligned}$$

This completes the proof of (ii).

(iii) This is a special case of Proposition 2.10 in [GM2]; we give a condensed version of the proof here. Equation (1) from 1.8(i) implies $f(\gamma n, V) = \Phi(\gamma)\cdot f(n,V)$ for f as above and $\gamma \in \Gamma$, $n \in N$. A short computation using the the fact that $\varphi(\log\gamma) \in \mathcal{Z}$ and the expression of f in terms of A, B shows that this implies

$$A(\gamma n, V) - A(n,V) = \varphi(\log\gamma) - [B(V), \log\gamma]. \tag{8}$$

It follows from 1.2 that the image of $\log\Gamma$ under the orthogonal projection from $\mathcal{N}$ to $\mathcal{V}$ is a lattice $\mathcal{L}$ of full rank in $\mathcal{V}$. Thus vectors V such that $sV \in \mathcal{L}$ for some $0 \neq s \in \mathbf{R}$ are dense in $\mathcal{V}$. By smoothness of φ and B it suffices to prove (iii) for such V. Fix V of this form. There exists $s \in \mathbf{R}$ and $z \in \exp\mathcal{Z}$ such that

$$\gamma := z\exp(sV) \in \Gamma. \tag{9}$$

For this γ, equation (8) becomes:

$$A(\gamma n, V) - A(n,V) = \varphi(sV) - [B(V), sV] \tag{10}$$

since both φ and $\mathrm{ad}B(V)$ vanish on $\mathcal{Z}$. Since the right hand side of (10) is independent of n, so is the left hand side. Thus $H_1(n) := A(n, V)$ and $H_2(n) := A(\gamma n, V)$ differ by a constant. If we can show this constant difference to be zero, then (10) will imply (iii) by linearity. By (8), H_1 and H_2 are $(\Gamma \cap \exp \mathcal{Z})$-periodic. Since $\Gamma \cap \exp \mathcal{Z}$ is cocompact in $\exp \mathcal{Z}$, the two maps, restricted to $\exp \mathcal{Z}$, have well-defined average values $\bar{H}_1$ and $\bar{H}_2$, and it suffices to show that $\bar{H}_1 = \bar{H}_2$.

By 1.1(ii), $G^s(n, U) = (n \exp(sU), U)$ for $U \in \mathcal{V}$. Proposition 1.9 and (2) thus imply $f(n \exp(sV), V) = f(n, V) \exp(sV)$; in particular $A(n \exp(sV), V) = A(n, V)$. Therefore, for z as in (9) and for all $n \in \exp \mathcal{Z}$, we have $H_2(n) = A(zn \exp(sV), V) = H_1(zn)$. Hence, taking averages over $\exp \mathcal{Z}$, we obtain $\bar{H}_1 = \bar{H}_2$. $\square$

We have now reduced the Main Theorem to the following (recall Remark 1.5):

2.4. THEOREM. – *Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with metric $\langle .\,, . \rangle$. Let $\mathcal{Z} := [\mathcal{N}, \mathcal{N}]$, and denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$. Let φ be an almost inner derivation of $\mathcal{N}$. Suppose there exists a differentiable map $B : \mathcal{V}\backslash\{0\} \to \mathcal{V}$ such that the differentials $dB_{|V}$ are symmetric with respect to $\langle .\,, . \rangle$, and $\varphi(V) = [B(V), V]$ for all $V \in \mathcal{V}\backslash\{0\}$. Then φ is an inner derivation.*

This theorem will be proved in §3.

3. Proof of the Lie algebraic result

We first need some preparations.

3.1 DEFINITION. – Let $(\mathcal{V}, \langle .\,, . \rangle)$ be a Euclidean vector space, and let $B : \mathcal{V}\backslash\{0\} \to \mathcal{V}$ be a differentiable map such that the differentials $dB_{|V}$ are symmetric for every $V \in \mathcal{V}$. An element $J \in so(\mathcal{V})$ is called *B-admissible* if the map

$$\mathcal{V}\backslash\{0\} \ni V \mapsto \langle B(V), JV \rangle \in \mathbf{R}$$

is linear in V (more precisely, if this map can be extended to a linear functional on $\mathcal{V}$ by letting $0 \mapsto 0$). In this case denote by $L(J) \in \mathcal{V}$ the unique vector which satisfies $\langle B(V), JV \rangle = \langle L(J), V \rangle$ for all $V \in \mathcal{V}\backslash\{0\}$.

