ANNALES SCIENTIFIQUES DE L'É.N.S.

MISHA VISHIK

**Incompressible flows of an ideal fluid with vorticity in
borderline spaces of Besov type**

Annales scientifiques de l'É.N.S. 4^e série, tome 32, n° 6 (1999), p. 769-812

http://www.numdam.org/item?id=ASENS_1999_4_32_6_769_0

© Gauthier-Villars (Éditions scientifiques et médicales Elsevier), 1999, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'É.N.S. » (<http://www.elsevier.com/locate/ansens>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

INCOMPRESSIBLE FLOWS OF AN IDEAL FLUID WITH VORTICITY IN BORDERLINE SPACES OF BESOV TYPE

BY MISHA VISHIK^(*)

ABSTRACT. – We prove a uniqueness theorem for the Euler equations for an ideal incompressible fluid under the condition that vorticity belongs to a space of Besov type. We also prove an existence theorem in dimension two. © Elsevier, Paris

RÉSUMÉ. – Nous démontrons un théorème d'unicité pour le système d'Euler pour un fluide incompressible idéal sous l'hypothèse que le tourbillon appartient à un espace de type Besov. Nous démontrons également un théorème d'existence en dimension deux. © Elsevier, Paris

0. Introduction

In this paper we study the nonstationary Euler equations of an ideal incompressible fluid

$$(0.1) \quad \begin{cases} \dot{v}_j(x, t) = - \sum_{i=1}^n v_i \partial_i v_j - \partial_j p, & 1 \leq j \leq n, \ x \in \mathbb{R}^n, \\ \operatorname{div} v = \sum_{j=1}^n \partial_j v_j = 0, \\ v(x, 0) = v_0(x). \end{cases}$$

Here $v(x, t) = (v_1, \dots, v_n)$ is the Eulerian velocity of a fluid flow. For incompressible fluids the key characteristic of the flow is vorticity

$$\omega_{ij}(x, t) = \partial_j v_i - \partial_i v_j, \quad 1 \leq i, j \leq n.$$

The mathematical theory of Euler equations (0.1) is an old subject. In [C1] J.-Y. Chemin develops the theory in detail and gives an account of more recent results including his work [C2] on the regularity of vortex patches for the two-dimensional Euler equations.

Existence and uniqueness theorems are obtained for the problem (0.1) (locally in time for $n \geq 3$) for vorticity in various function spaces with *supercritical* smoothness. Here

^(*) E-mail: vishik@math.utexas.edu

Supported in part by the NSF grant DMS-9531769 and TARP grant 003658-071.

we refer to the condition $s > n/p$ where s is smoothness of vorticity in the sense of a particular scale of function spaces based on L^p .

The papers of L. Lichtenstein [L], N. Gunther [G], Wolibner [W] that started the subject deal with Hölder classes. Some of the more recent papers include those of D. Ebin and J. Marsden [EM], J. Bourguignon and H. Brezis [BB], R. Temam [T], T. Kato and G. Ponce [KP] (Sobolev spaces).

In [V] we proved the global existence for $n = 2$ where vorticity belongs to the Besov space $B_{2/s,1}^s$.

In the present paper we continue to investigate the *borderline* case $s = n/p$ which corresponds to the critical case of Sobolev (-Besov-Lizorkin-Triebel) embedding.

V. Yudovich [Y1] proved the basic uniqueness theorem for the weak solutions of (0.1) under the condition (bounded domain in $\mathbb{R}^n$) $\omega \in L^\infty$. For $n = 2$ he was able to construct a weak solution in this class that exists globally in time. The existence of a weak solution in two dimensions (possibly without uniqueness) was proved for a wider class L^p , $1 < p < \infty$, by V. Yudovich [Y1] and by R. DiPerna and A. Majda [DM]. D. Chae [C] proved the existence theorem for vorticity in $L \log L(\mathbb{R}^2)$. J.-M. Delort [De] constructed a weak solution with initial vorticity arbitrary sign definite measure. A Bourbaki talk by P. Gérard [Ge] gives an account of this result as well as the results of [C2]. A different proof of J.-M. Delort's result was given by L.C. Evans and S. Müller [EvM] and by A. Majda [M].

V. Scheffer [S] and A. Shnirelman [Sh] constructed the first examples of *nonuniqueness* for weak solutions with $v(\cdot, \cdot) \in L_{loc}^2(\mathbb{R}^2 \times \mathbb{R})$.

Recently V. Yudovich [Y2] further improved his uniqueness theorem [Y1] allowing (case of a bounded domain) $\omega \in \bigcap_{p_0 \leq p < \infty} L^p$ so that $\|\omega\|_{L^p} \leq C\theta(p)$ and $\theta(p)$ grows *moderately* in p . For example $\theta(p) = \log p$ guarantees uniqueness.¹ More precisely, V. Yudovich proved that for $Z(\alpha) = \inf_{\varepsilon \in (0, p_0^{-1}]} \{\varepsilon^{-1} \alpha^\varepsilon \theta(\frac{1}{\varepsilon})\}$ the condition $\int_1^\infty \alpha^{-1} Z(\alpha) d\alpha = \infty$ implies uniqueness. This result holds for arbitrary n while the existence theorem was proved in [Y2] (see also [Y3]) only for $n = 2$. The proof of uniqueness in [Y2] is based on the energy method. In case the growth condition for $\theta(p)$ *fails* V. Yudovich constructed *counterexamples to this method* of proving uniqueness (if not to uniqueness per se).

We introduce here a different uniqueness class which is a variant of a *borderline* Besov space. To describe a particular case of the results of the paper we define

$$B_\Pi = \left\{ f \in S'(\mathbb{R}^n) \mid \sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} = O(\Pi(N)) \right\}.$$

Here the increasing function Π satisfies certain conditions (see (i)-(iii) below), $\Delta_j f$ denotes the terms in Littlewood-Paley decomposition of f . One possible choice is $\Pi(N) = (N+2) \log_2^\kappa(N+3)$, $0 \leq \kappa \leq 1$. Notice $f \in B_\Pi$ means the norm $\|f\|_{B_{\infty,1}^0}$ is generally divergent but in a controlled way.

We prove uniqueness for the problem (0.1), n arbitrary, under the condition

$$(0.2) \quad \int_1^\infty \Pi(\alpha)^{-1} d\alpha = \infty,$$

¹ "Not much stronger than linear" on p. 28 of [Y2] is a misprint. Logarithmic singularities that lead to linear in p growth of $\|\omega\|_{L^p}$ are in fact not allowed but singularities of the kind $\log \log |x|$ that produce $\log p$ growth of $\|\omega\|_{L^p}$ as $p \rightarrow \infty$ are covered by the results of [Y2].

$\omega \in L^{p_0}$, $p_0 \in (1, n)$ and the norm $\|\omega\|_{B_\Pi}$ is bounded. By the choice $\Pi(N) = N + 2$ this implies uniqueness for flows with vorticity in Hölder space $\Lambda^0 = B_{\infty, \infty}^0$. As a corollary, a solution with vorticity bounded in bmo is unique. Indeed, $\text{bmo} \hookrightarrow \Lambda^0$.

For Π as above the (unique) solution exists globally in time ($n = 2$) for $0 < \kappa \leq \frac{1}{2}$ and (possibly only) locally in time for $\frac{1}{2} < \kappa \leq 1$. The initial vorticity belongs to the space B_Γ with $\Gamma(N) = \log_2^\kappa(N + 3)$.

The existence theorem is proved in §9.

The proof of the uniqueness theorem given in §8 uses a “paradifferential analogue” of the Osgood uniqueness argument and seems *unrelated to the energy method*. The condition (0.2) in fact guarantees that the Lagrangian flow satisfies conditions of the Osgood uniqueness theorem. It also looks similar to the growth condition in [Y2]. On the other hand there are functions in B_Γ with a *compact support* that fail to belong to L^p for *all* sufficiently large p (see §2). Therefore the growth condition for $\theta(p)$ is violated.

The main tool we use to get an a priori estimate in B_Π are *wavelet decompositions* of vorticity. In fact we obtain a complete description of all spaces of this type in terms of wavelet coefficients. This allows to get the information about the action of a volume preserving homeomorphism (given by the fluid flow) on B_Γ .

We use the method of H. Bahouri and J.-Y. Chemin [BC] to obtain the necessary “paradifferential estimates” of the terms in Littlewood-Paley decomposition of the vorticity ω .

Some of the immediate problems left open include whether the time of existence is indeed finite for e.g., $\kappa = 1$ or this restriction is due to the weakness of the method. Also the relation with the transport theory developed by R. DiPerna and P.-L. Lions [DL], B. Desjardins [D1], H. Bahouri and J.-Y. Chemin [BC], J.-Y. Chemin and N. Lerner [CL], B. Desjardins [D2], especially for the flows that are not quasi Lipschitz, seems of interest.

1. Function spaces

Let $\Gamma : \mathbb{R} \rightarrow [1, \infty)$ be a locally Lipschitz continuous monotonically nondecreasing function satisfying the following conditions:

- (i) $\Gamma(\alpha) = 1$ for $\alpha \in (-\infty, -1]$, $\lim_{\alpha \rightarrow \infty} \Gamma(\alpha) = \infty$.
- (ii) There is a constant $C > 0$ such that $C^{-1}\Gamma(\beta) \leq \Gamma(\alpha) \leq C\Gamma(\beta)$ for $\alpha, \beta \in [-1, \infty)$, $|\alpha - \beta| \leq 1$.
- (iii) There is a constant $C > 0$ such that

$$C\Gamma(\alpha) \geq \int_{\alpha}^{\infty} 2^{-(\xi-\alpha)} \Gamma(\xi) d\xi, \alpha \in [-1, \infty).$$

EXAMPLE 1.1. $\Gamma(\alpha) = (N + 2)^\varepsilon \log_2^\kappa(\alpha + 3)$ for $\alpha \geq -1$, where $\varepsilon, \kappa \in [0, \infty)$, except $\varepsilon = \kappa = 0$.

We choose radial $\Phi \in S(\mathbb{R}^n)$ such that $\text{supp } \hat{\Phi} \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq 1\}$ and $|\hat{\Phi}(\xi)| \geq C > 0$ on $\{\xi \in \mathbb{R}^n \mid |\xi| \leq \frac{5}{6}\}$. We also choose radial $\varphi \in S(\mathbb{R}^n)$ with $\text{supp } \hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2\}$, $|\hat{\varphi}(\xi)| \geq C > 0$ on $\{\xi \in \mathbb{R}^n \mid \frac{3}{5} \leq |\xi| \leq \frac{5}{3}\}$. Let $\varphi_j(x) = 2^{jn} \varphi(2^j x)$ [i.e., $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$], $j \in \mathbb{Z}$.

DEFINITION 1.2. – Let for $f \in S'(\mathbb{R}^n)$

$$\begin{aligned}\Delta_{-1}f &= \hat{\Phi} \left(\frac{1}{i} \frac{\partial}{\partial x} \right) f = (\hat{\Phi} \cdot \hat{f})^\vee = \Phi * f ; \\ \Delta_j f &= \hat{\varphi}_j \left(\frac{1}{i} \frac{\partial}{\partial x} \right) f = (\hat{\varphi}_j \cdot \hat{f})^\vee = \varphi_j * f , \quad j \geq 0 ; \\ \Delta_j f &= 0 , \quad j \leq -2 ; \\ S_k f &= \sum_{j \leq k} \Delta_j f .\end{aligned}$$

DEFINITION 1.3. – Let Γ be the function satisfying (i)–(iii) above. Let

$$(1.1) \quad B_\Gamma = \left\{ f \in S'(\mathbb{R}^n) \mid \sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} \leq C\Gamma(N) , N \geq -1 \right\} .$$

The norm on B_Γ is defined as the best constant in (1.1):

$$\|f\|_\Gamma = \sup_{N \geq -1} \left(\sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} \right) \Gamma(N)^{-1} .$$

We give the characterization of B_Γ via the wavelet expansions. For standard Besov spaces see [FJ], [LM], [M].

Let radial $\psi \in S(\mathbb{R}^n)$ satisfy

$$\text{supp } \hat{\psi} \subset \left\{ \xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad |\hat{\psi}(\xi)| \geq C > 0 \text{ on } \left\{ \xi \in \mathbb{R}^n \mid \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \right\} .$$

Choose radial $\Psi \in S(\mathbb{R}^n)$ so that $\text{supp } \hat{\Psi} \subset \{ \xi \in \mathbb{R}^n \mid |\xi| \leq \pi \}$, $|\hat{\Psi}(\xi)| \geq C > 0$ on $\{ \xi \in \mathbb{R}^n \mid |\xi| \leq 1 \}$. We define the set of dyadic cubes as follows: for $m \in \mathbb{Z}$, $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$

$$Q_{m,k} = \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid k_i 2^{-m} \leq x_i < (k_i + 1) 2^{-m} ; i = 1, \dots, n \} .$$

We set for any dyadic cube $Q = Q_{m,k}$

$$\psi_{m,k}(x) = \psi(2^m x - k) .$$

PROPOSITION 1.4. – Let $f \in B_\Gamma$. There is the following representation

$$(1.2) \quad f(x) = \sum_{k \in \mathbb{Z}^n} a_k \Psi(x - k) + \sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}^n} a_{m,k} \psi_{m,k}(x) .$$

The series in (1.2) is convergent in $S'(\mathbb{R}^n)$. The coefficients a_k , $a_{m,k}$ satisfy the following estimate

$$(1.3) \quad C^{-1} \|f\|_\Gamma \leq \sup_{N \geq -1} \left(\sum_{m=-1}^N \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \right) \Gamma(N)^{-1} \leq C \|f\|_\Gamma ,$$

where the constant $C > 0$ does not depend on $f \in B_\Gamma$. For $m = -1$ we set $a_{-1,k} = a_k$, $k \in \mathbb{Z}^n$.

Proof. – It is possible to make the above choices of $\varphi, \Phi, \psi, \Psi$ so that

$$\hat{\Phi}(\xi)\hat{\Psi}(\xi) + \sum_{m=0}^{\infty} \hat{\varphi}(2^{-m}\xi)\hat{\psi}(2^{-m}\xi) = 1, \quad \xi \in \mathbb{R}^n.$$

Then, following [FJ]

$$(1.4) \quad f(\cdot) = \sum_{k \in \mathbb{Z}^n} (\Phi * f)(k) \Psi(\cdot - k) + \sum_{m=0}^{\infty} 2^{-mn} (\varphi_m * f)(2^{-m}k) \psi_m(\cdot - 2^{-m}k)$$

where $\psi_m(x) = 2^{mn} \psi(2^m x)$, $m \geq 0$. The series in the right side of (1.4) is convergent in $S'(\mathbb{R}^n)$. Therefore, we define $a_k = \Phi * f(k)$; $a_{m,k} = \varphi_m * f(2^{-m}k)$. By construction

$$\begin{aligned} |a_k| &\leq \|\Phi * f\|_{L^\infty} = \|\Delta_{-1}f\|_{L^\infty}, \\ |a_{m,k}| &\leq \|\varphi_m * f\|_{L^\infty} = \|\Delta_m f\|_{L^\infty} \end{aligned}$$

for $m \geq 0, k \in \mathbb{Z}^n$. This proves the second inequality in 1.3. It remains to prove the first inequality. We need the following estimate from [FJ].

LEMMA 1.5 ([FJ]). – *The following two inequalities hold true for any $M \in \mathbb{Z}_+$, $j, m \geq 0$*

$$(1.5) \quad |(\Delta_j \psi_{m,k})(x)| \leq C 2^{-(m-j)(n+1)} (1 + 2^j |x - k 2^{-m}|)^{-M} \text{ for } j \leq m, \text{ and}$$

$$(1.6) \quad |(\Delta_j \psi_{m,k})(x)| \leq C 2^{-(j-m)} (1 + 2^m |x - k 2^{-m}|)^{-M} \text{ for } m \leq j.$$

The same inequalities are valid for $j = -1, m \geq 0$ in case (1.5) and for $m = -1, j \geq 0$ in case (1.6) where $\psi_{-1,k}(\cdot) = \Psi(\cdot - k)$.

