ANNALES SCIENTIFIQUES DE L'É.N.S.

KEVIN P. KNUDSON

The homology of special linear groups over polynomial rings

Annales scientifiques de l'É.N.S. 4e série, tome 30, nº 3 (1997), p. 385-416

<http://www.numdam.org/item?id=ASENS_1997_4_30_3_385_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 30, 1997, p. 385 à 416.

THE HOMOLOGY OF SPECIAL LINEAR GROUPS OVER POLYNOMIAL RINGS (1)

BY KEVIN P. KNUDSON (2)

ABSTRACT. – We study the homology of $SL_n(F[t,t^{-1}])$ by examining the action of the group on a suitable simplicial complex. The E^1–term of the resulting spectral sequence is computed and the differential, d^1, is calculated in some special cases to yield information about the low-dimensional homology groups of $SL_n(F[t,t^{-1}])$. In particular, we show that if F is an infinite field, then $H_2(SL_n(F[t,t^{-1}]),\mathbb{Z}) = K_2(F[t,t^{-1}])$ for $n \geq 3$. We also prove an unstable analogue of homotopy invariance in algebraic K-theory; namely, if F is an infinite field, then the natural map $SL_n(F) \to SL_n(F[t])$ induces an isomorphism on integral homology for all $n \geq 2$.

RÉSUMÉ. – Nous étudions l'homologie de $SL_n(F[t,t^{-1}])$ en examinant l'action de ce groupe sur un complexe simplicial adéquat. Le terme E^1 de la suite spectrale associée est déterminé et la différentielle d^1 est calculée dans certains cas, ce qui permet alors de comprendre l'homologie du groupe $SL_n(F[t,t^{-1}])$ en bas degré. En particulier, nous montrons que si F est un corps infini, alors $H_2(SL_n(F[t,t^{-1}]),\mathbb{Z}) = K_2(F[t,t^{-1}])$ pour $n \geq 3$. Nous prouvons aussi un analogue instable de l'invariance homotopique en K-théorie algébrique : si F est un corps infini alors la flèche naturelle $SL_n(F) \to SL_n(F[t])$ induit un isomorphisme en homologie entière pour $n \geq 2$.

Since Quillen's definition of the higher algebraic K-groups of a ring [15], much attention has been focused upon studying the (co)homology of linear groups. There have been some successes –Quillen's computation [14] of the mod l cohomology of $GL_n(\mathbb{F}_q)$, Soulé's results [18] on the cohomology of $SL_3(\mathbb{Z})$– but few explicit calculations have been completed. Most known results concern the stabilization of the homology of linear groups. For example, van der Kallen [11], Charney [7], and others have proved quite general stability theorems for GL_n of a ring. Also, Suslin [19] proved that if F is an infinite field, then the natural map

$$H_i(GL_m(F)) \longrightarrow H_i(GL_n(F))$$

is an isomorphism for $i \leq m$. Other noteworthy results include Borel's computation of the stable cohomology of arithmetic groups [1], [2], the computation of $H^\bullet(SL_n(F),\mathbb{R})$ for F a number field by Borel and Yang [3], and Suslin's isomorphism [20] of $H_3(SL_2(F))$ with the indecomposable part of $K_3(F)$.

This paper is concerned with studying the homology of linear groups defined over the polynomial rings $F[t]$ and $F[t,t^{-1}]$. One motivation for this is an attempt to find unstable analogues of the fundamental theorem of algebraic K-theory [15]: If R is a regular ring,

(1) Received by the editors March 29, 1996.
(2) Supported by an Alfred P. Sloan Doctoral Dissertation Fellowship.

then there are natural isomorphisms

$$K_i(R[t]) \cong K_i(R) \tag{1}$$

and

$$K_i(R[t,t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R). \tag{2}$$

In this paper, we study the homology of $SL_n(F[t,t^{-1}])$. Before stating our main result, we first establish some notation.

The group $SL_n(F[t,t^{-1}])$ acts on a contractible $(n-1)$-dimensional building $\mathcal{X}$ with fundamental domain an $(n-1)$-simplex $\mathcal{C}$. This yields a spectral sequence converging to the homology of $SL_n(F[t,t^{-1}])$ with E^1-term satisfying

$$E^1_{p,q} = \bigoplus_{\dim \sigma = p} H_q(\Gamma_\sigma) \tag{3}$$

where Γ_σ denotes the stabilizer of the p-simplex σ in $SL_n(F[t,t^{-1}])$, and σ is contained in $\mathcal{C}$. The vertex stabilizers are isomorphic to $SL_n(F[t])$, and the other stabilizers break up into isomorphism classes in such a way that in each class, there is a group Γ_σ which fits into a split short exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_\sigma \xrightarrow{t=0} P_\sigma \longrightarrow 1$$

where P_σ is a parabolic subgroup of $SL_n(F)$ and K consists of the matrices in $SL_n(F[t])$ which are congruent to the identity modulo t. Our main result is the following.

THEOREM (cf. Theorem 5.1). – *If F is an infinite field, then the inclusion $P_\sigma \longrightarrow \Gamma_\sigma$ induces an isomorphism*

$$H_\bullet(P_\sigma, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_\sigma, \mathbb{Z}).$$

If σ is a vertex, we have $\Gamma_\sigma = SL_n(F[t])$ and $P_\sigma = SL_n(F)$. In this case the theorem reduces to the following unstable analogue of (1).

THEOREM (cf. Theorem 3.4). – *If F is an infinite field, then the inclusion $SL_n(F) \longrightarrow SL_n(F[t])$ induces an isomorphism*

$$H_\bullet(SL_n(F), \mathbb{Z}) \longrightarrow H_\bullet(SL_n(F[t]), \mathbb{Z}).$$

This theorem improves on a result of Soulé [17].

Theorem 5.1 completes the computation of the E^1-term of the spectral sequence (3). However, the differential d^1 is difficult to calculate in general. In Section 6 we compute the map in a few special cases and obtain information about the low dimensional homology groups of $SL_n(F[t,t^{-1}])$. In particular, we show that if F is an infinite field, then for $n \geq 3$, there is an isomorphism

$$H_2(SL_n(F[t,t^{-1}]), \mathbb{Z}) \cong K_2(F[t,t^{-1}]).$$

The homology of $SL_2(F[t,t^{-1}])$ was studied by the author in [12] using slightly different techniques than those used here. The main result of [12] is the following.

THEOREM (cf. [12, Theorem 5.1]). – *Let F be a number field and denote by r_1 (resp. r_2) the number of real (resp. conjugate pairs of complex) embeddings of F. Then for $k \geq 2r_1 + 3r_2 + 2$ there is a natural isomorphism*

$$H_k(SL_2(F[t,t^{-1}]),\mathbb{Q}) \cong H_{k-1}(F^\times,\mathbb{Q}).$$

The results of this paper reprove and generalize the results of [12]. In particular, Theorems 3.1 and 4.3 of [12] hold for infinite fields of arbitrary characteristic, not just fields of characteristic zero.

This paper is organized as follows:

In Section 1 we present the necessary background material on the Bruhat-Tits building $\mathcal{X}$. We also introduce a complex $\mathcal{Y}$ which will be used in subsequent sections.

In Section 2 we study the action of $SL_n(F[t,t^{-1}])$ on $\mathcal{X}$ and examine the structure of the various stabilizers.

In Section 3 we prove Theorem 3.4, the unstable version of (1). Even though this is a special case of Theorem 5.1, we prove it separately for two reasons. First, it is a striking result which deserves to be called a theorem in its own right, and second, the proof sets the stage for the proof of Theorem 5.1.

In Section 4 we find fundamental domains for the actions of the various stabilizers on the complex $\mathcal{Y}$ introduced in Section 1.

In Section 5 we prove Theorem 5.1.

Finally, in Section 6 we compute the d^1-map in the spectral sequence (3) in some special cases.

Notation. – If G is a group acting on a simplicial complex X and if σ is a simplex in X, we denote the stabilizer of σ in G by G_σ. If R is a ring, we denote the group of units by $R^\times$. The set of $n \times n$ matrices over R will be denoted by $\mathbb{M}_n(R)$. Unless otherwise stated, F will be an infinite field of arbitrary characteristic.

1. Preliminaries on buildings

In this section, we summarize the basic facts about the Bruhat-Tits building associated to a vector space over a field with discrete valuation. The building was constructed in [6]; more detailed information may be found there (or *see* Brown [4, Ch. V]).

Let K be a field with discrete valuation, v. Denote by $\mathcal{O}$ the valuation ring of v; that is,

$$\mathcal{O} = \{x \in K : v(x) \geq 0\}.$$

Choose a field element π satisfying $v(\pi) = 1$, and denote by k the residue field $\mathcal{O}/\pi\mathcal{O}$. By a *lattice* in K^n, we mean a finitely generated $\mathcal{O}$-submodule which spans K^n; such a submodule is free of rank n. Two lattices L, L' are called *equivalent* if there is some nonzero field element x such that $L' = xL$. Denote the equivalence class of the lattice L by $[L]$. If $v_1,\ldots,v_n$ are linearly independent elements of K^n, denote the equivalence class of the lattice they span by $[v_1,\ldots,v_n]$.

Assign a *type* to a lattice class as follows. If $[v_1, \ldots, v_n]$ is a lattice class, we define its type to be the element

$$v(\det(v_1, \ldots, v_n))$$

modulo n, where $\det(v_1, \ldots, v_n)$ denotes the determinant of the matrix having $v_1, \ldots, v_n$ as columns.

Construct a simplicial complex X in the following manner. The vertices of X are equivalence classes of lattices in K^n. A collection of vertices $\Lambda_0, \Lambda_1, \ldots, \Lambda_m$ forms an m-simplex if there exist representatives $L_0, L_1, \ldots, L_m$ satisfying

$$\pi L_m \subset L_0 \subset L_1 \subset \cdots \subset L_m.$$

Since $L_i/\pi L_m$ is a subspace of the n-dimensional k-vector space $L_m/\pi L_m$, the maximal simplices of X have n vertices; that is, $\dim X = n - 1$. Moreover, the complex X is contractible [4, p. 137]. There is an obvious action of $GL_n(K)$ on X. Note that this action is transitive on the vertices of X.

We now find a fundamental domain for the action of $SL_n(K)$ on X. Let $\mathcal{C}$ be the $(n-1)$-simplex with vertices $[e_1, \ldots, e_i, \pi e_{i+1}, \ldots \pi e_n], i = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the standard basis of K^n. Then we have the following result (*see* [4, p. 137]).

PROPOSITION 1.1. – *The $(n-1)$-simplex $\mathcal{C}$ is a fundamental domain for the action of $SL_n(K)$ on X.*

Proof. – Let $\mathcal{C}'$ be an arbitrary $(n-1)$-simplex with vertices $\Lambda_0, \ldots, \Lambda_{n-1}$, with Λ_i of type $n - i$. By the Invariant Factor Theorem, there is a basis $f_1, \ldots, f_n$ of K^n such that

$$\Lambda_0 = [f_1, \ldots, f_n], \qquad \Lambda_1 = [f_1, \pi f_2, \ldots, \pi f_n], \ldots, \qquad \Lambda_{n-1} = [f_1, \ldots, \pi f_n],$$

and $\det(f_1, \ldots, f_n) = \pi^{nr} u$ for some integer r and $u \in \mathcal{O}^\times$. Replacing f_1 by $\pi^{-r} u^{-1} f_1$, and f_i by $\pi^{-r} f_i$, $i = 2, \ldots, n$, we still have

$$\Lambda_0 = [f_1, \ldots, f_n], \ldots, \qquad \Lambda_{n-1} = [f_1, \ldots, \pi f_n],$$

but now $\det(f_1, \ldots, f_n) = 1$. Let g be the matrix having $f_1, \ldots, f_n$ as columns. Then g takes $\mathcal{C}$ to $\mathcal{C}'$. Since the action of $SL_n(K)$ preserves type, it follows that $\mathcal{C}$ is a fundamental domain. □

The stabilizer of $[e_1, \ldots, e_n]$ in $SL_n(K)$ is the subgroup $SL_n(\mathcal{O})$. Thus, the stabilizer of $[e_1, \ldots, e_i, \pi e_{i+1}, \ldots, \pi e_n]$ is

$$g_i SL_n(\mathcal{O}) g_i^{-1},$$

where

$$g_i = \operatorname{diag}(1, 1, \ldots, 1, \pi, \ldots, \pi),$$

the first π appearing in the $(i+1)$st column. The stabilizer of an edge is the intersection of the stabilizers of its vertices; the stabilizer of a 2-simplex is the intersection of the stabilizers of its edges, and so on.

