Ann. Scient. Éc. Norm. Sup.,
4^{e} série, t. 35, 2002, p. 759 à 772.

THE HARD LEFSCHETZ THEOREM AND THE TOPOLOGY OF SEMISMALL MAPS

BY MARK ANDREA A. DE CATALDO [1] AND LUCA MIGLIORINI [2]

Dedicato a Meeyoung

ABSTRACT. – A line bundle on a complex projective manifold is said to be *lef* if one of its powers is globally generated and defines a semismall map in the sense of Goresky–MacPherson. As in the case of ample bundles the first Chern class of lef line bundles satisfies the Hard Lefschetz Theorem and the Hodge–Riemann Bilinear Relations. As a consequence, we prove a generalization of the Grauert contractibility criterion: the *Hodge Index Theorem for semismall maps*, Theorem 2.4.1. For these maps the Decomposition Theorem of Beilinson, Bernstein and Deligne is equivalent to the non-degeneracy of certain intersection forms associated with a stratification. This observation, joint with the Hodge Index Theorem for semismall maps gives a new proof of the Decomposition Theorem for the direct image of the constant sheaf. A new feature uncovered by our proof is that the intersection forms involved are definite.

RÉSUMÉ. – On dit qu'un fibré en droites sur une variété projective est *lef* si l'une de ses puissances tensorielles est engendrée par ses sections globales et définit un morphisme "semismall" dans le sens de Goreski–MacPherson. On prouve que, comme dans le cas des fibrés amples, la première classe de Chern des fibrés *lef* satisfait le théorème de Lefschetz difficile et les relations bilinéaires de Hodge–Riemann. Comme conséquence, on démontre une généralisation du critère de contractibilité de Grauert, le *théorème de l'indice de Hodge pour les morphismes "semismall"* (Theorem 2.4.1). Pour ces morphismes, le théorème de décomposition de Beilinson, Bernstein et Deligne équivaut à la non dégénérescence de certaines formes d'intersection associées à une stratification ; en combinant cette observation avec le théorème de l'indice pour les morphismes "semismall", on en déduit une nouvelle preuve du théorème de décomposition pour l'image directe du faisceau constant, avec l'information supplémentaire que les formes d'intersection en question sont définies.

1. Introduction

After Goresky and MacPherson introduced intersection cohomology complexes, Gelfand and MacPherson conjectured that, given a proper algebraic map of complex algebraic varieties, the direct image of the intersection cohomology complex of the domain splits as a direct sum of shifted intersection cohomology complexes of local systems on the image. This conjecture was

[1] Partially supported by N.S.F. Grant DMS 9701779.
[2] Member of GNSAGA, supported by MURST funds.

proved in even greater generality by Beilinson, Bernstein, Deligne and Gabber in [1] and is known as the Decomposition Theorem.

The Decomposition Theorem is the deepest and most encompassing result concerning the homology of algebraic maps, and has been widely used in problems of geometry and representation theory. It implies in particular the invariant cycle theorems, the semisimplicity of monodromy, the degeneration of the Leray spectral sequence for smooth maps and is a powerful tool to compute intersection cohomology.

The proof given in [1] is of arithmetic character; it proceeds by reduction to positive characteristic and relies on the theory of weights of the action of the Frobenius automorphism on l-adic sheaves. The interest in giving a different proof rests therefore on the possibility of shedding some light on the geometric phenomena underlying the homological statement.

In this paper we make a step in this direction by giving a new proof, relying on Hodge theory, of the Decomposition Theorem for the direct image of the constant sheaf by a projective semismall map from a nonsingular projective variety (Theorem 3.4.1).

We show that this follows quite directly from a stronger statement which is a generalization of the Grauert–Mumford criterion for the contractibility of configurations of curves on a surface. More precisely, we consider, for every relevant stratum of the map, a bilinear form given by the intersection pairing defined on the components of the general fibre over the stratum and we show that the nondegeneracy of this form is precisely the condition to extend the decomposition across the stratum under consideration (Theorem 3.3.3).

We then show that these forms come from polarizations of Hodge structures and are therefore not only nondegenerate, but also definite.

Along the way, we prove a statement which, despite its simplicity, is of independent interest: we characterize pull-backs of ample line bundles by semismall maps as precisely those line bundles which are semiample, i.e. a power is generated by its global sections, and which satisfy the conclusion of the Hard Lefschetz Theorem (Proposition 2.2.7, Theorem 2.3.1).

We call these line bundles *lef* and prove that the spaces of primitive cohomology classes with respect to lef line bundles are polarized pure Hodge structures (Theorem 2.3.1).

This fact, coupled with the linear independence of the cohomology classes of the half dimensional fibres of semismall maps, which we prove using the theory of mixed Hodge structures, gives what we call *the Hodge index theorem for semismall maps* (Theorem 2.4.1).

This latter implies the definiteness of the bilinear forms mentioned above and Theorem 3.4.1 follows.

The case of semismall maps is natural in this context, for the semismall condition implies that the direct image of the constant sheaf is perverse, and that the splitting asserted by the Decomposition Theorem is canonical.

In addition there are many examples of semismall maps of great relevance and their geometry seemed sufficiently complex to make them, in our eyes, a significant example to work on in connection with the question of giving a topological proof of the Decomposition Theorem.

We believe that the relation between the Decomposition Theorem and the intersection forms associated with the strata is illuminating. Furthermore, the direct proof of the non-degeneracy of these forms, with the new additional information on the signatures, neither relying on reduction to positive characteristic nor on Saito's theory of mixed Hodge modules, sheds light on the geometry underlying the decomposition theorem and gives some indications on its possible extensions beyond the algebraic category.

2. The Hard Lefschetz Theorem for lef line bundles

In this section we introduce the notion of *lef* line bundle on a projective variety. It is a positivity notion weaker than ampleness but stronger than semiampleness and bigness combined. Lef line bundles satisfy many of the cohomological properties of ampleness. We prove that they also satisfy the Hard Lefschetz Theorem, the Lefschetz Decomposition and the Hodge Riemann Bilinear relations on the primitive spaces. These results are all false, in general, for line bundles which are simultaneously generated by their global sections and big. Finally, we prove our Hodge Index Theorem for semismall maps.