To prove Proposition 1.4 we have to estimate $\|\Delta_j f\|_{L^\infty}$ where f is given by (1.2). We have from Lemma 1.5

$$\begin{aligned} \|\Delta_{-1}f\|_{L^\infty} &\leq \left\| \Delta_{-1} \sum_{k \in \mathbb{Z}^n} a_k \Psi(\cdot - k) \right\|_{L^\infty} + \sum_{m=0}^{\infty} \left\| \Delta_{-1} \sum_{k \in \mathbb{Z}^n} a_{m,k} \psi_{m,k} \right\|_{L^\infty} \\ &\leq C \left\| \sum_{k \in \mathbb{Z}^n} a_k \Psi(\cdot - k) \right\|_{L^\infty} + \sum_{m=0}^{\infty} \left\| \sum_{k \in \mathbb{Z}^n} a_{m,k} \Delta_{-1} \psi_{m,k} \right\|_{L^\infty} \\ &\leq C \sup_{k \in \mathbb{Z}^n} |a_k| \cdot \sup_x \sum_{k \in \mathbb{Z}^n} (1 + |x - k|)^{-M} \\ &\quad + C \sum_{m=0}^{\infty} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \cdot 2^{-m(n+1)} \sup_x \sum_k (1 + |x - k 2^{-m}|)^{-M} \\ &\leq C \sup_{k \in \mathbb{Z}^n} |a_k| + C \sum_{m=0}^{\infty} 2^{-m-1} \sup_{k \in \mathbb{Z}^n} |a_{m,k}|. \end{aligned}$$

Likewise, for $j \geq 0$, applying Lemma 1.5

$$\begin{aligned}
 \|\Delta_j f\|_{L^\infty} &\leq \left\| \sum_{k \in \mathbb{Z}^n} a_k \Delta_j \Psi(\cdot - k) \right\|_{L^\infty} \\
 &\quad + \sum_{m=0}^j \left\| \sum_{k \in \mathbb{Z}^n} a_{m,k} \Delta_j \psi_{m,k} \right\|_{L^\infty} \\
 &\quad + \sum_{m=j+1}^{\infty} \left\| \sum_{k \in \mathbb{Z}^n} a_{m,k} \Delta_j \psi_{m,k} \right\|_{L^\infty} \\
 &\leq C 2^{-j} \sup_{k \in \mathbb{Z}^n} |a_k| \left\| \sum_{k \in \mathbb{Z}^n} (1 + |\cdot - k|)^{-M} \right\|_{L^\infty} \\
 &\quad + C \sum_{m=0}^j 2^{-(j-m)} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \left\| \sum_{k \in \mathbb{Z}^n} (1 + 2^m |\cdot - k 2^{-m}|)^{-M} \right\|_{L^\infty} \\
 &\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)(n+1)} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \left\| \sum_{k \in \mathbb{Z}^n} (1 + 2^j |\cdot - k 2^{-m}|)^{-M} \right\|_{L^\infty} \\
 &\leq C 2^{-j-1} \sup_{k \in \mathbb{Z}^n} |a_k| + C \sum_{m=0}^j 2^{-(j-m)} \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \\
 &\quad + C \sum_{m=j+1}^{\infty} 2^{-(m-j)} \sup_{k \in \mathbb{Z}^n} |a_{m,k}|.
 \end{aligned}$$

Let

$$\begin{aligned}
 \sup_{k \in \mathbb{Z}^n} |a_{m,k}| &= b_m, \quad m \geq -1; \\
 \sum_{j=-1}^m b_j &= d_m, \quad m \geq -1; \quad d_{-2} = 0.
 \end{aligned}$$

Then, for every $j \geq -1$, $N \geq -1$

$$\begin{aligned}
 \|\Delta_j f\|_{L^\infty} &\leq C \sum_{m=-1}^j 2^{-(j-m)} b_m + C \sum_{m=j+1}^{\infty} 2^{-(m-j)} b_m; \\
 \sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} &\leq C \sum_{m=-1}^N \left(\sum_{j=-1}^{m-1} 2^{-(m-j)} + \sum_{j=m}^N 2^{-(j-m)} \right) b_m \\
 &\quad + C \sum_{m=N+1}^{\infty} \left(\sum_{j=-1}^N 2^{-(m-j)} \right) b_m \\
 &\leq C \sum_{m=-1}^N b_m + C \sum_{m=N+1}^{\infty} 2^{-(m-N)} (d_m - d_{m-1}) \\
 &\leq C d_N + C 2^N \sum_{m=N+1}^{\infty} (2^{-m} - 2^{-(m+1)}) d_m \\
 &\leq C d_N + C \sum_{m=N+1}^{\infty} 2^{-(m-N)} d_m.
 \end{aligned}$$

Suppose for any $m \geq -1$, $d_m \leq C\Gamma(m)$. Then the right side of the last inequality is

$$\leq C\Gamma(N) + C \int_N^\infty 2^{-(\xi-N)} \Gamma(\xi) d\xi \leq C\Gamma(N)$$

because of the conditions (ii) and (iii) describing Γ . This concludes the proof of the proposition.

We now construct the predual space to B_Γ . Let H_Γ be defined as follows:

$$H_\Gamma = \left\{ f \in S'(\mathbb{R}^n) \mid \exists \{d_j\}_{j=-1}^\infty, d_j \geq 0, \sum_{j=-1}^\infty d_j < \infty \right. \\ \left. \text{such that } \|\Delta_m f\|_{L^1} \leq \sum_{j \geq m} d_j \Gamma(j)^{-1}, m \geq -1 \right\}.$$

We set

$$\|f\|_{\mathbb{T}} = \inf_{\{d_j\}} \sum_{j=-1}^\infty d_j.$$

PROPOSITION 1.6. – *The dual space to H_Γ is isomorphic to B_Γ :*

$$H'_\Gamma \approx B_\Gamma.$$

Proof. – We omit the proof which is fairly standard (cf. [P]).

PROPOSITION 1.7. – *Let $f \in H_\Gamma$. There is the following representation:*

$$(1.7) \quad f(x) = \sum_{k \in \mathbb{Z}^n} a_k \Psi(x - k) + \sum_{m=0}^\infty \sum_{k \in \mathbb{Z}^n} a_{m,k} 2^{mn} \psi_{m,k}(x).$$

The series in (1.7) is convergent in $S'(\mathbb{R}^n)$. Moreover there is a constant $C > 0$ which is independent of f such that

$$(1.8) \quad C^{-1} \|f\|_{\mathbb{T}} \leq \inf_{\{d_j\}_{j=-1}^\infty, d_j \geq 0, \sum_{j=-1}^\infty d_j < \infty} \sum_{j=-1}^\infty d_j \leq C \|f\|_{\mathbb{T}}.$$

$$\sum_{k \in \mathbb{Z}^n} |a_{m,k}| \leq \sum_{j=m}^\infty d_j \Gamma(j)^{-1}, m \geq -1$$

We omit the proof which is similar to the proof of Proposition 1.4.

REMARK 1.8. – In particular predual H_Γ of the Banach space B_Γ is separable, and $S \hookrightarrow H_\Gamma$ is a dense embedding.

REMARK 1.9. – Since the function spaces B_Γ and H_Γ do not depend upon the choice of Φ, φ satisfying standard conditions (see e.g., [P], [FJ]) Propositions 1.4, 1.7 provide a complete description of these spaces in terms of wavelet decompositions with respect to band-limited wavelets constructed by P. Lemarié and Y. Meyer [LM], [M2].

REMARK 1.10. – If $\Gamma(N) = N + 2$, $N \geq -1$ then obviously $L^\infty \hookrightarrow B_\Gamma$. In case $\Gamma(N)$ grows slower than $(N + 2)$, that is $\Gamma(N) = o(N + 2)$ as $N \rightarrow \infty$, the above embedding is no longer valid. It means for such a $\Gamma(N)$ the space B_Γ is not comparable with the space L^∞ . As we mentioned in the introduction V. Yudovich proved the global existence and uniqueness theorem [Y1] in dimension 2 for the Euler equations with vorticity in L^∞ (uniqueness for arbitrary $n \geq 2$).

2. A counterexample

PROPOSITION 2.1. – For any $\rho \in (1, \infty)$ there exists a function f such that $f \in B_\Gamma$, $f \in \bigcap_{p \in (1, \rho)} L^p(\mathbb{R}^n)$ and $f \notin L^p(\mathbb{R}^n)$ for any $p \geq \rho$.

Proof. – We use the following fundamental characterization of L^p -spaces [LM], [M2]. To simplify the notation we give the construction in dimension $n = 1$. Let ψ, Ψ be obtained from the MRA procedure [M2]. Let

$$f(x) = \sum_{k \in \mathbb{Z}} a_k \Psi(x - k) + \sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}} a_{m,k} \psi(2^m x - k) .$$

THEOREM 2.2 ([LM], [M2]). – Let $p \in (1, \infty)$. Then $f \in L^p$ if and only if the following two conditions are satisfied:

$$(2.1) \quad \{a_k\}_{k \in \mathbb{Z}} \in \ell^p ;$$

$$(2.2) \quad g = \left(\sum_{m=0}^{\infty} \sum_{k \in \mathbb{Z}} |a_{m,k}|^2 \chi_{Q_{m,k}} \right)^{1/2} \in L^p .$$

We note that the function g in (2.2) is a discrete analogue of the Littlewood-Paley quadratic function; $\chi_{Q_{m,k}}$ denotes the characteristic function of a dyadic cube $Q_{m,k}$.

To construct the example in Proposition 2.1 let $a_k = 0$ for $k \in \mathbb{Z}$ so that (2.1) is satisfied.

Since $\Gamma(\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$ there is a sequence $\{c_m\}_{m=0}^{\infty}$ satisfying the following 3 conditions:

$$(2.3) \quad c_m \rightarrow 0 \text{ as } m \rightarrow \infty ;$$

$$(2.4) \quad \sum_{m=0}^N |c_m| \leq CT(N) , \quad N \geq 0 ;$$

$$(2.5) \quad \sum_{m=0}^{\infty} |c_m|^2 = \infty .$$

Indeed we proceed by induction.

We construct a strictly monotonic sequence of nonnegative integers $\{n_i\}$, $n_1 = 0$ and set

$$c_m = \begin{cases} i^{-1/2} , & m = n_i \\ 0 , & m \neq n_i \text{ for all } i \geq 1. \end{cases}$$

Thus (2.3) and (2.5) are satisfied. We set $C = \Gamma(0)^{-1}$ and assume $n_1, \dots, n_q$ are already constructed so that (2.4) is satisfied for all $N \leq n_q$. Choose $n_{q+1} > n_q$ so that

$$\sum_{i=1}^{q+1} i^{-1/2} \leq CT(n_{q+1}) .$$

Such a choice could be made since $\Gamma(\alpha) \rightarrow \infty$. This takes care of (2.4). We fix a sequence $\{c_m\}_{m=0}^\infty$ satisfying (2.3)–(2.5) and let

$$f = \sum_{m=0}^{\infty} c_m \sum_{k=0}^{2^m-1} \varepsilon_{m,k} \psi(2^m x - k) ,$$

where $\varepsilon_{m,k}$ are taking on values 0 and 1.

We now assert that we can choose $\varepsilon_{m,k}$ in such a way that

$$(2.6) \quad g(x) = \begin{cases} 0 , & x \notin Q_{0,0} = [0, 1) \\ x^{-\rho^{-1}} , & x \in Q_{0,0} . \end{cases}$$

Indeed we start with

$$\varepsilon_{0,0} = \begin{cases} 1 , & \text{if } |c_0|^2 \leq 1 \\ 0 , & \text{otherwise.} \end{cases}$$

Assume we constructed $\varepsilon_{m,k}$, $m \leq \ell$ so that

$$(2.7) \quad \sum_{m=0}^{\ell} |c_m|^2 \sum_{k=1}^{2^m-1} \varepsilon_{m,k} \chi_{Q_{m,k}}(x) \leq x^{-2\rho^{-1}} , \quad x \in Q_{0,0} .$$

To do the induction step we simply set $\varepsilon_{\ell+1,\kappa} = 1$ in case

$$(2.8) \quad \sum_{m=0}^{\ell} |c_m|^2 \sum_{k=1}^{2^m-1} \varepsilon_{m,k} \chi_{Q_{m,k}}(x) + |c_{\ell+1,\kappa}|^2 \chi_{Q_{\ell+1,\kappa}} \leq x^{-2\rho^{-1}}$$

for $x \in Q_{\ell+1,\kappa}$, $0 \leq \kappa \leq 2^{\ell+1} - 1$ and $\varepsilon_{\ell+1,\kappa} = 0$ otherwise.

Then (2.7) is satisfied on the $(\ell + 1)$ -st step because of (2.8) and (2.6) follows from (2.3) and (2.5). Obviously $g(x)$ in (2.6) satisfies the conditions $g \in \bigcap_{p \in (1, \rho)} L^p$, $g \notin L^p$ for any $p \geq \rho$.

We conclude from the Theorem 2.2 that f satisfies the same properties. Furthermore, from Proposition 1.4 and (2.4) $f \in B_\Gamma$. This concludes the proof.

REMARK 2.3. – The function f in Proposition 2.1 could be chosen to have *compact support*. Indeed, using sufficiently regular wavelets with compact support [D], [M2] and repeating the same construction word for word produces such a function.

3. Estimates for the Biot-Savart law

Let the vorticity $\omega \in L^{p_0} \cap B_\Gamma$, $1 < p_0 < n$. We define the velocity as

$$u = \mathcal{K} * \omega ,$$

$\mathcal{K}$ being the kernel in the Biot-Savart law.

THEOREM 3.1. – *The function u is continuous and bounded on $\mathbb{R}^n$ with the following modulus of continuity:*

$$(3.1) \quad |u(x) - u(y)| \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_{\Gamma} \Gamma(-\log_2 |x - y|)) |x - y|, \quad x \neq y;$$

$$(3.2) \quad \|u\|_{L^\infty} \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_{\Gamma}), \quad 1 < p_0 < n.$$

Proof. – For any $N \geq -1$, $x \neq y$

$$(3.3) \quad \begin{aligned} |u(x) - u(y)| &\leq \sum_{j=-1}^N \|\Delta_j \nabla u\|_{L^\infty} |x - y| + 2 \sum_{j=N+1}^{\infty} \|\Delta_j u\|_{L^\infty} \\ &\leq C \sum_{j=-1}^N \|\Delta_j \nabla u\|_{L^\infty} |x - y| + C \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty}. \end{aligned}$$

We estimate the second term in the right side of (3.3) as follows. Let

$$d_m = \sum_{j=-1}^m \|\Delta_j \omega\|_{L^\infty}.$$

Then,

$$(3.4) \quad \begin{aligned} \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty} &\leq \sum_{j=N+1}^{\infty} 2^{-j} (d_j - d_{j-1}) \\ &= -2^{-(N+1)} d_N + \sum_{j=N+1}^{\infty} d_j (2^{-j} - 2^{-(j+1)}) \\ &\leq -2^{-(N+1)} d_N + \|\omega\|_{\Gamma} \sum_{j=N+1}^{\infty} \Gamma(j) 2^{-j} \\ &\leq C 2^{-N} \Gamma(N) \|\omega\|_{\Gamma} \end{aligned}$$

because of the conditions (ii) and (iii). We estimate now the first term in the right side of (3.3)

$$(3.5) \quad \begin{aligned} \sum_{j=-1}^N \|\Delta_j \nabla u\|_{L^\infty} &\leq \|\Delta_{-1} \nabla u\|_{L^\infty} + C \sum_{j=0}^N \|\Delta_j \omega\|_{L^\infty} \\ &\leq C \|\Delta_{-1} \nabla u\|_{L^{p_0}} + C \Gamma(N) \|\omega\|_{\Gamma} \\ &\leq C \|\Delta_{-1} \omega\|_{L^{p_0}} + C \Gamma(N) \|\omega\|_{\Gamma} \\ &\leq C \|\omega\|_{L^{p_0}} + C \Gamma(N) \|\omega\|_{\Gamma}. \end{aligned}$$

Here we used Bernstein's inequality and boundedness of the Calderón-Zygmund operator $\omega \mapsto \nabla u$ in L^{p_0} . The inequalities (3.3)–(3.5) yield

$$|u(x) - u(y)| \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_{\Gamma} \Gamma(N)) |x - y| + C 2^{-N} \Gamma(N) \|\omega\|_{\Gamma}.$$

We choose $N = \lceil -\log_2 |x - y| \rceil$ (in case $|x - y| \leq 2$, otherwise choose $N = -1$). This proves (3.1). It remains to prove (3.2). We have

$$\begin{aligned} \|u\|_{L^\infty} &\leq \|\Delta_{-1}u\|_{L^\infty} + \sum_{j=0}^{\infty} \|\Delta_j u\|_{L^\infty} \\ &\leq \|\Delta_{-1}u\|_{L^\infty} + C \sum_{j=0}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty} . \end{aligned}$$

The second term is $\leq C\|\omega\|_\Gamma$ as follows from (3.4) for $N = -1$. To estimate the first term we write for $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi \equiv 1$ near the origin,

$$\begin{aligned} (3.6) \quad \|\Delta_{-1}u\|_{L^\infty} &\leq \|(\chi\mathcal{K}) * \Delta_{-1}\omega\|_{L^\infty} + \|((1-\chi)\mathcal{K}) * \Delta_{-1}\omega\|_{L^\infty} \\ &\leq \|\chi\mathcal{K}\|_{L^1} \|\Delta_{-1}\omega\|_{L^\infty} + \|(1-\chi)\mathcal{K}\|_{L^{q_0}} \|\Delta_{-1}\omega\|_{L^{p_0}} \\ &\leq C \|\Delta_{-1}\omega\|_{L^{p_0}} \end{aligned}$$

since $\chi\mathcal{K} \in L^1$ and $(1-\chi)\mathcal{K} \in L^{q_0}$ ($\frac{1}{q_0} + \frac{1}{p_0} = 1$). This concludes the proof.