In this paper, we shall be interested in studying various group actions on two Bruhat-Tits buildings associated to two different fields associated to a field F.

EXAMPLE 1.2. – Denote by $\mathcal{L}$ the field of formal Laurent series over F. Define a valuation v on $\mathcal{L}$ by

$$v\Bigg(\sum_{i\geq n_0} a_i t^i\Bigg) = n_0, \qquad a_{n_0} \neq 0.$$

Here, we choose $\pi = t$. Observe that the ring $F[t,t^{-1}]$ is dense in $\mathcal{L}$. Denote by $\mathcal{X}$ the Bruhat-Tits building associated to $\mathcal{L}^n$.

EXAMPLE 1.3. – Denote by $F(t)$ the field of fractions of $F[t]$. Define a valuation v_∞ on $F(t)$ by

$$v_\infty(a/b) = \deg b - \deg a, \qquad b \neq 0.$$

In this case, we choose $\pi = 1/t$. Denote by $\mathcal{Y}$ the Bruhat-Tits building associated to $F(t)^n$.

Remark. – Denote by $\widehat{K}$ the completion of K with respect to the valuation v. Then the Bruhat-Tits buildings of K and $\widehat{K}$ are isomorphic. In particular, the completion $\widehat{F(t)}$ of $F(t)$ is isomorphic to $\mathcal{L}$ via the map $t \mapsto t^{-1}$. It follows that the complexes $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic. Although these complexes are isomorphic, it will be convenient to distinguish them when doing homological computations.

2. The action of $SL_n(F[t,t^{-1}])$ on $\mathcal{X}$

We now investigate the action of the group $SL_n(F[t,t^{-1}])$ on the complex $\mathcal{X}$ of Example 1.2. Since $F[t,t^{-1}]$ is a dense subring of the field $\mathcal{L}$, we have the following result.

LEMMA 2.1. – *The subgroup $SL_n(F[t,t^{-1}])$ is dense in $SL_n(\mathcal{L})$.*

Proof. – The closure of $SL_n(F[t,t^{-1}])$ in $SL_n(\mathcal{L})$ contains the subgroup of elementary matrices over $\mathcal{L}$. Since these matrices generate $SL_n(\mathcal{L})$, the result follows. □

Denote by V the vector space $\mathcal{L}^n$ and let $GL(V)^\circ$ denote the kernel of the homomorphism

$$v \circ \det : GL(V) \longrightarrow \mathbb{Z}.$$

Then we have the following (*cf.* [16, Thm. 2, p. 78]).

PROPOSITION 2.2. – *If G is a subgroup of $GL(V)^\circ$ whose closure contains $SL(V)$, then the $(n-1)$-simplex $\mathcal{C}$ (see Proposition 1.1) is a fundamental domain for the action of G on $\mathcal{X}$.*

Proof. – We know that $\mathcal{C}$ is a fundamental domain for the action of $SL(V)$ on $\mathcal{X}$. Let $\mathcal{C}'$ be an $(n-1)$-simplex in $\mathcal{X}$. There is an element s of $SL(V)$ with

$$s\mathcal{C} = \mathcal{C}'.$$

Let U be the subgroup of $GL_n(\mathcal{O})$ consisting of the matrices which are congruent to the identity mod t; this is an open subgroup of $GL(V)$. By hypothesis, there is an element u of U and an element g of G with $g = su$. Observe that u fixes each vertex of $\mathcal{C}$. Hence, we have the chain of equalities

$$g\mathcal{C} = su\mathcal{C} = s\mathcal{C} = \mathcal{C}',$$

and since G preserves type, it follows that $\mathcal{C}$ is a fundamental domain for the action of G on $\mathcal{X}$. □

The preceding two results imply that the $(n-1)$-simplex $\mathcal{C}$ is a fundamental domain for the action of $SL_n(F[t,t^{-1}])$ on $\mathcal{X}$.

We now identify the stabilizers in $SL_n(F[t,t^{-1}])$ of the simplices of $\mathcal{C}$. Label the vertices of $\mathcal{C}$ as

$$p_i = [e_1, \ldots, e_{i-1}, te_i, \ldots, te_n], \qquad i = 1, 2, \ldots, n.$$

Note that $p_1 = [te_1, \ldots, te_n] = [e_1, \ldots, e_n]$. Evidently, the stabilizer of p_1 in $SL_n(F[t,t^{-1}])$ is the subgroup

$$SL_n(F[t]) = SL_n(\mathcal{O}) \cap SL_n(F[t,t^{-1}]).$$

Denote by g_i the matrix

$$g_i = \operatorname{diag}(1, \ldots, 1, t, \ldots, t), \qquad i = 2, \ldots, n$$

where the first $i-1$ entries are equal to 1. Then the stabilizer of p_i in $SL_n(F[t,t^{-1}])$ is

$$g_i SL_n(F[t]) g_i^{-1}.$$

Denote by $\Gamma_{i_1,\ldots,i_k}$ the stabilizer of the $(k-1)$-simplex having vertices $p_{i_1}, \ldots, p_{i_k}$. The group $\Gamma_{i_1,\ldots,i_k}$ is the intersection of the stabilizers $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$ of the vertices of the simplex. Elements of $\Gamma_{i_1,\ldots,i_k}$ have the form

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix}$$

where we have

$$L_r \in \mathbb{M}_{i_r - i_{r-1}}(F[t]), \qquad 1 \le r \le k+1$$
$$V_{r,s} \in \mathbb{M}_{i_r - i_{r-1}, i_s - i_{s-1}}(F[t]), \qquad 1 \le r, s \le k+1$$

(here, we set $i_0 = 1$ and $i_{k+1} = n+1$).

Consider the stabilizers $\Gamma_{1,j_2,\dots,j_k}$. These are subgroups of $\Gamma_1 = SL_n(F[t])$. Elements of the group $\Gamma_{1,j_2,\dots,j_k}$ have the form

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k-1} & V_{1,k} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k-1} & V_{2,k} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k-1} & V_{3,k} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k,1} & tV_{k,2} & tV_{k,3} & \cdots & tV_{k,k-1} & L_k \end{pmatrix}$$

where we have

$$L_r \in \mathbb{M}_{j_{r+1}-j_r}(F[t]), \qquad 1 \le r \le k$$
$$V_{r,s} \in \mathbb{M}_{j_{r+1}-j_r, j_{s+1}-j_s}(F[t]), \qquad 1 \le r, s \le k$$

(here, we set $j_1 = 1$ and $j_{k+1} = n+1$).

These groups are related as follows.

PROPOSITION 2.3. – *The group $\Gamma_{i_1,\dots,i_k}$ is conjugate to $\Gamma_{1,(i_2-i_1+1),\dots,(i_k-i_1+1)}$ inside $GL_n(F[t,t^{-1}])$.*

Proof. – First conjugate $\Gamma_{i_1,\dots,i_k}$ by the element

$$g = \mathrm{diag}(t, t \dots, t, 1, \dots, 1)$$

where the first $i_1 - 1$ entries are equal to t. The resulting group has elements of the form

$$\begin{pmatrix} L_1 & tV_{12} & tV_{13} & \cdots & tV_{1,k} & V_{1,k+1} \\ V_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ V_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ V_{k,1} & tV_{k,2} & tV_{k,3} & \cdots & L_k & V_{k,k+1} \\ V_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix}$$

where the L_r and $V_{r,s}$ are as above. Now conjugate by the permutation matrix corresponding to the permutation

$$\begin{aligned} 1 &\mapsto n - i_1 + 2 \\ 2 &\mapsto n - i_1 + 3 \\ &\vdots \quad \vdots \\ i_1 - 1 &\mapsto n \\ i_1 &\mapsto 1 \\ &\vdots \quad \vdots \\ i_2 - 1 &\mapsto i_2 - i_1 \\ i_2 &\mapsto i_2 - i_1 + 1 \\ i_2 + 1 &\mapsto i_2 - i_1 + 2 \\ &\vdots \quad \vdots \\ n &\mapsto n - i_1 + 1. \end{aligned}$$

Note that if τ denotes the n-cycle $(12\cdots n)$, then this permutation is simply τ^{i_1-1}. The resulting group has the form

$$\begin{pmatrix} L_2 & V_{23} & V_{24} & \cdots & \cdots & V_{2,k+1} & V_{21} \\ tV_{32} & L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} \\ tV_{42} & tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} \\ \vdots & & & \ddots & & & \vdots \\ tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\ tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\ tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1 \end{pmatrix},$$

which is precisely the group $\Gamma_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)}$. □

If σ is a p-simplex in $\mathcal{C}$, denote by Γ_σ the stabilizer of σ in $SL_n(F[t,t^{-1}])$. Since the complex $\mathcal{X}$ is contractible, we have a spectral sequence converging to the homology of $SL_n(F[t,t^{-1}])$ with E^1-term

$$E^1_{p,q} = \bigoplus_{\dim\sigma=p} H_q(\Gamma_\sigma) \tag{4}$$

where σ ranges over the p-simplices of $\mathcal{C}$. By Proposition 2.3, we need only compute the homology of each $\Gamma_{1,j_2,\ldots,j_k}$; we do this in Section 5.

In the next section we single out the Γ_i, $i = 1,\ldots,n$ and compute their homology.

3. The vertex stabilizers. The homology of $SL_n(F[t])$

Notation. – For G a subgroup of $GL_n(R)$, R a commutative ring with unit, denote by $\overline{G}$ the subgroup $G \cap SL_n(R)$.

Consider the stabilizers $\Gamma_1,\ldots,\Gamma_n$ of the vertices of $\mathcal{C}$. Each of these is isomorphic to $SL_n(F[t])$. To compute homology we use the Bruhat-Tits building $\mathcal{Y}$ of Example 1.3. Recall that this is the building associated to the n-dimensional vector space $V = F(t)^n$.

There is an obvious left action of $SL_n(F[t])$ on $\mathcal{Y}$. Let $e_1,\ldots,e_n$ be the standard basis of V. Then the subcomplex $\mathcal{T}$ having vertices

$$[e_1t^{r_1}, e_2t^{r_2},\ldots,e_{n-1}t^{r_{n-1}}, e_n], \qquad \text{where} \quad r_1 \geq r_2 \geq \cdots \geq r_{n-1} \geq 0$$

is a fundamental domain for the action of $SL_n(F[t])$ on $\mathcal{Y}$ [17].

The complex $\mathcal{T}$ is an infinite wedge. Denote by v_0 the vertex $[e_1,\ldots,e_n]$ and by v_i the vertex $[e_1t, e_2t,\ldots,e_it, e_{i+1},\ldots,e_n], i = 1,2,\ldots,n-1$. For a k element subset $I = \{i_1,\ldots,i_k\}$ of $\{1,2,\ldots,n-1\}$, define $E_I^{(k)}$ to be the subcomplex of $\mathcal{T}$ which is the union of all rays with origin v_0 passing through the $(k-1)$-simplex $\langle v_{i_1},\ldots,v_{i_k}\rangle$. There are $\binom{n-1}{k}$ such $E_I^{(k)}$. Observe that if $I = \{1,2,\ldots,n-1\}$, then $E_I^{(n-1)} = \mathcal{T}$. When we write $E_J^{(l)}$, the superscript l denotes the cardinality of the set J.

