ANNALES SCIENTIFIQUES DE L'É.N.S.

GILBERT LEVITT
MARTIN LUSTIG

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Annales scientifiques de l'É.N.S. 4e série, tome 33, nº 4 (2000), p. 507-517
<http://www.numdam.org/item?id=ASENS_2000_4_33_4_507_0>

Ann. Scient. Éc. Norm. Sup.,
4e série, t. 33, 2000, p. 507 à 517.

MOST AUTOMORPHISMS OF A HYPERBOLIC GROUP HAVE VERY SIMPLE DYNAMICS

BY GILBERT LEVITT AND MARTIN LUSTIG

ABSTRACT. – Let G be a non-elementary hyperbolic group (e.g. a non-abelian free group of finite rank). We show that, for "most" automorphisms α of G (in a precise sense), there exist distinct elements X^+, X^- in the Gromov boundary ∂G of G such that $\lim_{n\to+\infty} \alpha^{\pm n}(g) = X^{\pm}$ for every $g \in G$ which is not periodic under α. This follows from the fact that the homeomorphism $\partial\alpha$ induced on ∂G has North–South (loxodromic) dynamics. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Soit G un groupe hyperbolique non élémentaire (par exemple un groupe libre non abélien de rang fini). Nous montrons que, pour "la plupart" des automorphismes α de G (en un sens bien précis), il existe deux éléments distincts X^+, X^- dans le bord de Gromov ∂G de G tels que $\lim_{n\to+\infty} \alpha^{\pm n}(g) = X^{\pm}$ pour tout $g \in G$ non périodique sous l'action de α. Ceci résulte du fait que l'homéomorphisme $\partial\alpha$ induit sur ∂G a une dynamique Nord–Sud (loxodromique). © 2000 Éditions scientifiques et médicales Elsevier SAS

0. Introduction and statement of results

Let α be an automorphism of a (word) hyperbolic group G. Fixing $g \in G$, we consider the sequence of iterates $\alpha^n(g)$, for $n \geqslant 1$. We assume that g is not α-periodic, so that $\alpha^n(g)$ goes off to infinity in G.

We will show that, for "most" automorphisms α of G (in a sense that will be made precise), there exists a point X^+ in the Gromov boundary ∂G such that $\alpha^n(g)$ converges to X^+ for every nonperiodic g. If G is free on a finite set A, this says that there exists a sequence of letters $a_k^{\pm 1} \in A \cup A^{-1}$ such that, for any non-periodic g, the kth letter of $\alpha^n(g)$ equals $a_k^{\pm 1}$ for n large.

This dynamical behavior is best expressed in terms of the homeomorphism $\partial\alpha$ induced by α on ∂G: for most $\alpha \in \operatorname{Aut} G$, the map $\partial\alpha$ has North–South dynamics in the following sense. We say that $\partial\alpha$, or α, has *North–South dynamics* (also called loxodromic dynamics) if $\partial\alpha$ has two distinct fixed points X^+, X^-, and $\lim_{n\to+\infty} \partial\alpha^{\pm n}(X) = X^{\pm}$ uniformly on compact subsets of $\partial G \setminus \{X^{\mp}\}$.

This implies (see Proposition 2.3) that *the set of α-periodic elements $g \in G$ is a virtually cyclic subgroup* (*possibly finite*), *and* $\lim_{n\to+\infty} \alpha^{\pm n}(g) = X^{\pm}$ *if $g \in G$ is not α-periodic*. For an arbitrary automorphism, it is proved in [15] that $\alpha^n(g)$ limits onto a finite subset of ∂G (that may depend on g).

If for instance α is conjugation i_m by $m \in G$, then $\partial\alpha$ is simply left-translation by m, and $\partial\alpha$ has North–South dynamics for all m outside of a finite set of conjugacy classes (those consisting of torsion elements), see [5,11,12].

In general, we consider an outer automorphism $\Phi \in \operatorname{Out} G$, viewed as a collection of ordinary automorphisms $\alpha \in \operatorname{Aut} G$. For a topological motivation, induce Φ by a continuous map $f: X \to X$ with $\pi_1 X \simeq G$. Automorphisms $\alpha \in \Phi$ correspond to lifts of f to the universal covering of X. Different lifts may have very different properties. On the other hand, conjugate maps have similar dynamical properties. This led Nielsen [18] to define lifts of f to be isogredient if they are conjugate by a covering transformation.

Going back to group automorphisms, we therefore define $\alpha, \beta \in \Phi$ to be *isogredient* if $\beta = i_h \circ \alpha \circ i_h^{-1}$ for some $h \in G$, with $i_h(g) = hgh^{-1}$ (the word "similar" was used in [9]).

We denote $\mathcal{S}(\Phi)$ the set of isogredience classes of automorphisms representing Φ. If $\Phi = 1$, then $\mathcal{S}(\Phi)$ may be identified to the set of conjugacy classes of G modulo its center. We say that $s \in \mathcal{S}(\Phi)$ has North–South dynamics if automorphisms $\alpha \in s$ have North–South dynamics on ∂G.

THEOREM 0.1. – *Let G be a hyperbolic group, and $\Phi \in \operatorname{Out} G$. Assume G is non-elementary (i.e. G is not virtually cyclic).*

(1) *All but finitely many $s \in \mathcal{S}(\Phi)$ have North–South dynamics.*

(2) *The set $\mathcal{S}(\Phi)$ of isogredience classes is infinite.*

Example 0.2. – When Φ is induced by a pseudo-Anosov homeomorphism φ of a closed surface Σ, the "exceptional" automorphisms $\alpha \in \Phi$ (those that do not have North–South dynamics) correspond to lifts of φ having a fixed point in the universal covering of Σ. The set of exceptional classes in $\mathcal{S}(\Phi)$ is in one-to-one correspondence with the set of fixed points of φ. It may be empty, see [8] for an explicit example. On the other hand, the number of fixed points of φ^k goes to infinity with k. Thus the number of exceptional isogredience classes cannot be bounded in terms of G only.