2.1. Semismall maps and lef line bundles

Let $f:X\to Y$ be a proper holomorphic map. For every integer k define

$$Y^k := \{y\in Y \,|\, \dim f^{-1}(y)=k\}.$$

The spaces Y^k are locally closed analytic subvarieties of Y whose disjoint union is Y. If a fiber is reducible, then it is understood that its dimension is the highest among the dimensions of its components.

DEFINITION 2.1.1. – We say that a proper holomorphic map $f:X\to Y$ of irreducible varieties is *semismall* if $\dim Y^k + 2k \leqslant \dim X$ for every k. Equivalently, f is semismall if and only if there is no irreducible subvariety $T\subseteq X$ such that $2\dim T - \dim f(T) > \dim X$.

Remark 2.1.2. – A semismall map is necessarily generically finite.

From now on we shall assume that semismall maps are proper and surjective.

DEFINITION 2.1.3. – We say that a line bundle M on a complex projective variety X is *lef* if a positive multiple of M is generated by its global sections and the corresponding morphism onto the image is semismall.

Remark 2.1.4. – If the map associated to a multiple of M generated by its global sections is semismall, then the map associated with any other multiple of M generated by its global sections is semismall as well. A lef line bundle is nef and big, but not conversely.

PROPOSITION 2.1.5 (Weak Lefschetz Theorem for lef line bundles). – *Let M be a lef line bundle on a smooth complex projective variety X. Assume that M admits a section $s\in H^0(X,M)$ whose reduced zero locus is a smooth divisor Y. Denote by $i:Y\to X$ the inclusion.*

The restriction map $i^*:H^r(X)\longrightarrow H^r(Y)$ is an isomorphism for $r<\dim X-1$ and it is injective for $r=\dim X-1$.

Proof. – The proof can be obtained by a use of the Leray spectral sequence coupled with the theorem on the cohomological dimension of constructible sheaves on affine varieties. See, for example, [17]. See also [10], Lemma 1.2. □

Remark 2.1.6. – The Weak Lefschetz Theorem has been considerably strengthened by Goresky and MacPherson [13], II.1.1.

PROPOSITION 2.1.7 (Bertini Theorem for lef line bundles). – *Let M be a lef line bundle on a nonsingular complex projective variety X. Assume that M is generated by its global sections. Let $W'\subseteq |M|$ be the set of divisors Y in the linear system of M such that Y is smooth and $M_{|Y}$ is lef. Then the set W' contains a nonempty and Zariski open subset $W\subseteq |M|$.*

Proof. – The standard Bertini Theorem implies that a generic divisor $D \in |M|$ is nonsingular. Let $f: X \to Y$ be the semismall map associated with $|M|$ and Y^k be the locally closed subvarieties mentioned above. The set of divisors containing at least one among the closed subvarieties $f^{-1}(\overline{Y^k})$ is a finite union of linear proper subspaces of $|M|$. The conclusion follows. □

2.2. The property HL

Let X be a smooth, compact, oriented manifold of even real dimension $2n$. We will use the notation $H^r(X)$ for $H^r(X,\mathbb{Q})$. The bilinear form on $H^*(X) := \bigoplus H^r(X)$ defined by $(\alpha,\beta) = \int_X \alpha \wedge \beta$ is non-degenerate by Poincaré duality.

Let $\omega \in H^2(X)$. We define a bilinear form on $H^{n-r}(X)$ by setting

$$\Psi(\alpha,\beta) = (-1)^{\frac{(n-r)(n-r-1)}{2}} \int_X \omega^r \wedge \alpha \wedge \beta,$$

for every $0 \leqslant r \leqslant n$. The form Ψ is non-degenerate precisely when the linear map $L^r = L^r_\omega : H^{n-r}(X) \to H^{n+r}(X)$, sending α to $\omega^r \wedge \alpha$, is an isomorphism.

DEFINITION 2.2.1. – We say that (X,ω) *has property* HL_r if the map

$$L^r_\omega : H^{n-r}(X) \to H^{n+r}(X)$$

given by the cup product with ω^r is an isomorphism.

We say that (X,ω) has property HL if it has property HL_r for every $0 \leqslant r \leqslant n$.

Note that property HL_0 is automatic and that property HL_n is equivalent to $\int_X \omega^n \neq 0$.

Define $H^{n-r}(X) \supseteq P^{n-r} = P^{n-r}_\omega := Ker L^{r+1}_\omega$ and call its elements *primitive* (with respect to ω). The following Lefschetz-type decomposition is immediate.

PROPOSITION 2.2.2. – *Assume that* (X,ω) *has property HL. For every* $0 \leqslant r \leqslant n$*, we have the following "primitive" decomposition*

$$H^{n-r}(X) = P^{n-r} \oplus L_\omega\big(H^{n-r-2}(X)\big).$$

There is a direct sum decomposition

$$H^{n-r}(X) = \bigoplus L^i_\omega P^{n-r-2i}.$$

The subspaces $L^i_\omega P^{n-r-2i}$ *are pairwise orthogonal in* $H^{n-r}(X)$.

Remark 2.2.3. – The projection of $H^{n-r}(X)$ onto P^{n-r} is given by

$$\alpha \to \alpha - L_\omega\big(L^{r+2}_\omega\big)^{-1} L^{r+1}_\omega \alpha,$$

where $(L^{r+2}_\omega)^{-1}$ denotes the inverse to $(L^{r+2}_\omega) : H^{n-r-2}(X) \to H^{n+r+2}(X)$.

DEFINITION 2.2.4. – A *rational Hodge structure of pure weight* r is a rational vector space H with a bigraduation of $H_\mathbb{C} = H \otimes \mathbb{C} = \bigoplus H^{p,q}$ for $p+q=r$ such that $H^{p,q} = \overline{H^{q,p}}$.

DEFINITION 2.2.5. – A *polarization* of the weight r Hodge structure H is a bilinear form Ψ on H, symmetric for r even, anti-symmetric for r odd, such that its $\mathbb{C}$-bilinear extension, to $H_\mathbb{C}$, still denoted by Ψ, satisfies:

(a) the spaces $H^{p,q}$ and $H^{s,t}$ are Ψ-orthogonal whenever either $p \neq t$, or $q \neq s$;
(b) $i^{p-q}\Psi(\alpha,\overline{\alpha}) > 0$, for every non-zero $\alpha \in H^{p,q}$.