Define a filtration $V^\bullet$ of $\mathcal{T}$ by setting $V^{(0)} = v_0$ and

$$V^{(k)} = \bigcup_I E_I^{(k)}, \qquad 1 \le k \le n-1 \tag{5}$$

where I ranges over all k-element subsets of $\{1, 2, \ldots, n-1\}$. Note that $V^{(n-1)} = \mathcal{T}$.

Evidently, the stabilizer of v_0 in $SL_n(F[t])$ is the subgroup $SL_n(F)$. For any other vertex $v = [e_1 t^{r_1}, e_2 t^{r_2}, \ldots, e_{n-1} t^{r_{n-1}}, e_n]$ in $\mathcal{T}$, let Γ_v denote the stabilizer of v in $SL_n(F[t])$. The subgroup Γ_v is the semidirect product of a reductive group L_v contained in $SL_n(F)$ and a unipotent group U_v contained in $SL_n(F[t])$. If p_{kl} denotes the polynomial in the kth row and lth column of an element of Γ_v, then we have $\deg p_{kl} \le r_k - r_l$. It follows that the subgroup Γ_v has a block form

$$\Gamma_v = \overline{\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1m} \\ & L_2 & V_{23} & \cdots & V_{2m} \\ & & \ddots & & \vdots \\ & 0 & & L_{m-1} & V_{m-1,m} \\ & & & & L_m \end{pmatrix}}$$

where the L_k and V_{kl} satisfy

$$\begin{aligned} &L_k \in GL_{i_k - i_{k-1}}(F), && \text{where } r_{i_{k-1}+1} = r_{i_{k-1}+2} = \cdots = r_{i_k} \\ &V_{kl} \in \mathbb{M}_{i_k - i_{k-1}, i_l - i_{l-1}}(F[t]), && \text{where } r_{i_{k-1}+1} = r_{i_{k-1}+2} = \cdots = r_{i_k} \\ & && \phantom{\text{where }} r_{i_{l-1}+1} = r_{i_{l-1}+2} = \cdots = r_{i_l} \end{aligned}$$

(we set $i_0 = 0$). Observe that the stabilizers $\Gamma_{v_i}, i = 1, 2, \ldots, n-1$, have the block form of the $n-1$ maximal parabolic subgroups in SL_n. If $I = \{i_1, \ldots, i_k\}$ and if v is a vertex in $E_I^{(k)}$ which does not lie in any $E_J^{(k-1)}$, where $J \subset I$, then Γ_v has the block form of the intersection $\Gamma_{v_{i_1}} \cap \cdots \cap \Gamma_{v_{i_k}}$. Observe that if v is a vertex of $\mathcal{T}$ not lying in any $E_J^{(n-2)}$, then the r_i are positive and distinct and hence the group Γ_v is upper triangular.

If e is an edge with vertices v, w, then the stabilizer Γ_e is simply the intersection $\Gamma_v \cap \Gamma_w$. Similarly, the stabilizer of a 2-simplex is the intersection of the edge stabilizers, and so on. It follows that if $l \le k$ and if σ is an l-simplex in $E_I^{(k)}$, where $I = \{i_1, \ldots, i_k\}$, not lying entirely in any $E_J^{(k-1)}$, where $J \subset I$, then Γ_σ has the block form of the intersection $\Gamma_{v_{i_1}} \cap \cdots \cap \Gamma_{v_{i_k}}$.

The case $n = 3$ is shown in Figure 1.

Since the complex $\mathcal{Y}$ is contractible, we have a spectral sequence converging to $H_\bullet(SL_n(F[t]), \mathbb{Z})$ with E^1-term satisfying

$$E^1_{p,q} = \bigoplus_{\dim \sigma = p} H_q(\Gamma_\sigma) \tag{6}$$

where σ ranges over the simplices of $\mathcal{T}$.

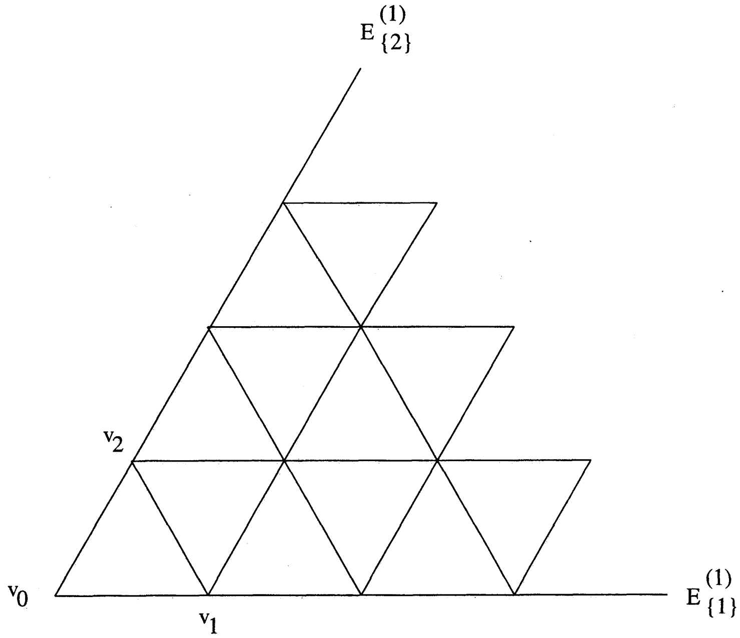

Fig. 1. – The fundamental domain $\mathcal{T}$ for $n = 3$.

3.1. The homology of the stabilizers

We now compute the homology of the groups Γ_σ. Suppose that A is an F-algebra. Let P be a subgroup of $GL_{n+m}(A)$ having block form

$$P = \begin{pmatrix} L_1 & M \\ 0 & L_2 \end{pmatrix}$$

where $L_1 \subseteq GL_n(A)$, $L_2 \subseteq GL_m(A)$, and M is a vector subspace of $\mathbb{M}_{n,m}(A)$ such that $L_1 M = M = ML_2$. Suppose that each L_i contains the group of diagonal matrices over F. Denote by L the subgroup of P defined by

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$

A proof of the following is deduced easily from [10, Lemma 9] by observing that the argument used works with F replaced by A. Recall that $\overline{G}$ denotes the intersection $G \cap SL_n(R)$.

PROPOSITION 3.1. – *If F is an infinite field, then the inclusion $\overline{L} \longrightarrow \overline{P}$ induces an isomorphism*

$$H_\bullet(\overline{L}, \mathbb{Z}) \longrightarrow H_\bullet(\overline{P}, \mathbb{Z}).$$ □

COROLLARY 3.2. – *Suppose that P is a subgroup of $GL_n(A)$ having block form*

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1m} \\ & L_2 & V_{23} & \cdots & V_{2m} \\ & & \ddots & & \vdots \\ & 0 & & L_{m-1} & V_{m-1,m} \\ & & & & L_m \end{pmatrix}$$

where each $L_i \subseteq GL_{n_i}(A)$ and each V_{ij} is a vector subspace of $\mathbb{M}_{n_i,n_j}(A)$ such that $L_iV_{ij} = V_{ij} = V_{ij}L_j$. Assume that each L_i contains the group of diagonal matrices over F. Denote by L the subgroup

$$L = \begin{pmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_m \end{pmatrix}$$

of P. Then the inclusion $\overline{L} \longrightarrow \overline{P}$ induces an isomorphism

$$H_\bullet(\overline{L}, \mathbb{Z}) \longrightarrow H_\bullet(\overline{P}, \mathbb{Z}).$$

Proof. – Consider the sequence of inclusions

$$\overline{L} \to \overline{\left(\begin{array}{ccc|c} L_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & L_{m-1} & V_{m-1,m} \\ \hline & 0 & & L_m \end{array}\right)} \to$$

$$\overline{\left(\begin{array}{ccc|cc} L_1 & & 0 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 0 & & L_{m-2} & V_{m-2,m-1} & V_{m-2,m} \\ \hline & & & L_{m-1} & V_{m-1,m} \\ 0 & & & 0 & L_m \end{array}\right)}$$

$$\cdots \to \overline{\left(\begin{array}{cc|ccc} L_1 & 0 & \cdots & \cdots & 0 \\ 0 & L_2 & V_{23} & \cdots & V_{2m} \\ \hline \vdots & & \ddots & & \vdots \\ \vdots & & & L_{m-1} & V_{m-1,m} \\ 0 & 0 & & & L_m \end{array}\right)} \longrightarrow \overline{P}.$$

By Proposition 3.1, each of these maps induces a homology isomorphism. It follows that the inclusion $\overline{L} \to \overline{P}$ induces an isomorphism

$$H_\bullet(\overline{L}, \mathbb{Z}) \longrightarrow H_\bullet(\overline{P}, \mathbb{Z}). \qquad \square$$

If σ is a simplex in $\mathcal{T}$, then the subgroup Γ_σ has a block form as in the corollary. We have an extension

$$1 \longrightarrow U_\sigma \longrightarrow \Gamma_\sigma \longrightarrow L_\sigma \longrightarrow 1$$

where U_σ is a unipotent group and L_σ is a reductive subgroup of $SL_n(F)$. The corollary implies that the inclusion $L_\sigma \to \Gamma_\sigma$ induces an isomorphism

$$H_\bullet(L_\sigma, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_\sigma, \mathbb{Z}).$$

Let $I = \{i_1, \ldots, i_k\}$ be a subset of $\{1, 2, \ldots, n-1\}$. If σ is a simplex in

$$E_I^{(k)} - \bigcup_{J \subset I} E_J^{(k-1)},$$

then Γ_σ has the block form of the intersection $\Gamma_{v_{i_1}} \cap \cdots \cap \Gamma_{v_{i_k}}$. If τ is another such simplex, then Γ_τ has the same block form. Thus, $L_\sigma = L_\tau$ and it follows that Γ_σ and Γ_τ have the same homology. Moreover, if σ is a face of τ, then the map $\Gamma_\tau \to \Gamma_\sigma$ induces an isomorphism on homology.

3.2. The homology of $SL_n(F[t])$

Given a coefficient system $\mathcal{M}$ on a simplicial complex Z (*i.e.*, a covariant functor from the simplices of Z to the category of abelian groups), we may define the chain complex $C_\bullet(Z, \mathcal{M})$ by setting

$$C_p(Z, \mathcal{M}) = \bigoplus_{\dim \sigma = p} \mathcal{M}(\sigma)$$

with boundary map the alternating sum of the maps induced by the face maps in Z.

We shall make use of the following result (compare with [18, Lemma 6]).

LEMMA 3.3. – *Suppose $F^{(0)} \subset F^{(1)} \subset \cdots \subset F^{(k)} = Z$ is a filtration of the simplicial complex Z by subcomplexes such that each $F^{(i)}$ and each component of $F^{(i)} - F^{(i-1)}$ is contractible. Suppose that $\mathcal{M}$ is a coefficient system on Z such that the restriction of $\mathcal{M}$ to each component of $F^{(i)} - F^{(i-1)}$ is constant. Then the inclusion $F^{(0)} \longrightarrow Z$ induces an isomorphism*

$$H_\bullet(F^{(0)}, \mathcal{M}) \longrightarrow H_\bullet(Z, \mathcal{M}).$$

Proof. – The filtration of Z induces a filtration of $C_\bullet(Z, \mathcal{M})$. This yields a spectral sequence converging to $H_\bullet(Z, \mathcal{M})$ with E^1-term having ith column

$$H_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M}).$$

Consider the relative chain complex $C_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M})$. By hypothesis, this chain complex is a direct sum of chain complexes with constant coefficients. Since each $F^{(i)}$ is contractible, it follows that

$$H_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M}) = 0, \qquad i \geq 1.$$

Thus, only the 0th column $H_\bullet(F^{(0)}, \mathcal{M})$ is nonzero. This proves the lemma. □

We may now compute $H_\bullet(SL_n(F[t]), \mathbb{Z})$. The argument in the proof below is used implicitly by Soulé in the proof of Theorem 5 of [17].