This example suggests the possibility of using exceptional isogredience classes to develop a fixed point theory for general outer automorphisms of free groups. Exceptional isogredience classes would be the algebraic analogue of Nielsen classes of fixed points, and there should be a (rational) zeta function obtained as a sum over exceptional classes of powers of Φ (compare [7]).

Example 0.3. – Suppose G is free. It follows from [3, Lemma 5.1] that some power of Φ contains an exceptional isogredience class. It may be shown using [4] and [14] that the isogredience class of α is the only exceptional class when α is the irreducible automorphism $a \mapsto abc$, $b \mapsto bab$, $c \mapsto cabc$ studied in [13].

The proof of the first assertion of Theorem 0.1 when G is not free requires the following fact, which is of independent interest:

PROPOSITION 0.4 (Quasiisometries of hyperbolic spaces have a quasi-fixed point or a quasi-axis). – *Let f be a (λ, C)-quasiisometry of a δ-hyperbolic proper geodesic metric space (E, d) to itself. There exists $M = M(\delta, \lambda, C)$, independent of E and f, with the following property: if $d(f(x), x) > M$ for all $x \in E$, then there exists a bi-infinite geodesic γ such that the Hausdorff distance between γ and $f(\gamma)$ is finite.*

Increasing M if necessary, we conclude (Corollary 1.4) that the action of f on ∂E has North–South dynamics, with fixed points the two endpoints of γ.

1. Quasiisometries of hyperbolic spaces

We start by proving Proposition 0.4. The proof may be seen as a generalization of the well-known argument constructing the axis of an isometry of an $\mathbf{R}$-tree having no fixed points (see [17]). We refer the reader to [5,11,12,23] for basic facts about hyperbolic spaces, quasiisometries, and hyperbolic groups.

Let (E,d) be a proper δ-hyperbolic geodesic metric space. Properness is assumed mostly for convenience, in particular E could be an $\mathbf{R}$-tree in what follows.

For $x,y \in E$, we denote $[x,y]$ any geodesic segment from x to y. Given a point z, any point $p \in [x,y]$ that is δ-close to both segments $[x,z]$ and $[y,z]$ will be called a projection of z onto $[x,y]$ (two projections are only a few δ's apart).

Recall that $f: E \to E$ is a (λ, C)-quasiisometry if

$$\frac{1}{\lambda} d(x,y) - C \leqslant d\big(f(x), f(y)\big) \leqslant \lambda d(x,y) + C$$

for all $x,y \in E$, and there exists g satisfying the same inequalities such that $f \circ g$ and $g \circ f$ are C-close to the identity. We let $\ell(f) = \inf_{x\in E} d(f(x),x)$ be the minimum displacement of f. Note that $\ell(g) \leqslant \lambda \ell(f) + 2C$.

The following lemma is left as an exercise.

LEMMA 1.1. – *If f is a (λ, C)-quasiisometry of a compact interval to itself, then $\ell(f) \leqslant C$.*

From now on, we fix δ, λ, C. The quantities C_1, M_1, C_2 introduced below depend only on these three numbers, not on E, f, or the points under consideration. We also say that two points x, y are close, or have bounded distance, if their distance may be bounded a priori by some number depending only on δ, λ, C.

The quasiisometry f has the following basic property: there exists C_1 such that, for any geodesic segment $[x,y]$, the image of $[x,y]$ is contained in the C_1-neighborhood of $[f(x), f(y)]$.

Consider a geodesic triangle $a, f(a), f^2(a)$. Let u be a projection of a onto $[f(a), f^2(a)]$, and v a projection of $f(u)$ onto $[f(a), f^2(a)]$.

LEMMA 1.2. – *There exists M_1 such that, if $\ell(f) > M_1$, then $v \in [u, f^2(a)]$.*

Proof. – Suppose $v \in [f(a), u]$. Since $f(v)$ is close to $[f^2(a), f(u)]$ and $f(u)$ is close to v, the point $f^2(u)$ is close to $[f^2(a), f(u)]$. Thus, up to a bounded error, the points u and $f^2(u)$ both lie on the segment $[f^2(a), f(u)]$. It follows that f or g is close to a map sending $[u, f(u)]$ into itself. Lemma 1.1 implies that some point of $[u, f(u)]$ is close to its image by f. □

We assume from now on that $\ell(f) > M_1$.

LEMMA 1.3. – *There exists C_2 with the following property: for any $a \in E$, there exist three points p, q, r, lying in this order on $[a, f^2(a)]$, such that*

(1) *q is C_2-close to a projection of $f(a)$ onto $[a, f^2(a)]$;*

(2) *p is C_2-close to $g(q)$;*

(3) *r is C_2-close to $f(q)$.*

Proof. – With the same notations as above, it follows from Lemma 1.2 that u is close to $[f(a), f(u)]$. Therefore $g(u)$ is close to $[a, u]$. We also know that $f(u)$ is close to $[u, f^2(a)]$. Let p, q, r be projections onto $[a, f^2(a)]$ of $g(u), u, f(u)$ respectively. Either they are in the correct order $a, p, q, r, f^2(a)$, or this may be achieved by moving them by a bounded amount. □

Note that $f(q)$ is close to $[f(p), f(r)]$, hence to $[q, f^2(q)]$.