PROPOSITION 3.2. – *Let $\omega \in L^{p_0} \cap L^{p_1}$ where $1 < p_0 < n$, $n < p_1 < \infty$. Then for all $x, y \in \mathbb{R}^n$, $|x - y| \leq 1$ we have*

$$(3.7) \quad |u(x) - u(y)| \leq C|x - y|^{1-\frac{n}{p_1}} \|\omega\|_{L^{p_1}} ,$$

where $C > 0$ depends on n and p_1 ,

$$(3.8) \quad \|u\|_{L^\infty} \leq C \|\omega\|_{L^{p_0}}^{\frac{p_0(p_1-n)}{n(p_1-p_0)}} \|\omega\|_{L^{p_1}}^{\frac{p_1(n-p_0)}{n(p_1-p_0)}} .$$

Proof. – For any $N \geq -1$

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{j=-1}^N |\Delta_j u(x) - \Delta_j u(y)| + 2 \sum_{j=N+1}^{\infty} \|\Delta_j u\|_{L^\infty} \\ &\leq \sum_{j=-1}^N \|\Delta_j \nabla u\|_{L^\infty} |x - y| + C \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \omega\|_{L^\infty} \\ &\leq \|\Delta_{-1} \nabla u\|_{L^\infty} |x - y| + C \sum_{j=0}^N \|\Delta_j \omega\|_{L^\infty} |x - y| \\ &\quad + C \sum_{j=N+1}^{\infty} 2^{-j+\frac{nj}{p_1}} \|\Delta_j \omega\|_{L^{p_1}} \\ &\leq C \|\Delta_{-1} \omega\|_{L^{p_1}} |x - y| + \sum_{j=0}^N 2^{nj/p_1} \|\omega\|_{L^{p_1}} |x - y| \\ &\quad + C 2^{-N(1-\frac{n}{p_1})} \|\omega\|_{L^{p_1}} \\ &\leq C \|\omega\|_{L^{p_1}} \left(|x - y| + 2^{Nn/p_1} |x - y| + 2^{-N(1-\frac{n}{p_1})} \right) . \end{aligned}$$

We choose $N = \lceil -\log_2 |x - y| \rceil$. This proves (3.7).

To estimate $\|u\|_{L^\infty}$ we proceed as in (3.6)

$$\|u\|_{L^\infty} \leq \|(1 - \chi)\mathcal{K}\|_{L^{q_0}} \|\omega\|_{L^{p_0}} + \|\chi\mathcal{K}\|_{L^{q_1}} \|\omega\|_{L^{p_1}} .$$

Replacing $\chi(\cdot)$ by $\chi(\cdot/R)$ and choosing the appropriate R yields (3.8).

PROPOSITION 3.3. – *Let $1 < p_0 < n$, $\omega \in L^{p_0}$. Then $u = \mathcal{K} * \omega \in L^{np_0/(n-p_0)}$ and*

$$(3.9) \quad \|u\|_{L^{np_0/(n-p_0)}} \leq C \|\omega\|_{L^{p_0}} .$$

Proof. – Follows immediately from the Hardy-Littlewood-Sobolev inequality.

4. Action of volume preserving homeomorphisms on B_Γ

Let $g : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ be a volume preserving homeomorphism. Assume there are decreasing functions $\sigma, \rho : \mathbb{R} \rightarrow (0, \infty)$, $\lim_{\xi \rightarrow \infty} \sigma(\xi) = 0$, $\lim_{\xi \rightarrow \infty} \rho(\xi) = 0$, satisfying the following

$$(4.1) \quad |g^{-1}(x) - g^{-1}(y)| \leq c_{g^{-1}} \sigma(-\log_2 |x - y|) ;$$

$$(4.2) \quad |g(x) - g(y)| \leq c_g \rho(-\log_2 |x - y|) ;$$

$$(4.3) \quad \sigma(\xi) = 2^{-\xi} \quad , \quad \rho(\xi) = 2^{-\xi} \quad \text{for } \xi \leq 0 ;$$

$$(4.4) \quad \log_2 \sigma(\xi) \quad , \quad \log_2 \rho(\xi) \quad \text{are convex.}$$

It follows from (4.3), (4.4) that

$$\sigma(\xi) \geq 2^{-\xi} \quad , \quad \rho(\xi) \geq 2^{-\xi} \quad , \quad \xi \in \mathbb{R} .$$

We need the following technical statement which is the main ingredient of the proof of the a priori estimate.

PROPOSITION 4.1. – *Assume g, ρ, σ satisfy the above properties and for $m \geq -1$*

$$(4.5) \quad f_m(x) = \sum_{k \in \mathbb{Z}^n} a_{m,k} \psi(2^m x - k) ,$$

where

$$(4.6) \quad \sup_{k \in \mathbb{Z}^n} |a_{m,k}| \leq 1 ,$$

$$\left[f_{-1}(x) = \sum_{k \in \mathbb{Z}^n} a_k \Psi(x - k) \right] .$$

Then there is a constant $\gamma = \gamma(n)$ such that for $j \leq m$

$$(4.7) \quad \|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} \leq C c_{g^{-1}}^n c_g^{n+1} 2^{\gamma j} \rho(m) .$$

Proof. – We have

$$(4.8) \quad \Delta_j(f_m \circ g^{-1})(x) = \sum_k a_{m,k} 2^{nj} \int \varphi(2^j y) \psi(2^m g^{-1}(x - y) - k) dy .$$

Since the statement is evident for $j = m = -1$,

$$\left[\left\| \sum_k a_k \Psi(x - k) \right\|_{L^\infty} \leq C \sup_k |a_k| \right]$$

we will assume that $m \geq 0$.

Let

$$(4.9) \quad \begin{aligned} h_{j,m,k}(x) &= 2^{nj} \int \varphi(2^j y) \psi(2^m g^{-1}(x - y) - k) dy \\ &= \int \psi(2^m g^{-1}(x - 2^{-j} y) - k) \varphi(y) dy , \end{aligned}$$

where we made a change of variables $y \rightarrow 2^{-j} y$. Then since $\int \psi(y) dy = 0$,

$$(4.10) \quad \begin{aligned} h_{j,m,k}(2^{-j} x + g(2^{-m} k)) &= \int \psi(2^m g^{-1}(2^{-j}(x - y) + g(2^{-m} k)) - k) \varphi(y) dy \\ &= \left(\int_{|x-y| < \frac{|x|}{2}} + \int_{|x-y| \geq \frac{|x|}{2}} \right) q(x - y) \int_0^1 \nabla \varphi(x + \tau(y - x)) \cdot (y - x) d\tau dy \\ &= I + J , \end{aligned}$$

where

$$q(z) = \psi(2^m g^{-1}(2^{-j} z + g(2^{-m} k)) - k) .$$

We estimate I first.

Since $\varphi \in S(\mathbb{R}^n)$ we have for any real M

$$(4.11) \quad \begin{aligned} |I| &\leq \int_{|x-y| < \frac{|x|}{2}} |q(x - y)| |x - y| \int_0^1 |\nabla \varphi(x + \tau(y - x))| d\tau dy \\ &\leq C \int_{|z| < \frac{|x|}{2}} |q(z)| |z| dz (1 + |x|)^{-M} . \end{aligned}$$

We substitute

$$(4.12) \quad z = 2^j (g(2^{-m}(w + k)) - g(2^{-m} k)) ;$$

$$g(2^{-m}(w + k)) = 2^{-j} z + g(2^{-m} k) ;$$

$$(4.13) \quad w = 2^m g^{-1}(2^{-j} z + g(2^{-m} k)) - k .$$

Since g is a volume preserving homeomorphism, for any real M_1, M_2 and since ρ is a decreasing function,

$$\begin{aligned}
 (4.14) \quad \int_{|z| < \frac{|x|}{2}} |q(z)| |z| dz &= 2^{(j-m)n+j} \int_{|z| < \frac{|x|}{2}} |\psi(w)| |g(2^{-m}(w+k)) - g(2^{-m}k)| dw \\
 &\leq 2^{(j-m)n+j} c_g \left(\int_{|w| < 2^m} + \int_{|w| \geq 2^m} \right) |\psi(w)| \rho(m - \log_2 |w|) dw \\
 &\leq 2^{(j-m)n+j} c_g \int_{|w| \geq 2^m} |\psi(w)| 2^{-m} |w| dw \\
 &\quad + 2^{(j-m)n+j} C c_g \sum_{\ell=-m}^{\infty} \rho(m+\ell) 2^{-n\ell} (1+2^{-\ell})^{-M_1} \\
 &\leq C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \left(\rho(0) 2^{-(M_1-n)m} \right. \\
 &\quad \left. + \rho(1) 2^{-(M_1-n)(m-1)} + \dots + \rho(m) 2^0 \right. \\
 &\quad \left. + \rho(m+1) 2^{-n} + \rho(m+2) 2^{-2n} \right. \\
 &\quad \left. + \dots + \rho(m+\ell) 2^{-\ell n} + \dots \right) \\
 &\leq C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \left(\rho(0) 2^{-(M_1-n)m} \right. \\
 &\quad \left. + \rho(1) 2^{-(M_1-n)(m-1)} + \dots + \rho(m) 2^0 \right).
 \end{aligned}$$

LEMMA 4.2. – Let ρ be any function that satisfies (4, 3), (4.4). Let $M_3 > M_4 > 0$. Then

$$\rho(0)^{M_4} 2^{-M_3 m} + \rho(1)^{M_4} 2^{-M_3(m-1)} + \dots + \rho(m)^{M_4} 2^0 \leq C \rho(m)^{M_4}.$$

Proof. – Since $\log_2 \rho$ is convex,

$$\begin{aligned}
 &\rho(0)^{M_4} 2^{-mM_3} + \rho(1)^{M_4} 2^{-(m-1)M_3} + \dots + \rho(m)^{M_4} 2^0 \\
 &\leq \rho(0)^{M_4} 2^{-mM_3} + \rho(0)^{\frac{m-1}{m}M_4} \rho(m)^{\frac{1}{m}M_4} 2^{-(m-1)M_3} + \dots + \rho(m)^{M_4} 2^0 \\
 &\leq \rho(m)^{M_4} \left\{ \left(\frac{\rho(0)}{\rho(m)} \right)^{\frac{m}{m}M_4} 2^{-M_3 m} + \left(\frac{\rho(0)}{\rho(m)} \right)^{\frac{m-1}{m}M_4} 2^{-M_3(m-1)} + \dots + \left(\frac{\rho(0)}{\rho(m)} \right)^0 2^0 \right\} \\
 &\leq \rho(m)^{M_4} \left(1 - 2^{-M_3} \rho(m)^{-M_4/m} \right)^{-1}.
 \end{aligned}$$

But $\rho(m) \geq 2^{-m}$; therefore, $\rho(m)^{1/m} \geq \frac{1}{2}$. This implies the statement.

REMARK 4.3. – In particular,

$$\sigma(0)^{M_4} 2^{-M_3 m} + \sigma(1)^{M_4} 2^{-M_3(m-1)} + \dots + \sigma(m)^{M_4} 2^0 \leq C \sigma(m)^{M_4}.$$

To continue the proof of the proposition we note that the right side of (4.14) is

$$\leq C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \rho(m) \leq C c_g 2^{(j-m)n+j} \rho(m)$$

in case $M_2 \geq 0$.

Choosing $M_2 = 0$, $M_1 = n + 2$ we obtain from (4.11), (4.14)

$$|I| \leq C c_g 2^{(j-m)n+j} \rho(m) (1 + |x|)^{-M}.$$

We now turn to J in the right side of (4.10)

$$(4.15) \quad |J| \leq \int_{|x-y| \geq \frac{|x|}{2}} |q(x-y)| \int_0^1 d\tau |\nabla \varphi(x + \tau(y-x))| |y-x| dy \\ \leq C \int_{|x-y| \geq \frac{|x|}{2}} |q(x-y)| |x-y| (1 + |x-y|)^{-1} dy$$

since

$$\int_0^1 |\nabla \varphi(x + \tau(y-x))| d\tau \leq C(1 + |x-y|)^{-1}.$$

The right side of (4.15) after the substitution (4.12), (4.13) becomes

$$(4.16) \quad \leq C \int_{|z| \geq \frac{|x|}{2}} |q(z)| |z| (1 + |z|)^{M-1} dz (1 + |x|)^{-M} \\ \leq C \int_{|z| \geq \frac{|x|}{2}} |q(z)| (|z| + |z|^M) dz (1 + |x|)^{-M} \\ \leq C 2^{(j-m)n} \int |\psi(w)| \{2^j |g(2^{-m}(w+k)) - g(2^{-m}k)| \\ + 2^{Mj} |g(2^{-m}(w+k)) - g(2^{-m}k)|^M\} dw (1 + |x|)^{-M} \\ \leq C 2^{(j-m)n} \left(\int_{|w| \geq 2^m} + \int_{|w| < 2^m} \right) |\psi(w)| \left((c_g 2^j \rho(m - \log_2 |w|) \right. \\ \left. + c_g^M 2^{Mj} \rho(m - \log_2 |w|)^M) dw (1 + |x|)^{-M} \right) \\ \leq C 2^{(j-m)n} \int_{|w| \geq 2^m} |\psi(w)| \left(2^{j-m} c_g |w| + 2^{M(j-m)} c_g^M |w|^M \right) dw (1 + |x|)^{-M} \\ + C 2^{(j-m)n} \left(\sum_{\ell=-m}^{\infty} \left\{ 2^{-n\ell} (1 + 2^{-\ell})^{-M_5} 2^j c_g \rho(m + \ell) \right. \right. \\ \left. \left. + 2^{-n\ell} (1 + 2^{-\ell})^{-M_5} 2^{Mj} c_g^M \rho(m + \ell)^M \right\} \right) (1 + |x|)^{-M} \\ \leq C c_g^M 2^{(j-m)(n+1)-M_6 m} (1 + |x|)^{-M} \\ + \left(C c_g 2^{(j-m)n+j} \rho(m) + C c_g^M 2^{(j-m)n+Mj} \rho(m)^M \right) (1 + |x|)^{-M} \\ \leq C 2^{(j-m)n} \left(c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M \right) (1 + |x|)^{-M}$$

for appropriate choice of M_5 , M_6 . Combining the estimates for I and J and using (4.10) we obtain ($m \geq j$)

$$\begin{aligned}
 (4.17) \quad \|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} &\leq \sup_x \sum_k |a_{m,k}| |h_{j,m,k}(x)| \\
 &\leq \kappa_{j,m} \sup_k |a_{m,k}| \sup_x \sum_k (1 + 2^j |x - g(2^{-m}k)|)^{-M} \\
 &\leq \kappa_{j,m} \sup_x \sum_k (1 + 2^j |x - g(2^{-m}k)|)^{-M},
 \end{aligned}$$

where

$$(4.18) \quad \kappa_{j,m} = C 2^{(j-m)n} \left(c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M \right).$$

Next we have to take care of the sum in the right side of (4.17). Let $x \in \mathbb{R}^n$ be fixed. Let

$$(4.19) \quad N(x, \lambda) = \# \{k \in \mathbb{Z}^n \mid |x - g(2^{-m}k)| \leq \lambda\}.$$

Then,

$$\begin{aligned}
 (4.20) \quad \sum_k \left(1 + 2^j |x - g(2^{-m}k)|\right)^{-M} &= \int_{-\infty}^{\infty} (1 + 2^j \lambda)^{-M} dN(x, \lambda) \\
 &= M 2^j \int_0^{\infty} (1 + 2^j \lambda)^{-M-1} N(x, \lambda) d\lambda.
 \end{aligned}$$

We need the estimate for $N(x, \lambda)$.

$$\begin{aligned}
 (4.21) \quad N(x, \lambda) &\leq \# \{k \in \mathbb{Z}^n \mid |g^{-1}(x) - 2^{-m}k| \leq c_{g^{-1}} \sigma(-\log_2 \lambda)\} \\
 &= \# \{k \in \mathbb{Z}^n \mid |2^m g^{-1}(x) - k| \leq 2^m c_{g^{-1}} \sigma(-\log_2 \lambda)\} \\
 &\leq C(1 + 2^{mn} c_{g^{-1}}^n \sigma(-\log_2 \lambda)^n).
 \end{aligned}$$

Combining (4.19)–(4.21) we obtain

$$\begin{aligned}
 (4.22) \quad \sum_k (1 + 2^j |x - g(2^{-m}k)|)^{-M} &\leq C 2^j \int_0^{\infty} (1 + 2^j \lambda)^{-M-1} (1 + 2^{mn} c_{g^{-1}}^n \sigma(-\log_2 \lambda)^n) d\lambda \\
 &\leq C + C c_{g^{-1}}^n 2^{mn} \int_0^{\infty} (1 + \lambda)^{-M-1} \sigma(j - \log_2 \lambda)^n d\lambda.
 \end{aligned}$$

To estimate the integral we split

$$\begin{aligned}
 (4.23) \quad &\left(\int_0^{2^j} + \int_{2^j}^{\infty} \right) (1 + \lambda)^{-M-1} \sigma(j - \log_2 \lambda)^n d\lambda \\
 &\leq 2^{-jn} \int_{2^j}^{\infty} (1 + \lambda)^{-M-1} \lambda^n d\lambda + C \sum_{\ell=-j}^{\infty} \sigma(j + \ell)^n 2^{-\ell} (1 + 2^{-\ell})^{-M-1} \\
 &\leq C 2^{-jM} + C \sigma(j)^n \leq C \sigma(j)^n,
 \end{aligned}$$

provided M is sufficiently large ($M > n$). From (4.17)–(4.23) and since $\sigma(j) \geq 2^{-j}$

$$(4.24) \quad \|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} \leq C 2^{(j-m)n} (c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M) (1 + 2^{mn} c_{g^{-1}}^n \sigma(j)^n) \\ \leq C c_{g^{-1}}^n 2^{jn} \sigma(j)^n (c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M) .$$

We now choose $M = (n + 1)$. The right side of (4.24) is

$$\leq C c_{g^{-1}}^n c_g^{n+1} 2^{(2n+1)j} \rho(m) .$$

We set $\gamma = 2n + 1$. This concludes the proof.