Let X be a nonsingular complex projective variety. For any ample line bundle M define $L_M := L_{c_1(M)} : H^{n-r}(X) \to H^{n-r+2}(X)$. Classical Hodge theory gives that

$$\Psi_M(\alpha,\beta) = (-1)^{\frac{(n-r)(n-r-1)}{2}} \int_X L_M^r(\alpha) \wedge \beta$$

is a polarization of the weight $(n-r)$ Hodge structure $P_M^{n-r} = Ker L_M^{r+1} \subseteq H^{n-r}(X)$.

We say that (X,M) has property HL_r (HL, resp.) if $(X, c_1(M))$ has property HL_r (HL, resp.).

The HL property for a pair (X,M) with X projective and M nef implies that Ψ_M is a polarization. In fact, the first Chern class of such a line bundle can be written as a limit of rational Kähler classes and the following proposition applies.

PROPOSITION 2.2.6. – *Let X be a compact connected complex Kähler manifold of dimension n and M be a line bundle such that (X,M) has property HL and $c_1(M) = \lim_{i\to\infty}\omega_i$, ω_i Kähler. The bilinear form $\Psi_M(\alpha,\beta) = (-1)^{\frac{(n-r)(n-r-1)}{2}} \int_X L_M^r(\alpha)\wedge\beta$ is a polarization of the weight $(n-r)$ Hodge structure $P_M^{n-r} = Ker L_M^{r+1} \subseteq H^{n-r}(X)$, for every $0 \leqslant r \leqslant n$.*

Proof. – The only thing that needs to be proved is the statement $i^{p-q}\Psi_M(\alpha,\overline{\alpha}) > 0$ for every non-zero $\alpha \in P_M^{n-r} \cap H^{p,q}(X)$.

Since the classes ω_i are Kähler, we have the decomposition

$$H^{n-r}(X) = P_{\omega_i}^{n-r} \oplus L_{\omega_i}\big(H^{n-r-2}(X)\big).$$

Let π_i denote the projection onto $P_{\omega_i}^{n-r}$,

$$\pi_i(\alpha) = \alpha - L_{\omega_i}\big(L_{\omega_i}^{r+2}\big)^{-1} L_{\omega_i}^{r+1}\alpha.$$

Since M satisfies the HL condition, the map $L_M^{r+2} : H^{n-r-2}(X) \to H^{n+r+2}(X)$ is invertible so that $\lim_{i\to\infty}(L_{\omega_i}^{r+2})^{-1} = (L_M^{r+2})^{-1}$. Identical considerations hold for the (p,q)-parts of these invertible maps. It follows that, if $\alpha \in P_M^{n-r} \cap H^{p,q}(X)$, then $\lim_{i\to\infty}\pi_i(\alpha) = \alpha$. Since the operators π_i are of type $(0,0)$, $\pi_i(\alpha) \in P_{\omega_i}^{n-r} \cap H^{p,q}(X)$. Therefore,

$$i^{p-q}\Psi_{\omega_i}\big(\pi_i(\alpha), \overline{\pi_i(\alpha)}\big) > 0.$$

It follows that $i^{p-q}\Psi_M(\alpha,\overline{\alpha}) \geqslant 0$. The HL property for M implies that Ψ_M is non-degenerate, therefore Ψ_M is a polarization of P_M^{n-r}. □

The following elementary fact highlights the connection between the HL property and lef line bundles. See [19] for related considerations.

PROPOSITION 2.2.7. – *Let $f : X \to Y$ be a surjective projective morphism from a nonsingular projective variety X, A be a line bundle on Y and $M := f^*A$.*

If M has property HL, then f is semismall.

Proof. – If f is not semismall, then there exists an irreducible subvariety $T \subseteq X$ such that $2\dim T - n > \dim f(T)$. Let $[T] \in H^{2(n-\dim T)}(X) = H^{n-(2\dim T-n)}(X)$ be the fundamental class of T. The class $c_1(M)^{2\dim T-n}$ can be represented by a $\mathbb{Q}$-algebraic cycle that does not intersect T. It follows that $c_1(M)^{(2\dim T-n)} \cdot [T] = 0$, i.e. M does not satisfy $HL_{2\dim T-n}$. □

2.3. The Hard Lefschetz Theorem and the signature of intersection forms

Our goal is to prove the following extension of the classical Hard Lefschetz Theorem which also constitutes a converse to Proposition 2.2.7. At the same time we prove that the Hodge–Riemann Bilinear Relations hold on the corresponding primitive spaces.

THEOREM 2.3.1. – *Let X be a nonsingular complex projective variety and M be a lef line bundle on X.*

The pair (X,M) has property HL. In addition, Ψ_M is a polarization of $P_M^{n-r} = Ker L_M^{r+1}$.

Remark 2.3.2. – Proposition 2.2.2 implies the decomposition of the singular cohomology of X into subspaces which are primitive with respect to M. It is immediate to check that $\dim_{\mathbb{C}} P_M^l = b_l - b_{l-2}$ and $\dim_{\mathbb{C}} P_M^{p+q} \cap H^{p,q}(X) = h^{p,q}(X) - h^{p-1,q-1}(X)$.

The proof of 2.3.1 is by induction on $\dim X$. The case $n=1$ is classical, for M is then necessarily ample. The statement is invariant under taking non-zero positive powers of the line bundle M. Withouth loss of generality we assume that $\dim X \geqslant 2$ and that M is generated by its global sections.

Let $s \in H^0(X,M)$ be a section with smooth zero locus Y such that $M_{|Y}$ is lef. Such a section exists by Proposition 2.1.7. Note that Y is necessarily connected by Bertini Theorem. Denote by $i : Y \to X$ the inclusion.

Let $\hat{L}_M^r := (L_{M_{|Y}})^r : H^{n-1-r}(Y) \to H^{n-1+r}(Y)$. The projection formula, coupled with Poincaré Duality, implies that $L_M^r = i_* \circ \hat{L}_M^{r-1} \circ i^*$.

LEMMA 2.3.3. – *If $(Y, M_{|Y})$ has property HL, then (X,M) has properties HL_r for $r=0$ and $2 \leqslant r \leqslant n$.*

If $(Y, M_{|Y})$ has property HL, then (X,M) has property HL if and only if the restriction of the intersection form on $H^{n-1}(Y)$ to the subspace $i^*H^{n-1}(X)$ is non-degenerate.