We now complete the proof of Proposition 0.4. View Lemma 1.3 as a way of assigning a point q to any point a. We construct a sequence q_n by iterating this process, with q_0 the point assigned by Lemma 1.3 to an arbitrary starting point $a \in E$. Since $f(q_n)$ is close to $[q_n, f^2(q_n)]$, the point q_{n+1} is close to $f(q_n)$.

Note that by construction $q_{n+1} \in [q_n, f^2(q_n)]$, while q_{n+2} is close to $f(q_{n+1})$, hence to $[q_n, f^2(q_n)]$ by assertion (3) of Lemma 1.3. Thus the broken geodesics $\gamma_n = [q_n, q_{n+1}] \cup [q_{n+1}, q_{n+2}]$ are uniformly quasigeodesic. Also note that by assertion (2) of Lemma 1.3 we have

$$d(q_n, q_{n+1}) \geqslant d\big(g(q_{n+1}), q_{n+1}\big) - C_2 \geqslant \ell(g) - C_2,$$

showing that the overlap between γ_n and γ_{n+1} is bounded below by a linear function of $\ell(f)$.

It follows from [5, Théorème 3.1.4] or [11, Théorème 5.25] that the sequence q_n is an infinite quasigeodesic γ^+ if $\ell(f)$ is large enough. Since $d(f(q_n), q_{n+1})$ is bounded, the point at infinity of γ^+ is fixed by ∂f (the homeomorphism induced by f on ∂E). The quasigeodesic may be extended in the other direction by applying the same construction to g, yielding a bi-infinite quasigeodesic, hence a second fixed point for ∂f. This proves Proposition 0.4.

COROLLARY 1.4. – *Let f be a (λ, C)-quasiisometry of a δ-hyperbolic proper geodesic metric space (E, d) to itself. There exists $N = N(\delta, \lambda, C)$, independent of E and f, with the following property: if $d(f(x), x) > N$ for all $x \in E$, then ∂f has North–South dynamics.*

Proof. – Suppose $\ell(f) > M$. Let γ be a bi-infinite geodesic joining two fixed points X_0, X_1 of ∂f. Consider $X \neq X_0, X_1$ in ∂E. Let θ be a projection of X onto γ. A projection θ' of $\partial f(X)$ is close to $f(\theta)$. If $\ell(f)$ is large enough, the distance from θ to θ' is bounded below and the oriented segment $\theta\theta'$ always points towards the same endpoint X_i of γ, independently of the choice of X. Applying this argument to both f and g, we deduce that ∂f has North–South dynamics. □

2. North–South dynamics

We first prove:

THEOREM 2.1. – *Let $\Phi \in \operatorname{Out} G$, with G hyperbolic. All but finitely many isogredience classes $s \in \mathcal{S}(\Phi)$ have North–South dynamics on ∂G.*

Proof. – Let E be the Cayley graph of G with respect to some finite generating set A, with the natural left-action of G. We identify the set of vertices of E with G, and ∂E with ∂G. We fix a "basepoint" $\alpha \in \Phi$, and we represent it by a quasiisometry $J : E \to E$ sending a vertex g to the vertex $\alpha(g)$, equivariant in the sense that $\alpha(h)J = Jh$ for every $h \in G$.

Given $\beta \in \Phi$, we write $\beta = i_m \circ \alpha$ and we consider the map $J_\beta = mJ$ (this involves a choice for m if the center of G is not trivial). Note that it maps a vertex g onto $m\alpha(g)$ (not onto $\beta(g) = m\alpha(g)m^{-1}$).

The map J_β satisfies $\beta(g)J_\beta = J_\beta g$, it induces $\partial\beta$ on ∂E (because a right-translation of G induces the identity on the boundary), and the maps J_β are uniformly quasiisometric (because they differ by left-translations).

If two maps J_β, J_γ, with $\beta, \gamma \in \Phi$, coincide at some point of E, then clearly $\beta = \gamma$. More generally:

LEMMA 2.2. – *Let $\beta, \gamma \in \Phi$. If there exist $g, h \in G$ with*

$$g^{-1}J_\beta(g) = h^{-1}J_\gamma(h),$$

then β and γ are isogredient.

Proof. – Writing $\beta = i_m \circ \alpha$ and $\gamma = i_n \circ \alpha$ we get

$$g^{-1}m\alpha(g) = h^{-1}n\alpha(h)$$

which we rewrite as

$$nm^{-1} = hg^{-1}m\alpha(gh^{-1})m^{-1} = hg^{-1}\beta(gh^{-1}),$$

showing that $\gamma = i_{nm^{-1}} \circ \beta = i_{hg^{-1}} \circ \beta \circ (i_{hg^{-1}})^{-1}$ is isogredient to β. □

By Corollary 1.4, there exists a number N (independent of β) such that, if J_β moves every point of E more than N, then $\partial\beta$ has North–South dynamics. Since E is a locally finite graph, Lemma 2.2 implies that this condition is fulfilled for all $\beta \in \Phi$ outside of a finite set of isogredience classes. This completes the proof of Theorem 2.1. □

Remark. – When G is a free group F_n, there is (using the notations of [4]) a one-to-one correspondence between $\mathcal{S}(\Phi)$ and the set of connected components of the graph $D(\varphi)$, for $\varphi \in \Phi$. In this case one may use Lemma 5.1 of [4] instead of Proposition 0.4 in the above proof. Also note that, as a corollary of Theorem 4 of [9], the map $\partial\beta$ has at most 4 fixed points for $\beta \in \Phi$ outside of at most $4n-4$ isogredience classes. Another remark: $\mathcal{S}(\Phi)$ is infinite when $\Phi \in \operatorname{Out} F_n$ fixes a nontrivial conjugacy class, by Proposition 5.4 of [4].

