Vincent Guirardel

Dynamics of out ($F_n$) on the boundary of outer space


<http://www.numdam.org/item?id=ASENS_2000_4_33_4_433_0>
DYNAMICS OF Out($F_n$) ON THE BOUNDARY
OF OUTER SPACE

BY VINCENT GUIRARDEL

ABSTRACT. – In this paper, we study the dynamics of the action of Out($F_n$) on the boundary $\partial CV_n$ of outer space: we describe a proper closed Out($F_n$)-invariant subset $\mathcal{F}_n$ of $\partial CV_n$ such that Out($F_n$) acts properly discontinuously on the complementary open set. Moreover, we prove that there is precisely one minimal non-empty closed invariant subset $\mathcal{M}_n$ in $\mathcal{F}_n$. This set $\mathcal{M}_n$ is the closure of the Out($F_n$)-orbit of any simplicial action lying in $\mathcal{F}_n$. We also prove that $\mathcal{M}_n$ contains every action having at most $n - 1$ ergodic measures. This makes us suspect that $\mathcal{M}_n = \mathcal{F}_n$. Thus $\mathcal{F}_n$ would be the limit set of Out($F_n$), the complement of $\mathcal{F}_n$ being its set of discontinuity. © 2000 Éditions scientifiques et médicales Elsevier SAS

RESUME. – Dans cet article, nous étudions la dynamique de l’action du groupe Out($F_n$) sur la frontière $\partial CV_n$ de l’outre-espace : nous décrivons un sous-ensemble fermé propre $\mathcal{F}_n$ de $\partial CV_n$ invariant sous l’action de Out($F_n$) et tel que Out($F_n$) agisse proprement discontinuément sur l’ouvert complémentaire. Nous prouvons ensuite qu’il existe un unique fermé non-vide invariant non vide $\mathcal{M}_n$ dans $\mathcal{F}_n$. Cet ensemble $\mathcal{M}_n$ est l’adhérence de l’orbite de toute action simpliciale appartenant à $\mathcal{F}_n$. Nous démontrons enfin que $\mathcal{M}_n$ contient toutes les actions ayant au plus $n - 1$ mesures ergodiques. Ce dernier résultat rend probable l’égalité de $\mathcal{M}_n$ et de $\mathcal{F}_n$, de sorte que $\mathcal{F}_n$ serait l’ensemble limite de Out($F_n$), le complémentaire de $\mathcal{F}_n$ étant son domaine de discontinuité. © 2000 Éditions scientifiques et médicales Elsevier SAS

Outer space $CV_n$ has been introduced by M. Culler and K. Vogtmann as an analogue of Teichmüller space for the group Out($F_n$) of outer automorphisms of the non-abelian free group $F_n$. Outer space is the set of minimal free isometric actions of $F_n$ on simplicial $\mathbb{R}$-trees modulo equivariant homothety. It has a natural compactification $\overline{CV}_n$ in the set of minimal isometric actions of $F_n$ on $\mathbb{R}$-trees. Both $CV_n$ and $\overline{CV}_n$ are endowed with a natural action of Out($F_n$) by precomposition.

Like the Teichmüller space $T_S$ of a closed surface $S$, $CV_n$ is a contractible space, the action of Out($F_n$) on $CV_n$ is properly discontinuous and not cocompact. The quotient being a finite disjoint union of open simplices, it may be thought of as having finite volume (see [9]). Moreover, every outer automorphism of $F_n$ fixes a point in $\overline{CV}_n$ (see [6,21]).

Outer space has proven to be useful in the study of Out($F_n$). M. Culler and K. Vogtmann computed the virtual cohomological dimension of Out($F_n$) ([9]) using outer space. Furthermore, M. Bestvina and M. Feighn showed that Out($F_n$) is $(2n - 3)$-connected at infinity by using some Morse theory on a bordified version of outer space [5]. However, outer space happens to be more complicated than Teichmüller space and not much is known about this space and its compactification.

Thurston theory shows that the mapping class group of a closed orientable surface $S$ acts with dense orbits on $\partial T_S$, the boundary of Thurston’s compactification of $T_S$ (see [13] for instance). When the surface is not orientable, there is an open invariant subspace of full measure in $\partial T_S$.  

© 2000 Éditions scientifiques et médicales Elsevier SAS. All rights reserved
consisting of measured foliations having a regular closed one-sided leaf [11]. The action is not properly discontinuous on this set since infinite order Dehn twists fix some of its points. Surprisingly, it seems to be unknown whether the mapping class group acts with dense orbits on the complementary closed set.

In this paper, we try to understand the analogous problem in outer space: does Out$(F_n)$ act with dense orbits on the boundary of outer space? The answer to this question is no.

**Definition.** - Let $O_n$ be the set of simplicial $F_n$-actions $T$ such that

- $T$ has trivial edge stabilizers,
- $T$ has cyclic vertex stabilizers,
- whenever $\text{Stab} v \neq \{1\}$, $\text{Stab} v$ acts transitively on the set of incident orbits.

Equivalently, $T$ lies in $O_n$ if and only if every non-trivial group in the graph of groups $T/F_n$ is cyclic and is attached to a terminal vertex of $T/F_n$.

Fig. 1 shows a typical action in $O_n$. Because of its friendly face with antennae, I was suggested to christen the actions in $O_n$ Martian actions (many thanks to Claire!). The set $O_n$ can also be seen to be the set of simplicial actions in $CV_n$ with finite stabilizer in Out$(F_n)$. It is clearly invariant under the action of Out$(F_n)$.

**Theorem 1.** - The set $O_n$ is open in $CV_n$ and Out$(F_n)$ acts properly discontinuously on $O_n$.

Since $CV_n \nsubseteq O_n$, the closed set $F_n = CV_n \setminus O_n$ is a proper invariant compact subset of $\partial CV_n$. M. Feighn pointed out that the intersection of the closure of the spine of outer space [9] with $\partial CV_n$ is a subset of $F_n$.

**Theorem 2.** - Let $n \geq 3$. Let $T$ be a simplicial action lying in $F_n$ and let $T'$ be a small action of $F_n$ on an $\mathbb{R}$-tree. Then there exists a sequence $\alpha_k$ of elements of Out$(F_n)$ such that

$$\lim_{k \to \infty} T' \cdot \alpha_k = T.$$

This theorem has an interesting corollary about the dynamics of Out$(F_n)$ on $\partial CV_n$:

**Corollary.** - For $n \geq 3$, there exists precisely one minimal non-empty closed invariant subset in $CV_n$. This set $M_n$ is the closure of the orbit of any simplicial action lying in $F_n$ under the action of Out$(F_n)$.

It would be interesting to know whether $F_n = M_n$. If the equality held, $F_n$ would be equal to the intersection of the closure of the spine of $CV_n$ with $\partial CV_n$. Moreover, $F_n$ could be thought of as a limit set of Out$(F_n)$ and $O_n$ as a domain of discontinuity like in the theory of Kleinian groups.

An argument by Bestvina and Feighn [4] shows that any action in $CV_n$ in which there exists an arc with non-trivial stabilizer lies in $M_n$. Furthermore, since any action in $CV_n$ can be decomposed as a graph of actions with dense orbits (see [19,14]), proving $F_n = M_n$ reduces
to showing that any action in $\overline{CV}_n$ with dense orbits belongs to $\mathcal{M}_n$. We prove that this is true under a technical condition:

**Theorem 3.** Let $n \geq 3$ and let $T \in \overline{CV}_n$ be an action of $F_n$ with dense orbits. Assume that the Lebesgue measure on $T$ is the sum of at most $n-1$ ergodic measures. Then $T$ lies in $\mathcal{M}_n$.

**Remark.** There is a bound coming from the topological dimension of $\partial CV_n$ for the number of ergodic measures of an action in $\partial CV_n$ (see [4,14] and Section 5.1). This bound can be seen to be $3n-4$. Therefore, we suspect that Theorem 3 still holds with no assumption on the Lebesgue measure so that $\mathcal{F}_n = \mathcal{M}_n$.

We also note that if $\alpha \in \text{Out}(F_n)$ is irreducible with irreducible powers, then it has only two fixed points in $\overline{CV}_n$ [22] which implies that they are uniquely ergodic and hence lie in $\mathcal{M}_n$.

After some definitions in Section 1, we introduce in Section 2 the folding to approximate technique to obtain approximations of a simplicial action. This technique rules out some natural candidates to be open, and leads to the definition of the set $\mathcal{O}_n$. In Section 3, we prove that $\mathcal{O}_n$ is open and that the action of $\text{Out}(F_n)$ on $\mathcal{O}_n$ is properly discontinuous (Theorem 1). In Section 4, we use the folding to approximate technique to study the dynamics of $\text{Out}(F_n)$ on $\mathcal{F}_n = \overline{CV}_n \setminus \mathcal{O}_n$ and prove Theorem 2. In Section 5, we introduce the tools of measure theory on $\mathbb{R}$-trees needed to prove Theorem 3.

This work is a part of a Ph-D thesis defended at the Université Toulouse III in January 1998. Many thanks to my advisor Gilbert Levitt who encouraged me, carefully checked my work, and suggested many improvements.

### 1. Preliminaries

**1.1. Group actions on $\mathbb{R}$-trees**

Basic facts about $\mathbb{R}$-trees may be found in [28,29].

**Definition.** An $\mathbb{R}$-tree is a metric space $T$ such that between two points $x, y \in T$, there exists precisely one topological arc (denoted by $[x, y]$), and this arc is isometric to an interval in $\mathbb{R}$.

In this paper, every $\mathbb{R}$-tree will be endowed with an isometric action of a finitely generated group. For simplicity, we will denote by the same letter $T$ the tree and the action. We will also simply say action to talk about an action on an $\mathbb{R}$-tree. Most often, the group considered will be the free group $F_n$ on $n$ letters. If an isometry $g$ of an $\mathbb{R}$-tree has no fixed point, then it has a translation axis isometric to $\mathbb{R}$ and we say that $g$ is hyperbolic. When $g$ has a fixed point, it is called elliptic. The characteristic set $\text{Char} g$ of $g$ is either its axis or the set of its fixed points, depending on whether $g$ is elliptic or hyperbolic.

An action on an $\mathbb{R}$-tree is said to be minimal if it has no proper invariant subtree and if it is not reduced to one point. If an action of a finitely generated group $\Gamma$ on an $\mathbb{R}$-tree $T$ has no global fixed point, then there is a unique invariant minimal subtree of $T$ and it is the union of the translation axes of hyperbolic elements in $\Gamma$. All the actions we consider are henceforth assumed to be minimal.

We will call simplicial $\mathbb{R}$-tree (or simply a simplicial tree) a connected simply-connected simplicial 1-complex together with a metric which makes it an $\mathbb{R}$-tree. For shortness' sake, we will say simplicial action to mean a simplicial isometric action on a simplicial $\mathbb{R}$-tree. We will always assume that a simplicial action has no inversion i.e. that no edge is flipped by any element of $\Gamma$ since one can reduce to this case by performing a barycentric subdivision.
A morphism of $\mathbb{R}$-trees $f : T \to T'$ is a continuous map such that every arc in $T$ may be subdivided into finitely many intervals which are isometrically embedded in $T'$ by $f$. We will also be interested in maps preserving alignment i.e. such that $a \in [b, c] \Rightarrow f(a) \in [f(b), f(c)]$. Note that a morphism of $\mathbb{R}$-trees which preserves alignment is an isometry.

In an $\mathbb{R}$-tree $T$, a germ at a point $x \in T$ is a germ of isometric applications $[0, \varepsilon] \to T$ sending 0 to $x$. The set of germs at a point $x \in T$ is in one to one correspondence with the connected components of $T \setminus \{x\}$. A point $x \in T$ is called a branch point if there are at least three germs at $x$. In a simplicial tree, the branch points are the vertices of valence at least three. We will sometimes use the projection of a point $x$ on a closed subtree $S$: it is the point in $S$ closest to $x$.