THEOREM 3.4. – *If F is an infinite field, then the natural inclusion $SL_n(F) \to SL_n(F[t])$ induces an isomorphism*

$$H_\bullet(SL_n(F), \mathbb{Z}) \longrightarrow H_\bullet(SL_n(F[t]), \mathbb{Z}).$$

Proof. – Recall the spectral sequence (6). The E^1-term satisfies

$$E^1_{p,q} = \bigoplus_{\dim \sigma = p} H_q(\Gamma_\sigma) \Longrightarrow H_{p+q}(SL_n(F[t])).$$

For each $q \geq 0$, define a coefficient system $\mathcal{F}_q$ on $\mathcal{T}$ by

$$\mathcal{F}_q(\sigma) = H_q(\Gamma_\sigma).$$

Then the qth row in the spectral sequence is simply $C_\bullet(\mathcal{T}, \mathcal{F}_q)$ and the d^1-map is the boundary map in this chain complex.

Recall the filtration $V^\bullet$ of $\mathcal{T}$ (5). For each simplex in

$$E_I^{(k)} - \bigcup_{J \subset I} E_J^{(k-1)},$$

the stabilizers have the same reductive part and hence have the same homology (*see* the discussion following the proof of Corollary 3.2). It follows that the restriction of $\mathcal{F}_q$ to each component of $V^{(i)} - V^{(i-1)}$ is constant. By Lemma 3.3, the inclusion $v_0 \longrightarrow \mathcal{T}$ induces an isomorphism

$$H_\bullet(v_0, \mathcal{F}_q) \longrightarrow H_\bullet(\mathcal{T}, \mathcal{F}_q).$$

Observe that

$$H_p(v_0, \mathcal{F}_q) = \begin{cases} H_q(SL_n(F)) & p = 0 \\ 0 & p > 0. \end{cases}$$

It follows that the E^2-term of the spectral sequence (6) satisfies

$$E^2_{p,q} = \begin{cases} H_q(SL_n(F)) & p = 0 \\ 0 & p > 0. \end{cases}$$ □

Remark. – Theorem 3.4 may be viewed as an unstable version of Quillen's homotopy invariance in algebraic K-theory [15].

Remark. – The $n = 2$ case of Theorem 3.4 was proved for fields of characteristic zero in [12] by considering the Mayer-Vietoris sequence associated to the amalgamated free product decomposition (due to Nagao [13])

$$SL_2(F[t]) \cong SL_2(F) *_{B(F)} B(F[t]) \tag{7}$$

where $B(R)$ denotes the upper triangular group over R. Proposition 3.2 of [12] shows that $B(F)$ and $B(F[t])$ are the same homologically. This implies that the Mayer-Vietoris sequence associated to (7) breaks into short exact sequences

$$0 \longrightarrow H_k(B(F)) \longrightarrow H_k(B(F[t])) \oplus H_k(SL_2(F)) \longrightarrow H_k(SL_2(F[t])) \longrightarrow 0,$$

from which it follows that $H_\bullet(SL_2(F),\mathbb{Z}) \cong H_\bullet(SL_2(F[t]),\mathbb{Z})$.

As an immediate consequence of Theorem 3.4 we have the following result.

COROLLARY 3.5. – *The natural inclusion $GL_n(F) \to GL_n(F[t])$ induces an isomorphism*

$$H_\bullet(GL_n(F),\mathbb{Z}) \longrightarrow H_\bullet(GL_n(F[t]),\mathbb{Z}).$$

Proof. – Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & SL_n(F) & \longrightarrow & GL_n(F) & \longrightarrow & F^\times & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & SL_n(F[t]) & \longrightarrow & GL_n(F[t]) & \longrightarrow & F^\times & \longrightarrow & 1. \end{array}$$

This yields a map of spectral sequences which by Theorem 3.4 is an isomorphism at the E^2-level. □

By applying a theorem of Suslin, we have the following stability result.

COROLLARY 3.6. – *If $n \leq m$, then the natural map*

$$H_i(GL_n(F[t]),\mathbb{Z}) \longrightarrow H_i(GL_m(F[t]),\mathbb{Z})$$

is an isomorphism for $i \leq n$.

Proof. – Consider the commutative diagram

$$\begin{array}{ccc} H_i(GL_n(F),\mathbb{Z}) & \longrightarrow & H_i(GL_m(F),\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_i(GL_n(F[t]),\mathbb{Z}) & \longrightarrow & H_i(GL_m(F[t]),\mathbb{Z}). \end{array}$$

By [19, 3.4], the top horizontal map is an isomorphism for $i \leq n$ and by Corollary 3.5, so is each of the two vertical maps. □

4. The level t congruence subgroup and a fundamental domain for the action of $\Gamma_{1,j_2,\ldots,j_k}$ on $\mathcal{Y}$

Consider the exact sequence

$$1 \longrightarrow K \longrightarrow SL_n(F[t]) \xrightarrow{t=0} SL_n(F) \longrightarrow 1$$

where K consists of those matrices which are congruent to the identity modulo t. In the preceding section we described a fundamental domain, $\mathcal{T}$, for the action of $SL_n(F[t])$ on the complex $\mathcal{Y}$ of Example 1.2. In order to find a fundamental domain for the action of

$\Gamma_{1,j_2,\ldots,j_k}$ on $\mathcal{Y}$, we proceed in steps. First, we find a fundamental domain for the action of K, then a fundamental domain for the action of $\Gamma_{1,2,\ldots,n}$, and finally, a fundamental domain for the action of $\Gamma_{1,j_2,\ldots,j_k}$.

Denote by $B_n(F)$ the upper triangular subgroup of $SL_n(F)$ and choose a set S of coset representatives for $SL_n(F)/B_n(F)$. Set

$$\mathcal{T}' = \bigcup_{s\in S} s\mathcal{T}.$$

PROPOSITION 4.1. – *The complex $\mathcal{T}'$ is a fundamental domain for the action of K on $\mathcal{Y}$.*

Proof. – Let σ be an $(n-1)$-simplex of $\mathcal{Y}$. There exists some x in $SL_n(F[t])$ and a unique simplex σ_0 of $\mathcal{T}$ such that $\sigma = x\sigma_0$. Write

$$x = ky, \qquad k \in K, \qquad y \in SL_n(F)$$

and

$$y = su, \qquad s \in S, \qquad u \in B_n(F).$$

Then

$$\sigma = ksu\sigma_0.$$

Note that u acts trivially on $\mathcal{T}$; *i.e.*, $u\sigma_0 = \sigma_0$. Hence, $\sigma = ks\sigma_0$, and thus

$$\sigma \equiv s\sigma_0 \bmod K.$$

It remains to show that no two vertices of $\mathcal{T}'$ are identified by K.

Suppose $x : s_1\Lambda_1 \longrightarrow s_2\Lambda_2$ where the s_i belong to S and x is some element of K. Then

$$s_1 s_2^{-1} x : s_1\Lambda_1 \longrightarrow s_1\Lambda_2.$$

Now, $s_1 s_2^{-1} x$ belongs to $SL_n(F[t])$ and the $s_1\Lambda_i$ are inequivalent modulo $SL_n(F[t])$ (*i.e.*, we could have taken $s_1\mathcal{T}$ as a fundamental domain). Hence, $\Lambda_1 = \Lambda_2$. Denote this common vertex by Λ. Moreover, $s_1 s_2^{-1} x$ stabilizes $s_1\Lambda$. Observe that the stabilizer of $s_1\Lambda$ in $SL_n(F[t])$ is

$$s_1(SL_n(F[t]))_\Lambda s_1^{-1}.$$

It follows that

$$s_1 s_2^{-1} x = s_1 \gamma s_1^{-1}$$

where γ stabilizes Λ. So,

$$x = s_2 \gamma s_1^{-1}. \tag{8}$$

We have a split exact sequence

$$1 \longrightarrow (K \cap (SL_n(F[t]))_\Lambda) \longrightarrow (SL_n(F[t]))_\Lambda \xrightarrow{t=0} P_\Lambda \longrightarrow 1$$

where P_Λ is a parabolic subgroup of $SL_n(F)$. Write $\gamma = kv$, where $k \in K$ and $v \in P_\Lambda$. Then

$$\begin{aligned} x &= s_2 k v s_1^{-1} \\ &= s_2(vs_1^{-1})(s_1 v^{-1})k(vs_1^{-1}). \end{aligned}$$

Since K is a normal subgroup of $SL_n(F[t])$, we have

$$(s_1 v^{-1})k(vs_1^{-1}) \in K.$$

Denote this element by k'. Then we may write

$$x = s_2(vs_1^{-1})k'$$

or

$$x(k')^{-1} = s_2(vs_1^{-1}). \tag{9}$$

Now, the element $x(k')^{-1}$ belongs to K while the element $s_2(vs_1^{-1})$ belongs to $SL_n(F)$. Since the groups K and $SL_n(F)$ intersect in the identity, both sides of equation (9) must equal 1. It follows that

$$s_2 = s_1 v^{-1}.$$

Since v^{-1} stabilizes Λ, we have

$$s_2\Lambda = (s_1 v^{-1})\Lambda = s_1\Lambda.$$

It follows that $\mathcal{T}'$ is a fundamental domain for the action of K on $\mathcal{Y}$. □

Remark. – When $n = 2$, Proposition 4.1 allows us to deduce the free product decomposition

$$K = *_{s\in\mathbb{P}^1(F)} sCs^{-1} \tag{10}$$

where

$$C = \left\{ \begin{pmatrix} 1 & tp(t) \\ 0 & 1 \end{pmatrix} : p(t) \in F[t] \right\}$$

(here, the set S of coset representatives of $SL_2(F)/B_2(F)$ may be identified with $\mathbb{P}^1(F)$). For further details *see* [12, 4.1].

Now consider the stabilizer $\Gamma_{1,2,\ldots,n}$ of the simplex $\mathcal{C}$ (*see* Proposition 1.1). We have a split short exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_{1,2,\ldots,n} \xrightarrow{t=0} B_n(F) \longrightarrow 1.$$

Choose a set of representatives for the permutation group Σ_n in $SL_n(F)$ (*e.g.*, we could take even permutations of the identity matrix along with odd permutations of the matrix $\text{diag}(-1, 1, \ldots, 1)$). Denote by $\mathcal{D}_{1,2,\ldots,n}$ the subcomplex of $\mathcal{Y}$ defined by

$$\mathcal{D}_{1,2,\ldots,n} = \bigcup_{p\in\Sigma_n} p\mathcal{T}.$$

PROPOSITION 4.2. – *The subcomplex $\mathcal{D}_{1,2,\ldots,n}$ is a fundamental domain for the action of $\Gamma_{1,2,\ldots,n}$ on $\mathcal{Y}$.*

Proof. – We have a split extension

$$1 \longrightarrow U \longrightarrow B_n(F) \xrightarrow{\pi} T \longrightarrow 1$$

where U is the unipotent radical of $B_n(F)$ and T is the diagonal subgroup. The composition of π with the map

$$\Gamma_{1,2,\ldots,n} \xrightarrow{t=0} B_n(F)$$

yields a split extension

$$1 \longrightarrow G \longrightarrow \Gamma_{1,2,\ldots,n} \longrightarrow T \longrightarrow 1.$$

Here, the group G consists of matrices of the form

$$\begin{pmatrix} 1+tp_{11} & p_{12} & \cdots & \cdots & p_{1n} \\ tp_{21} & 1+tp_{22} & \cdots & \cdots & p_{2n} \\ \vdots & & & & \vdots \\ tp_{n1} & \cdots & \cdots & tp_{n,n-1} & 1+tp_{nn} \end{pmatrix}$$

where the p_{ij} lie in $F[t]$. We first show that $\mathcal{D}_{1,2,\ldots,n}$ is a fundamental domain for the action of G on $\mathcal{Y}$.