PROPOSITION 2.3. – *Suppose $\partial\alpha$ has North–South dynamics, with attracting fixed point X^+ and repelling fixed point X^-. Then*:

(1) *The subgroup $P(\alpha) \subset G$ consisting of all α-periodic elements is either finite or virtually $\mathbf{Z}$ with limit set $\{X^+, X^-\}$.*
(2) *If $g \in G$ is not α-periodic, then* $\lim_{n\to+\infty} \alpha^{\pm n}(g) = X^\pm$.

Proof. – Given $g \in G$ of infinite order, we denote $g^{\pm\infty} = \lim_{n\to+\infty} g^{\pm n}$. These are distinct points of ∂G. Note that $\partial\alpha(g^{\pm\infty}) = \alpha(g)^{\pm\infty}$. The subgroup of G consisting of elements whose action on ∂G leaves $\{g^\infty, g^{-\infty}\}$ invariant is the maximal virtually cyclic subgroup N_g containing g. If $h \notin N_g$, then $\{g^\infty, g^{-\infty}\}$ is disjoint from its image by h. If $\partial\alpha(g^\infty) = g^\infty$, then N_g is α-invariant (i.e. $\alpha(N_g) = N_g$).

Suppose (1) is false. Then there exist two α-periodic elements g, h of infinite order generating a non-elementary group. The points $g^{\pm\infty}$ and $h^{\pm\infty}$ are four distinct periodic points of $\partial\alpha$, a contradiction.

To prove (2), first suppose G is virtually cyclic. Then G maps onto $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ with finite kernel (see [21]). From this one deduces that the periodic subgroup $P(\alpha)$ has index at most 2 and contains all elements of infinite order (an instructive example is conjugation by ab in $\langle a, b \mid a^2 = b^2 = 1\rangle$). Both ends of G are fixed by $\partial\alpha$; all non-periodic torsion elements (if any) converge towards one end under iteration of α, towards the other end under iteration of α^{-1}.

Now consider the general case. It suffices to show $\lim_{n\to+\infty} \alpha^n(g) = X^+$. Since X^+ is a fixed point of $\partial\alpha$, we are free to replace α by a power if needed. We first note that there exists a number C such that

$$(g, g^\infty) \geqslant \frac{1}{2}|g| - C$$

for every g of infinite order (where (,) denotes Gromov's scalar product based at the identity in the Cayley graph, and | | is word length). This follows easily from Lemma 3.5 of [20] (if G is free and A is a basis, $C = -1/2$ clearly works).

Suppose g is not α-periodic. Then $\lim_{n\to\infty} |\alpha^n(g)| = \infty$. If furthermore g has infinite order, applying the previous inequality to $\alpha^n(g)$ yields

$$\lim_{n\to+\infty} \alpha^n(g) = \lim_{n\to+\infty} \big(\alpha^n(g)\big)^\infty = \lim_{n\to+\infty} \partial\alpha^n(g^\infty) = X^+$$

(note that $g^\infty \neq X^-$, since otherwise $N(g)$ would be α-invariant and g would be periodic).

Now we consider a non α-periodic element g of finite order. We distinguish two cases. Suppose first that $\{X^+, X^-\}$ is the limit set of an infinite α-invariant virtually cyclic subgroup H. We may assume that H is maximal (it then contains all periodic elements). If $g \notin H$, choose $h \in H$ of infinite order, with $h^{\pm\infty} = X^{\pm}$. Replacing α by a power, we may assume $\alpha(h) = h$. We have $gh^{\infty} \neq h^{-\infty}$, and therefore $g_k = h^k g h^k$ has infinite order for k large enough. Since $\alpha^n(g_k)$ converges to X^+ as $n \to +\infty$, we find that $\alpha^n(g)$ converges to $h^{-k}X^+ = X^+$, as desired. There remains to rule out the possibility that non-periodic torsion elements $g \in H$ converge towards X^- under iteration of α. If this happens, choose $j \notin H$. Since j and gj are not α-periodic (they don't belong to H), we know that $\alpha^n(j)$ and $\alpha^n(gj)$ are close to X^+ for n large. But $\alpha^n(g)$ and $\alpha^n(g^{-1})$ are close to X^-. This is impossible.

If $\{X^+, X^-\}$ is not as above, then X^+ (respectively X^-) is an attracting (respectively repelling) fixed point for the action of $\alpha \cup \partial\alpha$ on the compact space $G \cup \partial G$ (see [14]). The desired result $\lim_{n\to+\infty} \alpha^n(g) = X^+$ follows from an elementary dynamical argument. Indeed, the sequence $\alpha^n(g)$, with $n > 0$, has some limit point $X \in \partial G$. We have $X \neq X^-$ because X^- is repelling on $G \cup \partial G$, and therefore $\partial\alpha^n(X)$ converges to X^+. We then deduce that X^+ is a limit point of $\alpha^n(g)$, and finally that $\alpha^n(g)$ converges to X^+ because X^+ is attracting on $G \cup \partial G$. □

3. Isogredience classes

The main result of this section is the infiniteness of $\mathcal{S}(\Phi)$ (but see also Proposition 3.7). We first study four different situations where we can reach this conclusion. For now, we only assume that G is any finitely generated group. We fix $\Phi \in \operatorname{Out} G$ and $\alpha \in \Phi$.