1.2. The topology on a set of actions on $\mathbb{R}$-trees

Two actions of a group $\Gamma$ on $\mathbb{R}$-trees $T$ and $T'$ are identified if there exists an equivariant isometry between $T$ and $T'$. Sometimes, in projectivised spaces, we will identify $T$ and $T'$ if there exists an equivariant homothety between them.

On any set of minimal actions of a fixed finitely generated group $\Gamma$, one can consider the translation lengths topology. This topology is based on the length function of an action $(T, \Gamma)$. It is the function $l_T : \Gamma \to \mathbb{R}_+$ defined by

$$l_T(\gamma) = \inf_{x \in T} d(x, \gamma x).$$

The translation lengths topology is the smallest topology that makes continuous the functions $T \mapsto l_T(\gamma)$ for $\gamma \in \Gamma$. An abelian action is an action whose length function is the absolute value of a morphism $\Gamma \to \mathbb{R}$. For sets of non-abelian actions of a finitely generated group, this topology is Hausdorff (see [8]).

A set of minimal actions of a fixed finitely generated group $\Gamma$ on $\mathbb{R}$-trees can also be equipped with the equivariant Gromov topology. This topology roughly says that two actions are close if they look the same metrically in restriction to a finite subtree while only considering the action of a finite subset of $\Gamma$. Here finite subtree means a subtree which is the convex hull of finitely many points. Let’s give a definition to make this more precise:

**DEFINITION.** – Consider two actions of a finitely generated group $\Gamma$ on two $\mathbb{R}$-trees $T$ and $T'$, and take $\varepsilon > 0$, a finite subset $F$ of $\Gamma$, and two finite subtrees $K \subset T$ and $K' \subset T'$. An $F$-equivariant $\varepsilon$-approximation between $K$ and $K'$ is a binary relation $R \subset K \times K'$ satisfying the three following conditions:

- for every point $x \in K$, there exists a point $x' \in K'$ such that $x Rx'$;
- for every point $x' \in K'$, there exists a point $x \in K$ such that $x Rx'$;
- if $x Rx'$ and $y Ry'$, then for all $g, h \in F$, the numbers $d_T(g.x, h.y)$ and $d_{T'}(g.x', h.y')$ are $\varepsilon$-close to each other.

When $x Rx'$, we say that $x'$ is an approximation point of $x$. If $T$ is an action, for any $\varepsilon > 0$, any finite subset $F$ of $\Gamma$, and any finite subtree $K \subset T$, consider the set $V_T(\varepsilon, F, K)$ consisting of actions $(T', \Gamma)$ such that there exists a finite subtree $K' \subset T'$ with an $F$-equivariant $\varepsilon$-approximation between $K$ and $K'$. By definition, the sets $V_T(\varepsilon, F, K)$ form a neighbourhood basis of $T$ in the equivariant Gromov topology. Note that $\varepsilon$-approximations behave nicely with respect to the Hausdorff topology: an $F$-equivariant $\varepsilon$-approximation between $K$ and $K'$ such that $K'$ is at a Hausdorff-distance $\gamma$ from $K_1$, naturally defines an $F$-equivariant $(\varepsilon + 2\delta)$-approximation between $K$ et $K'_1$.

The equivariant Gromov topology is always finer than the translation lengths topology and is equivalent to the equivariant Gromov topology on sets of non-abelian actions (see [25]).
A group is said to be small if it doesn’t contain any subgroup isomorphic to the free group $F_2$. An arc in an $\mathbb{R}$-tree is called non-degenerate if it contains more than one point. A small action is an action such that the stabilizer of any non-degenerate arc is small. Note that throughout this article, the stabilizer of a set is understood to be its pointwise stabilizer. We assume from now on that $\Gamma$ itself is not small. Then any small action of $\Gamma$ is non-abelian. Moreover, the projectivised space of small actions of $\Gamma$ is compact in both topologies (see [8,24]).

The set of actions of $\Gamma$ on $\mathbb{R}$-trees is naturally endowed with a right action of $\text{Aut}(\Gamma)$ by precomposition. Since two actions are identified if there exists an equivariant isometry between them, the subgroup $\text{Inn}(\Gamma)$ of inner automorphisms acts trivially on this set, hence we are left with an action of the group $\text{Out}(\Gamma) = \text{Aut}(\Gamma)/\text{Inn}(\Gamma)$ of outer automorphisms of $\Gamma$.

1.3. Outer space and very small actions

**Definition.** Outer space (sometimes called Culler–Vogtmann space) is the set $CV_n$ of free (minimal isometric) actions of $F_n$ on simplicial $\mathbb{R}$-trees modulo equivariant homothety.

$CV_n$ is invariant under the action of $\text{Out}(F_n)$. It is a disjoint union of open simplices obtained by equivariantly modifying the lengths of the edges of a tree in $CV_n$, and $\text{Out}(F_n)$ preserves this decomposition.

$CV_n$ is contained in the projectivised space of small actions of $F_n$. Its closure $\overline{CV_n}$ in this space is therefore compact. Moreover, Cohen and Lustig [7] and Bestvina and Feighn [4] have proved that $CV_n$ is exactly the space of very small actions of $F_n$.

**Definition.** An action of $F_n$ on an $\mathbb{R}$-tree $T$ is said to be very small if

- it is small,
- triod stabilizers are trivial (a triod is the convex hull of three points which are not aligned),
- for every $k \neq 0$ and every $g \in F_n$, $\text{Fix} g^k = \text{Fix} g$.

2. Origami: folding to approximate

2.1. Definitions of folds

The goal of this section is to describe a tool which will be fundamental in this paper: we use folds to get approximations of some simplicial trees. The idea of folding is not new, J.R. Stallings already used this technique in [30], and many others used this notion (see [3] and [12] for instance). However, we will consider not only edge-folding but rather path-folding in simplicial trees. We first need some technical conditions so that the folds behave nicely.

**Definition.** Let $T$ be a simplicial action of $F_n$ without inversion. Let $\alpha, \beta$ be two embedded edge-paths in $T$ starting from the same point $x$. We assume that $\alpha$ and $\beta$ run through the same number of edges and we denote by $\alpha_1, \ldots, \alpha_p$ and by $\beta_1, \ldots, \beta_p$ the edges of $\alpha$ and $\beta$. We say that $\alpha$ and $\beta$ satisfy the hypothesis (H) if

1. For all $i = 1, \ldots, p$, $\alpha_i$ and $\beta_i$ have the same length.
2. $\alpha_i$ and $\beta_i$ are distinct edges.
3. There exists an equivariant orientation of the edges of $T$ (called the folding orientation) such that $\alpha_i$ and $\beta_i$ are positively oriented for $i = 1, \ldots, p$. In this case, we say that $\alpha$ and $\beta$ are well oriented.

Clearly, (H3) means that there exists an orientation of the quotient graph $T/F_n$ such that the projections of $\alpha$ and $\beta$ are well oriented.
DEFINITION. - Let $T$ be a simplicial action of $F_n$ without inversion and let $\alpha_1$ and $\beta_1$ be two oriented edges satisfying (H).

The elementary fold between $\alpha_1$ and $\beta_1$ is the quotient of $T$ by the smallest equivariant equivalence relation in $T$ which identifies $\alpha_1$ with $\beta_1$ and also identifies their terminal vertices. The simplicial complex $T/\alpha_1\sim\beta_1$ thus obtained is a tree (see for instance [3]), it has a natural metric and an isometric action of $F_n$ without inversion. The quotient map $f : T \to T/\alpha_1\sim\beta_1$ is called the folding map.

DEFINITION. - Let $T$ be a simplicial action of $F_n$ without inversion and let $\alpha$ and $\beta$ be two edge paths satisfying (H).

The fold between $\alpha = \alpha_1 \ldots \alpha_p$ and $\beta = \alpha_1 \ldots \alpha_p$ is the quotient $T/\alpha\sim\beta$ of $T$ by the smallest equivariant equivalence relation in $T$ which identifies $\alpha_i$ with $\beta_i$ for $i = 1 \ldots p$. It is a composition of elementary folds $f_i$:

$$T \xrightarrow{f_1} T_1 \xrightarrow{f_2} T_2 = T_1/f_1(\alpha_2) \sim f_1(\beta_2) \xrightarrow{f_3} \ldots \xrightarrow{f_p} T_p = T/\alpha\sim\beta.$$ 

The elementary folds $f_i$ are called intermediate folds. We denote by $q_i = f_i \circ \ldots \circ f_1 : T \to T_i$ and by $q = f_p \circ \ldots \circ f_1 : T \to T/\alpha\sim\beta$ the folding map.

This decomposition shows that $T/\alpha\sim\beta$ is a simplicial tree with a natural isometric action of $F_n$ with no inversion.

2.2. Preimage of an edge

We define the preimage of an edge $e'$ of a simplicial tree $T'$ under a simplicial map $f : T \to T'$ to be the set $f^{-1}(e')$ of edges which map to $e'$ under $f$ (and not the set of points in $T$ which are mapped to a point of the closed edge $e$). The main interest of the hypothesis (H3) is the following remark.

LEMA 2.1. - If $f : T \to T'/\alpha_1\sim\beta_1$ is the elementary fold between the edges $\alpha_1$ and $\beta_1$ satisfying (H) then $f^{-1}(e')$ is either a single edge or a set of edges having the same origin according to the folding orientation. We say that these edges are centrifugal.

Remark. - If (H3) is not satisfied, then $f^{-1}(e')$ may be unbounded.

Proof. - $T'/\alpha_1\sim\beta_1$ is the quotient of $T$ by the equivalence relation generated by the binary relation $\sim_1$ described by $e \sim_1 e'$ if there exists $g \in F_n$ such that $\{g.e, g.e'\} = \{\alpha_1, \beta_1\}$. One needs only to notice that if $e \sim_1 e' \sim_1 e''$, then $e, e', e''$ are centrifugal. $\square$

Note that Lemma 2.1 implies that if $\alpha, \beta$ satisfy (H) then none of the intermediate folds can be isometries (i.e. the intermediate folds satisfy (H2)) because $f_i(\alpha_2) = f_i(\beta_2)$ would contradict the lemma.

The following corollary is the tool which allows us to get approximations from folds.

COROLLARY 2.2. - Take two edge paths $\alpha$ and $\beta$ in a simplicial tree satisfying (H), and denote by $q : T \to T'/\alpha\sim\beta$ the folding map. Suppose that each intermediate fold $f_i$ is a fold between two edges with trivial stabilizer.

If $e, e'$ are two adjacent edges in $T$ which are identified by $q$, then they are identified under the first intermediate fold $f_1$.

Remark. - The hypothesis on the intermediate folds $f_i$ can be weakened but the corollary is false with no hypothesis at all on $f_i$.
DYNAMICS OF $\text{Out}(F_n)$ ON THE BOUNDARY OF OUTER SPACE 439

Proof. Assume on the contrary that there exists an index $i > 0$ such that $q_i(e) \neq q_i(e')$ and $q_{i+1}(e) = q_{i+1}(e')$. This implies that $q_i(e) \neq q_i(e')$ are centrifugal.

Suppose first that $q_i(\alpha_{i+1})$ and $q_i(\beta_{i+1})$ don't lie in the same orbit of $T_i$. Then, the fact that \( \text{Stab} q_i(\alpha_{i+1}) = \text{Stab} q_i(\beta_{i+1}) = \{1\} \) implies that an edge is identified with $q_i(\alpha_{i+1})$ and $q_i(\beta_{i+1})$ through $f_{i+1}$ only if it equals $q_i(\alpha_{i+1})$ or $q_i(\beta_{i+1})$. Therefore, we can assume without loss of generality that $q_i(e) = q_i(\alpha_{i+1})$ and $q_i(e') = q_i(\beta_{i+1})$. Thanks to the previous corollary, $q_{i-1}(e)$ and $q_{i-1}(\alpha_{i+1})$ have the same origin, and similarly for $q_{i-1}(e')$ and $q_{i-1}(\beta_{i+1})$. But this prevents them from being adjacent, which is a contradiction.