Proof. – Standard; see [16]. □

Proof of Theorem 2.3.1. – Assume that Theorem 2.3.1 holds in dimension $\dim X - 1$. By Lemma 2.3.3 it is enough to show the statement of non-degeneracy on $i^*H^{n-1}(X)$. Consider $i_* : H^{n-1}(Y) \to H^{n+1}(X)$. We have $(i^*H^{n-1}(X))^{\perp} = Ker\, i_* \subseteq P^{n-1}_{M_{|Y}}(Y)$. By induction this last space is polarized by the intersection form. In particular, the intersection form is non-degenerate on $(i^*H^{n-1}(X))^{\perp} = Ker\, i_*$ so that it is non-degenerate on $i^*H^{n-1}(X)$. It follows that (X,M) has property HL. We conclude by Proposition 2.2.6. □

2.4. The Hodge Index Theorem for semismall maps

Let us record the following consequence of Theorem 2.3.1. Coupled with Corollary 2.4.2, it is a higher dimensional analogue of Grauert–Mumford contractibility test for curves on surfaces.

THEOREM 2.4.1 (Hodge Index Theorem for semismall maps). – *Let $f : X \to Y$ be a semismall map from a nonsingular complex projective variety of even dimension n onto a projective variety Y and $y \in Y$ be a point such that $\dim f^{-1}(y) = \frac{n}{2}$. Denote by Z_l, $1 \leqslant l \leqslant r$, the irreducible components of maximal dimension of $f^{-1}(y)$.*

Then the cohomology classes $[Z_l] \in H^n(X)$ are linearly independent and the symmetric matrix $(-1)^{\frac{n}{2}} ||Z_l \cdot Z_m||$ is positive definite.

Proof. – Let $M = f^*A$ be a lef line bundle, pull-back of an ample line bundle A on Y. The image of the cycle class map $cl : H_n^{BM}(f^{-1}(y)) \to H^n(X)$ belongs to the primitive space $P_M^n(X)$ which is polarized by the intersection form by virtue of Theorem 2.3.1.

We are left with showing that the map cl is injective.

Because of Poincaré Duality, the map cl is the transposed of the natural restriction map $r : H^n(X) \longrightarrow H^n(f^{-1}(y))$. We now prove that r is surjective. Let Y^0 be an affine neighborhood of $y \in Y$ such that $Supp(R^n f_* \mathbb{Q}_X)_{Y^0} = \{y\}$. The theorem on the cohomological dimension of constructible sheaves on affine varieties, coupled with the assumption "f is semismall" gives that $H^p(Y^0, (R^q f_* \mathbb{Q}_X)_{|Y^0}) = \{0\}$, for every $p+q>n$; see [10], Lemma 1.2 for example. The Leray Spectral sequence gives the surjection:

$$H^n\bigl(f^{-1}(Y^0)\bigr) \to \bigl(R^n f_* \mathbb{Q}_X\bigr)_y \simeq H^n\bigl(f^{-1}(y)\bigr).$$

By [8], Proposition 8.2.6, we conclude that the map r is surjective as well. □

COROLLARY 2.4.2. – *Let $f : X \to Y$ be a birational semismall map from a nonsingular quasi projective complex variety of even dimension n onto a quasi projective complex variety with isolated singularities $y_1, \ldots, y_l \in Y$, such that f is an isomorphism over $Y \setminus \{y_1, \ldots, y_l\}$ and $\dim f^{-1}(y_k) = \frac{n}{2}$, $\forall k = 1, \ldots, l$. Then the conclusions of Theorem* 2.4.1 *hold.*

Proof. – One finds a semismall projective completion $f'' : X'' \to Y''$ of f to which we apply Theorem 2.4.1. Since the bilinear form on the fibers is non-degenerate, the cycle classes of the fundamental classes of the fibers stay independent in $H^n(X)$. □

Remark 2.4.3. – If (Y, y) is a germ of a normal complex space of dimension two, and $f : X \to Y$ is a resolution of singularities, then Grauert Contractibility Criterion, see [15] Theorem 4.4, implies that the form in question is non-degenerate and negative definite.

The following is a natural question. A positive answer would yield a proof of the Decomposition Theorem for semismall holomorphic maps from complex manifolds and for constant coefficients; see Theorem 3.3.3.

Question 2.4.4. – Let $f : V \to W$ be a proper holomorphic semismall map from a complex manifold of even dimension n onto an analytic space Y. Assume that the fiber $f^{-1}(w)$ over a point $w \in W$ has dimension $\frac{n}{2}$.

Is the intersection form on $H_n^{BM}(f^{-1}(w))$ non-degenerate? Is it $(-1)^{\frac{n}{2}}$-positive definite?

3. The topology of semismall maps

We now proceed to a study of holomorphic semismall maps from a complex manifold. First we need to prove Proposition 3.1.2, a simple splitting criterion in derived categories for which we could not find a reference. We study the topology of these maps by attaching one stratum at the time. In doing so a symmetric bilinear form emerges naturally; see Proposition 3.2.4 and Lemma 3.2.5. We then prove that the Decomposition Theorem for these maps and for constant coefficients is equivalent to the non-degeneration of these forms; Theorem 3.3.3. Finally, we give a proof of the Decomposition Theorem when the domain and target are projective, Theorem 3.4.1. A new feature that we discover is that the forms are definite by virtue of our Hodge Index Theorem for semismall maps.

3.1. Homological algebra

Let $\mathcal{A}$ be an abelian category with enough injectives, e.g. sheaves of abelian groups on a topological space, and $C(\mathcal{A})$ be the associated category of complexes. Complexes and morphisms can be truncated. Given an integer t, we have two types of truncations: $\tau_{\leqslant t}A$ and $\tau_{\geqslant t}A$. The former is defined as follows:

$$(\tau_{\leqslant t}A)^i := A^i \text{ for } i \leqslant t-1, \quad (\tau_{\leqslant t}A)^t := Ker(A^t \to A^{t+1}), \quad (\tau_{\leqslant t}A)^i := \{0\} \text{ for } i > t.$$

The latter is defined as follows:

$$(\tau_{\geqslant t}A)^i := \{0\} \text{ for } i \leqslant t-1, \quad (\tau_{\geqslant t}A)^t := Coker(A^{t-1} \to A^t), \quad (\tau_{\geqslant t}A)^i := A^i \text{ for } i > t.$$

Let $h: A \to B$ be a morphism of complexes. The truncations $\tau_{\leqslant t}(h): \tau_{\leqslant t}A \to \tau_{\leqslant t}B$ and $\tau_{\geqslant t}(h) : \tau_{\geqslant t}A \to \tau_{\geqslant t}B$ are defined in the natural way. The operations of truncating complexes and morphisms of complexes induce functors in the derived category $D(\mathcal{A})$.