5. A priori estimate

Let Γ be a function $\mathbb{R} \rightarrow [1, \infty)$ satisfying the conditions (i)–(iii) above. Let $\Gamma_1(\alpha) = (\alpha + 2)\Gamma(\alpha)$ for $\alpha \in [-1, \infty)$, $\Gamma_1(\alpha) = 1$ for $\alpha \in (-\infty, -1)$. We assume

(iv) Γ_1 satisfies the condition (iii);

(v) Γ_1 is convex;

(vi) $\int_1^\infty \Gamma_1(\alpha)^{-1} d\alpha = \infty$.

The conditions (i), (ii) are automatically satisfied for Γ_1 . It is clear also that (ii)–(iv) follow from one condition

$$\lim_{\alpha \rightarrow \infty} \text{ess} \Gamma'(\alpha) \Gamma(\alpha)^{-1} = 0 .$$

Indeed, assuming this last condition, (ii) follows immediately while (iii) and (iv) are verified using integration by parts in the right side of (iii) (also Γ being replaced by Γ_1). For example, $\Gamma(\alpha) = \log_2^\kappa(\alpha + 3)$, $\alpha \geq -1$, where $\kappa \in (0, 1]$ satisfies all the conditions (i)–(vi). So does $\Gamma(\alpha) = \log_2(\alpha + 3) \log_2 \log_2(\alpha + 5)$, etc.

We fix $f \in B_\Gamma \cap L^{p_0} \cap L^{p_1}$ where $1 < p_0 < 2 < p_1 < \infty$ and consider the Euler equation for the vorticity of an incompressible fluid:

$$(5.1) \quad \dot{\omega}(x, t) = -(v, \nabla) \omega \equiv - \sum_{i=1}^2 v_i \partial_i \omega ; \quad (x, t) \in \mathbb{R}^2 \times [0, T_1] ,$$

$$(5.2) \quad v = \mathcal{K} * \omega ,$$

$$(5.3) \quad \omega(\cdot, 0) = f(\cdot) \in B_\Gamma \cap L^{p_0} \cap L^{p_1} ,$$

where $\mathcal{K}$ is the kernel in Biot-Savart law. We fix $T_1 > 0$ and assume we are given a flow

$$(5.4) \quad v(\cdot) \in \mathcal{K} * C([0, T_1]; B_{\Gamma_1} \cap L^{p_0} \cap L^{p_1}) .$$

According to Theorem 3.1 the classical trajectories are well defined. Indeed, the conditions of the Osgood uniqueness theorem are satisfied as follows from (vi). We refer to [CL] and

[C1] for an appropriate version of the Osgood uniqueness theorem. Since $\operatorname{div} v = 0$, the corresponding flow is volume preserving.

We denote the flow by $g_v(t) = g(t)$, i.e.

$$(5.5) \quad \dot{g}_v(x, t) = v(g_v(x, t), t); \quad g_v(x, 0) = x, \quad x \in \mathbb{R}^2.$$

For any $t \in [0, T_1]$, $g_v(\cdot, t)$ is a volume preserving homeomorphism $\mathbb{R}^2 \xrightarrow{\text{onto}} \mathbb{R}^2$.

Let $\lambda(t)$, $t \in [0, T]$, $0 < T \leq T_1$ be a continuous positive nondecreasing function defined as follows:

$$(5.6) \quad \dot{\lambda} = C\lambda^2,$$

$$(5.7) \quad \lambda(0) = \max(\|\omega(0)\|_{B_{\Gamma_1}}, 1)$$

The constant $C > 0$ will be chosen later.

THEOREM 5.1. — *Let $f \in B_{\Gamma} \cap L^{p_0} \cap L^{p_1}$. Let $v(\cdot)$ be a regular solution to (5.1)–(5.3) satisfying (5.4). Let*

$$(5.8) \quad (\alpha + 2)\Gamma'(\alpha) \leq C \text{ for a.e. } \alpha \in [-1, \infty);$$

$$(5.9) \quad \|f\|_{B_{\Gamma} \cap L^{p_0} \cap L^{p_1}} \leq C.$$

Then there is a $T > 0$, $T \leq T_1$ such that

$$(5.9') \quad \|\omega(t)\|_{B_{\Gamma_1}} \leq \lambda(t), \quad t \in [0, T].$$

Both $T > 0$ and the constant C in (5.6) that defines λ depend only on the constant in (5.9) and on Γ . In addition, $\omega(t)$ is equimeasurable with f .

REMARK 5.2. — Obviously (5.8) implies (vi).

Proof. — We define first $\lambda(\cdot)$ on the whole interval $[0, T_1]$ as follows:

$$\lambda(t) = \max\left(\sup_{0 \leq \tau \leq t} \|\omega(\tau)\|_{B_{\Gamma_1}}, 1\right), \quad t \in [0, T_1].$$

Let C_0 be a sufficiently large constant such that (see (3.7))

$$|w(x) - w(y)| \leq C_0|x - y|, \quad |x - y| \geq \frac{1}{2}$$

for any $w = \mathcal{K} * f_1$ where $f_1 \in L^{p_0} \cap L^{p_1}$ is equimeasurable with f . In particular

$$(5.10) \quad |v(x, t) - v(y, t)| \leq C_0|x - y|, \quad |x - y| \geq \frac{1}{2}, \quad t \in [0, T_1].$$

Also from (3.1) for C_0 sufficiently large

$$(5.11) \quad |v(x, t) - v(y, t)| \leq C_0\lambda(t)\Gamma_1(-\log_2|x - y|)|x - y|, \quad |x - y| \leq \frac{1}{2}.$$

We define

$$\tilde{\Gamma}_1(m, t) = \begin{cases} \lambda(t)^{-1}, & -\infty < m \leq m_1 \\ 1 + (m + 1)\Gamma'_1(-1+), & m_1 \leq m \leq -1 \\ \Gamma_1(m), & -1 < m < \infty \end{cases}$$

with $m_1 = -1 - (1 - \lambda(t)^{-1})\Gamma'_1(-1+)^{-1}$.

Since $\Gamma'_1(-1+) \geq 1$, we have $-2 \leq m_1 \leq 1$. It is clear that $\tilde{\Gamma}_1$ is *convex* on $\mathbb{R}$. It follows from (5.10), (5.11) that

$$(5.12) \quad |v(x, t) - v(y, t)| \leq C_0 \lambda(t) \tilde{\Gamma}_1(-\log_2 |x - y|, t) |x - y|, \quad x \neq y.$$

We estimate stretching in the flow given by the vector-field $v(x, t)$, $t \in [0, T_1]$.

We denote by $\mu(t) = \mu(m, t)$ solution to the Cauchy problem

$$(5.13) \quad \dot{\mu}(m, t) = -C_0(\log_2 e) \lambda(t) \tilde{\Gamma}_1(\mu(m, t), t), \quad \mu(m, 0) = m \in \mathbb{R}.$$

The solution exists and is unique for all $t \in [0, T_1]$ as follows easily from our construction of $\tilde{\Gamma}_1$.

PROPOSITION 5.2. – *Let $x, y \in \mathbb{R}^n$, $x \neq y$. Then*

$$|g_v(t)x - g_v(t)y| \leq 2^{-\mu(m, t)}, \quad t \in [0, T_1]$$

where $m = -\log_2 |x - y|$.

Proof. – Let $\xi(t) = |g_v(t)x - g_v(t)y|$. Then (5.12) yields

$$\begin{aligned} \frac{d}{dt} \xi(t) &\leq |v(g_v(t)x, t) - v(g_v(t)y, t)| \\ &\leq C_0 \lambda(t) \tilde{\Gamma}_1(-\log_2 \xi(t), t) \xi(t). \end{aligned}$$

Therefore,

$$\frac{d}{dt} (-\log_2 \xi(t)) \geq -C_0 \lambda(t) (\log_2 e) \tilde{\Gamma}_1(-\log_2 \xi(t), t).$$

Since the right side is *nonincreasing* on $\mathbb{R}$ as a function of $(-\log_2 \xi(t))$ it follows from the Gronwall type of argument that

$$-\log_2 \xi(t) \geq \mu(m, t), \quad t \in [0, T_1].$$

This proves the proposition.

PROPOSITION 5.3. – *For any fixed $t \in [0, T_1]$ the function*

$$m \longmapsto \mu(m, t)$$

is concave.

Proof. – To simplify the notation we change time and reduce the equation (5.13) to

$$\dot{\mu}(m, t) = -\tilde{\Gamma}_1(\mu(m, t), t), \quad \mu(m, 0) \equiv m.$$

Let $m_1, m_2 \in \mathbb{R}$, $m_1 \neq m_2$. For any $\theta \in (0, 1)$

$$\begin{aligned} (\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t))' &= -\theta\tilde{\Gamma}_1(\mu(m_1, t), t) \\ &- (1 - \theta)\tilde{\Gamma}_1(\mu(m_2, t), t) \leq -\tilde{\Gamma}_1(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t), t) . \end{aligned}$$

Hence,

$$-(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t))' \geq \tilde{\Gamma}_1(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t), t) .$$

Since the right side is a *nonincreasing* function of $-(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t))$ employing the same Gronwall's argument as before yields

$$-(\theta\mu(m_1, t) + (1 - \theta)\mu(m_2, t)) \geq -\mu(\theta m_1 + (1 - \theta)m_2, t) .$$

This completes the proof.

We now set

$$\tilde{\rho}(m, t) = 2^{-\mu(m, t)} .$$

It follows from Proposition 5.3 that $\log_2 \tilde{\rho}(m, t)$ is convex. For $m \leq -2$ $\mu(m, t) = m - C_0(\log_2 e)t$ thus $\tilde{\rho}(m, t) = e^{C_0 t} 2^{-m}$.

We now set

$$\begin{aligned} \rho(m, t) &= \frac{1}{4} e^{-C_0 t} \tilde{\rho}(m - 2, t) ; \\ c_{g_v(t)} &= 4e^{C_0 t} . \end{aligned}$$

Then all the conditions for ρ at the beginning of §4 are satisfied for $g = g_v(t)$, $\rho(m) = \rho(m, t)$. In particular, (4.2) follows from Proposition 5.2.

There are similar statements concerning σ . We just replace $v(t)$ by $-v(T_1 - t)$, $\tilde{\Gamma}_1(m, t)$ by $\tilde{\Gamma}_1(m, T_1 - t)$, etc. Let $\eta(m, t)$ solve the Cauchy problem

$$\dot{\eta}(m, t) = -C_0(\log_2 e)\lambda(T_1 - t)\tilde{\Gamma}_1(\eta(m, t), T_1 - t) , \quad \eta(m, 0) = m .$$

PROPOSITION 5.2'. – Let $x, y \in \mathbb{R}^n$, $x \neq y$. Then

$$|g_{-v(T_1 - \cdot)}(t)x - g_{-v(T_1 - \cdot)}(t)y| \leq 2^{-\eta(m, t)} , \quad t \in [0, T_1]$$

where $m = -\log_2 |x - y|$.

PROPOSITION 5.3'. – For any fixed $t \in [0, T_1]$ the function

$$m \mapsto \eta(m, t)$$

is concave.

We set

$$\sigma(m, T_1) = \frac{1}{4} e^{-C_0 T_1} 2^{-\eta(m - 2, T_1)} .$$

Then (4.1), (4.3) and (4.4) are satisfied for $g = g_v(T_1)$, $\sigma(m) = \sigma(m, T_1)$, $C_{g^{-1}} = 4e^{C_0 T_1}$. Replacing T_1 by arbitrary $t \in [0, T_1]$ we recover the desirable estimates for the homeomorphism $(g_v(t))^{-1}$.

To continue the proof of Theorem 5.1 we have to estimate $\|f \circ g_v(t)^{-1}\|_{B_{\Gamma_1}}$ for $t \in [0, T_1]$.

PROPOSITION 5.4. – *Let $f, g_v(t)$ $t \in [0, T_1]$ be as above. Then*

$$\|f \circ g_v(t)^{-1}\|_{\Gamma_1} \leq C \|f\|_{\Gamma} 2^C \int_0^t \lambda(\tau) d\tau, \quad t \in [0, T_1].$$

Proof. – Let $N \geq -1$. We use the “wavelet” decomposition (1.2) and Proposition 4.1. We have for arbitrary $m \geq N$ (see (4.5)):

$$\begin{aligned} (5.14) \quad & \sum_{j=-1}^N \|\Delta_j(f \circ g_v(t)^{-1})\|_{L^\infty} \\ & \leq \sum_{j=-1}^N \sum_{\ell=-1}^{\infty} \|\Delta_j(f_\ell \circ g_v(t)^{-1})\|_{L^\infty} \\ & = \sum_{j=-1}^N \left(\sum_{\ell=-1}^m + \sum_{\ell=m+1}^{\infty} \right) \|\Delta_j(f_\ell \circ g_v(t)^{-1})\|_{L^\infty} \\ & \leq C(N+2) \sum_{\ell=-1}^m \|f_\ell\|_{L^\infty} + \sum_{j=-1}^N C 2^{\gamma j} \sum_{\ell=m+1}^{\infty} \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}| \\ & \leq C(N+2) \Gamma(m) \|f\|_{\Gamma} + C 2^{\gamma N} \sum_{\ell=m+1}^{\infty} \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}|. \end{aligned}$$

We used in (5.14) that the constants $c_{g_v(t)^{\pm 1}}$ are uniformly bounded with respect to $t \in [0, T_1]$.

The last sum in (5.14) requires some attention. We proceed as follows. Let

$$d_\ell = \sum_{j=-1}^{\ell} \sup_{k \in \mathbb{Z}^n} |a_{j, k}|.$$

Then,

$$\begin{aligned} (5.15) \quad & \sum_{\ell=m+1}^{\infty} \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}| \\ & = \sum_{\ell=m+1}^{\infty} \rho(\ell, t) (d_\ell - d_{\ell-1}) \\ & = -d_m \rho(m+1, t) + \sum_{\ell=m+1}^{\infty} (-\rho(\ell+1, t) + \rho(\ell, t)) d_\ell \\ & \leq -d_m \rho(m+1, t) + \|f\|_{\Gamma} \sum_{\ell=m+1}^{\infty} (-\rho(\ell+1, t) + \rho(\ell, t)) \Gamma(\ell) \\ & \leq -d_m \rho(m+1, t) + \|f\|_{\Gamma} \sum_{\ell=m+1}^{\infty} \int_{\ell}^{\ell+1} -\partial_\xi \rho(\xi, t) d\xi \Gamma(\ell) \\ & \leq -d_m \rho(m+1, t) + C \|f\|_{\Gamma} \int_{m+1}^{\infty} -\partial_\xi \rho(\xi, t) \Gamma(\xi) d\xi \end{aligned}$$

since Γ satisfies (ii). Since (5.8) is satisfied we can estimate the integral in the right side of (5.15) as follows:

$$\begin{aligned} & \int_{m+1}^{\infty} -\partial_{\xi} \rho(\xi, t) \Gamma(\xi) \, d\xi \\ & \leq C \int_{m+1}^{\infty} \partial_{\xi} \mu(\xi - 2, t) 2^{-\mu(\xi-2, t)} \Gamma(\xi) \, d\xi \\ & = C \int_{m-1}^{\infty} \partial_{\xi} \mu(\xi, t) 2^{-\mu(\xi, t)} \Gamma(\xi + 2) \, d\xi \\ & \leq C \int_{m-1}^{\infty} \partial_{\xi} \mu(\xi, t) 2^{-\mu(\xi, t)} \Gamma(\xi) \, d\xi \\ & = C \int_{\mu(m-1, t)}^{\infty} 2^{-\mu} \Gamma(\xi(\mu, t)) \, d\mu . \end{aligned}$$

We have from (5.13)

$$(5.16) \quad \Gamma(\xi) - \Gamma(\mu) = C_0(\log_2 e) \int_0^t \lambda(\tau) \Gamma'(\mu(\tau)) \tilde{\Gamma}_1(\mu(\tau), \tau) \, d\tau .$$

This combined with (5.8) yields

$$\Gamma(\xi) \leq \Gamma(\mu) 2^{C \int_0^t \lambda(\tau) \, d\tau} .$$

Hence,

$$\begin{aligned} (5.17) \quad & \sum_{\ell=m+1}^{\infty} \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}| \\ & \leq C \|f\|_{\Gamma} 2^{C \int_0^t \lambda(\tau) \, d\tau} \Gamma(m) 2^{-\mu(m-1, t)} \\ & \leq C \|f\|_{\Gamma} 2^{C \int_0^t \lambda(\tau) \, d\tau} \Gamma(m) \rho(m+1, t) . \end{aligned}$$

The choice of m in (5.14) so far was arbitrary. We now choose m so that (assuming $N \geq 1$)

$$(5.18) \quad \mu(m-1, t) \geq \gamma N > \mu(m-2, t) .$$

As follows from (5.17) the second term in the right side of (5.14) is

$$(5.19) \quad \leq C \|f\|_{\Gamma} 2^{C \int_0^t \lambda(\tau) \, d\tau} \Gamma(m) .$$

We have to estimate $\Gamma(m)$ as above

$$\Gamma(m) \leq \Gamma(\mu(m, t)) 2^{C \int_0^t \lambda(\tau) \, d\tau} .$$

Furthermore, in general, for all m

$$\partial_m \mu(m, t) \leq 1 .$$

Therefore, using (5.8)

$$(5.20) \quad \begin{aligned} \Gamma(m) &\leq C\Gamma(\gamma N)2^C \int_0^t \lambda(\tau) d\tau \\ &\leq C\Gamma(N)2^C \int_0^t \lambda(\tau) d\tau . \end{aligned}$$

It follows from (5.14), (5.19), (5.20) that for $N \geq 1$

$$(5.21) \quad \sum_{j=-1}^N \|\Delta_j(f \circ g_v^{-1}(t))\|_{L^\infty} \leq C\Gamma_1(N)\|f\|_\Gamma 2^C \int_0^t \lambda(\tau) d\tau .$$

This concludes the proof of the Proposition.