• By definition, the automorphisms $\beta = i_m \circ \alpha$ and $\gamma = i_n \circ \alpha$ are isogredient if and only if there exists $g \in G$ with $\gamma = i_g \circ \beta \circ i_g^{-1}$, or equivalently $n = gm\alpha(g^{-1})c$ with c in the center of G. Though we will not use it, we note that *$\mathcal{S}(\Phi)$ is infinite if the center of G is finite and the action of Φ on $H_1(G; \mathbf{R})$ has 1 as an eigenvalue.*

Now assume that Φ preserves some $\mathbf{R}$-tree (see [6], [17], [22] for basics about $\mathbf{R}$-trees). This means that there is an $\mathbf{R}$-tree T equipped with an isometric action of G whose length function satisfies $\ell \circ \Phi = \lambda\ell$ for some $\lambda \geqslant 1$. We always assume that the action is minimal and irreducible (no global fixed point, no invariant line, no invariant end). We say $g \in G$ is hyperbolic if it is hyperbolic as an isometry of T. We shall use the following fact due to Paulin [19]: any segment $[a, b] \subset T$ is contained in the axis of some hyperbolic $g \in G$.

Because $\ell \circ \Phi = \lambda\ell$, it follows from [6] (see also [9], [16]) that, given $\alpha \in \Phi$, there is a (unique) map $H = H_\alpha : T \to T$ with the following properties: H is a homothety with stretching factor λ (i.e. $d(Hx, Hy) = \lambda d(x, y)$), and it satisfies $\alpha(g)H = Hg$ for every $g \in G$. If $\beta = i_m \circ \alpha$, then $H_\beta = mH_\alpha$. If $\beta = i_g \circ \alpha \circ i_g^{-1}$ is isogredient to α, then $H_\beta = gH_\alpha g^{-1}$ is conjugate to H_α.

• First consider the case when $\lambda = 1$. In this case the translation length of the isometry H_β is an isogredience invariant of β and we easily get:

PROPOSITION 3.1. – *Suppose $\ell \circ \Phi = \ell$, where ℓ is the length function of an irreducible action of G on an $\mathbf{R}$-tree. Then $\mathcal{S}(\Phi)$ is infinite.*

Proof. – Fix $\alpha \in \Phi$. Using Paulin's lemma, it is easy to construct $m \in G$ with the translation length of mH_α arbitrarily large. The corresponding automorphisms $i_m \circ \alpha$ are in distinct isogredience classes. □

• The case $\lambda > 1$ is harder.

PROPOSITION 3.2. – *Suppose $\ell \circ \Phi = \lambda\ell$, where $\lambda > 1$ and ℓ is the length function of an irreducible action of G on an $\mathbf{R}$-tree T. Assume that arc stabilizers are finite, and there exists*

$N_0 \in \mathbf{N}$ *such that, for every* $Q \in T$, *the action of* $\operatorname{Stab} Q$ *on* $\pi_0(T \setminus \{Q\})$ *has at most* N_0 *orbits. Then* $\mathcal{S}(\Phi)$ *is infinite.*

An arc stabilizer is the pointwise stabilizer of a nondegenerate segment $[a, b]$, and $\operatorname{Stab} Q$ denotes the stabilizer of Q.

Proof. – Fix $\alpha \in \Phi$ and consider $H = H_\alpha$. We choose a point $P \in T$ as follows. It is the unique fixed point of H if H has a fixed point in T. Otherwise H has a unique fixed point Q in the metric completion $\overline{T}$ of T, and a unique eigenray ρ (by definition, ρ is the image of an isometric embedding $\rho: (0, \infty) \to T$ such that $H\rho(t) = \rho(\lambda t)$ for all $t > 0$, see [9]). We let P be any point on ρ. In both cases $P \in [H^{-1}P, HP]$.

For further reference, we note that the stabilizer of any initial segment $\rho(0,t)$ of an eigenray is the same as the stabilizer of the whole eigenray, because $\operatorname{Stab} \rho(0,t)$ and $\operatorname{Stab} \rho(0, \lambda t) = \alpha(\operatorname{Stab} \rho(0,t))$ are finite groups with the same order. Suppose furthermore that H has two eigenrays ρ, ρ', and $g \in G$ maps an initial segment of ρ onto an initial segment of ρ'. From the basic equation $\alpha(g)H = Hg$ it follows that $g^{-1}\alpha(g)$ fixes an initial segment of ρ, hence all of ρ, and we deduce that g maps the *whole* of ρ onto ρ'.

Returning to the main line of proof, we want to find $v, w \in G$ generating a free subgroup of rank 2, such that:

(i) vP and wP belong to a component T^+ of $T \setminus \{P\}$.

(ii) $v^{-1}P$ and $w^{-1}P$ belong to another component T^-.

(iii) If $HP \neq P$, then $H^{\pm 1}P \in T^\pm$.

(iv) If $HP = P$, then $H(T^+) \neq T^-$.

Note that these conditions force v and w to be hyperbolic, with axes intersecting in a nondegenerate segment containing P in its interior. Furthermore, the two axes induce the same orientation on their intersection.

It is easy to construct v, w using Lemma 2.6 of [6] and Paulin's lemma, except in one "bad" situation where (iv) cannot be achieved: $HP = P$, and $T \setminus \{P\}$ has exactly two components, which are permuted by H.

If H is bad, we have to change our initial choice of $\alpha \in \Phi$. We use the following observation. Suppose H_1, H_2 are homotheties with the same dilation factor $\lambda > 1$ and distinct fixed points P_1, P_2; if H_1 (respectively H_2) does not send the component of $T \setminus \{P_1\}$ (respectively $T \setminus \{P_2\}$) containing P_2 (respectively P_1) into itself, then $H_2 H_1^{-1}$ is a hyperbolic isometry whose axis contains $[P_1, P_2]$.

We choose $m \in G$ acting on T as a hyperbolic isometry with axis not containing P, and we replace α by $\alpha' = i_m \circ \alpha$. Let $H' = mH = H_{\alpha'}$. We claim that H' cannot be bad (with respect to its fixed point P'). Indeed, this follows from the above observation because the axis of $H'H^{-1}$ does not contain P. Thus, when H is bad, we can find v, w satisfying the above conditions with respect to H'. For simplicity, we keep writing H, α rather than H', α'.