If A is a complex acyclic in degrees $l \neq t$ for some integer t, i.e. if $\tau_{\leqslant t}A \simeq \tau_{\geqslant t}A$, then $A \simeq \mathcal{H}^t(A)[-t]$.

The cone construction for a morphism of complexes $h: A \to B$ gives rise, in a non-unique way, to a diagram of morphism of complexes $A \stackrel{h}{\to} B \to M(h) \stackrel{[1]}{\to} A[1]$. A diagram of morphisms $X \to Y \to Z \stackrel{[1]}{\to} X[1]$ in $D(\mathcal{A})$ is called a distinguished triangle if it is isomorphic to a diagram arising from a cone.

A morphism $h: A \to B$ in $D(\mathcal{A})$ gives rise to a distinguished triangle $A \stackrel{h}{\to} B \to C \to A[1]$. If $h = 0$, then $C \simeq A[1] \oplus B$ and the induced morphism $A[1] \to A[1]$ is an isomorphism.

A morphism $h: A \to B$ in the derived category gives a collection of morphisms in cohomology $\mathcal{H}^l(h): \mathcal{H}^l(A) \to \mathcal{H}^l(B)$. A distinguished triangle $A \to B \to C \stackrel{[1]}{\to} A[1]$ gives rise to a cohomology long exact sequence:

$$\ldots \mathcal{H}^l(A) \to \mathcal{H}^l(B) \to \mathcal{H}^l(C) \to \mathcal{H}^{l+1}(A) \ldots$$

A non-zero morphism $h: A \to B$ in the derived category may nonetheless induce the zero morphisms between all cohomology groups. However, we have the following simple and standard.

LEMMA 3.1.1. – *Let t be an integer and A and B be two complexes such that $A \simeq \tau_{\leqslant t}A$ and $B \simeq \tau_{\geqslant t}B$. Then the natural map $Hom_{D(\mathcal{A})}(A,B) \to Hom_{\mathcal{A}}(\mathcal{H}^t(A), \mathcal{H}^t(B))$ is an isomorphism of abelian groups.*

Proof. – It is enough to replace B by an injective resolution placed in degrees no less than t. □

We shall need the following elementary splitting criterion.

PROPOSITION 3.1.2. – *Let $C \stackrel{u}{\longrightarrow} A \stackrel{v}{\longrightarrow} B \stackrel{[1]}{\longrightarrow} C[1]$ be a distinguished triangle and t be an integer such that $A \simeq \tau_{\leqslant t}A$ and $C \simeq \tau_{\geqslant t}C$.*
Then $\mathcal{H}^t(u): \mathcal{H}^t(C) \to \mathcal{H}^t(A)$ is an isomorphism iff

$$A \simeq \tau_{\leqslant t-1}B \oplus \mathcal{H}^t(A)[-t]$$

and the map v is the direct sum of the natural map $\tau_{\leqslant t-1}B \to B$ and the zero map.

Proof. – Assume that $\mathcal{H}^t(u)$ is an isomorphism. Apply the functor $Hom(A,-)$ to the distinguished triangle $\tau_{\leqslant t-1}B \stackrel{\nu_{t-1}}{\longrightarrow} \tau_{\leqslant t}B \stackrel{\pi}{\longrightarrow} \mathcal{H}^t(B)[-t] \stackrel{[1]}{\longrightarrow} \tau_{\leqslant t-1}B[1]$ and we get the following exact sequence:

$$\cdots \to Hom^{-1}\big(\tau_{\leqslant t}A, \mathcal{H}^t(B)[-t]\big) \to Hom^0(\tau_{\leqslant t}A, \tau_{\leqslant t-1}B)$$
$$\to Hom^0(\tau_{\leqslant t}A, \tau_{\leqslant t}B) \to Hom^0\big(\tau_{\leqslant t}A, \mathcal{H}^t(B)[-t]\big) \to \cdots$$

Since $\mathcal{H}^t(B)[-t]$ is concentrated in degree t, $Hom^{-1}(\tau_{\leqslant t}A, \mathcal{H}^t(B)[-t]) = \{0\}$. The morphism $\mathcal{H}^t(v) = 0$, for $\mathcal{H}^t(u)$ is surjective.

It follows that there exists a unique lifting v' of $\tau_{\leqslant t}(v)$, i.e. there exists a unique $v' : A \to \tau_{\leqslant t-1}B$ such that $\tau_{\leqslant t}(v) = \nu_{t-1} \circ v'$.

We complete v' to a distinguished triangle:

$$\tau_{\leqslant t}A \xrightarrow{v'} \tau_{\leqslant t-1}B \xrightarrow{v''} M(v') \xrightarrow{[1]} \tau_{\leqslant t}A[1]\ .$$

By degree considerations, the morphism $\mathcal{H}^l(v') = 0$ for $l \geqslant t$. Since v' is a lifting of $\tau_{\leqslant t}(v)$, the morphism $\mathcal{H}^l(v')$ is an isomorphism for $l \leqslant t-1$ and it is the zero map for $l \geqslant t$. This implies that $M(v') \simeq \mathcal{H}^t(A)[-t+1]$ and that $\mathcal{H}^{t-1}(v'') = 0$. By virtue of Lemma 3.1.1, we get that $v'' = 0$.

The desired splitting follows. The converse can be read off the long exact cohomology sequence. □

3.2. The bilinear forms associated with relevant strata

Let $f : X \to Y$ be a proper holomorphic semismall map with X nonsingular connected of dimension n. Let us summarize the results from stratification theory (cf. [13], Ch. 1) that we shall need in the sequel. They are based essentially on Thom First Isotopy Lemma.

There exists a collection of disjoint locally closed and *connected* analytic subvarieties $Y_i \subseteq Y$ such that:

(a) $Y = \coprod_i Y_i$ is a Whitney stratification of Y.

(b) $Y_i \cap \overline{Y_j} \neq \emptyset$ iff $Y_i \subseteq \overline{Y_j}$.