Consider the extension

$$1 \longrightarrow K \longrightarrow G \xrightarrow{t=0} U \longrightarrow 1.$$

Suppose that σ is an $(n-1)$-simplex in $\mathcal{Y}$. Then there exist $k \in K$, $s \in S$, and $\sigma_0 \in \mathcal{T}$ such that

$$\sigma = ks\sigma_0.$$

Recall the *Bruhat decomposition* of $SL_n(F)$ (*see e.g.*, [9, p. 172]):

$$SL_n(F) = \bigcup_{p\in\Sigma_n} UpB$$

(here, $B = B_n(F)$). From this it follows that if s is an element of the set S, then we may write $s = upv$ for some $u \in U$, $p \in \Sigma_n$, and $v \in B_n(F)$. Then we have the chain of equalities

$$\sigma = ks\sigma_0 = kupv\sigma_0 = kup\sigma_0.$$

The last equality follows since $B_n(F)$ acts trivially on $\mathcal{T}$. Now, ku lies in G. Hence,

$$\sigma \equiv p\sigma_0 \operatorname{mod} G.$$

It follows that $\mathcal{D}_{1,2,\ldots,n}$ is a fundamental domain for the action of G on $\mathcal{Y}$. Observe that the diagonal subgroup T acts trivially on $\mathcal{D}_{1,2,\ldots,n}$.

LEMMA 4.3. – *Suppose a group H acts on a simplical complex $\mathcal{Z}$, and that there is a split extension*

$$1 \longrightarrow N \longrightarrow H \longrightarrow Q \longrightarrow 1.$$

Suppose further that the subcomplex $\mathcal{A}$ is a fundamental domain for the action of N on $\mathcal{Z}$ and that Q acts trivially on $\mathcal{A}$. Then $\mathcal{A}$ is a fundamental domain for the action of H on $\mathcal{Z}$.

Proof. – It suffices to show that no two vertices of $\mathcal{A}$ are identified by the action of H. Suppose that v_1 and v_2 are vertices of $\mathcal{A}$ and that there is an element h in H with $hv_1 = v_2$. Write $h = nq$, where $n \in N$, and $q \in Q$. Then we have

$$v_2 = hv_1 = nqv_1 = nv_1.$$

Since the vertices of $\mathcal{A}$ are inequivalent modulo N, we must have $v_1 = v_2$. □

The lemma implies that $\mathcal{D}_{1,2,\ldots,n}$ is a fundamental domain for the action of $\Gamma_{1,2,\ldots,n}$ on $\mathcal{Y}$. This completes the proof of Proposition 4.2. □

Finally, consider the group $\Gamma_{1,j_2,\ldots,j_k}$. Note that $\Gamma_{1,j_2,\ldots,j_k}$ contains the subgroup H of Σ_n consisting of permutation matrices that are products of the form

$$\sigma_1\sigma_2\cdots\sigma_{k-1}$$

where σ_i is a permutation of the set

$$\{j_i, j_i+1, \ldots, j_{i+1}-1\}$$

(we take $j_1 = 1$). Let N be a set of coset representatives of $H\backslash\Sigma_n$ containing the identity. Define a subcomplex $\mathcal{D}_{1,j_2,\ldots,j_k}$ by

$$\mathcal{D}_{1,j_2,\ldots,j_k} = \bigcup_{p\in N} p\mathcal{T}.$$

PROPOSITION 4.4. – *The complex $\mathcal{D}_{1,j_2,\ldots,j_k}$ is a fundamental domain for the action of $\Gamma_{1,j_2,\ldots,j_k}$ on $\mathcal{Y}$.*

Proof. – Observe that $\Gamma_{1,j_2,\ldots,j_k}$ contains the group $\Gamma_{1,2,\ldots,n}$. It follows that a fundamental domain for the action of $\Gamma_{1,j_2,\ldots,j_k}$ on $\mathcal{Y}$ is no larger than $\mathcal{D}_{1,2,\ldots,n}$. If σ is an $(n-1)$-simplex in $\mathcal{Y}$, then there exist $g \in \Gamma_{1,2,\ldots,n}$, $p \in \Sigma_n$, and $\sigma_0 \in \mathcal{T}$ such that

$$\sigma = gp\sigma_0.$$

Write $p = hn$, where $h \in H$ and $n \in N$. Then we have the chain of equalities

$$\sigma = gp\sigma_0 = ghn\sigma_0.$$

Since gh lies in $\Gamma_{1,j_2,\ldots,j_k}$, it follows that

$$\sigma \equiv n\sigma_0 \bmod \Gamma_{1,j_2,\ldots,j_k},$$

and hence, $\mathcal{D}_{1,j_2,\ldots,j_k}$ is a fundamental domain for the action of $\Gamma_{1,j_2,\ldots,j_k}$ on $\mathcal{Y}$. □

5. The homology of $\Gamma_{1,j_2,\ldots,j_k}$

We now compute the homology of the various $\Gamma_{1,j_2,\ldots,j_k}$. This will complete the computation of the E^1-term of the spectral sequence (4) since by Proposition 2.3 each $\Gamma_{i_1,\ldots,i_k}$ is isomorphic to some $\Gamma_{1,j_2,\ldots,j_k}$.

We have a split short exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_{1,j_2,\ldots,j_k} \xrightarrow{t=0} P_{1,j_2,\ldots,j_k} \longrightarrow 1$$

where $P_{1,j_2,\ldots,j_k}$ is a parabolic subgroup of $SL_n(F)$.

THEOREM 5.1. – *The natural inclusion $P_{1,j_2,\ldots,j_k} \longrightarrow \Gamma_{1,j_2,\ldots,j_k}$ induces an isomorphism*

$$H_\bullet(P_{1,j_2,\ldots,j_k}, \mathbb{Z}) \longrightarrow H_\bullet(\Gamma_{1,j_2,\ldots,j_k}, \mathbb{Z}).$$

Proof. – Since the complex $\mathcal{Y}$ is contractible, we obtain a spectral sequence converging to the homology of $\Gamma_{1,j_2,\ldots,j_k}$ satisfying

$$E^1_{p,q} = \bigoplus_{\dim \sigma = p} H_q(G_\sigma) \tag{11}$$

where G_σ is the stabilizer of the p-simplex σ in $\Gamma_{1,j_2,\ldots,j_k}$ $(\sigma \subset \mathcal{D}_{1,j_2,\ldots,j_k})$.

Recall the filtration $V^\bullet$ of $\mathcal{T}$ (5) defined in Section 3. Define a filtration $W^\bullet$ of $\mathcal{D}_{1,j_2,\ldots,j_k}$ by setting

$$W^{(l)} = \bigcup_{p \in N} pV^{(l)}, \qquad 0 \le l \le n-1.$$

Note that $W^{(0)} = v_0$ and that the group G_{v_0} is precisely $P_{1,j_2,\ldots,j_k}$. Define a coefficient system $\mathcal{G}_q$ on $\mathcal{D}_{1,j_2,\ldots,j_k}$ by

$$\mathcal{G}_q(\sigma) = H_q(G_\sigma).$$

Then the qth row of the spectral sequence (11) is the chain complex

$$C_\bullet(\mathcal{D}_{1,j_2,\ldots,j_k}, \mathcal{G}_q).$$

On each component of $W^{(i)} - W^{(i-1)}$, the coefficient system $\mathcal{G}_q$ is constant (*i.e.*, the stabilizers in the translate $p\mathcal{T}$ are conjugate to the stabilizers in $\mathcal{T}$ and hence have isomorphic homology). So we may apply Lemma 3.3 to deduce that the inclusion $v_0 \longrightarrow \mathcal{D}_{1,j_2,\ldots,j_k}$ induces an isomorphism

$$H_\bullet(v_0, \mathcal{G}_q) \longrightarrow H_\bullet(\mathcal{D}_{1,j_2,\ldots,j_k}, \mathcal{G}_q).$$

Now the E^2-term of the spectral sequence (11) satisfies

$$E^2_{p,q} = \begin{cases} H_q(P_{1,j_2,\ldots,j_k}) & p = 0 \\ 0 & p > 0. \end{cases}$$

This completes the proof of Theorem 5.1. □

Remark. – Theorem 3.4 is the special case $\Gamma_1 = SL_n(F[t])$ and $P_1 = SL_n(F)$.

Remark. – In the case of $\Gamma_{1,2,\ldots,n}$ and $P_{1,2,\ldots,n} = B_n(F)$, it is not necessary to define the filtration $W^\bullet$ of $\mathcal{D}_{1,2,\ldots,n}$ to prove the result. Indeed, Corollary 3.2 implies that each G_σ is homologically equivalent to $B_n(F)$. It follows that the qth row of spectral sequence (11) is the chain complex

$$C_\bullet(\mathcal{D}_{1,2,\ldots,n}, H_q(B_n(F))).$$

Since $\mathcal{D}_{1,2,\ldots,n}$ is contractible, the homology of the complex vanishes except in dimension zero, where we get $H_q(B_n(F))$.

Remark. – When $n = 2$, we only have the group Γ_{12}. In this case, Theorem 5.1 states that

$$H_\bullet(\Gamma_{12}) \cong H_\bullet(B_2(F)).$$

This was proved in [12] for fields of characteristic zero by examining the Lyndon-Hochschild-Serre spectral sequence associated to the extension

$$1 \longrightarrow K \longrightarrow \Gamma_{12} \longrightarrow B_2(F) \longrightarrow 1.$$

The free product decomposition (10) for K allows us to deduce that

$$H_k(K) = \bigoplus_{s \in \mathbb{P}^1(F)} H_k(sCs^{-1}), \qquad k \geq 1.$$

Utilizing Shapiro's Lemma and a standard center kills argument, Proposition 4.4 of [12] shows that

$$H_\bullet(B_2(F), H_k(K)) = 0, \qquad k \geq 1.$$

The $n = 2$ case of Theorem 5.1 follows easily. In [12], we used the action of $B_2(F)$ to kill the homology of K rather than finding a fundamental domain for the action of Γ_{12} on $\mathcal{Y}$. This approach works well in that case, but fails for $n \geq 3$ since we no longer have the free product decomposition for K.

6. The d^1-map

Having completed the computation of the E^1-term of the spectral sequence (4), we now turn our attention to the differential, d^1. Unfortunately, the computation of this map is rather difficult as it depends upon computing the maps induced on homology by the various inclusions $P_I \longrightarrow P_J$, where P_I and P_J are parabolic subgroups of $SL_n(F)$. To get a feel for the oddities which may occur, we present the following two results. Recall that for a field F, we denote by $B_2(F)$ the subgroup of $SL_2(F)$ consisting of upper triangular matrices.

PROPOSITION 6.1. (Dupont-Sah[8]) – *The natural map*

$$H_2(B_2(\mathbb{C})) \longrightarrow H_2(SL_2(\mathbb{C}))$$

is surjective. □

The following result and its proof were communicated to me by J. Yang.