Now assume by way of contradiction that there are only K isogredience classes in $\mathcal{S}(\Phi)$. Given an integer p, consider the set W consisting of words in the letters v, w containing each letter exactly p times (we do not use v^{-1} or w^{-1}). We fix p such that W has more than Ks^2N_0 elements, where s is the order of the stabilizer of the arc $I = [P, vP] \cap [P, wP]$ and N_0 is defined in the statement of Proposition 3.2. We will consider the automorphisms $i_\sigma \circ \alpha$, for $\sigma \in W$, and the corresponding homotheties σH.

Consider $\sigma = u_1 \dots u_{2p} \in W$, with each u_i equal to v or w. The elements v, w were chosen in such a way that the points

$$P, u_1P, u_1u_2P, \dots, u_1 \dots u_{2p}P, u_1 \dots u_{2p}HP = \sigma HP$$

all lie in this order on the segment $[P, \sigma HP]$ (with the last two points possibly equal). Since P belongs to the axis of both v and w, we find that, for any $\sigma \in W$, the length of $[P, \sigma HP]$ equals

$$L = p\ell(v) + p\ell(w) + d(P, HP)$$

independently of σ. We also observe that, if $\sigma, \tau \in W$, then $[P, \sigma HP] \cap [P, \tau HP]$ contains the segment $I = [P, vP] \cap [P, wP]$.

Furthermore the intersection $[P, \sigma HP] \cap [\sigma HP, (\sigma H)^2 P]$ consists only of σHP: this follows from $P \in [u_{2p}^{-1} P, Hu_1 P]$ if $HP = P$, from $P \in [H^{-1}P, u_1 P]$ if $HP \neq P$. This implies that $[P, \sigma HP]$ is contained in an eigenray ρ_σ of the homothety σH. Let Q_σ denote the fixed point of σH in the completion $\overline{T}$ (the origin of ρ_σ).

Now we remark that $[P, \sigma HP]$ is *the only fundamental domain of length* L for the action of σH on its eigenray ρ_σ. In particular, $d(Q_\sigma, P) = \frac{L}{\lambda - 1}$ is independent of $\sigma \in W$.

Suppose for a moment that for every $\sigma \in W$ the map σH has only one eigenray (this happens in particular if $Q_\sigma \in \overline{T} \setminus T$). If i_c conjugates $i_\sigma \circ \alpha$ and $i_\tau \circ \alpha$ (with $\sigma, \tau \in W$ and $c \in G$), then c conjugates σH and τH. Therefore c sends ρ_σ onto ρ_τ, and the fundamental domain $[P, \sigma HP]$ onto $[P, \tau HP]$. Since these segments both contain I, we find $c \in \operatorname{Stab} I$. This contradicts the choice of p in this special case, since we obtain $|W|/s$ distinct isogredience classes in $\mathcal{S}(\Phi)$.

In general, if i_c conjugates $i_\sigma \circ \alpha$ and $i_\tau \circ \alpha$, we can only say that c sends Q_σ to Q_τ. Since $|W| > Ks^2 N_0$, we can find distinct elements $\sigma, \tau(1), \ldots, \tau(s^2+1)$ in W such that some $i_{c(j)}$ conjugates $i_\sigma \circ \alpha$ and $i_{\tau(j)} \circ \alpha$, and some element $h(j) \in \operatorname{Stab} Q_{\tau(j)}$ sends an initial segment of the $[\tau(j)H]$-eigenray $c(j)\rho_\sigma$ onto an initial segment of $\rho_{\tau(j)}$.

We have pointed out earlier that $h(j)$ sends the whole eigenray $c(j)\rho_\sigma$ onto $\rho_{\tau(j)}$. Therefore $h(j)c(j) \in \operatorname{Stab} I$. Thus there are at least $s+1$ values of j for which the maps $\tau(j)H$ have a common eigenray containing I. This is a contradiction because at most s elements of G can have the same action on I. This completes the proof of Proposition 3.2. □

• We also need:

PROPOSITION 3.3. – *$\mathcal{S}(\Phi)$ is infinite if G is hyperbolic, non-elementary, and Φ has finite order in* $\operatorname{Out} G$.

Proof. – Let J be the subgroup of $\operatorname{Aut} G$ consisting of all automorphisms whose image in $\operatorname{Out} G$ is a power of Φ. The exact sequence $\{1\} \to K \to J \to \langle \Phi \rangle \to \{1\}$, with $K = G/\text{Center}$ and $\langle \Phi \rangle$ finite, shows that J is hyperbolic, non-elementary. The set of automorphisms $\alpha \in \Phi$ is a coset of J mod K. If $\alpha, \beta \in \Phi$ are isogredient, they are conjugate in J. The proof of Proposition 3.3 is therefore concluded by applying the following fact, due to T. Delzant. □

LEMMA 3.4. – *Let J be a non-elementary hyperbolic group. Let K be a normal subgroup with abelian quotient. Every coset of J mod K contains infinitely many conjugacy classes.*

Proof. – Fix u in the coset C under consideration. Suppose for a moment that we can find $c, d \in K$, generating a free group of rank 2, such that $uc^\infty \neq c^{-\infty}$ and $ud^\infty \neq d^{-\infty}$ (recall that we denote $g^{\pm\infty} = \lim_{n \to +\infty} g^{\pm n}$ for g of infinite order). Consider $x_k = c^k u c^k$ and $y_k = d^k u d^k$. For k large, the above inequalities imply that these two elements have infinite order, and do not generate a virtually cyclic group because $x_k^{\pm\infty}$ (respectively $y_k^{\pm\infty}$) is close to $c^{\pm\infty}$ (respectively $d^{\pm\infty}$). Fix k, and consider the elements $z_n = x_k^{n+1} y_k^{-n}$. They belong to the coset C, because J/K is abelian, and their stable norm goes to infinity with n. Therefore C contains infinitely many conjugacy classes.