Suppose now that there exists an $h \in F_n$ such that $h \cdot q_i(\alpha_{i+1}) = q_i(\beta_{i+1})$ (this $h$ is unique because $\text{Stab} q_i(\alpha_{i+1}) = \{1\}$). In this case, the set of edges which are identified with $q_i(\alpha_{i+1})$ by $f_{i+1}$ is exactly $h^2 \cdot q_i(\alpha_{i+1})$ so we can assume that $q_i(e) = q_i(\alpha_{i+1})$ and $q_i(e') = q_i(h^k \cdot \alpha_{i+1})$ for some $k \neq 0$. Now let $A$ and $B$ be the preimages of $q_i(\alpha_{i+1})$ and $q_i(\beta_{i+1})$ under $f_i$. The previous corollary implies that $A$ and $B$ are two sets of centrifugal edges whose centers are $p_A$ and $p_B$, the terminal points of $q_{i-1}(\alpha_i)$ and $q_{i-1}(\beta_i)$. Now since $q_{i-1}(e') \in h^k \cdot A$, and because $q_{i-1}(e)$ and $q_{i-1}(e')$ are centrifugal, $h^k \cdot A$ is a set of centrifugal edges with center $p_A$. Therefore, $h^k$ fixes $p_A$ and $h$ sends $p_A$ to $p_B$, so $h$ fixes the midpoint of $[p_A, p_B]$ which is the origin of $q_{i-1}(\alpha_i)$. Hence $h^k$ fixes $q_{i-1}(\alpha_i)$ which contradicts the assumption on the fold $f_i$. \( \square \)

2.3. Folding to approximate

We are now ready to prove the folding to approximate lemma.

FOLDING TO APPROXIMATE LEMMA. Let $T$ be a simplicial action of $F_n$ without inversion. Let $\alpha$ and $\beta$ be two paths in $T$ with origin $x$ satisfying the $(H)$ condition such that $\text{Stab} x$ is infinite. Let $w_k$ be a sequence of distinct elements in $\text{Stab} x$ and let $T^{(k)} = T/\alpha \sim \sim w_k \beta$. Assume that each intermediate fold is a fold between edges with trivial stabilizer.

Then $T^{(k)}$ converges to $T$ as $k \to \infty$.

Proof. We only need to prove that two incident edges $e, e'$ of $T$ are identified by only finitely many folds $q^{(k)} : T \to T^{(k)}$. As a matter of fact, this will imply that any finite subtree of $T$ isometrically embeds in $T^{(k)}$ under $q^{(k)}$. To prove the convergence in the equivariant Gromov topology, take $K$ to be a finite subtree of $T$ and $F$ a finite subset of $F_n$, and let $K'$ be the convex hull of $K$ and $F \cdot K$. For $k$ large enough, $q^{(k)}$ is an isometry in restriction to $K'$ hence it gives an $F$-equivariant $0$-approximation between $K$ and $q^{(k)}(K)$.

Now we prove that two incident edges $e, e'$ of $T$ are identified by only finitely many folds $q^{(k)}$. Thanks to the previous corollary, we only need to check that they are identified by finitely many of the elementary folds $f_{i}^{(k)}$ between $\alpha_i$ and $w_k \beta_i$. ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE
When $\alpha_1$ and $\beta_1$ are not in the same orbit, $e$ and $e'$ are identified by $f_1^{(k)}$ if and only if there exists $g \in F_n$ such that $g \cdot (e, e') = \{\alpha_1, w_k, \beta_1\}$. This occurs for at most one $k$ since $\text{Stab } \alpha_1 = \{1\}$, $g$ is unique.

When $\beta_1 = h \cdot \alpha_1$ ($h$ is then unique), $e$ and $e'$ are identified by $f_1^{(k)}$ for some index $k$ if and only if there exists $g \in F_n$ such that $g \cdot e = \alpha_1$ and $g \cdot e' = (w_k h)^{i_k} \cdot \alpha_1$ for some $i_k \in \mathbb{Z} \setminus \{0\}$. If $k_0$ and $k$ are such indices, then $g \cdot e' = (w_k h)^{i_k} \cdot \alpha_1 = (w_k h)^{i_k} \cdot \alpha_1$ so $(w_k h)^{i_k} = (w_k h)^{i_k}$ and $w_k h$ lies in the finite set of roots of $(w_k h)^{i_k}$ which can hold for at most finitely many $k$. 

3. An open invariant subset of outer space

3.1. Looking for an open invariant subset

This folding to approximate lemma will be the cornerstone of Section 4. But first, this lemma will show us that some natural candidates for open and invariant sets are in fact not open.

The first candidate for open set in $\overline{CV}_n$ is the set $C_n$ of very small actions $T$ in which there is a non-degenerate arc $I$ containing no branch point of $T$ and such that $\text{Stab } I = \{1\}$. It is a natural candidate because the set of systems of isometries which give an action in $C_n$ is precisely the set of systems of isometries whose suspension have a family of compact simply-connected leaves, and this property is stable under perturbation (see Proposition IV.1 in [18]). Moreover, thanks to the exhaustive study of $CV_2$ by M. Culler and K. Vogtmann, it is easy to check that $C_2$ is open in $\overline{CV}_2$. However, $C_n$ is not open for $n \geq 3$: for instance if $T \in C$ is the action shown on Fig. 2, a folding operation allows us to approximate $T$ by a very small simplicial action whose edge stabilizers are not trivial. Thus, this approximating action doesn’t lie in $C_n$.

In view of this example, we see that the presence of a non cyclic vertex stabilizer allows many approximations, so we may consider a second candidate for open set: the set $C'_n$ of very small simplicial actions with cyclic edge and vertex stabilizers. Once again, it is natural because one can prove that the set of systems of isometries which define an action in $C'_n$ is open ([15, Theorem 4.4.5]). One also checks that $C'_2$ is open in $\overline{CV}_2$. But for $n \geq 3$, the folding to approximate lemma shows that $C'_n$ is not open (see Fig. 3). The reason is that one can perform folds at a vertex with non-trivial stabilizer which is not terminal in the quotient graph $T/F_n$ (a vertex is terminal if it has valence 1).

This leads us to consider the following set $\mathcal{O}_n$:

DEFINITION. – We define $\mathcal{O}_n$ to be the set of simplicial $F_n$-actions $T$ such that

- $T$ has trivial arc stabilizers,
- $T$ has cyclic vertex stabilizers,
- whenever $\text{Stab } v \neq \{1\}$, $\text{Stab } v$ acts transitively on the set of incident edges.

![Fig. 2. $T \in C_n$ is approximated by $T^{(k)} = T/\alpha \sim w_k \cdot \alpha \notin \mathcal{C}_n$, $C_n$ is not open in $\overline{CV}_n$ for $n \geq 3$.](image-url)
DYNAMICS OF $\text{Out}(F_n)$ ON THE BOUNDARY OF OUTER SPACE

$$T \in \mathcal{O}_n'$$

Fig. 3. $T \in \mathcal{O}_n'$ is approximated by $T^{(k)} = T/\alpha \sim \beta, \beta \notin \mathcal{O}_n'$, $\mathcal{O}_n'$ is not open in $\overline{CV_n}$ for $n \geq 3$.

$$\langle a \rangle \xrightarrow{\alpha} \langle b \rangle \xrightarrow{\beta} \langle c \rangle$$

$$\langle a, b^t c b^t \rangle \xrightarrow{\alpha = \beta, \beta} \langle c, b^t ab^t \rangle$$

$\mathcal{O}_n$ is open in $\overline{CV_n}$

**Theorem 1.** The set $\mathcal{O}_n$ is open in $\overline{CV_n}$ and $\text{Out}(F_n)$ acts properly discontinuously on $\mathcal{O}_n$.

**Remark.** Using the folding to approximate lemma, one proves that $\mathcal{O}_n$ is not open in the whole set of small actions.

Equivalently, $T$ lies in $\mathcal{O}_n$ if the edge groups of the graph of groups $T/F_n$ are trivial, and if the only non-trivial vertex groups are cyclic and are attached to terminal vertices of $T/F_n$.

**3.2. $\mathcal{O}_n$ is open in $\overline{CV_n}$**

**Proof of Theorem 1.** With no additional work, we will prove the well known fact that $\overline{CV_n}$ is open in the set of all actions of $F_n$. The proof of Theorem 1 goes as follows: we start with an action $T \in \mathcal{O}_n$ and consider a fundamental domain $D$ for this action. Given an action $T' \in \overline{CV_n}$ close enough to $T$ in the equivariant Gromov topology, there is a finite subtree $D'$ in $T'$ which approximates $D$. The main step is to build a fundamental domain $\Delta$ for $T'$ starting from $D'$.

**Fundamental domain and adapted basis for an action in $\mathcal{O}_n$**

Let $T$ be an action in $\mathcal{O}_n$ and consider the quotient metric graph of groups $Q = T/F_n$. Let $\tau$ be a maximal subtree of $Q$ and $\tilde{\tau}$ a preferred lift so that we get (using Bass–Serre theory) an identification between $F_n$ and $\pi_1(Q, \tau)$ and an equivariant isometry between $T$ and the universal cover of $Q$. Now choose an orientation for every edge in $Q \setminus \tau$ and one generator for every non-trivial vertex group of $Q$. The set of elements in $\pi_1(Q, \tau)$ corresponding to the edges of $Q \setminus \tau$ with the chosen orientations and to the chosen generators of the vertex groups provides a preferred basis $B$ of $F_n$. 
This basis $B$ has the following property: for every $\gamma \in B \cup B^{-1}$, either $\gamma, \tau \cap \tau$ is a single point that we denote by $\chi_{\gamma}$ (this happens when $\gamma$ corresponds to a vertex group generator) or there is an edge joining $\tau$ to $\gamma, \tau$ (when $\gamma$ comes from an edge in $Q \setminus \tau$) in which case we call $\chi_{\gamma}$ the midpoint of this edge. We define $D$ to be the union of $\tau$ and of the segments joining $\chi_{\gamma}$ to $\tau$ ($\gamma \in B \cup B^{-1}$). The following properties clearly hold:

**Lemma 3.1.** For every $\gamma \in B \cup B^{-1}$,
- $\gamma \cdot D \cap D = \{\chi_{\gamma}\}$,
- $\chi_{\gamma}$ is a terminal point of $D$,
- $\chi_{\gamma} = \chi_{\gamma}^{-1}$ if and only if $\gamma$ is elliptic in which case $\chi_{\gamma}$ is the only fixed point of $\gamma$.

Observe also that $D$ is a fundamental domain for $T$ in the following sense:

**Lemma 3.2.** $D$ meets every orbit in $T$ and if $x, y \in D$ are such that $y = w \cdot x$ for some $w \in F_n \setminus \{1\}$, then either
- $x = \chi_{\gamma}^{-1}$, $y = \chi_{\gamma}$, and $w = \gamma$ for some hyperbolic generator $\gamma \in B \cup B^{-1}$,
- or $x = y = \chi_{\gamma}$ and $w = \gamma^{-p}$ for an elliptic $\gamma \in B \cup B^{-1}$.

**Constructing $\Delta$ from $D'$**

We now define $V_\varepsilon(T)$ a neighbourhood of $T$: we set $F = \{1\} \cup B \cup B^{-1}$ and set $V_\varepsilon(T) = V_\varepsilon(F, D)$ to be the set of actions $T' \in CV_n$ such that there exists a finite subtree $D'$ in $T'$ with an $F$-equivariant $\varepsilon$-approximation between $D$ and $D'$. We are going to show that if $T' \in V_\varepsilon(T)$ for some small enough $\varepsilon$, then $T' \in O_n$.