PROPOSITION 6.2. – *If F is a number field, then the natural map*

$$j : H_2(B_2(F), \mathbb{Q}) \longrightarrow H_2(SL_2(F), \mathbb{Q})$$

is trivial. □

Proof. – If F is a number field, then the group $K_2(F)$ is torsion. Since the map $H_2(B_2(F), \mathbb{Z}) \to H_2(SL_2(F), \mathbb{Z})$ factors through the map $H_2(B_2(F), \mathbb{Z}) \to K_2(F)$, it follows that after tensoring with $\mathbb{Q}$, the map j is trivial. □

In light of these results, it seems to be a difficult question to compute the map

$$H_k(P_I) \longrightarrow H_k(P_J)$$

in general. Still, we are able to compute some special cases. In particular, we shall compute the maps $d^1_{*,0}$ and $d^1_{*,1}$.

6.1. The $q = 0$ case

Since the group $H_0(\Gamma_\sigma) = \mathbb{Z}$ for each simplex σ of $\mathcal{C}$, the $q = 0$ row of the spectral sequence (4) is simply the simplicial chain complex $S_\bullet(\mathcal{C})$. Since the simplex $\mathcal{C}$ is contractible, we have

$$E^2_{p,0} = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0. \end{cases}$$

6.2. The $q = 1$ case

Because we can find explicit representatives for elements of the various $H_1(\Gamma_\sigma)$, we are able to compute the map $d^1_{*,1}$. We begin by writing down the map explicitly.

Consider the group $\Gamma_{1,j_2,\ldots,j_k}$. By Theorem 5.1, we have

$$H_1(\Gamma_{1,j_2,\ldots,j_k}) \cong H_1(P_{1,j_2,\ldots,j_k}).$$

By Corollary 3.2, the group $P_{1,j_2,\ldots,j_k}$ has the same homology as its reductive part $L_{1,j_2,\ldots,j_k}$. The group $L_{1,j_2,\ldots,j_k}$ has the form

$$\begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix}$$

where each $B_i = GL_{j_{i+1}-j_i}(F)$ (*see* section 2). Now, for each i, $H_1(B_i) = F^\times$ (via the determinant map) and hence by the Künneth formula, $H_1(B_1 \times B_2 \times \cdots \times B_k) = (F^\times)^k$. It follows that

$$H_1(L_{1,j_2,\ldots,j_k}) \cong (F^\times)^{k-1},$$

via the map

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix} \mapsto (\det A_1, \det A_2, \ldots, \det A_{k-1}).$$

Since each $\Gamma_{i_1,\ldots,i_k}$ is conjugate to some $\Gamma_{1,j_2,\ldots,j_k}$, it follows that

$$H_1(\Gamma_{i_1,\ldots,i_k}) \cong (F^\times)^{k-1}.$$

Denote the simplex with vertices $i_1, i_2, \ldots, i_k$ by $\sigma_{i_1\cdots i_k}$. We now compute the map

$$H_1(\Gamma_{i_1,\ldots,i_k}) \longrightarrow H_1(\Gamma_{i_1,\ldots\widehat{i_l},\ldots,i_k})$$

induced by the face map $\sigma_{i_1\cdots i_k} \longrightarrow \sigma_{i_1\cdots\widehat{i_l}\cdots i_k}$.

LEMMA 6.3. – *Let $\sigma_{i_1\cdots i_k}$ be a $(k-1)$-simplex in $\mathcal{C}$ and suppose that $\sigma_{i_1\cdots\widehat{i_l}\cdots i_k}$ is a face of $\sigma_{i_1\cdots i_k}$. Then the map*

$$H_1(\Gamma_{i_1,\ldots,i_k}) \longrightarrow H_1(\Gamma_{i_1,\ldots\widehat{i_l},\ldots,i_k})$$

is the map

$$(F^\times)^{k-1} \longrightarrow (F^\times)^{k-2}$$

defined by

$$(\alpha_1, \ldots, \alpha_{k-1}) \mapsto \begin{cases} (\alpha_2, \alpha_3, \ldots, \alpha_{k-1}) & l = 1 \\ (\alpha_1, \ldots, \alpha_{l-1}\alpha_l, \widehat{\alpha_l}, \ldots, \alpha_{k-1}) & 2 \le l \le k-2 \\ (\alpha_1, \alpha_2, \ldots, \alpha_{k-2}) & l = k-1. \end{cases}$$

Proof. – To compute the map, we must chase elements around the following diagram: (for $2 \le l \le k-1$)

$$\begin{array}{ccccc} \Gamma_{i_1,\ldots,i_k} & \rightarrow & \Gamma_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)} & \rightarrow & L_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)} \\ \downarrow & & & & \\ \Gamma_{i_1,\ldots\widehat{i_l},\ldots,i_k} & \rightarrow & \Gamma_{1,\ldots,(\widehat{i_l-i_1}+1),\ldots,(i_k-i_1+1)} & \rightarrow & L_{1,\ldots,(\widehat{i_l-i_1}+1),\ldots,(i_k-i_1+1)} \end{array}$$

$$\begin{array}{ccc} \cdots & \rightarrow & (F^\times)^{k-1} \\ & & \downarrow \\ \cdots & \rightarrow & (F^\times)^{k-2} \end{array}$$

Consider first the case $2 \le l \le k-2$. Here the first maps are the same in each row. We follow elements around the diagram. In the first row, we have

$$
\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\
tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\
tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\
\vdots & & & \ddots & & \vdots \\
\vdots & & & & \ddots & \vdots \\
tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1}
\end{pmatrix}
$$

$$
\mapsto \begin{pmatrix}
L_2 & V_{23} & V_{24} & \cdots & \cdots & V_{2,k+1} & V_{21} \\
tV_{32} & L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} \\
tV_{42} & tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} \\
\vdots & & & \ddots & & & \vdots \\
tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\
tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\
tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1
\end{pmatrix}
$$

$$
\mapsto \begin{pmatrix}
L_2 & & & & & & 0 \\
& L_3 & & & & & \\
& & \ddots & & & & \\
& & & L_{l-1} & 0 & & \\
& & & 0 & L_l & & \\
& & & & & \ddots & \\
& & & & & L_{k+1} & V_{k+1,1} \\
0 & & & & & V_{1,k+1} & L_1
\end{pmatrix}
$$

$$
\mapsto (\det L_2, \det L_3, \ldots, \det L_k).
$$

In the second row, we have

$$
\begin{pmatrix}
L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\
tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\
tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\
\vdots & & & \ddots & & \vdots \\
\vdots & & & & \ddots & \vdots \\
tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1}
\end{pmatrix}
$$

$$
\mapsto \begin{pmatrix}
L_2 & V_{23} & V_{24} & \cdots & \cdots & V_{2,k+1} & V_{21} \\
tV_{32} & L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} \\
tV_{42} & tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} \\
\vdots & & & \ddots & & & \vdots \\
tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\
tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\
tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1
\end{pmatrix}
$$

$$\mapsto \begin{pmatrix} L_2 & & & & & & & 0 \\ & L_3 & & & & & & \\ & & \ddots & & & & & \\ & & & L_{l-1} & V_{l-1,l} & & & \\ & & & 0 & L_l & & & \\ & & & & & \ddots & & \\ & & & & & & L_{k+1} & V_{k+1,1} \\ 0 & & & & & & V_{1,k+1} & L_1 \end{pmatrix}$$

$$\mapsto (\det L_2, \dots, \det \begin{pmatrix} L_{l-1} & V_{l-1,l} \\ 0 & L_l \end{pmatrix}, \dots, \det L_k)$$
$$= (\det L_2, \dots, \det L_{l-1} \det L_l, \det L_{l+1}, \dots, \det L_k).$$

So we see that the map $(F^{\times})^{k-1} \longrightarrow (F^{\times})^{k-2}$ is given by

$$(\alpha_1, \dots, \alpha_{k-1}) \mapsto (\alpha_1, \dots, \alpha_{l-1}\alpha_l, \widehat{\alpha_l}, \dots, \alpha_{k-1}).$$

Next, consider the case $l = k-1$. Here the map in the second row is as follows:

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix}$$

$$\mapsto \begin{pmatrix} L_2 & V_{23} & V_{24} & \cdots & \cdots & V_{2,k+1} & V_{21} \\ tV_{32} & L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} \\ tV_{42} & tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} \\ \vdots & & & \ddots & & & \vdots \\ tV_{k,2} & tV_{k,3} & tV_{k,4} & \cdots & L_k & V_{k,k+1} & V_{k,1} \\ tV_{k+1,2} & tV_{k+1,3} & tV_{k+1,4} & \cdots & tV_{k+1,k} & L_{k+1} & V_{k+1,1} \\ tV_{12} & tV_{13} & tV_{14} & \cdots & tV_{1,k} & V_{1,k+1} & L_1 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} L_2 & & & & \\ & L_3 & & & \\ & & \ddots & & \\ & & & L_{k-1} & \\ & & & & \begin{pmatrix} L_k & V_{k,k+1} & V_{k,1} \\ 0 & L_{k+1} & V_{k+1,1} \\ 0 & V_{1,k+1} & L_1 \end{pmatrix} \end{pmatrix}$$

$$\mapsto (\det L_2, \dots \det L_{k-1}).$$

So, the map $(F^{\times})^{k-1} \longrightarrow (F^{\times})^{k-2}$ is simply

$$(\alpha_1, \dots, \alpha_{k-1}) \mapsto (\alpha_1, \dots, \alpha_{k-2}).$$

Finally, consider the case $l = 1$. In this case, we are omitting the first vertex i_1. Thus, we use different conjugation maps in the isomorphisms

$$\Gamma_{i_1,\ldots,i_k} \longrightarrow \Gamma_{1,(i_2-i_1+1),\ldots,(i_k-i_1+1)}$$

and

$$\Gamma_{i_2,\ldots,i_k} \longrightarrow \Gamma_{1,(i_3-i_2+1),\ldots,(i_k-i_2+1)}.$$

Now the second row of the diagram looks like

$$\begin{pmatrix} L_1 & V_{12} & V_{13} & \cdots & V_{1,k} & t^{-1}V_{1,k+1} \\ tV_{21} & L_2 & V_{23} & \cdots & V_{2,k} & V_{2,k+1} \\ tV_{31} & tV_{32} & L_3 & \cdots & V_{3,k} & V_{3,k+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ tV_{k+1,1} & tV_{k+1,2} & tV_{k+1,3} & \cdots & tV_{k+1,k} & L_{k+1} \end{pmatrix}$$

$$\mapsto \begin{pmatrix} L_3 & V_{34} & \cdots & \cdots & V_{3,k+1} & V_{31} & V_{32} \\ tV_{43} & L_4 & \cdots & \cdots & V_{4,k+1} & V_{41} & V_{42} \\ & & \ddots & & & & \\ tV_{k,3} & tV_{k,4} & & L_k & V_{k,k+1} & V_{k,1} & V_{k,2} \\ tV_{k+1,3} & tV_{k+1,4} & & & L_{k+1} & V_{k+1,1} & V_{k+1,2} \\ tV_{13} & tV_{14} & & & V_{1,k+1} & L_1 & V_{12} \\ tV_{23} & tV_{24} & & & V_{2,k+1} & V_{21} & L_2 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} L_3 & & & & \\ & L_4 & & & \\ & & \ddots & & \\ & & & L_k & \\ & & & & \begin{pmatrix} L_{k+1} & V_{k+1,1} & V_{k+1,2} \\ V_{1,k+1} & L_1 & V_{12} \\ V_{2,k+1} & V_{21} & L_2 \end{pmatrix} \end{pmatrix}$$

$$\mapsto (\det L_3, \ldots, \det L_k).$$

Hence, the map $(F^\times)^{k-1} \longrightarrow (F^\times)^{k-2}$ is given by

$$(\alpha_1, \ldots, \alpha_{k-1}) \mapsto (\alpha_2, \ldots, \alpha_{k-1}).$$

This completes the proof of Lemma 6.3. □

Denote the element $(\alpha_1, \ldots, \alpha_{k-1})$ of $H_1(\Gamma_{i_1,\ldots,i_k})$ by $\sigma_{i_1\cdots i_k} \otimes [\alpha_1, \ldots \alpha_{k-1}]$. Then the d^1-map is given by the formula

$$(12) \qquad \begin{aligned} d^1 : \sigma_{i_1\cdots i_k} \otimes [\alpha_1, \ldots, \alpha_{k-1}] \\ \mapsto \sigma_{i_2\cdots i_k} \otimes [\alpha_2, \ldots, \alpha_{k-1}] \\ + \sum_{l=2}^{k-1} (-1)^{l-1} \sigma_{i_1\cdots\widehat{i_l}\cdots i_k} \otimes [\alpha_1, \ldots, \alpha_{l-1}\alpha_l, \widehat{\alpha_l}, \ldots, \alpha_{k-1}] \\ + (-1)^{k-1} \sigma_{i_1\cdots i_{k-1}} \otimes [\alpha_1, \ldots \alpha_{k-2}]. \end{aligned}$$