We can now finish the proof of the Theorem. Let $\tilde{\lambda}(\cdot)$ satisfy the following ordinary differential equation:

$$(5.22) \quad \dot{\tilde{\lambda}} = C\tilde{\lambda}^2, \quad \tilde{\lambda}(0) = \lambda(0), \quad t \in [0, T],$$

where the constant C is sufficiently large. The time $T > 0$ is chosen to be less than the blow-up time for (5.22). Then (5.9') is satisfied because (5.21) implies $\lambda(t) \leq \tilde{\lambda}(t)$, $t \in [0, T]$. In addition to this $f \circ g_v(t)^{-1}$ is equimeasurable with f .

This concludes the proof.

We now give an a priori estimate on an infinite interval.

THEOREM 5.5. — *Let $f \in B_\Gamma \cap L^{p_0} \cap L^{p_1}$ where Γ satisfies (i)–(v). Let $v(t)$ be a regular solution to (5.1)–(5.3) such that*

$$v(\cdot) \in \mathcal{K} * C_{\text{loc}}([0, \infty); B_{\Gamma_1} \cap L^{p_0} \cap L^{p_1}) .$$

Assume

$$(5.23) \quad \Gamma'(\alpha)\Gamma_1(\alpha) \leq C \quad \text{a.e. } \alpha \geq -1 ;$$

$$(5.24) \quad \|f\|_{B_\Gamma \cap L^{p_0} \cap L^{p_1}} \leq C .$$

Then there exists a positive continuous nondecreasing function $\lambda(\cdot)$ on $[0, \infty)$ such that

$$(5.25) \quad \|\omega(t)\|_{B_{\Gamma_1}} \leq \lambda(t), \quad t \in [0, \infty)$$

This function depends only on the constant in (5.24) and on Γ .

Proof. — We have from (5.16), (5.23)

$$(5.26) \quad \Gamma(\xi) \leq \Gamma(\mu) + C \int_0^t \lambda(\tau) d\tau .$$

Therefore,

$$(5.27) \quad \int_{m+1}^\infty -\partial_\xi \rho(\xi, t) \Gamma(\xi) d\xi \leq C\rho(m+1, t) \left\{ \Gamma(\mu(m, t)) + \int_0^t \lambda(\tau) d\tau \right\} .$$

This yields

$$\sum_{\ell=m+1}^{\infty} \rho(\ell, t) \sup_{k \in \mathbb{Z}^n} |a_{\ell, k}| \leq C \|f\|_{\Gamma} \rho(m+1, t) \left\{ \Gamma(\mu(m, t)) + \int_0^t \lambda(\tau) d\tau \right\}.$$

We choose m so that (5.18) is satisfied. Then the second term in the right side of (5.14) is

$$\begin{aligned} (5.28) \quad & \leq C \|f\|_{\Gamma} \left\{ \Gamma(\mu(m, t)) + \int_0^t \lambda(\tau) d\tau \right\} \\ & \leq C \|f\|_{\Gamma} \left\{ \Gamma(N) + \int_0^t \lambda(\tau) d\tau \right\}. \end{aligned}$$

Using (5.26), (5.28), (5.14) we get for $N \geq 1$

$$\begin{aligned} \sum_{j=-1}^N \|\Delta_j(f \circ g_v(t)^{-1})\|_{L^\infty} & \leq C \|f\|_{\Gamma} N \left(\Gamma(N) + \int_0^t \lambda(\tau) d\tau \right) \\ & \leq C \left(1 + \int_0^t \lambda(\tau) d\tau \right) \|f\|_{\Gamma} \Gamma_1(N). \end{aligned}$$

We fix an arbitrary $T > 0$ and choose $\tilde{\lambda}$ to satisfy the initial value problem

$$\dot{\tilde{\lambda}} = C \tilde{\lambda}, \quad \tilde{\lambda}(0) = C, \quad t \in [0, T],$$

where C is a sufficiently large constant. Then

$$\|f \circ g_v(t)^{-1}\|_{B_{\Gamma_1}} \leq \lambda(t) \leq \tilde{\lambda}(t), \quad t \in [0, T].$$

This concludes the proof.

EXAMPLE 5.6. – The function

$$\Gamma(\alpha) = \log_2^\kappa(\alpha + 3), \quad \alpha \geq -1$$

satisfies (5.23) for $0 < \kappa \leq \frac{1}{2}$.

6. Commutator estimates

Let u be a vector field, $\operatorname{div} u = 0$, $u \in L^\infty$, $\nabla u \in B_\Pi$. Let the function $\Pi : \mathbb{R} \rightarrow [1, \infty)$ satisfies the conditions (i)–(iii) above. Let (e.g.) $w \in L^\infty$. We define as in [BC] for $j \geq 1$

$$R_j(u, w) = \Delta_j(u, \nabla)w - (S_{j-2}u, \nabla)\Delta_j w.$$

We use the normalization $\hat{\Phi}(\xi) + \sum_{j=0}^{\infty} \hat{\varphi}(2^{-j}\xi) = 1$ with the same conditions on the support as in §1. Then $-\hat{\Phi}(\xi/2) + \hat{\Phi}(\xi) = \hat{\varphi}(\xi)$.

For $|j - j'| \geq 2$ we have $\Delta_j \Delta_{j'} = 0$. We also use the standard notation (when a and b are functions or distributions)

$$T_a b = \sum_{j=-1}^{\infty} S_{j-2} a \Delta_j b ;$$

$$R(a, b) = \sum_{|j-j'| \leq 1} \Delta_j a \Delta_{j'} b .$$

Then Bony's [B] formula (see also [M1]) reads

$$a \cdot b = T_a b + T_b a + R(a, b) .$$

THEOREM 6.1. – *There is an absolute constant M_0 so that the following inequality holds true:*

$$(6.1) \quad \|R_j(u, w)\|_{L^\infty} \leq C \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u\|_{L^\infty} + \|S_{j'-2} \nabla u\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right\} \\ + C 2^j \sum'_{\substack{j' \geq j - M_0 \\ |j' - j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} .$$

When $j' = -1$ the factor $\|\Delta_{j'} \nabla u\|_{L^\infty}$ in the last sum ought to be replaced by $\|\Delta_{-1} u\|_{L^\infty}$.

Here and below the notation $\sum'$ is to indicate this convention: $\|\Delta_{-1} u\|_{L^\infty}$ instead of $\|\Delta_{-1} \nabla u\|$ (also for u_1, w in the next section).

REMARK 6.2. – The first term in the first sum could be estimated as follows:

$$\|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u\|_{L^\infty} \leq C \|S_{j'-2} w\|_{L^\infty} \|\Delta_{j'} \nabla u\|_{L^\infty} , \quad j' \geq -1 .$$

Notice that both the left side and the right side vanish for $j' \leq 0$.

Proof. – The proof will follow very closely the arguments of H. Bahouri and J.-Y. Chemin [BC]. We have

$$(6.2) \quad R_j(u, w) = \sum_{\ell=1}^4 R_j^\ell(u, w) ,$$

where

$$(6.3) \quad \begin{cases} R_j^1(u, w) = \sum_{k=1}^n \Delta_j T_{\partial_k w} u_k \\ R_j^2(u, w) = - \sum_{k=1}^n [T_{u_k} \partial_k, \Delta_j] w \\ R_j^3(u, w) = \sum_{k=1}^n T_{u_k - S_{j-2} u_k} \partial_k \Delta_j w \\ R_j^4(u, w) = \sum_{k=1}^n \{ \Delta_j R(u_k, \partial_k w) - R(S_{j-2} u_k, \Delta_j \partial_k w) \} . \end{cases}$$

Indeed,

$$\begin{aligned}
 \Delta_j(u, \nabla)w &= \Delta_j \sum_{k=1}^n u_k \partial_k w \\
 &= \Delta_j \sum_{k=1}^n \{T_{\partial_k w} u_k + T_{u_k} \partial_k w + R(u_k, \partial_k w)\} \\
 &= \sum_{\ell=1}^2 R_j^\ell(u, w) + \sum_{k=1}^n \{T_{u_k} \partial_k \Delta_j w + \Delta_j R(u_k, \partial_k w)\} \\
 &= \sum_{\ell=1}^3 R_j^\ell(u, w) + \sum_{k=1}^n \{T_{S_{j-2}u_k} \partial_k \Delta_j w + \Delta_j R(u_k, \partial_k w)\} \\
 &= \sum_{\ell=1}^4 R_j^\ell(u, w) + \sum_{k=1}^n \left\{ T_{S_{j-2}u_k} \partial_k \Delta_j w + T_{\partial_k \Delta_j w} S_{j-2}u_k \right. \\
 &\quad \left. + R(S_{j-2}u_k, \Delta_j \partial_k w) - T_{\partial_k \Delta_j w} S_{j-2}u_k \right\} \\
 &= \sum_{\ell=1}^4 R_j^\ell + (S_{j-2}u, \nabla) \Delta_j w - \sum_{k=1}^n T_{\partial_k \Delta_j w} S_{j-2}u_k .
 \end{aligned}$$

We claim the last term is vanishing.

Indeed,

$$\sum_{k=1}^n T_{\partial_k \Delta_j w} S_{j-2}u_k = \sum_{k=1}^n \sum_{j'} S_{j'-2} \partial_k \Delta_j w \cdot \Delta_{j'} S_{j-2}u_k .$$

The second factor vanishes for $j' \geq j$. The first factor vanishes for $j' \leq j$. This proves (6.2). We estimate the terms in the right side of (6.2) in L^∞ . We have

$$R_j^1(u, w) = \sum_{k=1}^n \Delta_j \sum_{j'} S_{j'-2} \partial_k w \Delta_{j'} u_k .$$

All the terms in this sum with $|j - j'| > M_0$ vanish. Hence,

$$(6.4) \quad \|R_j^1(u, w)\|_{L^\infty} \leq \sum_{k=1}^n \sum_{|j'-j| \leq M_0} \|S_{j'-2} \partial_k w\|_{L^\infty} \|\Delta_{j'} u_k\|_{L^\infty} .$$

We may assume $j' \geq 0$ in the right side of (6.4) since otherwise $S_{j'-2} \partial_k w = 0$. Hence,

$$(6.5) \quad \|\Delta_{j'} u_k\|_{L^\infty} \leq C 2^{-j} \|\Delta_{j'} \nabla u\|_{L^\infty} .$$

Also

$$(6.6) \quad \|S_{j'-2} \partial_k w\|_{L^\infty} \leq C 2^j \|S_{j-2} w\|_{L^\infty} .$$

The combination of (6.4)–(6.6) yields

$$\begin{aligned}
 (6.7) \quad \|R_j^1(u, w)\|_\infty &\leq C \sum_{|j'-j| \leq M_0} \|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u\|_{L^\infty} \\
 &\leq C \sum_{|j'-j| \leq M_0} \|S_{j'-2} w\|_{L^\infty} \|\Delta_{j'} \nabla u\|_{L^\infty} , \quad j \geq -1 .
 \end{aligned}$$

We now study $R_j^2(u, w)$

$$\begin{aligned} R_j^2(u, w) &= - \sum_{j'} \sum_{k=1}^n [S_{j'-2} u_k \partial_k \Delta_{j'}, \Delta_j] w \\ &= - \sum_{k=1}^n \sum_{j'} [S_{j'-2} u_k, \Delta_j] \partial_k \Delta_{j'} w \\ &= - \sum_{k=1}^n \sum_{|j'-j| \leq M_0} [S_{j'-2} u_k, \Delta_j] \partial_k \Delta_{j'} w . \end{aligned}$$

We use as in [BC] the following explicit representation of the operator in the right side

$$\begin{aligned} R_j^2(u, w)(x) &= - \sum_{k=1}^n \sum_{|j'-j| \leq M_0} 2^{jn} \int \varphi(2^j(x-y)) (S_{j'-2} u_k(x) - S_{j'-2} u_k(y)) \partial_k \Delta_{j'} w(y) dy \\ &= - \sum_{|j'-j| \leq M_0} \sum_{k=1}^n 2^{j(n+1)} \int \partial_k \varphi(2^j(x-y)) (S_{j'-2} u_k(x) - S_{j'-2} u_k(y)) \Delta_{j'} w(y) dy \end{aligned}$$

since $\operatorname{div} S_{j'-2} u(y) = S_{j'-2} \operatorname{div} u(y) = 0$.

Applying the Taylor's formula we obtain

$$\begin{aligned} R_j^2(u, w)(x) &= - \sum_{|j'-j| \leq M_0} \sum_{k=1}^n 2^{j(n+1)} \int \partial_k \varphi(2^j(x-y)) \\ &\quad \sum_{m=1}^n \int_0^1 S_{j'-2} \partial_m u_k(x + \tau(y-x)) \cdot (x_m - y_m) d\tau \Delta_{j'} w(y) dy \\ &= - \sum_{|j'-j| \leq M_0} \sum_{k=1}^n \sum_{m=1}^n \int \partial_k \varphi(z) z_m \int_0^1 S_{j'-2} \partial_m u_k(x - \tau 2^{-j} z) \Delta_{j'} w(x - 2^{-j} z) d\tau dz . \end{aligned}$$

Minkowski integral inequality yields:

$$\begin{aligned} (6.8) \quad \|R_j^2(u, w)\|_{L^\infty} &\leq C \sum_{m=1}^n \sum_{k=1}^n \sum_{|j'-j| \leq M_0} \|S_{j'-2} \partial_m u_k\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \\ &\leq C \sum_{|j'-j| \leq M_0} \|S_{j'-2} \nabla u\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} . \end{aligned}$$

Next we estimate the term $R_j^3(u, w)$. The argument similar to the one used for R_j^1 leads to the estimate

$$(6.9) \quad \|R_j^3(u, w)\| \leq C \sum_{m=j-1}^j \|\Delta_m \nabla u\|_{L^\infty} \|\Delta_j w\|_{L^\infty} , \quad j \geq -1 ,$$

except when $m = -1$ the factor $\|\Delta_{-1}\nabla u\|_{L^\infty}$ ought to be replaced by $\|\Delta_{-1}u\|_{L^\infty}$. We now turn to $R_j^4(u, w)$:

$$(6.10) \quad R_j^4(u, w) = R_j^{4,1}(u, w) + R_j^{4,2}(u, w) ,$$

$$(6.11) \quad R_j^{4,1}(u, w) = \sum_{k=1}^n \Delta_j \partial_k R(u_k - S_{j-2}u_k, w) ,$$

$$(6.12) \quad R_j^{4,2}(u, w) = \sum_{k=1}^n \Delta_j R(S_{j-2}u_k, \partial_k w) - R(S_{j-2}u_k, \Delta_j \partial_k w) .$$

We have:

$$R_j^{4,1}(u, w) = \sum_{k=1}^n \partial_k \Delta_j \sum_{|j'-j''|\leq 1} \Delta_{j'}(u_k - S_{j-2}u_k) \Delta_{j''} w .$$

Therefore,

$$(6.13) \quad \begin{aligned} \|R_j^{4,1}(u, w)\|_{L^\infty} &\leq C 2^j \sum_{j'\geq j-M_0} \sum_{|j'-j''|\leq 1} \|\Delta_{j'}u\|_{L^\infty} \|\Delta_{j''}w\| \\ &\leq C 2^j \sum_{j'\geq j-M_0} \sum_{|j'-j''|\leq 1} 2^{-j'} \|\Delta_{j'}\nabla u\|_{L^\infty} \|\Delta_{j''}w\|_{L^\infty} , \end{aligned}$$

except when $j' = -1$ the factor $\|\Delta_{j'}\nabla u\|_{L^\infty}$ ought to be replaced by $\|\Delta_{-1}u\|_{L^\infty}$. We now estimate the term $R_j^{4,2}(u, w)$:

$$(6.14) \quad \begin{aligned} R_j^{4,2}(u, w) &= \sum_{k=1}^n \sum_{|j'-j''|\leq 1} \left\{ \Delta_j((\Delta_{j'}S_{j-2}u_k) \cdot \Delta_{j''}\partial_k w) \right. \\ &\quad \left. - (\Delta_{j'}S_{j-2}u_k)(\Delta_{j''}\Delta_j\partial_k w) \right\} \\ &= \sum_{k=1}^n \sum_{j-1\geq j'\geq j-M_0} \sum_{|j'-j''|\leq 1} [\Delta_j, \Delta_{j'}S_{j-2}u_k] \Delta_{j''}\partial_k w . \end{aligned}$$

Using the explicit integral representation as above and proceeding as with the term R_j^2 we arrive at

$$(6.15) \quad \begin{aligned} \|R_j^{4,2}(u, w)\|_{L^\infty} &\leq C \sum_{\substack{|j'-j|\leq M_0 \\ |j'-j''|\leq 1}} \|\Delta_{j'}S_{j-2}\nabla u\|_{L^\infty} \|\Delta_{j''}w\|_{L^\infty} \\ &\leq C \sum'_{\substack{|j'-j|\leq M_0 \\ |j'-j''|\leq 1}} \|\Delta_{j'}\nabla u\|_{L^\infty} \|\Delta_{j''}w\|_{L^\infty} \end{aligned}$$

Adding up the estimates (6.7), (6.8), (6.9), (6.13) and (6.15) and using (6.2), (6.3), (6.10)–(6.12) yield the statement of the Theorem.