Denote by $d$ the length of the shorted edge in $D$, and assume that $\varepsilon$ is small enough compared to $d$. Then any element $\gamma \in B \cup B^{-1}$ which is hyperbolic in $T$ must be hyperbolic in $T'$. Note however that an elliptic element in $B \cup B^{-1}$ may be hyperbolic in $T$ (but its translation length must be small compared to $d$).

We start by changing $D'$ into $D_1'$ such that $\gamma \cdot D_1' \cap D_1' = \emptyset$ for $\gamma \in B \cup B^{-1}$.

**Definition.** Let $K$ be a finite tree and $\delta > 0$. We call $\delta$-interior of $K$ the set $\text{ints}(K)$ of points of $K$ which are the midpoint of a segment of $K$ of length $2\delta$.

Clearly, $\text{ints}(K)$ is a finite subtree of $K$ and if $K$ has diameter at least $2\delta$, $\text{ints}(K)$ is non-empty and $K$ lies in the $\delta$-neighbourhood of $\text{ints}(K)$.

**Lemma 3.3.** Let $\delta = 3\varepsilon$, and let $D_1' = \text{ints}(D')$. If $\varepsilon$ is small enough compared to $d$, then

$\forall \gamma \in B \cup B^{-1}$ $\gamma \cdot D_1' \cap D_1' = \emptyset$.

**Proof.** Assume on the contrary that there exists $y' = \gamma \cdot x' \in D_1' \cap \gamma \cdot D_1'$ for some $\gamma \in B \cup B^{-1}$ and argue towards a contradiction. By definition of the $\delta$-interior, there are some points $x_1', x_2', y_1', y_2' \in D'$ such that $x_1', x', x_2'$ (respectively $y_1', y', y_2'$) are aligned in this order and $\delta$-far from each other (we say that $a, b, c$ are aligned in this order if $b \in [a, c]$).

Consider some approximation points $x_1, x_2, y_1, y_2$ in $D$ of $x_1', x_2', y_1', y_2'$, respectively. Since $y' = \gamma \cdot x'$, $d(\gamma \cdot x, y) \leq \varepsilon$. But $\chi_{\gamma} \in [\gamma \cdot x, y]$ since $\gamma \cdot x$ and $y$ lie in the two subtrees $\gamma \cdot D$ and $D$ which intersect only in $\{\chi_{\gamma}\}$. Therefore, $x$ is $\varepsilon$-close to $\chi_{\gamma}^{-1}$ and every branch point of $D$ is at least $(d - \varepsilon)$-far from $x$.

The distance from $x$ to its projection $p$ on the segment $[x_1, x_2]$ is at most $3\varepsilon/2$. As a matter of fact, in an $R$-tree the distance from a point $a$ to its projection on a segment $[b, c]$ is the Gromov product

$$(b|c)_a = \frac{1}{2} (d(a, b) + d(a, c) - d(b, c)).$$
Now the fact that \( d(x, x_1) \) and \( d(x, x_2) \) are greater than \( \delta - \varepsilon > 3\varepsilon \) implies that \( p \) has to be distinct from \( x_1 \) and \( x_2 \). Moreover, \( p \) cannot be a branch point of \( D \) if \( \varepsilon \) is small compared to \( d \). This means that \( x \) must lie in \( [x_1, x_2] \). But then, since \( [x_1, x_2] \) doesn’t contain any branch point of \( D \), either \( \chi_{\gamma^{-1}}, x_1, x, x_2 \) or \( \chi_{\gamma^{-1}}, x_2, x, x_1 \) are aligned in this order. Since \( d(x, x_1), d(x, x_2) \geq \delta - \varepsilon > \varepsilon \), this prevents \( x \) from being \( \varepsilon \)-close to \( \chi_{\gamma^{-1}} \) which gives a contradiction. □

Since \( D' \) is in the \( \delta \)-neighbourhood of \( D_0' \), there is an \( F \)-equivariant \( \varepsilon_1 \)-approximation between \( D \) and \( D_0' \) for \( \varepsilon_1 = \varepsilon + 2\delta = 7\varepsilon \). Hence, we forget the approximation between \( D \) and \( D' \) and we concentrate on the approximation between \( D \) and \( D_0' \). For every \( \gamma \in B \cup B^{-1} \) we choose an approximation point \( \chi_\gamma' \in D_0' \) of \( \chi_\gamma \).

It will now be easy to construct a fundamental domain \( \Delta \) for \( T' \) by adding to \( D_0' \) the segments \( I_\gamma \), defined as follows (see Fig. 5):

**Definition.** For every \( \gamma \in B \cup B^{-1} \), we define the points \( \pi_\gamma \) in \( T' \) and the segments \( I_\gamma \subset T' \) as follows:
- If \( \gamma \) is elliptic in \( T' \), we call \( \pi_\gamma = \pi_{\gamma^{-1}} \) the projection of \( D_0' \) on \( \text{Fix}_{T'} \gamma \) and we take \( I_\gamma = I_{\gamma^{-1}} \) to be the segment joining \( D_0' \) to \( \pi_{\gamma^{-1}} \).
- If \( \gamma \) is hyperbolic in \( T' \), we call \( \pi_\gamma \) the midpoint of the intersection of \( \text{Axis}_{T'}(\gamma) \) with the segment joining \( D_0' \) to \( \gamma.D_0' \) (so that \( \gamma.\pi_{\gamma^{-1}} = \pi_{\gamma} \)). We then take \( I_\gamma \) to be the segment joining \( D_0' \) to \( \pi_{\gamma} \).

We then define \( \Delta = D_0' \cup \bigcup_{\gamma \in B \cup B^{-1}} I_\gamma \).

**Remark.** Note that by minimality, \( \Delta \) meets every orbit of \( T' \); the fact that for all \( \gamma \in B \cup B^{-1} \), \( \gamma.\Delta \cap \Delta \neq \emptyset \) implies that \( F_{\pi_\gamma} \Delta \) is connected and invariant, and hence must be equal to \( T' \).

**Lemma 3.4.** The arc \( I_\gamma \) is contained in the \( \varepsilon_1 \)-neighbourhood of \( \chi_\gamma' \).

**Proof.** Since \( I_\gamma \) is contained in the segment joining \( D_0' \) to \( \gamma.D_0' \), every segment \([p, q]\) with \( p \in D_0' \) and \( q \in \gamma.D_0' \) contains \( I_\gamma \). Hence \( I_\gamma \subset [\chi_{\gamma'}', \gamma.\chi_{\gamma'}^{-1}] \), but since \( \chi_{\gamma'} = \gamma.\chi_{\gamma^{-1}} \), we get \( d(\chi_{\gamma'}', \gamma.\chi_{\gamma^{-1}}) \leq \varepsilon_1 \). □

This lemma implies that there is an \( F \)-equivariant \( (\varepsilon_2 = 3\varepsilon_1) \)-approximation between \( D \) and \( \Delta \) for which \( \pi_\gamma \) is an approximation point of \( \chi_{\gamma} \).

**\( \Delta \) is a fundamental domain for \( T' \)**

**Lemma 3.5.** If \( \varepsilon \) is small enough compared to \( d \), then \( \pi_\gamma \) is a terminal point of \( \Delta \), and for all \( \gamma \in B \cup B^{-1} \),

\[ \gamma.\Delta \cup \Delta = \{ \pi_\gamma \} . \]
Moreover, one has $\pi_\gamma = \pi_{\gamma'}$ if and only if $\gamma = \gamma'$ or $\gamma = \gamma'^{-1}$ with $\gamma$ elliptic in $T'$. Finally, if $\gamma$ is elliptic in $T'$, then the germ of $\Delta$ at $\pi_\gamma$ is not fixed by $\gamma$, and if $\gamma$ is hyperbolic, the germ of $\Delta$ at $\pi_\gamma$ points towards the negative half-axis of $\gamma$.

Proof. - By construction, it is clear that for all $\gamma \in B \cup B^{-1}$,

$$\gamma. (D_\gamma^1 \cup I_\gamma \cup I_{\gamma^{-1}}) \cap (D_\gamma^1 \cup I_\gamma \cup I_{\gamma^{-1}}) = \{ \pi_\gamma \}.$$ 

To prove that $\gamma. \Delta \cap \Delta = \{ \pi_\gamma \}$ we just have to check that

$$\gamma. I_{\gamma'} \cap \Delta = \emptyset$$

for $\gamma' \in B \cup B^{-1} \setminus \{ \gamma, \gamma^{-1} \}$. But since $I_{\gamma'}$ is contained in the $\varepsilon_1$-neighbourhood of $\pi_{\gamma'}$, and since $\gamma. x_{\gamma'}$ is far apart from $D$ (at least at a distance $d$), we see that $\gamma. \pi_{\gamma'}$ must be far apart from $\Delta$.

Now by construction, $\pi_\gamma$ is a terminal point of $D_\gamma^1 \cup I_\gamma \cup I_{\gamma^{-1}}$ and must remain terminal in $\Delta$ since the intervals added to $D_\gamma^1 \cup I_\gamma \cup I_{\gamma^{-1}}$ are far from $I_\gamma \cup I_{\gamma^{-1}}$. The last claims follow immediately. \qed

To prove that $\Delta$ is a fundamental domain for $T'$, we use the same technique as [10]. We first introduce some notations.

DEFINITION. - Let $\gamma \in B \cup B^{-1}$.
- If $\gamma$ is hyperbolic in $T'$, we call $\eta_\gamma$ the germ of the positive axis of $\gamma$ at $\pi_\gamma$. We also set

$$S_\gamma = \{ x \in T' \setminus \{ \pi_\gamma \} | \text{germ}_{\pi_\gamma} ([\pi_\gamma, x]) = \eta_\gamma \}.$$ 

- If $\gamma$ is elliptic in $T'$, we call $\eta_\gamma = \eta_{\gamma^{-1}}$ the germ of $\Delta$ at $\pi_\gamma$. We also set

$$S_\gamma = \{ x \in (T', F_n) \setminus \{ \pi_\gamma \} | \exists k > 0 \text{ s.t. } \text{germ}_{\pi_\gamma} ([\pi_\gamma, x]) = \gamma^k. \eta_\gamma \}.$$ 

LEMMA 3.6. - If $T'$ is very small, or if $T$ is free simplicial, and if $\varepsilon$ is small enough, then the sets $S_\gamma$ are pairwise disjoint and do not meet $\Delta$.

Proof. - Assume that $\gamma \in B \cup B^{-1}$ is hyperbolic in $T'$. Since $\eta_\gamma$ and the germ of $\Delta$ at $\pi_\gamma$ point respectively towards the positive and negative half-axis of $\gamma$, $S_\gamma$ does not intersect $\Delta$.

Assume now that $\gamma \in B \cup B^{-1}$ is elliptic in $T'$. Then $\gamma$ has to be elliptic in $T$ (so that $T$ is not free). Here we use the hypothesis that $T'$ is very small: since $\gamma$ does not fix $\eta_\gamma$, $\gamma^k$ does not fix $\eta_\gamma$ so the germs $\gamma^k. \eta$ are all distinct and $S_\gamma, S_{\gamma^{-1}}$ and $\Delta$ are pairwise disjoint.