Let us now construct c, d as above. Choose $a, b \in K$ generating a free group of rank 2. We first explain how to get c. There is a problem only if $ua^\infty = a^{-\infty}$ and $ub^\infty = b^{-\infty}$. In that case there exist integers p, q with $ua^pu^{-1} = a^{-p}$ and $ub^qu^{-1} = b^{-q}$. We take $c = a^pb^q$, noting that $ucu^{-1} = a^{-p}b^{-q}$ is different from $c^{-1} = b^{-q}a^{-p}$.

Once we have c, we choose $c' \in K$ with $\langle c, c' \rangle$ free of rank 2, and we obtain d by applying the preceding argument using c' and cc' instead of a and b. The group $\langle c, d \rangle$ is free of rank 2 because d is a positive word in c' and cc'. □

Remark. – As pointed out by Delzant, similar arguments show that $\mathcal{S}(\Phi)$ is infinite when Φ has infinite order but is hyperbolic in the sense of [1] (because J is hyperbolic, see [1]).

We can now prove:

THEOREM 3.5. – *For every $\Phi \in \operatorname{Out} G$, with G a non-elementary hyperbolic group, the set $\mathcal{S}(\Phi)$ is infinite.*

Proof. – By Proposition 3.3, we may assume that Φ has infinite order. By Paulin's theorem [20], it preserves some $\mathbf{R}$-tree T with a nontrivial minimal small action of G (recall that an action of G is small if all arc stabilizers are virtually cyclic; the action of G on T is always irreducible).

If $\lambda = 1$, we use Proposition 3.1. If $\lambda > 1$, we apply Proposition 3.2. The existence of N_0 follows from work of Bestvina and Feighn [2] (alternatively, one could for G torsion-free use ad hoc trees as in [15]). Finiteness of arc stabilizers is stated as the next lemma. □

LEMMA 3.6. – *Suppose $\ell \circ \Phi = \lambda \ell$, where ℓ is the length function of a nontrivial small action of a hyperbolic group G on an $\mathbf{R}$-tree T. If $\lambda > 1$, then T has finite arc stabilizers.*

Proof. – This is proved in [9, Lemma 2.8] when G is free. We sketch the proof of the general case. We may assume that the action is minimal. Let $c \subset T$ be an arc with infinite stabilizer S. Let p be the index of S in the maximal virtually cyclic subgroup $\overline{S}$ that contains it. Fix $\alpha \in \Phi$, and denote by H the associated homothety of T.

Since there is a finite union of arcs whose union meets every orbit, we can find, for k large, disjoint subarcs $c_0, \dots, c_p$ of $H^k(c)$ such that $c_i = v_ic_0$ for some $v_i \in G$. For each i, the stabilizer of c_i lies between $\alpha^k(S) = \operatorname{Stab} H^k(c)$ and $\alpha^k(\overline{S})$. From $\operatorname{Stab} c_i = v_i \operatorname{Stab} c_0 v_i^{-1}$ we get $\alpha^k(\overline{S}) = v_i\alpha^k(\overline{S})v_i^{-1}$, hence $v_i \in \alpha^k(\overline{S})$. This is a contradiction since $1, v_1, \dots, v_p$ all lie in different cosets of $\alpha^k(\overline{S})$ modulo $\alpha^k(S)$. □

If G is a free group F_n, we also prove:

PROPOSITION 3.7. – *There exists a number C_n such that, for any $\Phi \in \operatorname{Out} F_n$ and any integer $k \geqslant 2$, the natural map $\mathcal{S}(\Phi) \to \mathcal{S}(\Phi^k)$ is at most C_n-to-one.*

Proof. – Let α_i $(1 \leqslant i \leqslant N)$ be pairwise non-isogredient automorphisms in Φ having isogredient kth powers. We want to bound N in terms of n only. We may assume that α_i^k is a fixed automorphism β.

Let T be an $\mathbf{R}$-tree with trivial arc stabilizers preserved by Φ (see [9, Theorem 2.1]), and H_i the homothety associated to α_i. The H_i's all have the same kth power H_β. For $i \neq j$, we have $H_i = g_{ij}H_j$ for some nontrivial $g_{ij} \in F_n$. Note that H_i and H_j cannot coincide on more than one point since F_n acts on T with trivial arc stabilizers.

First suppose $\lambda > 1$. Then H_β and all maps H_i fix the same point $Q \in \overline{T}$. The stabilizer $\operatorname{Stab} Q \subset F_n$ is α_i-invariant and has rank $< n$ by [10] (see [9]).

If $\operatorname{Stab} Q$ is trivial (in particular if $Q \in \overline{T} \setminus T$), then $g_{ij} = 1$ and $\alpha_i = \alpha_j$.

If $\operatorname{Stab} Q$ has rank $\geqslant 2$, we use induction on n since the restrictions of the α_i's to $\operatorname{Stab} Q$ are non-isogredient automorphisms representing the same outer automorphism [9, Lemma 5.1].

If $\operatorname{Stab} Q$ is cyclic, generated by some u, we note that g_{ij} is a power of u and $\alpha_i(u)$ is independent of i. If $\alpha_i(u) = u$, then H_i commutes with u and $H_i^k = H_j^k$ implies $g_{ij} = 1$. If

$\alpha_i(u) = u^{-1}$, we write $u^{2p} = u^p \alpha_i(u^p)^{-1}$, showing that α_i is isogredient to α_j whenever g_{ij} is an even power of u.