7. Uniqueness theorem

Let Π be a function $\mathbb{R} \rightarrow [1, \infty)$ that satisfies the conditions (i)–(iii) above. In this section the dimension $n \geq 2$ is *arbitrary*.

THEOREM 7.1 (Uniqueness theorem). – *Let the following conditions be satisfied:*

$$(7.1) \quad \int_1^\infty \Pi(\alpha)^{-1} d\alpha = \infty ;$$

$$(7.1') \quad \Pi(\alpha)2^{-\alpha} \text{ is nonincreasing for } \alpha \geq C, \quad \lim_{\alpha \rightarrow \infty} \Pi(\alpha)2^{-\alpha} = 0 .$$

Let for $t \in [0, T]$, $u_1(x, t)$, $u_2(x, t)$ be two vector fields in $\mathbb{R}^n$ and $\omega_{1,2} = \text{curl } u_{1,2}$. Assume the following:

$$(7.2) \quad \omega_{1,2} \in L^\infty([0, T]; L^{p_0}) , \quad \|\omega_{1,2}(\cdot)\|_\Pi \in L^\infty([0, T]) ;$$

$$(7.3) \quad u_{1,2} = \mathcal{K} * \omega_{1,2} ;$$

$$(7.4) \quad \text{div} u_{1,2} = 0 ;$$

$$(7.5) \quad \dot{u}_{1,2} = -(u_{1,2}, \nabla)u_{1,2} - \nabla p_{1,2} ;$$

$$(7.6) \quad \omega_{1,2}(\cdot, 0) = f(\cdot) \in B_\Pi \cap L^{p_0} ,$$

where $p_0 \in (1, n)$.

Then, $u_1 = u_2$ on $[0, T]$.

Proof. – Let $w = u_1 - u_2$, $\omega = \text{curl } w$, $p = p_1 - p_2$. Then,

$$(7.7) \quad \dot{w} = -(u_1, \nabla)w - (w, \nabla)u_2 - \nabla p , \quad t \in [0, T] ;$$

$$(7.8) \quad \text{div} w = 0 ;$$

$$(7.9) \quad w|_{t=0} = 0 .$$

Note that $w \in L^\infty(\mathbb{R}^n \times [0, T])$ as follows from (7.2), (7.3) and Theorem 3.1.

It follows from (7.7) that for any $j \geq -1$

$$(7.10) \quad \begin{aligned} \Delta_j \dot{w} = & -(S_{j-2}u_1, \nabla)\Delta_j w - R_j(u_1, w) \\ & - (S_{j-2}w, \nabla)\Delta_j u_2 - R_j(w, u_2) \\ & - \Delta_j \nabla p . \end{aligned}$$

We have according to Theorem 6.1

$$(7.11) \quad \|R_j(u_1, w)\|_{L^\infty} \leq C \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} \nabla u_1\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right. \\ \left. + \|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u_1\|_{L^\infty} \right\} \\ + C 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} ,$$

where $\sum'$ indicates the same agreement as in Theorem 6.1.

Likewise,

$$(7.12) \quad \|R_j(w, u_2)\|_{L^\infty} \leq C \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} \nabla w\|_{L^\infty} \|\Delta_{j'} u_2\|_{L^\infty} \right. \\ \left. + \|S_{j'-2} \nabla u_2\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right\} \\ + C 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} .$$

We have to estimate $\Delta_j \nabla p$.

Taking div of both sides of (7.10) we get

$$(7.12') \quad \Delta_j \Delta p = -\operatorname{div} R_j(u_1, w) - \operatorname{div} R_j(w, u_2) \\ - \operatorname{tr}(\nabla \Delta_j w \cdot \nabla S_{j-2} u_1) - \operatorname{tr}(\nabla \Delta_j u_2 \cdot \nabla S_{j-2} w) .$$

Here Δ in the left side stands for Laplacian and we have used the solenoidality condition for w and u_2 .

We consider two cases.

Case I. $j \geq 0$. Then from (7.10), (7.8), (7.12') since

$$-\nabla \Delta_j p(x) = \mathcal{F}_{\xi \rightarrow x}(i\xi|\xi|^{-2}(\Delta_j \Delta p)^\wedge(\xi)) ,$$

we can use a standard Littlewood-Paley argument in conjunction with the Bernstein's inequality and arrive at

$$(7.13) \quad \|\Delta_j \nabla p\|_{L^\infty} \leq C \|R_j(u_1, w)\|_{L^\infty} + C \|R_j(w, u_2)\|_{L^\infty} \\ + C 2^{-j} \|S_{j-2} \nabla u_1\|_{L^\infty} \|\Delta_j \nabla w\|_{L^\infty} \\ + C 2^{-j} \|S_{j-2} \nabla w\|_{L^\infty} \|\Delta_j \nabla u_2\|_{L^\infty} \\ \leq C \|R_j(u_1, w)\|_{L^\infty} + C \|R_j(w, u_2)\|_{L^\infty} \\ + C \|S_{j-2} \nabla u_1\|_{L^\infty} \|\Delta_j w\|_{L^\infty} \\ + C \|S_{j-2} \nabla w\|_{L^\infty} \|\Delta_j u_2\|_{L^\infty} .$$

Case II. $j = -1$. Then from (7.7) applying the Bernstein's inequality, choosing arbitrary $p_2 \in [\frac{np_0}{n-p_0}, \infty)$ and using boundedness of H. Weyl projection onto gradients we get

$$\begin{aligned}
 (7.14) \quad \|\Delta_{-1} \nabla p\|_{L^\infty} &\leq C \|\Delta_{-1} \nabla p\|_{L^{p_2}} \\
 &\leq C \left\| - \sum_{k=1}^n \partial_k \Delta_{-1} \{u_{1k} w + w_k u_2\} \right\|_{L^{p_2}} \\
 &\leq C \|\Delta_{-1}(u_1 \otimes w)\|_{L^{p_2}} + C \|\Delta_{-1}(w \otimes u_2)\|_{L^{p_2}} \\
 &\leq C \sum_{|j'-j''| \leq M_0} (\|\Delta_{j'} u_1\|_{L^{p_2}} + \|\Delta_{j'} u_2\|_{L^{p_2}}) \|\Delta_{j''} w\|_{L^\infty} .
 \end{aligned}$$

We now fix an $N \geq -1$ and estimate the sum

$$\begin{aligned}
 (7.15) \quad \sum_{j=-1}^N \|R_j(u_1, w)\|_{L^\infty} + \sum_{j=-1}^N \|R_j(w, u_2)\|_{L^\infty} \\
 + \sum_{j=-1}^N \|\Delta_j \nabla p\|_{L^\infty} + \sum_{j=-1}^N \|(S_{j-2} w, \nabla) \Delta_j u_2\|_{L^\infty} \equiv Q_1 + Q_2 + Q_3 + Q_4 .
 \end{aligned}$$

We will prove below that

$$Q_j \leq C \Pi(N) \|w\|_{B_{\infty,1}^0}, \quad j = 1, \dots, 4 .$$

Here the norm in Besov space $B_{\infty,1}^0$ is defined as follows:

$$B_{\infty,1}^0 = \left\{ w \in S' \mid \|w\|_{B_{\infty,1}^0} = \sum_{j=-1}^{\infty} \|\Delta_j w\|_{L^\infty} < \infty \right\} .$$

The inequality (7.11) yields (see Remark 6.2)

$$\begin{aligned}
 (7.16) \quad Q_1 &\leq C \sum_{j=-1}^N \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2} \nabla u_1\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} + \|S_{j'-2} w\|_{L^\infty} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \right\} \\
 &+ C \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} .
 \end{aligned}$$

The first sum in (7.16) can be taken care of as follows. Using (7.2), (7.3) and Proposition 3.3 we obtain:

$$\begin{aligned}
 (7.17) \quad \sum_{j=-1}^N \sum_{|j'-j| \leq M_0} \{ \|S_{j'-2} \nabla u_1\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} + \|S_{j'-2} w\|_{L^\infty} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \} \\
 \leq C \left(\sup_{-1 \leq j' \leq N+M_0} \|S_{j'-2} \nabla u_1\|_{L^\infty} \right) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty}
 \end{aligned}$$

$$\begin{aligned}
& + C \left(\sum_{j'=-1}^{N+M_0} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \right) \sup_{-1 \leq j' \leq N+M_0} \|S_{j'-2} w\|_{L^\infty} \\
& \leq C \|\omega_1\|_{B_{\Pi} \cap L^{p_0}} \Pi(N+M_0) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty} \\
& \leq C \Pi(N) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty} .
\end{aligned}$$

We now turn to the second sum in the right side of (7.16)

$$\begin{aligned}
(7.18) \quad & \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} \\
& \leq C \sum_{j''=-1}^{\infty} \|\Delta_{j''} w\|_{L^\infty} \left(\sum_{j=-1}^{\min(N, j''+M_0+1)} 2^{j-j''} \right) \sum'_{|j'-j''| \leq 1} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \\
& \leq C 2^{\min(N, j'')-j''} \sum'_{|j'-j''| \leq 1} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \left(\sum_{j''=-1}^{\infty} \|\Delta_{j''} w\|_{L^\infty} \right) .
\end{aligned}$$

We estimate the coefficient in front of $\sum_{j''=1}^{\infty} \|\Delta_{j''} w\|_{L^\infty}$. We consider two cases. In case $j'' \leq N$

$$(7.19) \quad 2^{\min(N, j'')-j''} \sum'_{|j'-j''| \leq 1} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \leq C \Pi(N) \|\omega_1\|_{B_{\Pi} \cap L^{p_0}} \leq C \Pi(N) .$$

In case $j'' > N$

$$\begin{aligned}
(7.20) \quad & 2^{N-j''} \sum'_{|j'-j''| \leq 1} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \\
& \leq C 2^{N-j''} \Pi(j'') \|\omega_1\|_{B_{\Pi} \cap L^{p_0}} \\
& \leq C 2^{N-j''} \Pi(j'') \leq C \int_N^{\infty} 2^{N-\xi} \Pi(\xi) d\xi \\
& \leq C \Pi(N) .
\end{aligned}$$

The combination of (7.18)–(7.20) yields

$$(7.21) \quad \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla u_1\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} \leq C \Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{L^\infty} .$$

It follows from (7.16), (7.17), (7.21) that

$$(7.22) \quad Q_1 \leq C \Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{L^\infty} .$$

We now estimate Q_2 . From (7.12)

$$\begin{aligned}
 (7.23) \quad Q_2 &\leq C \sum_{j=-1}^N \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2}w\|_{L^\infty} \|\Delta_{j'} \nabla u_2\|_{L^\infty} \right. \\
 &\quad \left. + \|S_{j'-2} \nabla u_2\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right\} \\
 &\quad + C \sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} .
 \end{aligned}$$

We take care of the first sum in the right side of (7.23) as follows:

$$\begin{aligned}
 (7.24) \quad &\sum_{j=-1}^N \sum_{|j'-j| \leq M_0} \left\{ \|S_{j'-2}w\|_{L^\infty} \|\Delta_{j'} \nabla u_2\|_{L^\infty} + \|S_{j'-2} \nabla u_2\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \right\} \\
 &\leq C \left(\sup_{-1 \leq j' \leq N+M_0} \|S_{j'-2}w\|_{L^\infty} \right) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} \nabla u_2\|_{L^\infty} \\
 &\quad + C \left(\sup_{-1 \leq j' \leq N+M_0} \|S_{j'-2} \nabla u_2\|_{L^\infty} \right) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'} w\|_{L^\infty}
 \end{aligned}$$

using the same argument as in (7.17). To estimate the second sum in (7.23) we notice that

$$\begin{aligned}
 (7.25) \quad &\sum_{j=-1}^N 2^j \sum'_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} 2^{-j'} \|\Delta_{j'} \nabla w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} \\
 &\leq C \sum_{j=-1}^N 2^j \sum_{\substack{j' \geq j-M_0 \\ |j'-j''| \leq 1}} \|\Delta_{j'} w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} \\
 &\leq C \sum_{j'=-1}^{\infty} \|\Delta_{j'} w\|_{L^\infty} \left(\sum_{j=-1}^{\min(N, j'+M_0)} 2^j \right) \sum_{|j''-j'| \leq 1} \|\Delta_{j''} u_2\|_{L^\infty} \\
 &\leq C \sum_{j=-1}^{\infty} \|\Delta_{j'} w\|_{L^\infty} 2^{\min(N, j')} \sum_{|j''-j'| \leq 1} \|\Delta_{j''} u_2\|_{L^\infty} .
 \end{aligned}$$

In case $j' \leq N$,

$$\begin{aligned}
 (7.26) \quad &2^{\min(N, j')} \sum_{|j''-j'| \leq 1} \|\Delta_{j''} u_2\|_{L^\infty} \leq C 2^{j'} 2^{-j'} \Pi(j') \|\omega_2\|_{B_{\Pi} \cap L^{p_0}} \\
 &\leq C \Pi(N) .
 \end{aligned}$$

In the opposite case $j' > N$,

$$\begin{aligned}
 (7.27) \quad 2^{\min(N, j')} \sum_{|j'' - j'| \leq 1} \|\Delta_{j''} u_2\|_{L^\infty} &\leq C 2^{N-j'} \Pi(j') \|\omega_2\|_{B_{\Pi} \cap L^{p_0}} \\
 &\leq C 2^{N-j'} \Pi(j') \leq C \int_N^\infty 2^{N-\xi} \Pi(\xi) d\xi \\
 &\leq C \Pi(N) .
 \end{aligned}$$

We conclude from (7.23)–(7.27) that

$$(7.28) \quad Q_2 \leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} .$$

We turn now to Q_3 in the right side of (7.15). From (7.13), (7.14), (7.15), (7.22), (7.28) [notice that the last two terms in the right side of (7.13) are already present in the estimates (7.11), (7.12) for $\|R_j(u_1, w)\|_{L^\infty}$, $\|R_j(w, u_2)\|_{L^\infty}$] we obtain:

$$\begin{aligned}
 (7.29) \quad Q_3 &\leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} + C \|\Delta_{-1} \nabla p\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} .
 \end{aligned}$$

Indeed, the factor in the right side on (7.14) is taken care of as follows:

$$\|\Delta_{j'} u_1\|_{L^{p_2}} + \|\Delta_{j'} u_2\|_{L^{p_2}} \leq C \|u_1\|_{L^{p_2}} + C \|u_2\|_{L^{p_2}} .$$