4$^e$ SÉRIE - TOME 33 - 2000 - N° 4
There remains only to prove that $\mathcal{S}_\gamma \cap \mathcal{S}_{\gamma'} = \emptyset$ when $\pi_\gamma \neq \pi_{\gamma'}$. But then $\mathcal{S}_\gamma = \mathcal{S}_\gamma \cup \mathcal{S}_{\gamma'}$ is a subtree of $T'$ which cannot intersect $\mathcal{S}_{\gamma'}$ because otherwise their union would be connected hence would contain $[\pi_\gamma, \pi_{\gamma'}] \subset \Delta$ which is impossible. ∎

Remark. − We note the following facts:
• $\gamma_{i-1} = \pi_{\gamma_i}.$
• $\gamma_i(\Delta \setminus \{\pi_{\gamma_i}\}) \subset \mathcal{S}_\gamma.$
• $\gamma_i \mathcal{S}_{\gamma'} \subset \mathcal{S}_\gamma$ for all $\gamma_i \in (B \cup B^{-1}) \setminus \{\gamma_i^{-1}\}.$

**Corollary 3.7.** − The finite tree $\Delta$ is a fundamental domain for $T'$ in the following sense:
• $\Delta$ meets every orbit in $T'$;
• Assume that $x, y \in \Delta$ are such that $x = w.y$ with $w \neq 1.$ Then
  - either $x \neq y$ in which case $x = \pi_{\gamma_i}, y = \pi_{\gamma_j}$ and $w = \gamma$ for some $\gamma \in B \cup B^{-1}$ hyperbolic in $T'$,
  - or $x = y = \pi_{\gamma_i}$ and $w = \gamma^p$ for some $\gamma \in B \cup B^{-1}$ elliptic in $T'$.

**Proof.** − We have already noted that $\Delta$ meets every orbit. Now write $w = \gamma_p \cdots \gamma_1$ as a reduced word with $\gamma_i \in B \cup B^{-1}.$ The previous remark shows that $\gamma_i \cdots \gamma_1.x \in \mathcal{S}_\gamma \cup \{\pi_{\gamma_i}\}. $ Moreover, if for some index $i$, $\gamma_i \cdots \gamma_1.x \neq \pi_{\gamma_i},$ then $\gamma_i \cdots \gamma_1.x \in \mathcal{S}_{\gamma_{i+1}}.$ We thus get inductively that $\gamma_p \cdots \gamma_1.x \in \mathcal{S}_\gamma,$ so $\gamma_p \cdots \gamma_1.x$ can’t lie in $\Delta.$ Therefore, $x = \pi_{\gamma_i}$ and $w = \gamma_{1}$ if $p = 1.$ We are done. Otherwise, we have $\pi_{\gamma_i} = \pi_{\gamma_{i+1}}$, which implies $\gamma_1 = \gamma_{i}.$ Since $w$ is reduced, we obtain recursively that $w = \gamma^p,$ and $\gamma_{1}$ is elliptic in $T'$ since $\pi_{\gamma_1} = \pi_{\gamma_{i+1}}. $ ∎

$\mathcal{O}_n$ is open in $\overline{CV}_n$

**Proposition 3.8.** − $\mathcal{O}_n$ is open in $\overline{CV}_n$ and $CV_n$ is open in the set of all actions of $F_n$ on $\mathbb{R}$-trees.

**Proof.** − Take $T'$ close enough to an action $T$ in $\mathcal{O}_n$ and assume moreover that $T'$ is very small in the case when $T$ is not free. Then the set $\Delta$ constructed above is a fundamental domain in the sense of Corollary 3.7.

First of all, $T'$ is simplicial because $T'$ is the union of the translates of $\Delta$ and $w.\Delta$ may only meet $w_0.\Delta$ in some $w_0.\pi_{\gamma_i}.$ Moreover, if $T$ is free, Corollary 3.7 shows that $T'$ is free.

If $T$ is not free, Corollary 3.7 implies that edge stabilizers are trivial and that vertex stabilizers are cyclic. Now if $x$ has non-trivial stabilizer, we may assume (up to the action of $F_n$) that $x = \pi_{\gamma_i}$ and $\text{Stab} x = \gamma^\Gamma_{\gamma_i}.$ This implies that $\gamma_1$ acts transitively on the set of edges incident to $x.$ We thus conclude that $T' \in \mathcal{O}_n. $ ∎

**The stabilizer in Out$(F_n)$ of every $T \in \mathcal{O}_n$ is finite**

**Lemma 3.9.** − The stabilizer in Out$(F_n)$ of every $T \in \mathcal{O}_n$ is finite.

**Proof.** − Assume that $\alpha \in \text{Aut}(F_n)$ fixes $T$ i.e. that there is an equivariant homothety $h$ between $T$ and $T.\alpha.$ This homothety naturally induces a homothety of the finite metric graph $T/F_n,$ which implies that $h$ must be an isometry. Since $\text{Id}: T \to T.\alpha$ is $\alpha$-equivariant (i.e. $\text{Id}(g.x) = \alpha(g).\text{Id}(x)$) for $x \in T \to T$ is $\alpha$-equivariant.

In [1], it is proved that $f$ induces an automorphism $\varphi$ of the graph of groups $Q = T/F_n$ in the following sense:

**Definition.** − If $Q$ is a graph of groups, an automorphism $\varphi$ of $Q$ consists in
• an automorphism $\varphi$ of the underlying graph (an isometry in our case),
• an isomorphism $\varphi_v : T_v \to \Gamma_{\varphi(v)}$ for each vertex $v \in Q$,
• for every oriented edge $e$ of $Q$, an isomorphism, $\varphi_e : \Gamma_e \to \Gamma_{\varphi(e)}$ such that $\varphi_e = \varphi_e.$
• for each vertex $v \in Q$, an element $\gamma_v \in \pi_1(Q, \varphi(v))$,
• for every oriented edge $e$ of $Q$, an element $\gamma_e \in \pi_1(Q, \varphi(o(e)))$ such that $\delta_e := \gamma_{o(e)}\gamma_e \in \Gamma_{o(e)}$.

such that the following diagram commutes:

\[
\begin{array}{ccc}
\Gamma_e & \xrightarrow{\varphi_e} & \Gamma_{o(e)} \\
\downarrow & & \downarrow \\
\Gamma_{\varphi(e)} & \xrightarrow{\varphi_{\varphi(e)}} & \Gamma_{\varphi(o(e))}
\end{array}
\]

Remark. – When edge stabilizers are trivial, the diagram automatically commutes.

Such a morphism $\varphi$ induces $\alpha$ on the fundamental group of $Q$ in the following sense: one defines $\varphi^*: \pi_1(Q, v) \to \pi_1(Q, \varphi(v))$ by setting, for every loop $(g_0, e_1, e_2, \ldots, e_n, g_n)$ in the graph of groups $Q$

$\varphi^*(g_0e_1 \ldots e_ng_n) = (\gamma_{v_0}\varphi(v_0)(g_0)\gamma_{v_0}^{-1}) \gamma_{v_1}\varphi(e_1)\gamma_{v_1}^{-1} \cdots (\gamma_{v_n}\varphi(v_n)(g_n)\gamma_{v_n}^{-1})$.

Then there exists a path $p_v$ in the graph of groups $Q$ joining $v$ to $\varphi(v)$ such that the induced morphism $I_{p_v} : \pi_1(Q, \varphi(v)) \to \pi_1(Q, v)$ is such that $I_{p_v} \circ \varphi^*$ induces $\alpha$ on $\pi_1(Q, v)$ (see [1]).

Denote by $\operatorname{Aut}(Q)$ the group of automorphisms of the graph of groups $Q$ and $\operatorname{Aut}_0(Q)$ the finite index subgroup of $\operatorname{Aut}(Q)$ consisting of automorphisms inducing the identity on the underlying graph of $Q$. We only have to prove that $\operatorname{Aut}(Q)$ has finite image in $\operatorname{Out}(\pi_1(Q, v))$.

Let $(g_0, e_1, e_2, \ldots, e_n, g_n)$ be a loop based at $v$ in the graph of groups $Q$. If $v_i$ is a terminal vertex of $Q$, $e_{i-1} = e_i$ and since $\Gamma_{v_i}$ is abelian, $\gamma_{v_{i-1}}\gamma_{v_i}$ commutes with $\varphi_{v_i}(g_i)$. This implies

$$(\gamma_{v_{i-1}}e_{i-1}\gamma_{v_{i-1}}^{-1})(\gamma_{v_i}\varphi_{v_i}(g_i)\gamma_{v_i}^{-1})(\gamma_{v_i}\varphi_{e_i}(\gamma_{e_i}^{-1}) = \gamma_{v_{i-1}}e_{i-1}\varphi_{v_i}(g_i)e_i\gamma_{e_i}^{-1}.$$ 

If $v_i$ is not a terminal point in $Q$ then $\gamma_{e_{i-1}}^{-1}\gamma_{v_i} \in \Gamma_{v_i} = \{1\}$ so

$$(\gamma_{v_{i-1}}e_{i-1}\gamma_{e_{i-1}}^{-1})(\gamma_{v_i}\varphi_{v_i}(g_i)\gamma_{v_i}^{-1})(\gamma_{v_i}\varphi_{e_i}(\gamma_{e_i}^{-1}) = \gamma_{v_{i-1}}e_{i-1}\varphi_{v_i}(g_i)e_i\gamma_{e_i}^{-1}.$$ 

Therefore, when $v = v_0 = v_n$ is not terminal (which we may assume without loss of generality), we derive that

$$\varphi^*(g_0e_1 \ldots e_ng_n) = \gamma_{v_0}\varphi(v_0)(g_0)\varphi(e_1)\varphi_{v_1}(g_1)\ldots \varphi_{v_{n-1}}(g_{n-1})\varphi_{v_n}(g_n)\gamma_{v_0}^{-1}.$$ 

Therefore, the image in $\operatorname{Out}(\pi_1(Q, v))$ of $\operatorname{Aut}_0(Q)$ is a quotient of the direct product of the automorphism groups of the vertex groups $\Gamma_{v_i}$, which are finite since $\Gamma_{v_i}$ are cyclic.

The action of $\operatorname{Out}(\Gamma)$ on $\mathcal{O}_n$ is properly discontinuous

There is a natural decomposition of $\mathcal{O}_n$ into open simplices: if $T \in \mathcal{O}_n$, we call $\sigma(T) \subset \mathcal{O}_n$ the set of actions obtained by changing equivariantly the lengths of the edges of $T$ (each length remaining non-zero). These open simplices form a partition of $\mathcal{O}_n$ which is preserved by $\operatorname{Out}(\Gamma)$. Moreover, there are finitely many orbits of simplices since such an orbit corresponds to an unmarked graph of groups (with no metric) which appears as a quotient of an action in $\mathcal{O}_n$.

We also consider the set $\tilde{\sigma}(T)$ (which may not be a closed simplex) of actions in $\mathcal{O}_n$ obtained by changing equivariantly the lengths of the edges of $T$ (here 0 is allowed but the action obtained must lie in $\mathcal{O}_n$). We then call $\tilde{S}(T)$ the star of $T$, i.e. the set of actions $T'$ such that $T \in \tilde{\sigma}(T')$. Equivalently, $\tilde{S}(T)$ is the set of simplicial actions $T'$ such that there exists an equivariant
application from $T'$ to $T$ which preserves alignment. Of course, $St(T)$ is a union of open simplices and if $T' \in St(T)$, $St(T) \subset St(T')$. This union is finite because if $\sigma_n$ was an infinite sequence of such simplices, then up to taking a subsequence, there would exist $\alpha_n \in Out(F_n)$ sending $\sigma_n$ to $\sigma_0$ and stabilizing $\sigma(T)$, but this contradicts the fact that the stabilizer of the barycenter of $\sigma(T)$ is finite.

**Proposition 3.10.** For all $T \in O_n$, $St(T)$ is open in $CV_n$.

**Remark.** This proposition implies that an action $T \in O_n \setminus CV_n$ may only be approximated in $CV_n$ by actions $T'$ whose quotient graphs have a separating edge. Indeed, for $T' \in CV_n$ close enough to $T$, $T'$ lies in $\sigma(T)$ which means that there is an equivariant map $T' \to T$ preserving alignment. The preimage of a terminal edge of $T/F_n$ is a separating edge in $T'/F_n$. Since the quotient graphs of the actions contained in the spine of outer space have no separating edge, this means that the closure in $CV_n$ of the spine of outer space is contained in $F_n = CV_n \setminus O_n$.