(c) The induced maps $f_i : f^{-1}(Y_i) \to Y_i$ are stratified submersions; in particular they are topologically locally trivial fibrations.

We call such data *a stratification of the map* f.

DEFINITION 3.2.1. – A stratum Y_i is said to be *relevant* if $2\dim f^{-1}(Y_i) - \dim Y_i = n$. Let $I' \subseteq I$ be the set of indices labeling relevant strata.

Let $i \in I$ be any index and $d_i := \dim Y_i$. Define $\mathcal{L}_i := (R^{n-d_i} f_* \mathbb{Q}_X)_{|Y_i}$. It is a local system on Y_i.

Remark 3.2.2. – If Y_i is not relevant, then $\mathcal{L}_i$ is the zero sheaf. If Y_i is relevant, then the stalks $(\mathcal{L}_i^*)_{y_i} \simeq H^{BM}_{n-d_i}(f^{-1}(y_i))$ of the dual local system are generated exactly by the fundamental classes of the irreducible and reduced components of maximal dimension of the fiber over y_i.

Remark 3.2.3. – Since the monodromy acts by permuting the irreducible components, it follows that the local systems split as direct sums of irreducible local systems: $\mathcal{L}_i \simeq \bigoplus_{j=1}^{m_i} \mathcal{L}_{ij}$.

Let $S := Y_i$, $d := \dim S$ and $\mathcal{L}_S := \mathcal{L}_i$. We now proceed to associating with S a symmetric bilinear form on the local system $\mathcal{L}_S^*$.

Let $s \in S$ and choose a small-enough euclidean neighborhood U of s in Y such that (a) $S' := S \cap U$ is contractible and (b) the restriction $i^* : H^{n-d}(f^{-1}(U)) \to H^{n-d}(f^{-1}(s))$ is an isomorphism.

Let $F_1, \ldots, F_r$ be the irreducible and reduced components of maximal dimension of $f^{-1}(S')$. By virtue of (a) above and of the topological triviality over S', the intersections $f_j := f^{-1}(s) \cap F_j$

are exactly the irreducible and reduced components of maximal dimension of $f^{-1}(s)$. The analogous statement is true for every point $s' \in S'$ and the components for the point s can be canonically identified with the ones of s'. The specialization morphism

$$i_s^! : H_{n+d}^{BM}\big(f^{-1}(S')\big) \to H_{n-d}^{BM}\big(f^{-1}(s)\big),$$

associated with the regular imbedding $i_s : \{s\} \to S'$, sends the fundamental class of a component F_l to the fundamental class of the corresponding f_l and it is an isomorphism; see [11], Ch. 10. We have the following sequence of maps:

$$H_{n-d}^{BM}\big(f^{-1}(s)\big) \xrightarrow{(i_s^!)^{-1}} H_{n+d}^{BM}\big(f^{-1}(S')\big) \xrightarrow{(\cap\mu_{f^{-1}(U)})^{-1}} H^{n-d}\big(f^{-1}(U), f^{-1}(U\setminus S')\big)$$
$$\overset{nat}{\to} H^{n-d}\big(f^{-1}(U)\big) \xrightarrow{i^*} H^{n-d}\big(f^{-1}(s)\big) \xrightarrow{\kappa} H_{n-d}^{BM}\big(f^{-1}(s)\big)^*.$$

The second map is the inverse to the isomorphism given by capping with the fundamental class $\mu_{f^{-1}(U)}$ (cf. [14], IX.4). The third map is the natural map in relative cohomology. The fourth map is an isomorphism by virtue of condition (b) above. The map κ is an isomorphism by the compactness of $f^{-1}(s)$.

We denote the composition, which is independent of the choice of U:

$$\rho_{S,s} : H_{n-d}^{BM}\big(f^{-1}(s)\big) \to H_{n-d}^{BM}\big(f^{-1}(s)\big)^*.$$

We have that

$$\rho_{S,s}(f_h)(f_k) = \deg F_h \cdot f_k,$$

where the refined intersection product takes place in $f^{-1}(U)$ and has values in $H_0^{BM}(f^{-1}(s))$.

Since the map f is locally topologically trivial along S, the maps $\rho_{S,s}$ define a map of local systems

$$\rho_S : \mathcal{L}_S^* \to \mathcal{L}_S.$$

We record the following fact for future use.

PROPOSITION 3.2.4. – *If S is not relevant, then ρ_S is the zero map between trivial local systems. Let $s \in S$ be a point. The map $\rho_{S,s}$ is an isomorphism iff the natural map $r_k : H^k(f^{-1}(U)) \to H^k(f^{-1}(U\setminus S'))$ is an isomorphism for every $k \leqslant n-d-1$, iff the natural map $s_k : H^k(f^{-1}(U\setminus S')) \to H^{k+1}(f^{-1}(U), f^{-1}(U\setminus S'))$ is an isomorphism for every $k \geqslant n-d$.*

Proof. – The domain and the range of $\rho_{S,s}$ are dual to each other. The statement follows from the relative cohomology sequence for the pair $(f^{-1}(U), f^{-1}(U\setminus S'))$, the isomorphisms $H^k(f^{-1}(U)) \simeq H^k(f^{-1}(s))$, $H^k(f^{-1}(U), f^{-1}(U\setminus S')) \simeq H_{2n-k}^{BM}(f^{-1}(S'))$ and the fact that $\dim f^{-1}(s) \leqslant \frac{n-d}{2}$, $\dim f^{-1}(S) \leqslant \frac{n+d}{2}$. □

Since $f_i : f^{-1}(S) \to S$ is a stratified submersion, given any point $s \in S$, we can choose an analytic normal slice $N(s)$ to S at s such that $f^{-1}(N(s))$ is a locally closed complex submanifold of X of dimension $n-d$. We now use this fact to express the map $\rho_{S,s}$ in terms of the refined intersection pairing on $f^{-1}(N(s))$.