Now suppose $\lambda = 1$. If H_β has no fixed point, then $N = 1$ since all H_i's coincide on the axis of H_β. Assume therefore that H_β has fixed points. If all maps H_i have a common fixed point Q, we can argue as above. We complete the proof by showing how to reduce to this situation.

Let Q_i be a fixed point of H_i, and e_i some edge containing Q_i and fixed by H_β. The action of F_n on pairs (Q_i, e_i) has at most $6n - 6$ orbits (twice the number of edges of the quotient graph T/F_n). After possibly dividing N by $6n - 6$ we may assume there is only one orbit. Note that the action on T of the element $c_{ij} \in F_n$ sending (Q_i, e_i) to (Q_j, e_j) commutes with H_β since e_i and e_j are both fixed by H_β. This implies that β fixes c_{ij}, and we can change α_i within its isogredience class so as to make all points Q_i the same, while retaining the property $\alpha_i^k = \beta$. □

Acknowledgements

We thank T. Delzant and F. Paulin for helpful comments about Proposition 0.4, and Delzant for Lemma 3.4.

REFERENCES

[1] BESTVINA M., FEIGHN M., A combination theorem for negatively curved groups, *J. Differential Geom.* **35** (1992) 85–101.
[2] BESTVINA M., FEIGHN M., Bounding the complexity of group actions on real trees, unpublished manuscript.
[3] BESTVINA M., FEIGHN M., HANDEL M., Solvable subgroups of $\mathrm{Out}(F_n)$ are virtually abelian, Preprint.
[4] COHEN M.M., LUSTIG M., On the dynamics and the fixed subgroup of a free group automorphism, *Invent. Math.* **196** (1989) 613–638.
[5] COORNAERT M., DELZANT T., PAPADOPOULOS A., *Géométrie et Théorie des Groupes*, Lecture Notes, Vol. **1441**, Springer, 1990.
[6] CULLER M., MORGAN J.W., Group actions on $\mathbf{R}$-trees, *Proc. London Math. Soc.* **55** (1987) 571–604.
[7] CURTILLET J.-C., *Geodäten auf flachen Flächen und eine Zetafunktion für Automorphismen von freien Gruppen, Ph.D. Thesis, Bochum*, 1997.
[8] DICKS W., LLIBRE J., Orientation-preserving self-homeomorphisms of the surface of genus two have points of period at most two, *Proc. Amer. Math. Soc.* **124** (1996) 1583–1591.
[9] GABORIAU D., JAEGER A., LEVITT G., LUSTIG M., An index for counting fixed points of automorphisms of free groups, *Duke Math. J.* **93** (1998) 425–452.
[10] GABORIAU D., LEVITT G., The rank of actions on $\mathbf{R}$-trees, *Ann. Sci. ENS* **28** (1995) 549–570.
[11] Ghys E., de la Harpe P. (Eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, Vol. **83**, Birkhäuser, 1990.
[12] GROMOV M., Hyperbolic groups, in: Gersten S.M. (Ed.), *Essays in Group Theory*, MSRI Publ., Vol. **8**, Springer, 1987, pp. 75–263.
[13] JÄGER A., LUSTIG M., Free group automorphisms with many fixed points at infinity, Math. Z. (to appear).
[14] LEVITT G., LUSTIG M., Periodic ends, growth rates, Hölder dynamics for automorphisms of free groups, Comment. Math. Helv. (to appear) (available from http://picard.ups-tlse.fr/~levitt/).
[15] LEVITT G., LUSTIG M., Dynamics of automorphisms of free groups and hyperbolic groups, Preprint (available from http://picard.ups-tlse.fr/~levitt/).
[16] LUSTIG M., Automorphisms, train tracks and non-simplicial $\mathbf{R}$-tree actions, Comm. in Alg. (to appear).
[17] MORGAN J.W., SHALEN P.B., Valuations, trees, and degenerations of hyperbolic structures, I, *Ann. Math.* **120** (1984) 401–476.

[18] NIELSEN J., Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen, *Acta Math.* **50** (1927) 189–358; English transl. in: Collected Mathematical Papers, Birkhäuser, 1986.

[19] PAULIN F., The Gromov topology on $\mathbf{R}$-trees, *Topology Appl.* **32** (1989) 197–221.

[20] PAULIN F., Sur les automorphismes extérieurs des groupes hyperboliques, *Ann. Sci. ENS* **30** (1997) 147–167.

[21] SCOTT P., WALL T., Topological methods in group theory, in: Wall T. (Ed.), *Homological Group Theory*, LMS Lect. Notes, Vol. **36**, Camb. Univ. Press, 1979, pp. 137–203.

[22] SHALEN P.B., Dendrology of groups: an introduction, in: Gersten S.M. (Ed.), *Essays in Group Theory*, MSRI Publ., Vol. **8**, Springer, 1987, pp. 265–319.

[23] SHORT H. ET AL., Notes on word hyperbolic groups, in: Ghys E., Haefliger A., Verjovsky A. (Eds.), *Group Theory from a Geometrical Viewpoint*, World Scientific, 1991, pp. 3–63.

(Manuscript received June 28, 1999.)

Gilbert LEVITT
Laboratoire Émile-Picard, UMR CNRS 5580,
Université Paul-Sabatier,
31062 Toulouse cedex 4, France
E-mail: levitt@picard.ups-tlse.fr

Martin LUSTIG
Laboratoire de mathématiques fondamentales et appliquées,
Université d'Aix-Marseille-III,
13397 Marseille cedex 20, France
E-mail: Martin.Lustig@math.u-3mrs.fr