**Proof.** We consider $T'$ close enough to $T$ and $\Delta \subset T'$ the fundamental domain of $T'$ constructed above. We note that $\Delta$ is the convex hull of the points $\pi_\gamma$ for $\gamma \in B \cup B^{-1}$ (because this convex hull meets every orbit of $T'$ by minimality of $T'$).

To prove the proposition we have to find an equivariant map from $T'$ to $T$ preserving alignment. The following lemma gives a map from $\Delta$ to $D$, linear on each edge of $\Delta$, which preserves alignment and sends $\pi_\gamma$ to $\chi_\gamma$. Because of Corollary 3.7, such an application naturally extends to an equivariant map from $T'$ to $T$ which preserves alignment.

**Lemma 3.11.** Let $D$ and $\Delta$ be two finite trees together with an $\varepsilon$-approximation between them. Assume that for every terminal vertex $\pi_\gamma$ of $\Delta$ there is an approximation point $\chi_\gamma$ which is terminal in $D$. If $\varepsilon$ is small compared to the length $d$ of the shortest edge of $D$, then there exists a natural application $f: \Delta \to D$ linear on the edges of $\Delta$, preserving alignment and sending $\pi_\gamma$ to $\chi_\gamma$.

**Proof.** We first define $f$ on the terminal vertices of $\Delta$ by sending $\pi_\gamma$ to $\chi_\gamma$. To extend $f$ to a branch point $b$ of $\Delta$, we consider a triod $(\pi_{\gamma_1}, \pi_{\gamma_2}, \pi_{\gamma_3})$ such that $\{b\} = [\pi_{\gamma_1}, \pi_{\gamma_2}] \cap [\pi_{\gamma_2}, \pi_{\gamma_3}] \cap [\pi_{\gamma_3}, \pi_{\gamma_1}]$ and we want to set $\{f(b)\} = [\chi_{\gamma_1}, \chi_{\gamma_2}] \cap [\chi_{\gamma_2}, \chi_{\gamma_3}] \cap [\chi_{\gamma_3}, \chi_{\gamma_1}]$. Note that $f(b)$ is either a branch point or a terminal point of $D$ (this happens when $f$ identifies two terminal points of $\Delta$) hence it is a vertex of $D$.

The point $f(b)$ is independent of the choice of the triod because $f(b)$ is $(3\varepsilon/2)$-close of an approximation point $b'$ of $b$ and two vertices of $D$ are at least at a distance $d$. By construction, $f$ preserves alignment in restriction to the set of branch points and terminal vertices of $D$, just extend $f$ linearly on edges to conclude.

**Proposition 3.12.** The action of $Out(F_n)$ on $O_n$ is properly discontinuous.

**Proof.** Let $K$ be a compact subset of $O_n$. Since $K$ is covered by a finite number of stars $St(T_i)$ each of which is a finite union of open simplices, $K$ is covered by finitely many open simplices. Since the decomposition of $O_n$ into simplices is equivariant, the proposition reduces to proving that the stabilizer of an open simplex is finite. Thus, the proposition follows from the fact that the barycenter of a simplex in $O_n$ has finite stabilizer.

This completes the proof of Theorem 1.

**4. Dynamics of $Out(F_n)$ on $F_n$**

In this section, we study the dynamics of $Out(F_n)$ on the closed invariant subset $F_n = \overline{CV_n \setminus O_n}$ of the boundary of outer space.
THEOREM 2. – Let \( T \) be simplicial in \( \mathcal{F}_n \) and let \( T' \) be any small action \((n \geq 3)\). Then there exists a sequence \( \alpha_k \) of elements of \( \text{Out}(\mathbb{F}_n) \) such that

\[
\lim_{k \to \infty} T' \cdot \alpha_k = T.
\]

The following corollary is a straightforward consequence of Theorem 2:

COROLLARY 4.1. – For \( n \geq 3 \), there exists precisely one minimal non-empty closed invariant subset of outer space. This set \( \mathcal{M}_n \) is the closure of the orbit of any simplicial action lying in \( \mathcal{F}_n \) under the action of \( \text{Out}(\mathbb{F}_n) \).

Remark. – In [14], D. Gaboriau and G. Levitt show that the topological dimension of \( \partial \mathcal{C} \mathcal{V}_n \) equals \( 3n - 5 \), thus refining a theorem by Bestvina and Feighn [4]. It is easy to find a simplex of simplicial actions of dimension \( 3n - 5 \) in \( \mathcal{F}_n \) and hence in \( \mathcal{M}_n \) (see Fig. 8). Therefore, the topological dimension of every open set in \( \mathcal{M}_n \) equals \( 3n - 5 \).

Another easy consequence of Theorem 2 is that the set of actions in \( \mathcal{M}_n \) having trivial stabilizer in \( \text{Out}(\mathbb{F}_n) \) is a dense \( G \delta \) in \( \mathcal{M}_n \).

The proof of Theorem 2 is analogous to the proof of the minimality of the action of the mapping class group of an orientable surface on the boundary of its Teichmüller space. The first step is a theorem by Cohen–Lustig about dynamics of Dehn twists in \( \overline{\mathcal{C} \mathcal{V}_n} \) which is the analogue of the fact that if \( i(c, \mathcal{F}) \neq 0 \) for a curve \( c \) and a measured foliation \( \mathcal{F} \), then iterating Dehn twists around \( c \) on \( \mathcal{F} \) makes it converge to \( c \). In a second step, we introduce a particular kind of action which we call “special curve”. A special curve \( T \) has the property that given any action \( T' \in \overline{\mathcal{C} \mathcal{V}_n} \), there exists an automorphism \( \alpha_0 \in \text{Out}(\mathbb{F}_n) \) such that “\( i(T, T' \cdot \alpha_0) \neq 0 \)”. Using the dynamics of Dehn twists, we see that the \( \text{Out}(\mathbb{F}_n) \)-orbit of \( T' \) accumulates on \( T \). In a third step, using the folding to approximate technique, we show that any simplicial action \( T \in \mathcal{F}_n \) may be approximated by a “special curve”, which proves Theorem 3.
4.1. Dynamics of Dehn twists

In [7], M. Cohen and M. Lustig study the dynamics of multiple Dehn twists. This theorem will be the engine enabling us to show that Out($F_n$)-orbits accumulate on some actions.

**Definition ([7]).** Let $Q$ be a graph of groups and $e_0$ be an oriented edge of $Q$. Consider an element $z_0$ of the center of the edge group $\pi_1(Q,e)$. We still denote by $e$ the element of the Bass group $\beta(Q)$ corresponding to an edge $e$. The single Dehn twist with twistor $z_0$ is the automorphism of $\pi_1(Q,v)$ induced by the automorphism $D$ of $\beta(Q)$ defined by

• $D(e_0) = e_0.e_0(z_0)$, $D(e_0^{-1}) = e_0^{-1}.e_0^{-1}(z_0)^{-1}$,
• $D(e) = e$ for every edge $e$ distinct from $e_0,e_0^{-1}$,
• $D(r) = r$ for every element $r$ of a vertex group.

The non-oriented edge corresponding to $e_0$ is called the twisted edge. A multiple Dehn twist of $Q$ is the (commuting) composition of single Dehn twists $D^k$ on distinct edges.

Recall that a graph of actions on $\mathbb{R}$-trees $Q$ is a graph of groups $Q$ together with the following data: for every vertex $v \in Q$ there is an action of the vertex group $\Gamma_v$ on an $\mathbb{R}$-tree $T_v$ (which may be reduced to one point), and for every oriented edge $e$, an attaching point $p_e \in T_{(e)}$ fixed by $i_e(\Gamma_e)$. A graph of actions naturally defines an $\mathbb{R}$-tree $T_Q$ endowed with an action of $\pi_1(Q)$ (see [19], or [7, combination lemma]).

If $Q$ is a graph of groups, we denote by triv($Q$) the set of edges of $Q$ with trivial edge group (it can be identified with the union of the open edges with trivial group in the geometric realization of the graph underlying $Q$). If $Q'$ is a subtree of $T$, triv($Q'$) is the set of edges of $Q'$ with trivial stabilizer.

**Cohen and Lustig’s Theorem about Dynamics of Dehn Twists ([7]).**

**The data.** Let $T$ be a very small simplicial action of $F_n$ and let $Q = T/F_n$ be the quotient graph of groups whose fundamental group is identified with $F_n$. Consider a union $A$ of connected components of $Q \setminus \text{triv}(Q)$. For every component $A_0$ of $A$, we consider an $\mathbb{R}$-tree $T_{A_0}$ endowed with a small action of $\Gamma_{A_0} = \pi_1(A_0)$ and an attaching point $p_e \in T_{A_0}$ for each edge in $Q \setminus A$ incident on $A_0$.

**The construction.** Let $T'$ be the simplicial action obtained by collapsing to one point every connected component of the preimage of $A$ in $T$. The graph of groups $Q' = T'/F_n$ is obtained by collapsing each connected component $A_0$ of $A$ and the corresponding vertex group is $\Gamma_{A_0} = \pi_1(A_0)$. We denote by $\frac{1}{k}T_{A_0}$ the action obtained by dividing by $k$ the metric of $T_{A_0}$. We denote by $Q_k$ the graph of actions obtained from $Q'$ by attaching to a vertex $A_0$ the tree $\frac{1}{k}T_{A_0}$, and the trivial action for any vertex of $Q'$ which is not the image of a component of $A$. The attaching points are the points $p_e$. We denote by $T_k$ the $F_n$-action corresponding to $Q_k$.

**The result.** Under the hypothesis that every edge group in $A_0$ has no fixed point in $T_{A_0}$, there exists Dehn twists $D_k$ on $Q$ such that the sequence of actions $T_k = T_k.D_k$ converges to $T$ as $k \to \infty$ (in the projectivised space).

Here is a consequence of this theorem: let $T$ be a very small simplicial action for which none of the edge stabilizers is trivial, and assume that $T'$ is a small action such that for every edge $e$ of $T$, the stabilizer of $e$ has no fixed point in $T'$. Then there exists Dehn twists $D_k$ such that $T'.D_k \xrightarrow{k \to \infty} T$ (just take $A = T/F_n$). Therefore, we may think of an action $T \in CV_n$ whose edge stabilizers are all non-trivial as an analogue of a (non-connected) curve in a surface. In this analogy, assuming that every edge stabilizer of $T$ has no fixed point in $T'$ corresponds to supposing that the intersection number of every connected component of a (non-connected) curve ($\sim T$) with a measured foliation ($\sim T'$) is non-zero. So, by analogy, (and to make notations...
lighter), we will say \( "i(T, T') \neq 0" \) when it is satisfied. So we may restate this particular case of the theorem as follows:

**Corollary 4.2.** - If \( T \in \overline{CV}_n \) is a "curve" and if \( T' \) is a small action such that \( "i(T, T') \neq 0" \), then there exist Dehn twists \( D_k \) such that \( T'.D_k \) converges to \( T \).

**Remark.** - We have no satisfactory definition of what could be an intersection number \( "i(T, T') \) for reasonable actions \( T, T' \in \overline{CV}_n \), that's why we keep quotes in the notation \( "i(T, T') \neq 0" \).

### 4.2. “Special curves”: some actions on which every \( \text{Out}(F_n) \)-orbit accumulates

**Definition.** - We say that an action \( T \in \overline{CV}_n \) is a "special curve" if it is a "curve" (i.e. it is simplicial, very small, and every edge stabilizer is non-trivial) and if there exists a basis \( \langle a_1, \ldots, a_n \rangle \) of \( F_n \) such that for all \( g \in F_n \) fixing an edge in \( T \), \( g \) or \( g^{-1} \) is a conjugate of a positive word in the \( a_i \)'s.

This condition is essentially technical. However, it is useful in the following proposition:

**Proposition 4.3.** - Let \( T \in \overline{CV}_n \) be a "special curve". Then the \( \text{Out}(F_n) \)-orbit of any small action accumulates on \( T \).