LEMMA 3.2.5. – *If $s \in S$, then $\rho_{S,s}(f_h)(f_k) = \deg f_h \cdot f_k$, where the refined intersection product on the r.h.s. takes place in $f^{-1}(N(s))$ and has values in $H_0^{BM}(f^{-1}(s))$. In particular, the map $\rho_S : \mathcal{L}_S^* \longrightarrow \mathcal{L}_S$ is symmetric.*

Proof. – Since $f_i : f^{-1}(S) \to S$ is a stratified submersion and $N(s)$ is a normal slice to S at s, F_j meets $f^{-1}(N(s))$ transversally at the general point of f_j. It follows that the refined intersection product $f^{-1}(N(s)) \cdot F_j$ is the fundamental class of f_j in $H^{BM}_{n-d}(f^{-1}(s))$. The result follows by applying [11], 8.1.1.a) to the maps $f^{-1}(s) \to f^{-1}(N(s)) \to f^{-1}(U)$. □

3.3. Inductive study of semismall analytic maps

Let $f : X \to Y$ and $\{Y_j\}$, $i \in I$, S be as in Section 3.2. We assume, for simplicity, the Y_i to be connected and I to be finite. There is no loss of generality, for strata of the same dimension do not interfere with each other from the point of view of the analysis that follows and could be treated simultaneously. As usual, we define a partial order on the index set I by setting $i \prec j$ iff $Y_i \subseteq \overline{Y_j}$. We fix a total order $I = \{i_1 < \cdots < i_\iota\}$ which is compatible with the aforementioned partial order and define the open sets $U_{\geqslant i} := \coprod_{j \geqslant i} Y_j$. Similarly, $U_{>i} := \coprod_{j>i} Y_j$. Let $\alpha_i : U_{>i} \to U_{\geqslant i}$ be the open imbedding. We can define the intermediate extension of a complex of sheaves $K^\bullet$ on $U_{>i}$ to a complex of sheaves on $U_{\geqslant i}$ by setting

$$\alpha_{i!*} K^\bullet = \tau_{\leqslant -\dim Y_i - 1} R\alpha_{i*} K^\bullet.$$

See [1]. The construction is general and can be iterated so that one can form the intermediate extension of a complex of sheaves on any Y_i to a complex on $\overline{Y_i} \cap U_{>j}$ for $j < i$. In particular, let $\mathcal{L}$ be a local system on Y_i. The intermediate extension of $\mathcal{L}[\dim Y_i]$ to $\overline{Y}_i \cap U_{>j}$ for $j < i$ is called the *intersection cohomology complex* associated with $\mathcal{L}$ and is denoted by $IC_{\overline{Y}_i \cap U_{>j}}(\mathcal{L})$.

DEFINITION 3.3.1. – Let $f : X \to Y$ be a proper holomorphic semismall map from a nonsingular connected complex manifold X of dimension n. We say that *the Decomposition Theorem holds for* f if there is an isomorphism

$$Rf_* \mathbb{Q}_X[n] \simeq \bigoplus_{k \in I'} IC_{\overline{Y_k}}(\mathcal{L}_k) \simeq \bigoplus_{k \in I'} \bigoplus_{m=1}^{m_k} IC_{\overline{Y_k}}(\mathcal{L}_{km}),$$

where the $\mathcal{L}_{km}$ are as in Remark 3.2.3.

Remark 3.3.2. – The Decomposition Theorem holds, in the sense defined above, for X, Y and f algebraic (cf. [1]) and for f a Kähler morphism (cf. [18]). In both cases, a far more general statement holds. As observed in [3], Section 1.7, in the case of semismall maps these results can be expressed in the convenient form of Definition 3.3.1. Note, however, that [3] does not prove the Decomposition Theorem for semismall maps.

We now proceed to show that the non-degeneracy of the forms ρ_S associated with the strata Y_i implies the Decomposition Theorem.

Recall that $I' \subseteq I$ is the subset labeling relevant strata. For ease of notation set

$$V := U_{>i}, \qquad V' := U_{\geqslant i}, \qquad S := Y_i$$

and let

$$V \xrightarrow{\alpha} V' \xleftarrow{\beta} S$$

be the corresponding open and closed imbeddings.

THEOREM 3.3.3. – *Assume that the Decomposition Theorem holds over V. The map ρ_S is an isomorphism iff the Decomposition Theorem holds over V' and the corresponding isomorphism restricts to the given one over V.*

Proof. – Denote by g the map $f_| : f^{-1}(V) \to V$. By cohomology and base change,

$$\big(Rf_*\mathbb{Q}_X[n]\big)_{|V} \simeq Rg_*\mathbb{Q}_{f^{-1}(V)}[n].$$

Similarly for V'. Clearly, we have $(Rf_*\mathbb{Q}_X[n])_{|V} \simeq \alpha^*[(Rf_*\mathbb{Q}_X[n])_{|V'}]$.

There is a distinguished "attaching" triangle, see [2], 5.14:

$$\beta_*\beta^!\big(Rf_*\mathbb{Q}_X[n]_{|V'}\big) \xrightarrow{u} \big(Rf_*\mathbb{Q}_X[n]\big)_{|V'} \xrightarrow{v} R\alpha_*\big(Rf_*\mathbb{Q}_X[n]_{|V}\big)$$
$$\xrightarrow{w\,[1]} \beta_*\beta^!\big(Rf_*\mathbb{Q}_X[n]_{|V'}\big)[1].$$

On the open set V the complex $\beta_*\beta^!(Rf_*\mathbb{Q}_X[n])_{|V'}$ is isomorphic to zero and the map v restricts to an isomorphism. Recalling the notation in Section 3.2, the long exact sequence of cohomology sheaves is, stalk-wise along the points of S, the long exact sequence for the cohomology of the pair $(f^{-1}(U), f^{-1}(U \setminus S'))$. In addition the map $\mathcal{H}^{-d}(u)$ is identified, stalk-wise along the points of S, with the map $\rho_{S,s}$. The statement follows from Proposition 3.2.4 which allows us to apply Proposition 3.1.2. □

Remark 3.3.4. – In the algebraic and Kähler case, the results [1] and [18], coupled with Theorem 3.3.3, imply that the forms ρ_S are non-degenerate for every $i \in I$; see also [4], Theorem 8.9.14. To our knowledge these results have no implications as to the sign of the intersection forms. Surprisingly, in the projective case we can determine that these forms are definite; see Section 3.4.