A linear subspace $\mathcal{J}$ of $so(\mathcal{V})$ is called *B-admissible* if every $J \in \mathcal{J}$ is B-admissible. Note that in this case the corresponding map $L : \mathcal{J} \to \mathcal{V}$ is linear.

3.2 LEMMA. – *Let $J, J' \in so(\mathcal{V})$ be B-admissible. Then for every $V \in \mathcal{V}\backslash\{0\}$ the following holds:*

(i) $dB_{|V} \cdot JV = J \cdot B(V) + L(J)$,

(ii) $\langle B(V), [J, J'] \cdot V \rangle = \langle J \cdot L(J') - J' \cdot L(J), V \rangle$.

Proof.

(i) Differentiating the equation $\langle B(V), JV \rangle = \langle L(J), V \rangle$ with respect to V we get

$$(*) \qquad \langle dB_{|V} \cdot U, JV \rangle + \langle B(V), JU \rangle = \langle L(J), U \rangle$$

for all $V \in \mathcal{V}\backslash\{0\}$, $U \in \mathcal{V}$. By the symmetry of $dB_{|V}$ and the skew-symmetry of J, statement (i) follows.

(ii) Letting $U := J'V$ in equation $(*)$ we get $\langle dB_{|V} \cdot J'V, JV\rangle + \langle B(V), JJ'V\rangle = \langle L(J), J'V\rangle$ for all $V \in \mathcal{V}\backslash\{0\}$. Applying (i) to J' gives $\langle J' \cdot B(V) + L(J'), JV\rangle + \langle B(V), JJ'V\rangle = \langle L(J), J'V\rangle$. Statement (ii) now follows from the skew-symmetry of J and J'. $\square$

3.3 Corollary. – *Let $\mathcal{J}$ be a B-admissible linear subspace of $so(\mathcal{V})$, and let $\tilde{\mathcal{J}}$ be the Lie subalgebra of $so(\mathcal{V})$ which is generated by $\mathcal{J}$. Then $\tilde{\mathcal{J}}$ is B-admissible, too. The corresponding linear map $L : \tilde{\mathcal{J}} \to \mathcal{V}$ satisfies*

$$(C) \qquad L([J, J']) = J \cdot L(J') - J' \cdot L(J)$$

for all $J, J' \in \tilde{\mathcal{J}}$.

3.4. Proposition. – *If $\mathcal{J}$ is a B-admissible linear subspace of $so(\mathcal{V})$ then there exists a $B_0 \in \mathcal{V}$ such that*

$$(E) \qquad L(J) = J \cdot B_0 \quad \textit{for all } J \in \mathcal{J}.$$

Proof. – First suppose that there is an element $V_0 \in \mathcal{V}\backslash\{0\}$ such that $JV_0 = 0$ for all $J \in \mathcal{J}$. In this case, Lemma 3.2(i) tells us that equation (E) holds for $B_0 := -B(V_0)$. Hence we can assume that no nonzero element of $\mathcal{V}$ is annihilated by $\mathcal{J}$.

Let $\tilde{\mathcal{J}}$ be the Lie algebra generated by $\mathcal{J}$. By Corollary 3.3, $\tilde{\mathcal{J}}$ is B-admissible, and $L : \tilde{\mathcal{J}} \to \mathcal{V}$ satisfies equation (C). Note that in the language of Lie algebra cohomology of the Lie algebra $\tilde{\mathcal{J}}$ with coefficients in the representation space $\mathcal{V}$, this equation just says that L is a 1-cocycle. On the other hand, the existence of a $B_0 \in \mathcal{V}$ such that equation (E) holds for every $J \in \tilde{\mathcal{J}}$ is equivalent to L being exact. But that L is indeed exact follows from the well-known Lemma 3.5 below. Thus we know that there is a $B_0 \in \mathcal{V}$ which satisfies (E) for every $J \in \tilde{\mathcal{J}}$, in particular for every $J \in \mathcal{J}$. $\square$