**Proof.** - Using Corollary 4.2 we only have to prove that there exists \( \alpha \in \text{Out}(F_n) \) such that \( "i(T, T'.\alpha) \neq 0" \). We know that there exists a basis \( B \) of \( F_n \) such that every edge stabilizer of \( T \) is generated by a conjugate of a positive word in \( B \). Therefore, the problem reduces to showing that there exists a basis \( B' \) of \( F_n \) such that every non-trivial positive word in this basis is hyperbolic in \( T' \): one can just take \( \alpha \) to be the automorphism sending \( B \) to \( B' \). So we just have to prove the following lemma. \( \square \)

**Lemma 4.4.** - For any small action \( T \) of \( F_n \), there exists a basis in which every non-trivial positive word in this basis is hyperbolic in \( T \).

**Proof.** - We first show that for any action of \( F_n \) with no global fixed point, there exists a basis of \( F_n \) containing a hyperbolic element. Start with any basis \( \langle a_1, \ldots, a_n \rangle \) of \( F_n \) and assume that every \( a_i \) is elliptic. Then \( a_i a_j \) is elliptic if and only if \( \text{Fix } a_i \cap \text{Fix } a_j \neq \emptyset \). If for every \( i \neq j \), the basis \( \langle a_1, \ldots, a_{i-1}, a_i a_j, a_{i+1}, \ldots, a_n \rangle \) is composed of elliptic elements only, then \( T \) has a global fixed point (Serre’s lemma).

We now prove that any small action has a basis composed of hyperbolic elements. We first notice that if \( a \) is hyperbolic and \( b \) is elliptic, \( ab \) is hyperbolic unless \( \text{Fix } b \cap \text{Axis } a \) contains exactly one point \( x \). Moreover, if \( b.\text{Axis } a \neq \text{Axis } a \), one easily checks that for \( k \) large enough, \( a^k b \) is hyperbolic. But if \( b.\text{Axis } a = \text{Axis } a \), then \( b^2 \) and \( ab^2 a^{-1} \) fix this axis and don’t commute which contradicts the fact that \( T \) is small. Now, starting with a basis \( \langle a_1, \ldots, a_n \rangle \) of \( F_n \) such that \( a_1 \) is hyperbolic, there exist integers \( k_i \) such that the basis \( \langle a_1, a_1^{k_2} a_2, \ldots, a_1^{k_n} a_n \rangle \) consists of hyperbolic elements.

From this basis \( \langle b_1, \ldots, b_n \rangle \), we can deduce another basis consisting of hyperbolic elements whose axes have a common non-degenerate segment \( I \) and whose orientations coincide: take \( I \) to be any non-degenerate interval in \( \text{Axis } (b_1) \) and note that for \( k_i \) large enough, \( b_1^{k_i} b_i b_1^{k_i} \) is hyperbolic and its axis contains \( I \). So one may take a basis of the form \( \langle b_1, (b_1^{k_2} b_2 b_1^{k_2} \pm 1), \ldots, (b_1^{k_n} b_n b_1^{k_n} \pm 1) \rangle \). To prove that every positive word in this basis is hyperbolic, we just have to notice that if \( a \) and \( b \) are hyperbolic isometries such that the intersection of their axes contains a non-degenerate interval \( I \) and whose orientations coincide, then \( ab \) is hyperbolic, its axis contains \( I \) and its orientation coincides with those of \( a \) and \( b \). \( \square \)
4.3. Approximation of a simplicial action in $\mathcal{F}_n$ by a “special curve”

Because of Proposition 4.3, the proof of Theorem 2 reduces to the following proposition:

**PROPOSITION 4.5.** - The set of “special curves” is dense in the set of simplicial actions in $\mathcal{F}_n$.

To prove this proposition, we will proceed in three steps. In the first step, we essentially approximate a simplicial action $T \in \mathcal{F}_n$ by a simplicial action $T' \in \mathcal{F}_n$ with trivial edge stabilizers and cyclic vertex stabilizers using the dynamic of Dehn twists. In the second step, using the folding to approximate technique, we approximate $T'$ by an action $T''$ with trivial edge stabilizers and whose quotient graph is a tree. Finally in the third step, using once again the folding to approximate technique, we approximate $T''$ by a “special curve”.

**First step: approximation to get rid of components of $Q \setminus \text{triv}(Q)$ with non-cyclic fundamental group**

Remember that $\text{triv}(Q)$ denotes the set of edges of $Q$ with trivial edge group.

**PROPOSITION 4.6.** - Any simplicial action $T \in \mathcal{F}_n$ may be approximated by a simplicial action $T' \in \mathcal{F}_n$ whose quotient graph of groups $Q' = T'/\mathcal{F}_n$ has the following properties:

- Every component of $Q' \setminus \text{triv}(Q')$ has cyclic fundamental group (as a graph of groups).
- At most one component of $Q' \setminus \text{triv}(Q')$ is not reduced to one point.

**Proof.** - Consider the union $A$ of the components of $Q \setminus \text{triv}(Q)$ whose fundamental group is not cyclic (as a graph of groups). Consider a component $A_0$ of $A$, $\Gamma_{A_0}$ its fundamental group and $m \geq 2$ the rank of this free group. We consider a simplicial action $T'_{A_0}$ of $A_0$ whose quotient graph of groups is a $(m - 1)$-rose whose edge stabilizers are trivial, and whose vertex stabilizer is infinite cyclic. We choose any attaching points for edges of $Q \setminus A$ incident on $A_0$.

To apply Cohen and Lustig’s theorem, we need the edge groups of $A_0$ not to fix any point in $T_{A_0}$. To achieve this, we apply the following lemma to a set $\{g_1, \ldots, g_k\}$ consisting of one generator of each edge group of $A_0$ and we change $T_{A_0}$ to $T_{A_0, \alpha}$.

**LEMMA 4.7.** - Let $g_1, \ldots, g_k$ be a finite set of elements of $\Gamma_{A_0}$. If $T_{A_0}$ is an action as above, there exists an automorphism $\alpha$ of $\Gamma_{A_0}$ such that $g_1, \ldots, g_k$ are hyperbolic in $T'_{A_0, \alpha}$.

**Proof.** - There is a natural basis $(a_1, \ldots, a_m)$ of $\Gamma_{A_0}$ such that $g \in \Gamma_{A_0}$ is elliptic in $T_{A_0}$ if and only if $g$ is conjugate to $a_1$. If $\varphi$ is the automorphism of $\Gamma_{A_0}$ fixing $a_2, \ldots, a_m$ and sending $a_1$ to $a_1 a_2$ then for every $g \in \Gamma_{A_0}$ there exists at most one $p \in \mathbb{Z}$ such that $\varphi^p(g)$ is conjugate to $a_1$, so that we can take $\alpha$ to be a power of $\varphi$. □

Now consider the sequence of actions $T'_k$ constructed in the theorem about dynamics of Dehn twists. We know that $T'_k$ converges to $T$. But $T'_k$ is simplicial, very small, and lies in $\mathcal{F}_n$ since there exists a non-terminal vertex of $Q'_k = T'_k/\mathcal{F}_n$ with non-trivial group (by choice of the actions $T_{A_0}$). By construction, no component of $Q'_k \setminus \text{triv}(Q'_k)$ has a non-cyclic fundamental group. This proves the first part of Proposition 4.6.

Now assume that $T$ satisfies the first part of Proposition 4.6 and not the second one. Then take $A$ to be the union of all but one connected components of $Q \setminus \text{triv}(Q)$ not reduced to one point. For every component $A_0$ of $A$, we consider a free action of the cyclic group $\pi_1(A_0)$ on a line $T_{A_0}$. Using the theorem about dynamics of Dehn twists, we get an approximation of $T$ which is very small and which lies in $\mathcal{F}_n$ (because it has a non-trivial edge stabilizer). This concludes the proof of Proposition 4.6. □
Second step: approximation by an action whose quotient graph is a tree

**Proposition 4.8.** - For \( n \geq 3 \), any simplicial action in \( F_n \) may be approximated by a simplicial action \( T' \in F_n \) with trivial edge stabilizers such that \( T'/F_n \) is a tree.

**Proof.** - Thanks to Proposition 4.6, we may assume that the quotient graph of groups \( Q = T/F_n \) satisfies the following: every component of \( Q \setminus \text{triv}(Q) \) has cyclic fundamental group and \( Q \setminus \text{triv}(Q) \) has at most one component not reduced to one point. Therefore, either \( T \) has trivial edge stabilizers or \( Q \setminus \text{triv}(Q) \) contains exactly one component not reduced to one point, and this component has cyclic fundamental group (as a graph of groups). First of all, we approximate \( T \) so that the lengths of its edges are all rational. We then multiply the metric by an integer and maybe subdivide some edges so that every edge in \( T \) has length 1. We first consider the case when \( T \) has trivial edge stabilizers: the treatment of the other case is similar but is a bit more technical.

**When \( T \) has trivial edge stabilizers**

Since \( T \) has trivial edge stabilizers and cyclic vertex stabilizers, the hypothesis \( T \in F_n \) means that there exists a non-terminal vertex \( x \in Q \) with non-trivial stabilizer. We prove the proposition by induction on the number of edges in \( T/F_n \): the idea is to perform on \( T \) a folding to approximate operation from such a vertex and to choose the folding paths so that the folded action has trivial edge stabilizers and contains a vertex with non-trivial stabilizer whose projection in the quotient graph of groups is not terminal. The following lemma tells us about the length of paths required to perform such an interesting fold.

**Lemma for folding sub-paths.** - Let \( T \) be a simplicial action with trivial edge stabilizers and whose edges all have length 1. Let \( \alpha = \alpha_1\alpha_2 \ldots \) and \( \beta = \beta_1\beta_2 \ldots \) be two (maybe infinite) paths in \( Q = T/F_n \) with same origin \( x \) and well oriented with respect to an orientation of \( Q \). Assume that the vertex \( x \) has non-trivial group and that \( \alpha_1 \neq \beta_1 \). We also suppose that one of the following conditions is satisfied:

1. \( \alpha \) and \( \beta \) are infinite;
2. \( \alpha \) is strictly longer than \( \beta \) and the terminal vertex of \( \beta \) has a non-trivial group.

Then there exist sub-paths \( \alpha', \beta' \) of \( \alpha, \beta \) with the same (non-zero) length such that for any lift of \( \alpha' \) and \( \beta' \), the actions \( T' = T/\alpha' \sim w_k, \beta' \) converge to \( T \), \( T' \) has trivial edge stabilizers, and its quotient graph of groups \( T'/F_n \) has a non-terminal vertex with non-trivial group (in particular, \( T' \in F_n \)).

**Remark.** - The action \( T' \) still has edges of length 1 and its quotient graph \( T'/F_n \) has strictly fewer edges than \( T \).

**Proof.** - We denote by \( T' = (T/F_n, \alpha, \beta, Q_1) \) the action obtained after the \( i \)-th intermediate fold of \( \alpha \) with \( w_k, \beta \), \( Q_1 = Q/F_n \), \( q_i : T_i \to T' \) the folding map and \( q_i : Q \to Q' \) the induced application.

We first notice that if for \( i > 1 \) the edges \( q_{i-1} \alpha_i \) and \( q_{i-1} \beta_i \) lie in the same orbit, then their common vertex \( x_i \) is non-trivial stabilizer, and its projection \( x_{i-1} \in Q_1 \) is not a terminal vertex since it belongs to the interior of the well oriented hence immersed path \( q_{i-1} \alpha_i \cup \alpha_i \). Moreover, the fact that those edges \( q_{i-1} \alpha_i \) and \( q_{i-1} \beta_i \) lie in the same orbit does not depend on \( w_k \) since the fold between two paths is the quotient by the smallest equivariant relation identifying them.