We choose $p_2 = \frac{np_0}{n-p_0}$. Then from (3.9)

$$\begin{aligned}
 (7.29') \quad \|u_{1,2}\|_{L^{p_2}} &\leq C \|\omega_{1,2}\|_{L^{p_0}} \leq C . \\
 Q_4 &\leq \sum_{j=-1}^N \|S_{j-2} w\|_{L^\infty} \|\Delta_j \nabla u_2\|_{L^\infty} \\
 &\leq \sup_{-1 \leq j \leq N} \|S_{j-2} w\|_{L^\infty} \sum_{j=-1}^N \|\Delta_j \nabla u_2\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j=-1}^{N-2} \|\Delta_j w\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} .
 \end{aligned}$$

It follows from (7.15), (7.22), (7.28), (7.29), (7.29') that

$$\begin{aligned}
 (7.30) \quad &\sum_{j=-1}^N \|R_j(u_1, w)\|_{L^\infty} + \sum_{j=-1}^N \|R_j(w, u_2)\|_{L^\infty} \\
 &+ \sum_{j=-1}^N \|\Delta_j \nabla p\|_{L^\infty} + \sum_{j=-1}^N \|(S_{j-2} w, \nabla) \Delta_j u_2\|_{L^\infty} \\
 &\leq C \Pi(N) \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} .
 \end{aligned}$$

Solving the equation (7.10) along characteristics and using (7.9) $[\Delta_j w]_{t=0} = 0$ we obtain, for any $t \in [0, T]$ and any $j \geq -1$,

$$(7.31) \quad \|\Delta_j w(t)\|_{L^\infty} \leq C \int_0^t \left\{ \|R_j(u_1(\tau), w(\tau))\|_{L^\infty} + \|R_j(w(\tau), u_2(\tau))\|_{L^\infty} \right. \\ \left. + \|\Delta_j \nabla p(\tau)\|_{L^\infty} + \|(S_{j-2} w(\tau), \nabla) \Delta_j u_2(\tau)\|_{L^\infty} \right\} d\tau .$$

We have for arbitrary $t \in [0, T]$, $N \geq -1$

$$(7.32) \quad \sum_{j=-1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} = \sum_{j=-1}^N \|\Delta_j w(t)\|_{L^\infty} + \sum_{j=N+1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} \\ \leq \sum_{j=-1}^N \|\Delta_j w(t)\|_{L^\infty} + C \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \nabla w(t)\|_{L^\infty} .$$

We estimate the second term in the right side of (7.32). Let

$$(7.33) \quad d_m = \sum_{k=-1}^m \|\Delta_k \nabla w(t)\|_{L^\infty} .$$

Then, from (7.2)

$$(7.34) \quad \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \nabla w(t)\|_{L^\infty} = \sum_{j=N+1}^{\infty} 2^{-j} (d_j - d_{j-1}) \\ = -2^{-(N+1)} d_N + \sum_{j=N+1}^{\infty} (2^{-j} - 2^{-(j+1)}) d_j \\ \leq C \sum_{j=N+1}^{\infty} 2^{-j} \Pi(j) \|\omega(t)\|_{B_\Pi \cap L^{p_0}} \\ \leq C 2^{-N} \Pi(N) .$$

Using (7.30)–(7.34) we get:

$$(7.35) \quad \sum_{j=-1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} \leq C \Pi(N) \int_0^t \sum_{j=-1}^{\infty} \|\Delta_j w(\tau)\|_{L^\infty} d\tau + C 2^{-N} \Pi(N) .$$

We denote by $\zeta(t)$ the following function on $[0, T]$

$$(7.36) \quad \zeta(t) = \int_0^t \sum_{j=-1}^{\infty} \|\Delta_j w(\tau)\|_{L^\infty} d\tau .$$

Since

$$\left\| \sum_{j=-1}^{\infty} \|\Delta_j w(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} \leq C \left\| \|\omega(\cdot)\|_{\Pi} \right\|_{L^\infty([0, T])} + C \|\omega\|_{L^\infty([0, T]; L^{p_0})} \leq C ,$$

$\zeta(t)$ is monotonically nondecreasing absolutely continuous function. Since $\zeta(0) = 0$ there exists a $t_0 \in [0, T]$ such that

$$\zeta(t) \equiv 0 \text{ on } [0, t_0] \quad ; \quad \zeta(t) > 0 \text{ on } (t_0, T] .$$

If $t_0 = T$ then the uniqueness follows. Therefore, we assume $t_0 < T$. Let ε be small enough so that $t_0 + \varepsilon < T$ and $\zeta(t) < 2^{-M_1-1}$ on $(t_0, t_0 + \varepsilon)$. We will choose M_1 later. For any $t \in (t_0, t_0 + \varepsilon)$ we choose N in (7.35) as follows:

$$N = [-\log_2 \zeta(t)],$$

where the square bracket means the entire part of a real number. Then (7.35) implies:

$$(7.37) \quad \dot{\zeta}(t) \leq C\Pi(-\log_2 \zeta(t)) \cdot \zeta(t) , \quad \zeta(0) = 0 .$$

Notice that from (7.1)

$$\int_0^{1/2} \zeta^{-1} \Pi(-\log_2 \zeta)^{-1} d\zeta = (\log_2 e)^{-1} \int_1^\infty \Pi(\alpha)^{-1} d\alpha = \infty .$$

Therefore, the Osgood uniqueness theorem applies to the initial value problem

$$(7.38) \quad \dot{\eta}(t, \delta) = C\Pi(-\log_2 \eta)\eta , \quad \eta(t_0, \delta) = \delta ,$$

where $\delta > 0$ is small. The solution η exists on $(t_0, t_0 + \varepsilon)$ and depends continuously on the parameter δ . The simple Gronwall's argument now works for comparison of ζ and η . Then $\zeta(t) < \eta(t, \delta)$ for sufficiently small δ and for all $t \in [t_0, t_0 + \varepsilon)$. For otherwise let $t_1 = \min_{t \in [t_0, t_0 + \varepsilon)} \{t \mid \zeta(t) = \eta(t, \delta)\}$. From (7.37), (7.38)

$$(7.39) \quad \begin{aligned} \zeta(t_1) &\leq C \int_{t_0}^{t_1} \Pi(-\log_2 \zeta(\tau)) \zeta(\tau) d\tau \\ &< \delta + C \int_{t_0}^{t_1} \Pi(-\log_2 \eta(\tau, \delta)) \eta(\tau, \delta) d\tau = \eta(t_1, \delta) . \end{aligned}$$

We have to use in (7.39) that the function $\zeta \mapsto \Pi(-\log_2 \zeta)\zeta$ is monotonically decreasing to 0 as $\zeta \rightarrow 0+$. This is just a reformulation of (7.1').

We choose M_1 in such a way that $\Pi(\alpha)2^{-\alpha}$ is nonincreasing for $\alpha \geq M_1$. Now (7.39) contradicts the definition of t_1 . Passing to the limit as $\delta \rightarrow 0+$ we have to conclude that $\zeta \equiv 0$ on $[t_0, t_0 + \varepsilon)$. But this contradicts the definition of t_0 . The proof of uniqueness is now completed.

COROLLARY 7.2. – Assume (7.3), (7.4), (7.5). Let, in addition to this,

$$\begin{aligned} \omega_{1,2} &\in L^\infty([0, T]; L^{p_0}) ; \\ \|\omega_{1,2}(\cdot)\|_{\text{bmo}} &\in L^\infty([0, T]) ; \\ \omega_{1,2}(\cdot, 0) &= f(\cdot) \in \text{bmo} \cap L^{p_0} , \end{aligned}$$

where $p_0 \in (1, n)$. Then $u_1 = u_2$ on $[0, T]$.

Proof. – Let $\Pi(N) = (N + 2)$, $N \geq -1$. Then $\text{bmo} \hookrightarrow B_\Pi$. But

$$\int_1^\infty (N + 2)^{-1} dN = \infty$$

so (7.1) is satisfied. Condition (7.1') is satisfied as well. Therefore uniqueness follows from the Theorem 7.1.

REMARK 7.3. – The same proof works for the space $\Lambda^0 = B_{\infty, \infty}^0$ replacing the space bmo .

8. Construction of the flow

In this section $n = 2$. Let Γ satisfy the conditions (i)–(v).

THEOREM 8.1. – Assume (5.8). Let

$$(8.1) \quad f \in B_\Gamma \cap L^{p_0} \cap L^{p_1}, \quad 1 < p_0 < 2 < p_1 < \infty.$$

Then there exists a $T > 0$ and a solution $\omega(\cdot)$ to the Euler equations

$$(8.2) \quad \dot{u} = -(u, \nabla)u - \nabla p, \quad \omega(0) = f,$$

$$(8.3) \quad u = \mathcal{K} * \omega, \quad \operatorname{div} u = 0,$$

satisfying the following condition

$$(8.4) \quad \omega(\cdot) \in L^\infty([0, T]; L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, T]; B_{\Gamma_1}),$$

where $C_{w^*}([0, T]; B_{\Gamma_1})$ stands for the space of weak* continuous functions with values in B_{Γ_1} in the sense of duality $H'_{\Gamma_1} = B_{\Gamma_1}$.

THEOREM 8.2. – Assume (5.23), (8.1). Then there exists a solution

$$\omega(\cdot) \in L^\infty_{\text{loc}}([0, \infty); L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, \infty); B_{\Gamma_1})$$

to the problem (8.2), (8.3).

REMARK 8.3. – The solution $\omega(\cdot)$ in both theorems is unique provided (7.1') is satisfied. Indeed, (5.8) implies (7.1) and uniqueness follows from the Theorem 7.1 with $\Pi = \Gamma_1$. Likewise (5.23) implies (7.1) with the same Π .

REMARK 8.4. – See the definition of H_{Γ_1} in Section 1 (also the Proposition 1.6).

To construct the solution in Theorems 8.1, 8.2 we use approximation by regular solutions. We need therefore the following result.

THEOREM 8.5 ([C1]). – Let $n = 2$, $p \in (1, \infty)$, $f \in \cap_{r>0} \Lambda^r \cap L^p$. Then there exists a (unique) solution

$$\omega(\cdot) \in \cap_{r>0} L^\infty_{\text{loc}}([0, \infty); \Lambda^r \cap L^p)$$

to the problem (8.2), (8.3).

Here Λ^r , $r \in \mathbb{R}$ denotes the Hölder space. For convenience we recall the definition.

DEFINITION 8.6. – Let $r \in \mathbb{R}$.

$$\Lambda^r = B^\infty_{\infty, \infty} = \left\{ f \in S' \mid \|f\|_{\Lambda^r} = \sup_{j \geq -1} 2^{rj} \|\Delta_j f\|_{L^\infty} < \infty \right\}.$$

In particular for $r > 0$, $\Lambda^r \subset L^\infty$.

PROPOSITION 8.7. – For $r > 0$, $\Lambda^r \subset B_\Gamma$.

Proof.

$$\sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} \leq \sum_{j=-1}^N 2^{-rj} \|f\|_{\Lambda^r} \leq (1 - 2^{-r})^{-1} \|f\|_{\Lambda^r} \leq CT(N), \quad N \geq -1.$$

Proof of Theorem 8.1. – We construct for any $m \geq 1$ the solution

$$(8.5) \quad \omega_m(\cdot) \in \cap_{r>0} L_{\text{loc}}^\infty([0, \infty); \Lambda^r \cap L^{p_0})$$

provided by Theorem 8.5 such that

$$(8.6) \quad \omega_m(0) = S_m f \in \cap_{r>0} \Lambda^r.$$

It follows from Littlewood-Paley theory and (8.1) that

$$(8.7) \quad \|\omega_m(0)\|_{L^{p_0}} \leq C, \quad \|\omega_m(0)\|_{L^{p_1}} \leq C.$$

We also have

$$(8.8) \quad \|\omega_m(0)\|_\Gamma = \sup_{N \geq -1} \sum_{j=-1}^N \|\Delta_j \omega_m(0)\|_{L^\infty} \Gamma(N)^{-1}.$$

But $\Delta_j S_m = 0$ for $j \geq m+2$; $\Delta_j S_m = \Delta_j$ for $j \leq m-2$ and in any case

$$(8.9) \quad \|\Delta_j S_m f\|_{L^\infty} = \|S_m \Delta_j f\|_{L^\infty} \leq C \|\Delta_j f\|_{L^\infty}.$$

Using properties of Γ we conclude from (8.8) that

$$(8.10) \quad \|\omega_m(0)\|_\Gamma \leq C \|f\|_\Gamma.$$

In addition (8.5) together with Proposition 8.7 imply

$$(8.11) \quad \omega_m(\cdot) \in L_{\text{loc}}^\infty([0, \infty); B_{\Gamma_1}).$$

From Theorem 5.1 using (8.7), (8.10), (8.11) we find a $T > 0$ and a constant $C > 0$ independent of m such that

$$(8.12) \quad \|\omega_m\|_{L^\infty([0, T]; L^{p_0} \cap L^{p_1})} \leq C, \quad \|\omega_m\|_{L^\infty([0, T]; B_{\Gamma_1})} \leq C.$$

Let $m > \ell \geq 1$ are two indices. We use the estimates of §7 where by abusing notation we set $\Pi = \Gamma_1$,

$$(8.13) \quad u_m = \mathcal{K} * \omega_m, \quad u_\ell = \mathcal{K} * \omega_\ell,$$

$$(8.14) \quad w = u_m - u_\ell = \mathcal{K} * (\omega_m - \omega_\ell).$$

The estimate (7.31) ought to be replaced by

$$(8.15) \quad \begin{aligned} \|\Delta_j w(t)\|_{L^\infty} \leq & \|\Delta_j w(0)\|_{L^\infty} + C \int_0^t \left\{ \|R_j(u_m(\tau), w(\tau))\|_{L^\infty} \right. \\ & + \|R_j(w(\tau), u_\ell(\tau))\|_{L^\infty} + \|\Delta_j \nabla p(\tau)\|_{L^\infty} \\ & \left. + \|(S_{j-2} w(\tau), \nabla) \Delta_j u_\ell(\tau)\|_{L^\infty} \right\} d\tau \end{aligned}$$

since $\Delta_j w(0)$ does not have to vanish. Combining (7.30), (8.15), (7.32)–(7.34) yields:

$$(8.16) \quad \sum_{j=-1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} \leq \sum_{j=-1}^N \|\Delta_j w(0)\|_{L^\infty} \\ + C\Pi(N) \int_0^t \sum_{j=-1}^{\infty} \|\Delta_j w(\tau)\|_{L^\infty} d\tau + C2^{-N}\Pi(N)$$

for any $N \geq -1$.

We use the same function $\zeta(t)$ as in (7.36). The estimate (8.16) implies

$$(8.17) \quad \dot{\zeta}(t) \leq \sum_{j=-1}^{\infty} \|\Delta_j w(0)\|_{L^\infty} + C\Gamma_1(-\log_2 \zeta(t))\zeta(t)$$

for $t \in [0, T]$ by making the same choice of N as in §7.

We now observe that

$$(8.18) \quad \kappa_{m,\ell} = \sum_{j=-1}^{\infty} \|\Delta_j w(0)\|_{L^\infty} = \sum_{j=-1}^{\infty} \|\Delta_j \mathcal{K} * (S_m - S_\ell)f\|_{L^\infty} \\ = \sum_{j=-1}^{\infty} \|\Delta_j \mathcal{K} * \sum_{q=\ell+1}^m \Delta_q f\|_{L^\infty} \\ \leq \sum_{q=\ell+1}^m \sum_{|j-q| \leq 1} \|\mathcal{K} * \Delta_j \Delta_q f\|_{L^\infty} \\ \leq C \sum_{q=\ell+1}^{\infty} \|\mathcal{K} \Delta_q f\|_{L^\infty} \\ \leq C \sum_{q=\ell+1}^{\infty} 2^{-q} \|\Delta_q f\|_{L^\infty} .$$

Let $d_m = \sum_{j=-1}^m \|\Delta_j f\|_{L^\infty}$. Then (8.18) implies

$$(8.19) \quad \kappa_{m,\ell} \leq C \sum_{q=\ell+1}^{\infty} 2^{-q} (d_q - d_{q-1}) \\ = -C2^{-(\ell+1)} d_\ell + C \sum_{q=\ell+1}^{\infty} 2^{-q-1} d_q \\ \leq C \sum_{q=\ell+1}^{\infty} 2^{-q} \Gamma(q) \leq C \int_{\ell+1}^{\infty} 2^{-\xi} \Gamma(\xi) d\xi \\ \leq C2^{-\ell} \Gamma(\ell) .$$

In (8.19) we used properties of Γ . Integrating both sides of (8.17) and using (8.18), (8.19) we obtain

$$\zeta(t) \leq C2^{-\ell} \Gamma(\ell) + C \int_0^t \Gamma_1(-\log_2 \zeta(\tau)) \zeta(\tau) d\tau .$$

Using monotonicity of $\Gamma_1(-\log \zeta) \cdot \zeta$ for small $\zeta \geq 0$ as in §8 we apply a similar Gronwall's argument to conclude

$$(8.20) \quad \zeta(t) \leq \eta(t, C2^{-\ell}\Gamma(\ell)) , \quad t \in [0, T] .$$

Going back to (8.17) and using again (8.18)–(8.19) and monotonicity of $\Gamma_1(-\log \zeta) \cdot \zeta$ for small $\zeta \geq 0$ we obtain

$$\begin{aligned} \dot{\zeta}(t) &\leq C2^{-\ell}\Gamma(\ell) + C \int_0^t \Gamma_1(-\log_2 \eta(\tau, C2^{-\ell}\Gamma(\ell))) \\ &\quad \cdot \eta(\tau, C2^{-\ell}\Gamma(\ell)) d\tau , \quad t \in [0, T] . \end{aligned}$$

Therefore, $\{u_m(\cdot)\}_{m=1}^\infty$ is a Cauchy sequence in the Banach space $L^\infty([0, T]; B_{\infty,1}^0)$. We remind that the norm in the Besov space $B_{\infty,1}^0$ is defined as follows

$$B_{\infty,1}^0 = \left\{ w \in S' \mid \|w\|_{B_{\infty,1}^0} = \sum_{j=-1}^\infty \|\Delta_j w\|_{L^\infty} < \infty \right\} .$$

Therefore, there exists a strong limit in $L^\infty([0, T]; B_{\infty,1}^0)$

$$(8.21) \quad u_m \rightarrow u \in L^\infty([0, T]; B_{\infty,1}^0) \quad \text{as } m \rightarrow \infty .$$

We claim now that in fact $\omega = \operatorname{curl} u$ satisfies

$$(8.22) \quad \|\omega(\cdot)\|_{\Gamma_1} \in L^\infty([0, T]) .$$

Let $N \geq -1$ be an arbitrary integer.