Let A be an abelian group (written additively). Denote by $Q_\bullet^{(n)}$ the chain complex defined as follows. To each $(k-1)$-simplex $\sigma_{i_1\cdots i_k}$ of $\mathcal{C}$ we assign the group A^{k-1}. The boundary map $d: Q_{k-1}^{(n)} \longrightarrow Q_{k-2}^{(n)}$ is given by formula (12) above. We will compute the homology of $Q_\bullet^{(n)}$ for any abelian group A. Taking $A = F^\times$ we obtain the terms $E_{*,1}^2$ of the spectral sequence (4).

To compute the homology of the complex $Q_\bullet^{(n)}$, we realize $Q_\bullet^{(n)}$ as a quotient of another complex $C_\bullet^{(n)}$. We shall then compute $H_\bullet(C_\bullet^{(n)})$ and use this along with a long exact homology sequence to obtain $H_\bullet(Q_\bullet^{(n)})$.

Construct the chain complex $C_\bullet^{(n)}$ by assigning to each $(k-1)$-simplex $\sigma_{i_1\cdots i_k}$ of $\mathcal{C}$ the group A^k. Define the boundary map ∂ by

$$\partial : \sigma_{i_1\cdots i_k} \otimes (a_1, \ldots, a_k) \mapsto \sum_{l=1}^{k} (-1)^{l-1} \sigma_{i_1\cdots \widehat{i_l} \cdots i_k} \otimes (a_1, \ldots, \widehat{a_l}, \ldots, a_k). \tag{13}$$

Observe that for each $n \geq 2$, $C_\bullet^{(n)}$ is a subcomplex of $C_\bullet^{(n+1)}$.

Denote by $B_\bullet^{(n)}$ the standard simplicial chain complex for $\mathcal{C}$ with coefficients in A. Embed the complex $B_\bullet^{(n)}$ into $C_\bullet^{(n)}$ via

$$\sigma_{i_1\cdots i_k} \otimes a \mapsto \sigma_{i_1\cdots i_k} \otimes (a, \ldots, a).$$

Then we have the following.

LEMMA 6.4. – *The quotient complex $C_\bullet^{(n)}/B_\bullet^{(n)}$ is isomorphic to the complex $Q_\bullet^{(n)}$.*

Proof. – Denote the quotient complex by $D_\bullet^{(n)}$. In $D_\bullet^{(n)}$, we have assigned to each simplex $\sigma_{i_1\cdots i_k}$ the group $A^k/A\cdot(1,\ldots,1) \cong A^{k-1}$. We need only check that the boundary map is the same as that for $Q_\bullet^{(n)}$. We take our isomorphism $A^k/A\cdot(1,\ldots,1) \cong A^{k-1}$ to be the map

$$(a_1, \ldots, a_k) \mapsto (a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}).$$

To compute the boundary map in $D_\bullet^{(n)}$, we lift elements to $C_\bullet^{(n)}$, apply ∂, and then project back to $D_\bullet^{(n)}$. Denote the projection map $C_\bullet^{(n)} \longrightarrow D_\bullet^{(n)}$ by π. Then we have

$$\begin{aligned}\pi : \sigma_{i_1\cdots i_k} &\otimes (0, a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1}) \\ &\mapsto \sigma_{i_1\cdots i_k} \otimes [a_1, \ldots, a_{k-1}]\end{aligned}$$

and

$$\begin{aligned}\partial : \sigma_{i_1\cdots i_k} &\otimes (0, a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1}) \\ &\mapsto \sum_{l=1}^{k} (-1)^{l-1} \sigma_{i_1\cdots \widehat{i_l} \cdots i_k} \otimes (0, a_1, \ldots, \widehat{a_1 + \ldots + a_{l-1}}, \ldots, a_1 + \ldots + a_{k-1}).\end{aligned}$$

Applying π to the right hand side of this equation, we see that the boundary map in $D_\bullet^{(n)}$ is the map

$$\begin{aligned}\sigma_{i_1\cdots i_k} &\otimes [a_1,\ldots,a_{k-1}]\\ &\mapsto \sigma_{i_2\cdots i_k}\otimes[a_2,\ldots,a_{k-1}]\\ &+\sum_{l=2}^{k-1}(-1)^{l-1}\sigma_{i_1\cdots\widehat{i_l}\cdots i_k}\otimes[a_1,\ldots,a_{l-1}+a_l,\widehat{a_l},\ldots,a_{k-1}]\\ &+(-1)^{k-1}\sigma_{i_1\cdots i_{k-1}}\otimes[a_1,\ldots,a_{k-2}].\end{aligned}$$

It follows that $D_\bullet^{(n)}$ is isomorphic to $Q_\bullet^{(n)}$. □

We now have a short exact sequence of chain complexes

$$0\longrightarrow B_\bullet^{(n)}\longrightarrow C_\bullet^{(n)}\longrightarrow Q_\bullet^{(n)}\longrightarrow 0.$$

The homology of $B_\bullet^{(n)}$ is easily computed (since $\mathcal{C}$ is contractible). We now compute the homology of $C_\bullet^{(n)}$.

PROPOSITION 6.5. – *The complex $C_\bullet^{(n)}$ is contractible. Hence, $H_\bullet(C_\bullet^{(n)})=0$.*

Proof. – If n is even, we define a contracting homotopy h for $C_\bullet^{(n)}$ by

$$\begin{aligned}h:\sigma_{i_1\cdots i_k}&\otimes(a_1,\ldots,a_k)\\ \mapsto&\sum_{l=1}^{i_1-1}\sigma_{li_1\ldots i_k}\otimes(0,(-1)^{i_1+l+1}a_1,(-1)^{i_2+l+1}a_2,\ldots,(-1)^{i_k+l+1}a_k)\\ -&\sum_{l=i_1+1}^{i_2-1}\sigma_{i_1li_2\ldots i_k}\otimes((-1)^{i_1+l+1}a_1,0,(-1)^{i_2+l+1}a_2,\ldots,(-1)^{i_k+l+1}a_k)\\ +&\cdots\\ +&(-1)^k\sum_{l=i_k+1}^{n}\sigma_{i_1\ldots i_kl}\otimes((-1)^{i_1+l+1}a_1,\ldots,(-1)^{i_k+l+1}a_k,0).\end{aligned}$$

If n is odd, then $n-1$ is even. So if $\sigma_{i_1\cdots i_k}$ is a simplex in $\mathcal{C}$ with $i_k<n$, then we may view $\sigma_{i_1\cdots i_k}\otimes(a_1,\ldots,a_k)$ as belonging to the subcomplex $C_\bullet^{(n-1)}$. Thus, we may use the formula above. We extend h to simplices with $i_k=n$ as follows. If $i_{k-1}<n-1$, then we define h to be

$$\begin{aligned}h:\sigma_{i_1\ldots i_{k-1}n}&\otimes(a_1,\ldots,a_k)\\ \mapsto&\sum_{l=1}^{i_1-1}\sigma_{li_1\ldots i_{k-1}n}\otimes(0,(-1)^{i_1+l+1}a_1,\ldots,(-1)^{n+l+1}a_k)\\ -&\sum_{l=i_1+1}^{i_2-1}\sigma_{i_1li_2\ldots i_{k-1}n}\otimes((-1)^{i_1+l+1}a_1,0,(-1)^{i_2+l+1}a_2,\ldots,(-1)^{n+l+1}a_k)\\ +&\cdots\end{aligned}$$

$$+(-1)^{k-1}\sum_{l=i_{k-1}+1}^{n-1}\sigma_{i_1\dots i_{k-1}ln}\otimes((-1)^{i_1+l+1}a_1,\dots,0,(-1)^{n+l+1}a_k)$$
$$-\sum_{l=1}^{i_1-1}\sigma_{li_1\dots i_{k-1}n}\otimes(0,\dots,0,(-1)^l a_k)$$
$$+\sum_{l=i_1+1}^{i_2-1}\sigma_{i_1li_2\dots i_{k-1}n}\otimes(0,\dots,0,(-1)^l a_k)$$
$$+\cdots$$
$$+(-1)^k\sum_{l=i_{k-1}+1}^{n-2}\sigma_{i_1\dots i_{k-1}ln}\otimes(0,\dots,0,(-1)^l a_k).$$

If $i_{k-1} = n-1$, then

$$h:\sigma_{i_1\dots i_{k-2},n-1,n}\otimes(a_1,\dots,a_k)$$
$$\mapsto\sum_{l=1}^{i_1-1}\sigma_{li_1\dots i_{k-2},n-1,n}\otimes(0,(-1)^{i_1+l+1}a_1,\dots,(-1)^{n+l+l}a_k)$$
$$-\sum_{i_1+1}^{i_2-1}\sigma_{i_1li_2\dots i_{k-2},n-1,n}\otimes((-1)^{i_1+l+1}a_1,0,\dots,(-1)^{n+l+1}a_k)$$
$$+\cdots$$
$$+(-1)^{k-2}\sum_{i_{k-2}+1}^{n-2}\sigma_{i_1\dots i_{k-2},n-1,n}$$
$$\otimes((-1)^{i_1+l+1}a_1,\dots,0,(-1)^{(n-1)+l+1}a_{k-1},(-1)^{n+l+1}a_k)$$
$$-\sum_{l=1}^{i_1-1}\sigma_{li_1\dots i_{k-2},n-1,n}\otimes(0,\dots,0,(-1)^l a_k)$$
$$+\sum_{l=i_1+1}^{i_2-1}\sigma_{i_1li_2\dots i_{k-2},n-1,n}\otimes(0,\dots,0,(-1)^l a_k)$$
$$+\cdots$$
$$+(-1)^{k-1}\sum_{l=i_{k-2}+1}^{n-2}\sigma_{i_1\dots i_{k-2}l,n-1,n}\otimes(0,\dots,0,(-1)^l a_k).$$

One checks that $\partial h + h\partial =$ identity. This completes the proof of the proposition. □

COROLLARY 6.6. – *The homology of the complex $Q_\bullet^{(n)}$ is given by*

$$H_k(Q_\bullet^{(n)}) = \begin{cases} A & k=1 \\ 0 & k\neq 1. \end{cases}$$

Proof. – Since $C_\bullet^{(n)}$ is contractible, the long exact homology sequence implies that

$$H_k(Q_\bullet^{(n)}) \cong H_{k-1}(B_\bullet^{(n)}).$$

The result follows since

$$H_k(B_\bullet^{(n)}) = \begin{cases} A & k=0 \\ 0 & k \neq 0. \end{cases} \qquad \square$$

Taking $A = F^\times$, we obtain the following.