3.4. Signature and Decomposition Theorem in the projective case

In this section we use Theorem 2.4.1, the previous inductive analysis and a Bertini-type argument to give a proof of the following theorem

THEOREM 3.4.1. – *Let $f : X \to Y$ be a semismall map from a nonsingular complex projective variety of dimension n onto a complex projective variety. The Decomposition Theorem holds for f, i.e. there is a canonical isomorphism*

$$Rf_*\mathbb{Q}_X[n] \simeq \bigoplus_{k\in I'} IC_{\overline{Y_k}}(\mathcal{L}_k) \simeq \bigoplus_{k\in I'} \bigoplus_{m=1}^{m_k} IC_{\overline{Y_k}}(\mathcal{L}_{km}).$$

For every relevant stratum S of dimension d the associated intersection form is non-degenerate and $(-1)^{\frac{n-d}{2}}$-definite.

Proof. – By virtue of Proposition 3.3.3 we are reduced to checking that the intersection form associated with a relevant d-dimensional stratum S is non-degenerate and $(-1)^{\frac{n-d}{2}}$-definite.

If $d = 0$, then the conclusion follows from Proposition 2.4.1.

Let $d > 0$. Let A be a very ample divisor on Y. The line bundle $M := f^*A$ is lef and generated by its global sections. By virtue of Proposition 2.1.7, we can choose d general sections $H_1, \ldots, H_d$ in the linear system $|A|$ such that their common zero locus H has the property that $f^{-1}(H)$ is nonsingular of dimension $n-d$, $f^{-1}(H) \to H$ is semismall, H meets S at a non-empty finite set of points $s_1, \ldots, s_r$ so that, for at least one index $1 \leqslant l \leqslant r$, a small neighborhood of a point s_l in H is a normal slice to S at s_l. By virtue of Theorem 2.4.1 the intersection form of $f^{-1}(s_l) \subseteq f^{-1}(H)$ has the required properties at the point s_l, and therefore at every point $s \in S$. We conclude by applying Lemma 3.2.5. □

Remark 3.4.2. – Theorem 3.4.1 can be applied even when the spaces are not complete, in the presence of a suitable completion of the morphism: one for which the domain is completed to a projective manifold, the target to a projective variety and the map to a semismall one. In general this may not be possible, but it can be done in several instances, e.g. the Springer resolution of the nilpotent cone of a complex semisimple Lie algebra, the Hilbert scheme of points on an algebraic surface mapping on the corresponding symmetric product, isolated singularities (see below), certain contraction of holomorphic symplectic varieties ....

COROLLARY 3.4.3. – *Let $f: X \to Y$ be a birational semismall map from a nonsingular quasi projective complex variety of dimension n onto a quasi projective complex variety Y with isolated singularities. Assume that f is an isomorphism outside the isolated singularities. The Decomposition Theorem holds for f.*

Proof. – We can reduce the statement to the complete projective case: see Corollary 2.4.2. □

Acknowledgements

Parts of this work have been done while the first author was visiting Hong Kong University in January 2000 and the Max-Planck Institut für Mathematik in the summer of 2000 and while the second author was visiting Harvard University in April 1999 and SUNY at Stony Brook in May 2000. We would like to thank D. Massey, A.J. Sommese and J.A. Wisniewski for useful correspondences.

REFERENCES

[1] BEILINSON A.A., BERNSTEIN J.N., DELIGNE P., Faisceaux pervers, *Astérisque* **100** (1982).
[2] BOREL A., et al., *Intersection Cohomology*, in: Progress in Mathematics, Vol. **50**, Birkhäuser, Boston, 1984.
[3] BORHO W., MACPHERSON R., Partial resolutions of nilpotent varieties, *Astérisque* **101–102** (1983) 23–74.
[4] CHRISS N., GINZBURG V., *Representation Theory and Complex Geometry*, Birkhäuser, Boston, 1997.
[5] DE CATALDO M., MIGLIORINI L., The Douady space of a complex surface, *Adv. in Math.* **151** (2000) 283–312.
[6] DELIGNE P., Théorème de Lefschetz et critères de dégénérescence de suites spectrales, *Publ. Math. IHES* **35** (1969) 107–126.
[7] DELIGNE P., Théorie de Hodge, II, *Publ. Math. IHES* **40** (1971) 5–57.
[8] DELIGNE P., Théorie de Hodge, III, *Publ. Math. IHES* **44** (1974) 5–78.
[9] DELIGNE P., La conjecture de Weil, II, *Publ. Math. IHES* **52** (1980) 138–252.
[10] ESNAULT H., VIEHWEG E., Vanishing and non-vanishing theorems, *Actes du Colloque de Théorie de Hodge, Astérisque* **179–180** (1989) 97–112.
[11] FULTON W., *Intersection Theory*, in: Ergebnisse der Mathematik, 3.folge. Band 2, Springer-Verlag, Berlin, 1984.
[12] GORESKY M., MACPHERSON R., Intersection homology II, *Inv. Math.* **71** (1983) 77–129.
[13] GORESKY M., MACPHERSON R., *Stratified Morse Theory*, in: Ergebnisse der Mathematik, 3.folge. Band 2, Springer-Verlag, Berlin, 1988.
[14] IVERSEN B., *Cohomology of Sheaves*, in: Universitext, Springer-Verlag, Berlin, 1986.
[15] LAUFER H., *Normal Two-Dimensional Singularities*, in: Annals of Mathematics Studies, Vol. **71**, Princeton University Press, 1971.
[16] LOOIJENGA E., Cohomology and intersection hohomology of algebraic varieties, in: Kollár J. (Ed.), *Complex Algebraic Geometry*, in: IAS/Park City Mathematics Series, Vol. **3**, American Mathematical Society, 1997, pp. 221–264.

[17] MIGLIORINI L., A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial, *J. Algebraic Geometry* **4** (1995) 353–361.
[18] SAITO M., Decomposition theorem for proper Kähler morphisms, *Tohoku Math. J. (2)* **42** (2) (1990) 127–147.
[19] WIŚNIEWSKI J.A., Cohomological invariants of complex manifolds coming from extremal rays, *Asian J. Math.* **2** (2) (1998) 289–302.

(Manuscrit reçu le 10 octobre 2001 ;
accepté, après révision, le 19 novembre 2001.)

Mark Andrea A. DE CATALDO
Department of Mathematics,
SUNY at Stony Brook,
Stony Brook, NY 11794, USA
E-mail: mde@math.sunysb.edu

Luca MIGLIORINI
Dipartimento di Matematica,
Università di Bologna,
Piazza di Porta S. Donato 5,
40126 Bologna, Italy
E-mail: migliori@dm.unibo.it