Consider the seminorm on $L^\infty([0, T]; B_{\infty,1}^0)$

$$\rho_N(v) = \left\| \sum_{j=-1}^N \|\Delta_j v(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} .$$

Obviously $\rho_N(u_m - u) \rightarrow 0$ as $m \rightarrow \infty$. On the other hand, from Bernstein's inequality

$$(8.23) \quad \left\| \sum_{j=-1}^N \|\Delta_j(\omega_m - \omega)(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} \leq C2^N \rho_N(u_m - u) .$$

Hence,

$$\begin{aligned} (8.24) \quad &\left\| \sum_{j=-1}^N \|\Delta_j \omega_m(\cdot)\|_{L^\infty} - \sum_{j=-1}^N \|\Delta_j \omega(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} \\ &\leq \left\| \sum_{j=-1}^N \left| \|\Delta_j \omega_m(\cdot)\|_{L^\infty} - \|\Delta_j \omega(\cdot)\|_{L^\infty} \right| \right\|_{L^\infty([0, T])} \\ &\leq \left\| \sum_{j=-1}^N \|\Delta_j(\omega_m - \omega)(\cdot)\|_{L^\infty} \right\|_{L^\infty([0, T])} \\ &\leq C2^N \rho_N(u_m - u) \rightarrow 0 \quad \text{as } m \rightarrow \infty , \end{aligned}$$

where on the last step we used (8.23). From (8.12) we have

$$(8.25) \quad \sum_{j=-1}^N \|\Delta_j \omega_m(t)\|_{L^\infty} \leq CT_1(N) ,$$

where the constant C does not depend on m . Passing to the limit as $m \rightarrow \infty$ in (8.24) and using (8.25) yields

$$\left\| \sum_{j=-1}^N \|\Delta_j \omega(\cdot)\|_{L^\infty} \right\|_{L^\infty([0,T])} \leq CT_1(N) .$$

This implies (8.22).

In fact since $\{\omega_m\}$ is a Cauchy sequence in $C([0,T]; B_{\infty,1}^0)$ this argument shows that $\omega(t) \in B_{\Gamma_1}$ for all $t \in [0,T]$. We note that (8.21) implies

$$(8.26) \quad u_m \rightarrow u \text{ strongly in } L^\infty(\mathbb{R}^2 \times [0,T]) \cap C(\mathbb{R}^2 \times [0,T]) .$$

We have in addition the following facts after possibly choosing a subsequence

$$(8.27) \quad \omega_m \rightarrow \omega \text{ weak* in } L^\infty([0,T]; L^{p_0}) \text{ and in } L^\infty([0,T]; L^{p_1}) ;$$

$$(8.28) \quad \dot{u}_m \rightarrow \dot{u} \text{ weak* in } L^\infty([0,T]; L^{p_0}) \text{ and in } L^\infty([0,T]; L^{p_1}) .$$

Let $\rho \in S$, $\operatorname{div} \rho = 0$ is a test function. Also let $\theta \in \mathcal{D}([0,T])$. Then, by construction

$$\langle u_m(0), \rho \rangle \theta(0) + \int_0^T \langle u_m(\tau), \rho \rangle \dot{\theta}(\tau) d\tau + \int_0^T \langle u_m(\tau), (u_m(\tau), \nabla) \rho \rangle \theta(\tau) d\tau = 0 .$$

We have $\langle u_m(0), \rho \rangle \rightarrow \langle \mathcal{K} * f, \rho \rangle$ as $m \rightarrow \infty$ because of (8.1), (8.6). Also from (8.26)

$$\begin{aligned} \int_0^T \langle u_m(\tau), \rho \rangle \dot{\theta}(\tau) d\tau &\rightarrow \int_0^T \langle u(\tau), \rho \rangle \dot{\theta}(\tau) d\tau ; \\ \int_0^T \langle u_m(\tau), (u_m(\tau), \nabla) \rho \rangle \theta(\tau) d\tau &\rightarrow \int_0^T \langle u(\tau), (u(\tau), \nabla) \rho \rangle \theta(\tau) d\tau \text{ as } m \rightarrow \infty . \end{aligned}$$

Therefore, the limit u satisfies the Euler equation and the initial condition

$$\begin{cases} \dot{u} = -(u, \nabla)u - \nabla p , \\ \operatorname{div} u = 0 , \quad u(0) = \mathcal{K} * f . \end{cases}$$

The remaining part of the proof is contained in the following Lemma.

LEMMA 8.4. – *The vorticity constructed above $\omega(\cdot)$ is weak* continuous with values in $B_{\Gamma_1} = H'_{\Gamma_1}$.*

Proof. – Since $\|u - u_m\|_{C([0,T]; B_{\infty,1}^0)} \rightarrow 0$ as $m \rightarrow \infty$ we have

$$(8.29) \quad \|\omega - \omega_m\|_{C([0,T]; B_{\infty,1}^{-1})} \rightarrow 0 \text{ as } m \rightarrow \infty .$$

Let $h \in H_{\Gamma_1}$. Consider the function $t \mapsto \langle \omega(t), h \rangle = \varphi(t)$ on $[0, T]$. Let $\varphi_m(t) = \langle \omega_m(t), h \rangle$. Then, for any $t_0 \in [0, T]$

$$(8.30) \quad \varphi(t) - \varphi(t_0) = (\varphi - \varphi_m)(t) - (\varphi - \varphi_m)(t_0) + (\varphi_m(t) - \varphi_m(t_0)) .$$

For a fixed m $\varphi_m(t) - \varphi_m(t_0) \rightarrow 0$ as $t \rightarrow t_0$ since $t \mapsto \omega_m(t)$ is (strongly) continuous with values in B_{Γ_1} . To obtain the statement of the lemma we estimate the first and the second term in (8.30). We have

$$|(\varphi - \varphi_m)(t)| \leq |\langle (\omega - \omega_m)(t), h - \tilde{h} \rangle| + |\langle (\omega - \omega_m)(t), \tilde{h} \rangle| ,$$

where $\tilde{h} \in H_{\Gamma_1}$.

We choose $\tilde{h} \in B_{1,1+\delta}^1$ so that

$$(8.31) \quad \|\tilde{h} - h\|_{\Gamma_1} < \varepsilon .$$

Obviously $B_{1,1+\delta}^1 \subset B_{1,\infty}^1 \subset H_{\Gamma_1}$. Indeed, if

$$\|\Delta_j f\|_{L^1} \leq C 2^{-j} = C 2^{-j} \Gamma_1(j) \Gamma_1(j)^{-1} ,$$

this implies

$$\|f\|_{\Gamma_1} \leq C \sum_{j=-1}^{\infty} 2^{-j} \Gamma_1(j) \leq C$$

according to (ii)-(iv). It is clear that $B_{1,1+\delta}^1$ is dense in H_{Γ_1} since it contains all functions from L^1 with a bounded Fourier spectrum.

Since $\|\omega(t)\|_{\Gamma_1}$, $\|\omega_m(t)\|_{\Gamma_1}$ are uniformly bounded on $[0, T]$,

$$(8.32) \quad |\langle (\omega - \omega_m)(t), h - \tilde{h} \rangle| < C\varepsilon .$$

Also, $B_{\infty,1}^{-1} \hookrightarrow B_{\infty,1+\delta}^{-1}$ and thus

$$(8.33) \quad |\langle (\omega - \omega_m)(t), \tilde{h} \rangle| \leq C \|\omega - \omega_m\|_{C([0,T]; B_{\infty,1}^{-1})} \|\tilde{h}\|_{B_{1,1+\delta}^1} .$$

We need here the duality $(B_{1,1+\delta}^1)' = B_{\infty,1+\delta}^{-1}$ (see e.g., [P], [Tr]). We now fix any $\delta > 0$ (say $\delta = 1$). For any $\varepsilon > 0$ we choose $\tilde{h}$ to satisfy (8.31) and using (8.29) choose m so that the right side of (8.33) becomes $< \varepsilon$.

Then from (8.31)–(8.33) $\overline{\lim}_{t \rightarrow t_0} |\varphi(t) - \varphi(t_0)| \leq C\varepsilon$. This concludes the proof of the lemma.

Theorem 8.1 is proved.

Proof of Theorem 8.2. – We repeat the proof of Theorem 8.1 word for word except now the choice of $T > 0$ is arbitrary. Here we have to refer to Theorem 5.5 instead of Theorem 5.1. Thus condition (5.23) has to be satisfied for this argument to work. This completes the proof.

ACKNOWLEDGEMENTS

The author thanks Haskell Rosenthal, Ralph Showalter, and Victor Yudovich for fruitful discussions. The author is extremely grateful to the anonymous referee for his thoughtful comments that improved the exposition as well as the author's understanding of the paper. He is grateful to Margaret Combs for her excellent typing of the paper.

REFERENCES

- [BC] H. BAHOURI and J.-Y. CHEMIN, Equations de transport relatives à des champs de vecteurs non-lipschitziens et mécanique des fluides, *Arch. Rational Mech. Anal.*, Vol. 127, 1994, pp. 159–181.
- [BKM] T. BEALE, T. KATO and A. MAJDA, Remarks on the breakdown of smooth solution for the 3-D Euler equation, *Comm. Math. Phys.*, Vol. 94, 1984, pp. 61–66.
- [B] J.-M. BONY, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Ann. de l'Ecole Norm. Sup.*, Vol. 14, 1981, pp. 209–246.
- [BB] J. BOURGUIGNON and H. BREZIS, Remarks on the Euler equation, *J. Funct. Anal.*, Vol. 15, 1974, no. 4, pp. 341–363.
- [C] D. CHAE, Weak solutions of 2D Euler equation with initial vorticity in $L \log L(R^2)$, *J. Diff. Equat.*, Vol. 103, 1993, pp. 323–337.
- [C1] J.-Y. CHEMIN, Fluides parfaits incompressibles, *Astérisque no. 230*, 1995.
- [C2] J.-Y. CHEMIN, Persistance de structures géométriques dans les fluides incompressibles bidimensionnelles, *Ann. Ecole Norm. Sup.*, Vol. 26, 1993, no. 4, pp. 1–26.
- [CL] J.-Y. CHEMIN and N. LERNER, Flot de champs de vecteurs non Lipschitziens et équations de Navier-Stokes, *J. Diff. Equat.*, Vol. 121, 1995, pp. 314–328.
- [D] I. DAUBECHIES, Ten lectures on wavelets, *SIAM*, 1992.
- [De] J.-M. DELORT, Existence de nappes de tourbillon en dimension deux, *J. Amer. Math. Soc.*, Vol. 4, 1991, pp. 553–586.
- [D1] B. DESJARDINS, A few remarks on ordinary differential equations, *Commun. in PDE's*, Vol. 21, 1966, pp. 1667–1703.
- [D2] B. DESJARDINS, Linear transport equations with initial values in Sobolev spaces and application to Navier-Stokes equations, *Diff. and Int. Equat.*, Vol. 10, 1997, no. 3, pp. 577–586.
- [DL] R. DI PERNA and P.-L. LIONS, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.*, Vol. 98, 1989, no. 3, pp. 511–549.
- [DM] R. DI PERNA and A. MAJDA, Concentrations in regularizations for 2-D incompressible flows, *Comm. Pure Appl. Math.*, Vol. 40, 1987, pp. 301–345.
- [EM] D. EBIN and J. MARS DEN, Groups of diffeomorphismes and the motion of an incompressible fluid, *Ann. of Math.*, Vol. 92, 1970, pp. 102–163.
- [EvM] L.C. EVANS and S. MÜLLER, Hardy spaces and the two-dimensional Euler equations with non-negative vorticity, *J. Amer. Math. Soc.*, Vol. 7, 1994, pp. 199–220.
- [FJ] M. FRAZIER and B. JAWERTH, Decomposition of Besov spaces, *Indiana Math. J.*, Vol. 34, 1985, pp. 777–799.
- [Ge] P. GÉRARD, Résultats récents sur les fluides parfaits incompressibles bidimensionnels [d'après J.-Y. Chemin et J.-M. Delort], *Seminar Bourbaki, 1991/1992, no. 757, Astérisque*, Vol. 206, 1992, pp. 411–444.
- [G] N. GUNTHER, On the motion of fluid in a moving container, *Izvestia Akad. Nauk USSR, Ser. Fiz.-Mat.*, Vol. 20, 1927, pp. 1323–1348, 1503–1532; Vol. 21, 1927, pp. 521–526, 735–756; Vol. 22, 1928, pp. 9–30.
- [KP] T. KATO and G. PONCE, Well posedness of the Euler and Navier-Stokes equations in Lebesgue spaces $L_p^s(R^2)$, *Revista Ibero-Americana*, Vol. 2, 1986, pp. 73–88.
- [LM] P. LEMARIÉ and Y. MEYER, Ondelettes et bases hilbertiennes, *Revista Ibero-Americana*, Vol. 2, 1986, pp. 1–18.

- [L] L. LICHTENSTEIN, Über einige Existenzprobleme der Hydrodynamik homogenerun-zusammendrückbarer, reibungsloser Flüssigkeiten und die Helmholtzschen Wirbelsätze, *Math. Z.*, Vol. 23, 1925, pp. 89–154; Vol. 26, 1927, pp. 196–323; Vol. 28, 1928, pp. 387–415; Vol. 32, 1930, pp. 608–725.
- [M] A. MAJDA, Remarks on weak solutions for vortex sheets with a distinguished sign, *Indiana Univ. Math. J.*, Vol. 42, 1993, pp. 921–939.
- [M1] Y. MEYER, Remarques sur un théorème de J.-M. Bony, *Suppl. Rend. Circ. Mat. Palermo*, Vol. 1, 1981, pp. 1–20.
- [M2] Y. MEYER, *Wavelets and operators*, Cambridge University Press, 1992.
- [P] J. PEETRE, *New Thoughts on Besov Spaces*, Duke Univ. Press, 1976.
- [S] V. SCHEFFER, An inviscid flow with compact support in space-time, *J. of Geom. Anal.*, Vol. 3, 1993, no. 4, pp. 343–401.
- [Sh] A. SHNIRELMAN, On the non-uniqueness of weak solution of the Euler equations, *Comm. Pure Appl. Math.*, Vol. 50, 1997, no. 12, pp. 1261–1286.
- [T] R. TEMAM, Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluid, *Lecture Notes in Math.*, Vol. 565, 1976, pp. 184–194.
- [Tr] H. TRIEBEL, *Theory of Function Spaces 2*, Birkhauser, 1992.
- [Y1] V. YUDOVICH, Nonstationary flow of an ideal incompressible liquid, *Zh. Vych. Mat.*, Vol. 3, 1963, pp. 1032–1066.
- [Y2] V. YUDOVICH, Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid, *Mathematical Research Letters*, Vol. 2, 1995, pp. 27–38.
- [Y3] V. YUDOVICH, Some bounds of solutions of elliptic equations, *Mat. Sbornik*, Vol. 59, 1962, pp. 229–244.
- [V] M. VISHIK, Hydrodynamics in Besov spaces, *Arch. for Rat. Mech. and Anal.*, Vol. 145, 1998, pp. 197–214.
- [W] W. WOLIBNER, Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long, *Math. Z.*, Vol. 37, 1933, pp. 698–627.

(Manuscript received December 1st 1997;
accepted, after revision, January 11 1999.)

M. VISHIK
Department of Mathematics,
The University of Texas at Austin,
Austin, Texas 78712-1082

Last minute addition: A result very similar to [Y2] was obtained independently in Philippe Serfati, Pertes de régularité pour le laplacien et l'équation d'Euler sur $\mathbb{R}^n$, preprint, 15 pp., 1994.