COROLLARY 6.7. – *The spectral sequence* (4) *satisfies*

$$E^2_{p,1} = \begin{cases} F^\times & p=1 \\ 0 & p \neq 1. \end{cases} \qquad \square$$

6.3. The second homology and cohomology groups

COROLLARY 6.8. – *There is an exact sequence*

$$0 \longrightarrow \operatorname{coker}\{d^1_{1,2} : E^1_{1,2} \to E^1_{0,2}\} \longrightarrow H_2(SL_n(F[t,t^{-1}])) \longrightarrow F^\times \longrightarrow 1.$$

Proof. – Since $E^2_{p,0} = E^2_{p,1} = 0$ for $p > 1$, we have $E^2_{0,2} = E^\infty_{0,2}$. The group $E^2_{0,2}$ is precisely the cokernel of $d^1 : E^1_{1,2} \longrightarrow E^1_{0,2}$. Since $E^2_{1,1} = F^\times$, the result follows. $\square$

COROLLARY 6.9. – *Let F be a number field and denote the number of real embeddings of F by r_1. Then*

$$H_2(SL_2(F[t,t^{-1}]),\mathbb{Q}) \cong (F^\times \otimes \mathbb{Q}) \oplus \mathbb{Q}^{2r_1}.$$

Proof. – By Borel-Yang [3], we have

$$H_2(SL_2(F),\mathbb{Q}) = \mathbb{Q}^{r_1}.$$

It follows that $E^1_{0,2} = \mathbb{Q}^{2r_1}$. By Proposition 6.2, the map $d^1 : E^1_{1,2} \to E^1_{0,2}$ is trivial. Hence, we have an exact sequence

$$0 \longrightarrow \mathbb{Q}^{2r_1} \longrightarrow H_2(SL_2(F[t,t^{-1}]),\mathbb{Q}) \longrightarrow F^\times \otimes \mathbb{Q} \longrightarrow 0. \qquad \square$$

We now investigate the map $d^1_{1,2}$.

PROPOSITION 6.10. – *If $n \geq 3$, then the cokernel of the map $d^1_{1,2} : E^1_{1,2} \longrightarrow E^1_{0,2}$ is isomorphic to $H_2(SL_n(F),\mathbb{Z})$.*

Proof. – The term $E^1_{0,2}$ is equal to

$$\bigoplus_{i=1}^{n} H_2(\Gamma_i).$$

Since each Γ_1 is conjugate to $SL_n(F[t])$ in $GL_n(F[t,t^{-1}])$, by Theorem 3.4 we have

$$E^1_{0,2} \cong H_2(SL_n(F),\mathbb{Z})^{\oplus n}.$$

Consider the map

$$p : H_2(SL_n(F), \mathbb{Z})^{\oplus n} \longrightarrow H_2(SL_n(F), \mathbb{Z})$$

defined by

$$p(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i.$$

The map p is surjective with kernel consisting of those elements of

$$H_2(SL_n(F), \mathbb{Z})^{\oplus n}$$

whose entries sum to zero. We show that the image of $d^1_{1,2}$ coincides with the kernel of p. Given a pair of integers i, j with $1 \leq i < j \leq n$, we have maps

$$H_2(\Gamma_{ij}) \longrightarrow H_2(\Gamma_i) \qquad \text{and} \qquad H_2(\Gamma_{ij}) \longrightarrow H_2(\Gamma_j)$$

induced by inclusion. The map $d^1_{1,2}$ is the alternating sum of these maps. To compute the image of $d^1_{1,2}$ as a subgroup of $H_2(SL_n(F), \mathbb{Z})^{\oplus n}$, we make use of the diagrams

$$\begin{array}{ccccc} H_2(\Gamma_{ij}) & \xrightarrow{\cong} & H_2(\Gamma_{1,j-i+1}) & \longrightarrow & H_2(\Gamma_1) \\ & & & \searrow & \uparrow \cong \\ & & & & H_2(\Gamma_{j-i+1}) \end{array}$$

to see that the image of $H_2(\Gamma_{ij})$ in $H_2(\Gamma_i)$ is isomorphic (via the identifications $\Gamma_i \cong \Gamma_1$) to the image of $H_2(\Gamma_{ij})$ in $H_2(\Gamma_j)$. Since $d^1_{1,2}$ maps $H_2(\Gamma_{ij})$ to $H_2(\Gamma_i)$ with a negative sign and to $H_2(\Gamma_j)$ with a positive sign, we see that the image of $d^1_{1,2}$ in $H_2(SL_n(F), \mathbb{Z})^{\oplus n}$ lies in the kernel of p.

To see that the image is all of the kernel, we use a result of Hutchinson [10, p. 200] which states that if F is an infinite field, then the map

$$H_2(\Gamma_{12}) \longrightarrow H_2(\Gamma_1)$$

is surjective for $n \geq 3$. It follows that the maps

$$H_2(\Gamma_{i,i+1}) \longrightarrow H_2(\Gamma_i) \qquad \text{and} \qquad H_2(\Gamma_{i,i+1}) \longrightarrow H_2(\Gamma_{i+1})$$

are surjective for $i = 1, \ldots, n-1$. Thus, the image of $d^1_{1,2}$ contains all elements of the form

$$(-a, a, 0, \ldots, 0),\ (0, -a, a, 0, \ldots, 0), \ldots,\ (0, \ldots, 0, -a, a)$$

and it follows that the image of $d^1_{1,2}$ coincides with the kernel of p. □

COROLLARY 6.11. – *If F is an infinite field, then for $n \geq 3$,*

$$H_2(SL_n(F[t, t^{-1}]), \mathbb{Z}) = H_2(SL_n(F), \mathbb{Z}) \oplus F^{\times}.$$

Proof. – The spectral sequence (4) gives an exact sequence

$$0 \longrightarrow H_2(SL_n(F), \mathbb{Z}) \xrightarrow{\phi} H_2(SL_n(F[t,t^{-1}]), \mathbb{Z}) \longrightarrow F^\times \longrightarrow 0.$$

Observe that the map $p : E^1_{1,2} \longrightarrow E^1_{0,2}$ is split by inclusion onto the first factor. It follows that the map ϕ is induced by the canonical inclusion $SL_n(F) \longrightarrow SL_n(F[t,t^{-1}])$. Observe that this map is split by the map

$$SL_n(F[t,t^{-1}]) \xrightarrow{t=1} SL_n(F).$$

It follows that $H_2(SL_n(F), \mathbb{Z})$ is a direct summand of $H_2(SL_n(F[t,t^{-1}]), \mathbb{Z})$. This proves the corollary. □

Remark. – Since $K_2(F[t,t^{-1}]) = K_2(F) \oplus K_1(F)$ and since

$$K_2(F) = H_2(SL_n(F), \mathbb{Z}) \qquad n \geq 3,$$

Corollary 6.11 implies that $H_2(SL_n(F[t,t^{-1}]), \mathbb{Z})$ stabilizes at $n = 3$; *i.e.*, for $n \geq 3$ we have an isomorphism

$$H_2(SL_n(F[t,t^{-1}]), \mathbb{Z}) \cong K_2(F[t,t^{-1}]).$$

COROLLARY 6.12. – *If* $n \geq 3$, *then*

$$H^2(SL_n(F[t,t^{-1}]), \mathbb{Z}) \cong H^2(SL_n(F), \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(F^\times, \mathbb{Z}).$$

Proof. – By the Universal Coefficient Theorem,

$$\begin{aligned} H^2(SL_n(F[t,t^{-1}]), \mathbb{Z}) &\cong \mathrm{Hom}_{\mathbb{Z}}(H_2(SL_n(F[t,t^{-1}]), \mathbb{Z}), \mathbb{Z}) \\ &\quad \oplus \mathrm{Ext}_{\mathbb{Z}}(H_1(SL_n(F[t,t^{-1}]), \mathbb{Z}), \mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(H_2(SL_n(F), \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(F^\times, \mathbb{Z}) \oplus 0 \\ &\cong H^2(SL_n(F), \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(F^\times, \mathbb{Z}). \end{aligned}$$ □

ACKNOWLEDGEMENTS

It is a pleasure to thank those who provided valuable insight during the preparation of this paper. In particular, I wish to thank Howard Garland, John Harer, Jun Yang, and Andrei Suslin for many helpful conversations. I am especially indebted to my advisor, Richard Hain, for his guidance and support during my graduate career.

This paper is dedicated to Ellen Wall.

REFERENCES

[1] A. Borel, *Stable real cohomology of arithmetic groups* (*Ann. Scient. Éc. Norm. Sup.* (4), Vol. 7, 1974, pp. 235-272).

[2] A. Borel, *Stable real cohomology of arithmetic groups II* (*Prog. Math.*, Vol. 14, 1981, pp. 21-55, Birkhäuser-Boston).

[3] A. Borel and J. Yang, *The rank conjecture for number fields* (*Mathematical Research Letters*, Vol. 1, 1994, pp. 689-699).

[4] K. Brown, *Buildings*, Springer-Verlag, Berlin, Heidelberg, New York, 1989.

[5] K. Brown, *Cohomology of Groups*, Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[6] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local I: Données radicielles valuées* (*Publ. IHES*, Vol. 41, 1972, pp. 5-252).

[7] R. Charney, *Homology stability for GL_n of a Dedekind domain* (*Invent. Math.*, Vol. 56, 1980, pp. 1-17).

[8] J. Dupont and H. Sah, *Scissors congruences II* (*J. Pure and Appl. Algebra*, Vol. 25, 1982, pp. 159-195).

[9] J. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, Berlin, Heidelberg, New York, 1981.

[10] K. Hutchinson, *A new approach to Matsumoto's Theorem* (*K-Theory*, Vol. 4, 1990, pp. 181-200).

[11] W. van der Kallen, *Homology stability for linear groups* (*Invent. Math.*, Vol. 60, 1980, pp. 269-295).

[12] K. Knudson, *The homology of $SL_2(F[t,t^{-1}])$* (*J. Alg.*, Vol. 180, 1996, pp. 87-101).

[13] H. Nagao, *On $GL(2, K[x])$* (*J. Poly. Osaka Univ.*, Vol. 10, 1959, pp. 117-121).

[14] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field* (*Ann. Math.*, Vol. 96, 1972, pp. 552-586).

[15] D. Quillen, *Higher algebraic K-theory: I*, in K-Theory, H. Bass, ed., Springer Lecture Notes in Mathematics, Vol. 341, 1976, pp. 85-147.

[16] J-P. Serre, *Trees*, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[17] C. Soulé, *Chevalley groups over polynomial rings*, in Homological Group Theory (C.T.C. Wall, ed.) (*London Math. Soc. Lecture Notes*, Vol. 36, Cambridge University Press, Cambridge, 1979, pp. 359-367).

[18] C. Soulé, *The cohomology of $SL_3(\mathbb{Z})$* (*Topology*, Vol. 17, 1978, pp. 1-22).

[19] A. Suslin, *Homology of GL_n, characteristic classes and Milnor K-theory* (*Springer Lecture Notes in Mathematics*, Vol. 1046, 1984, pp. 357-375).

[20] A. Suslin, *K_3 of a field and Bloch's group* (*Proceedings of the Steklov Institute of Mathematics*, 1991, Issue 4).

[21] A. Suslin, *On the structure of the special linear group over polynomial rings* (*Math. USSR Izvestija*, Vol. 11, 1977, pp. 221-238).

(Manuscript received April 4, 1996.)

K. P. Knudson
Department of Mathematics, Duke University,
Durham, NC 27708-0320.
Department of Mathematics,
Northwestern University,
Evanston, IL 60208.
E-mail: knudson@math.nwu.edu