3.5. Lemma – *For every subalgebra $\mathcal{G}$ of $so(\mathcal{V})$ which does not annihilate any nontrivial subspace of $\mathcal{V}$, the first cohomology group of $\mathcal{G}$ with coefficients in the representation space $\mathcal{V}$ vanishes: $H^1(\mathcal{G}; \mathcal{V}) = 0$.*

We do not know an explicit reference for the statement in this specific form. However, the proof of Whitehead's Lemma in [Ja], p˙ 77ff., can easily be imitated here. Compare also [OV], p˙ 16, Theorem 3.1. For the convenience of the reader we give an elementary sketch of the proof at the end of this section.

Proof of Theorem 2.4. – For $Z \in \mathcal{Z}$, define $J_Z \in so(\mathcal{V})$ by $\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle$ for all $X, Y \in \mathcal{V}$. Consider $\mathcal{J} := \{J_Z \mid Z \in \mathcal{Z}\}$. We know that $\langle B(V), J_Z V\rangle = -\langle [B(V), V], Z\rangle = -\langle \varphi(V), Z\rangle$ for every $V \in \mathcal{V}\backslash\{0\}$ and $Z \in \mathcal{Z}$. The last expression is obviously linear in V, hence $\mathcal{J}$ is B-admissible. By Proposition 3.4 there is a $B_0 \in \mathcal{V}$ such that $\langle B(V), J_Z V\rangle = \langle J_Z B_0, V\rangle$ for all $V \in \mathcal{V}\backslash\{0\}$, $Z \in \mathcal{Z}$. Thus $-\langle \varphi(V), Z\rangle = \langle [B_0, V], Z\rangle$ for all $V \in \mathcal{V}$, $Z \in \mathcal{Z}$, which implies that $\varphi = -\mathrm{ad}B_0$ is an inner derivation. $\square$

Proof of Lemma 3.5. – Consider the standard scalar product $(J, J') := -\mathrm{tr}(JJ')$ on $so(\mathcal{V})$, and let $\{E_1, \ldots, E_m\}$ be an orthonormal basis of $\mathcal{G}$ with respect to $(.,.)$. Define the "Casimir operator"

$$C := \sum_{i=1}^{m} E_i^2 \in \mathrm{End}(\mathcal{V}).$$

Using the facts that the E_i form an orthonormal basis of $\mathcal{G}$ and that the maps $\mathrm{ad}J : \mathcal{J} \to \mathcal{J}$ are skew-symmetric with respect to $(.,.)$, one easily checks that C commutes with every $J \in \mathcal{G}$. Suppose the kernel $\mathcal{V}_0$ of C were nontrivial. Then $\mathcal{V}_0$ would be invariant under every $J \in \mathcal{G}$. Note that $0 = \mathrm{tr}(C_{|\mathcal{V}_0}) = \sum_{i=1}^m \mathrm{tr}((E_i|_{\mathcal{V}_0})^2)$. But the $(E_i|_{\mathcal{V}_0})^2$ are negative semidefinite symmetric endomorphisms, hence by this equation their traces must all be zero, and therefore $E_i|_{\mathcal{V}_0} = 0$ for all i. This implies that $\mathcal{V}_0$ is annihilated by $\mathcal{G}$, in contradiction to our assumptions. Hence we know that C is nonsingular. Now let $L : \mathcal{G} \to \mathcal{V}$ be a cocycle, i.e., L satisfies $L([J, J']) = J \cdot L(J') - J' \cdot L(J)$ for all $J, J' \in \mathcal{G}$. Define

$$B_0 := C^{-1} \cdot \sum_{i=1}^{m} E_i \cdot L(E_i).$$

Using the linearity and the cocycle property of L and the facts that the E_i form an orthonormal basis, that the maps $\mathrm{ad}J : \mathcal{J} \to \mathcal{J}$ are skew-symmetric, and that C (and therefore C^{-1}) commutes with every $J \in \mathcal{G}$, it is easy to check that $J \cdot B_0 = L(J)$ for every $J \in \mathcal{G}$. Thus L is indeed exact. □

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