Therefore, if there exists an index \( i \) such that \( q_{i-1} \alpha_i \) and \( q_{i-1} \beta_i \) lie in the same orbit, we consider the smallest such \( i_0 \) (\( i_0 > 1 \)) and take \( \alpha' \) and \( \beta' \) to be the restriction of \( \alpha \) and \( \beta \) to the first \( i_0 - 1 \) edges. Therefore, every \( T' \) has trivial edge stabilizers for \( i < i_0 \) and we can apply the folding to approximate lemma to conclude.
If there is no such index $i$, we can apply the folding to approximate lemma to any of the $T_i^{(k)}$, but we are looking for an index $i$ such that $Q_i^{(k)}$ has a non-terminal vertex with non-trivial group. In this situation, every intermediate fold reduces by one the number of edges of the quotient graph which prevents $\alpha$ and $\beta$ from being both infinite. Hence the second assumption must hold. This implies that we can take $i = |\beta|$ (i.e. $\beta' = \beta$ and $\alpha'$ is the restriction of $\alpha$ having the same length as $\beta$). Indeed, if $v$ denotes the terminal vertex of $\beta$, $q_i^{(k)}(v)$ lies in the interior of the well oriented arc $q_i^{(k)}(\alpha)$ so its projection in $Q_i^{(k)}$ is non-terminal and has non-trivial group. 

Now, going back to the proof of Proposition 4.8 in the case that $T$ has trivial edge stabilizers (and edges with length 1), we argue by induction on the number of edges in $T/F_n$ using the lemma for folding sub-paths.

To apply this lemma, it is sufficient to find an infinite path $\bar{\alpha}$ in $Q = T/F_n$ which is well oriented for some orientation of the edges of $\bar{\alpha}$ in $Q$, starting at a vertex $\bar{x}$ with non-trivial group and such that there exists an edge $\bar{e}$ in $Q$ with origin $\bar{x}$ not contained in $\bar{\alpha}$. As a matter of fact, we can then inductively construct a well oriented path $\bar{\beta}$ starting at $\bar{e}$, by following any edge with the right orientation (whenever it is already in $\bar{\alpha}$ or $\bar{\beta}$). The only case where this can’t be done is when $\bar{\beta}$ reaches a terminal vertex of $Q$, but the stabilizer of this vertex has to be non-trivial by minimality. Therefore, in this situation, we can apply the lemma for folding sub-paths to conclude.

We may assume that $Q$ is not a tree because otherwise there is nothing to prove. Hence, there exists an embedded circle in $Q$. If there exists such a circle $C$ not containing $\bar{x}$, we define $\bar{\alpha}$ to be the path following a simple arc joining $\bar{x}$ to $C$ before turning around $C$ infinitely many times. Since $\bar{x}$ is not terminal in $Q$, there exists an edge $\bar{e}$ with origin $\bar{x}$ which is not in $\bar{\alpha}$. A similar argument works if $\bar{x}$ has valence at least 3 and $\bar{x} \in C$.

Therefore, the only remaining case is when $\bar{x}$ has valence 2 and every embedded circle in $Q$ contains $\bar{x}$ which may only happen when $Q$ has the homotopy type of a circle (as a simple graph). Let $C$ be the unique embedded circle in $Q$ ($\bar{x} \in C$ by hypothesis). We distinguish two cases. If there exists a vertex $\bar{v} \neq \bar{x}$ in $Q$ whose group is non-trivial and such that the length of the two simple paths $\bar{\alpha}$ and $\bar{\beta}$ joining $\bar{x}$ to $\bar{v}$ are distinct, we can apply the lemma for folding sub-paths.

If no such vertex exists, it means that $Q$ can be obtained from $C$ by gluing finitely many trees (maybe 0) on the point $\bar{u}$ which is antipodal to $\bar{x}$ in $C$ ($\bar{u}$ may not be a vertex if $|C|$ is odd in which case $Q = C$). In this case, we take $\bar{\alpha}$ and $\bar{\beta}$ to be the two simple paths joining $\bar{x}$ to $\bar{u}$ in $C$ (maybe in the barycentric subdivision of $Q$). We consider two lifts $\alpha$ and $\beta$ of $\bar{\alpha}$ and $\bar{\beta}$ with same origin $x$ and $w_k$ a sequence of distinct elements in Stab $x$. It is clear that every intermediate fold of $q^{(k)} : T \to T^{(k)} = T/\sim_{\alpha \sim \beta}$ is a fold between edges with trivial stabilizers so we can apply the folding to approximate lemma. Moreover, $Q^{(k)} = T^{(k)}/F_n$ is a tree so we just have to check that $T^{(k)} \notin \mathcal{O}_n$. If $q^{(k)}$ denotes the map induced by $q^{(k)}$ on the level of quotient graphs, the group of $q^{(k)}(u)$ is always non-trivial. It is even non-cyclic when the group of $\bar{u}$ is non-trivial. Therefore, $T^{(k)} \notin \mathcal{F}_n$ as soon as the group of $\bar{u}$ is non-trivial or when $Q$ is not a circle (because $q^{(k)}(\bar{u})$ is not terminal in $Q^{(k)}$). In the remaining situation where $Q$ is a circle and the group of $\bar{u}$ is trivial, the only vertex with non-trivial group is $\bar{x}$. This implies that the group of $\bar{x}$ is free of rank $n - 1$, and hence cannot be cyclic for $n \geq 3$. Therefore the stabilizer of $q^{(k)}(x)$ is non-cyclic and $T^{(k)} \in \mathcal{F}_n$ which concludes the proof of Proposition 4.8 in the case when $T$ has trivial edge stabilizers.

**When $T$ has non-trivial edge stabilizers**

In this case, we show how to approximate $T$ by an action satisfying the hypotheses of the previous case, i.e. an action with trivial edge stabilizers and such that there exists a non-terminal vertex in its quotient graph with non-trivial group.

We denote by $Q$ the quotient graph of groups $T/F_n$ as usual. As before, we can assume that edges of $T$ have length 1, that every component of $Q \setminus \text{triv}(Q)$ has cyclic fundamental group and...
\(Q \setminus \text{triv}(Q)\) has exactly one component \(I\) which is not reduced to one point. Since the fundamental group of \(I\) as a graph of groups is cyclic, and since its edge groups are non-trivial, \(\pi_1(I)\) has a global fixed point in \(T\), so \(I\) must be a tree (an HNN extension is never trivial). Moreover, the fact that \(T\) is very small says that \(I\) is an interval and that every edge morphism is an isomorphism onto the corresponding vertex group which means that the connected components of the preimage of \(I\) in \(T\) are intervals. We are going to prove a folding to approximate lemma for edge stabilizers by performing a fold on an action obtained thanks to Cohen–Lustig’s theorem about dynamics of Dehn twists.

In order to apply Cohen–Lustig’s theorem, we set \(A = I\), and take \(T_I\) to be a line with a non-trivial action of \(\pi_1(I) \cong \mathbb{Z}\) by translations. We choose a point \(x\) in \(T_I\) and take every attaching point \(p_e\) to be \(x\). We consider \(T_k\) and \(T'_k = T_k / D_k\) as in Cohen and Lustig’s theorem. Note that the graphs of groups \(Q_k = T_k / F_n\) and \(Q'_k = T'_k / F_n\) may be obtained from \(Q = T / F_n\) by collapsing \(I\) to a vertex \(v_i\) and by adding an edge \(e_I\) and gluing its endpoints to \(v_i\).

**Folding to approximate lemma for edge stabilizers.** – Let \(T\) be a very small simplicial action in \(F_n\) such that every component of \(Q \setminus \text{triv}(Q)\) has cyclic fundamental group and \(Q \setminus \text{triv}(Q)\) has exactly one component \(I\) which is not reduced to one point. Let \(T'_k\) be the sequence of actions constructed in Cohen and Lustig’s theorem with \(A = I\) as above.

Assume that \(\bar{\alpha}\) and \(\bar{\beta}\) are two paths in \(Q'_k \setminus e_I\) with origin \(v_I\). We choose some lifts \(\alpha^{(k)}\) and \(\beta^{(k)}\) of \(\bar{\alpha}\) and \(\bar{\beta}\) and we assume that they satisfy condition (H) and that when folding \(\alpha^{(k)}\) on \(\beta^{(k)}\), every intermediate fold is a fold between edges with trivial stabilizer.

Under the assumption that the first edges \(\alpha_1\) and \(\beta_1\) of \(\bar{\alpha}\) and \(\bar{\beta}\) correspond to edges in \(Q\) with distinct initial points, the actions \(T^{(k)}\) obtained from \(T'_k\) by folding \(\alpha^{(k)}\) along \(\beta^{(k)}\) converge to \(T\) as \(k \to \infty\).

**Proof.** – We prove convergence in the translation lengths topology. The theorem about dynamics of Dehn twists tells us that \(T'_k\) converges to \(T\).

If \(g \in F_n\) has a fixed point in \(T\) then its translation length in \(T'_k\) approaches 0 when \(k\) tends to infinity, and since a folding map decreases distances, \(l_{T^{(k)}(g)}(k) \to 0\).

If \(g \in F_n\) is hyperbolic in \(T\), then for large enough \(k\), it is hyperbolic in \(T'_k\). Moreover, because \(\alpha_1\) and \(\beta_1\) correspond to edges in \(Q\) with distinct initial points, a path in \(Q\) entering \(I\) from (the edge corresponding to) \(\alpha_1\) and leaving \(I\) through \(\beta_1\) will be twisted more and more so that the corresponding path in \(Q'_k\) will go through \(e_I\) more and more often between \(\alpha_1^{-1}\) and \(\beta_1\). A similar fact holds for a path entering \(I\) from \(\beta_1^{-1}\) and leaving \(I\) through \(\alpha_1\). This implies that for \(k\) large enough, the projection of the axis of \(g\) in \(Q'_k\) never successively runs through \(\alpha_1^{-1}\) and \(\beta_1\) or \(\beta_1^{-1}\) and \(\alpha_1\). Corollary 2.2 says that two adjacent edges which are not identified by the first elementary fold are not identified in \(T^{(k)} = T / \alpha^{(k)} \sim \beta^{(k)}\). Therefore, for large enough \(k\), the folding map isometrically embeds the axis of \(g\) into \(T^{(k)}\) so \(l_{T^{(k)}(g)} = l_{T'_k}(g)\) and therefore converges to \(l_T(g)\).

The **folding to approximate lemma for edge stabilizers** allows us to prove a version of the lemma for folding sub-paths for edge stabilizers:

**Lemma for folding sub-paths for edge stabilizers.** – Let \(T\) be a simplicial action whose edges have length 1, such that every component of \(Q \setminus \text{triv}(Q)\) has cyclic fundamental group and \(Q \setminus \text{triv}(Q)\) has exactly one component \(I\) which is not reduced to one point. As above, consider \(T'_k\) an approximation of \(T\) provided by Cohen and Lustig’s theorem.

Let \(\bar{\alpha} = \bar{\alpha}_1 \bar{\alpha}_2 \ldots\) and \(\bar{\beta} = \bar{\beta}_1 \bar{\beta}_2 \ldots\) be two (possibly infinite) paths in \(Q'_k \setminus e_I\) with the same origin \(x\), well oriented with respect to an orientation of \(Q\) and such that \(\bar{\alpha}_1\) and \(\bar{\beta}_1\) correspond to edges in \(Q\) with distinct initial points.

We also suppose that one of the following conditions is satisfied:
(1) \( \bar{\alpha} \) and \( \bar{\beta} \) are infinite;
(2) \( \bar{\alpha} \) is strictly longer than \( \bar{\beta} \) and the terminal vertex of \( \bar{\beta} \) has non-trivial stabilizer.

Then there exist sub-paths \( \bar{\alpha}', \bar{\beta}' \) of \( \bar{\alpha}, \bar{\beta} \) with the same (non-zero) length such that for any lift \( \alpha' \) and \( \beta' \) of \( \bar{\alpha}' \) and \( \bar{\beta}' \) with the same initial vertex \( x \), the actions \( T^{(k)} = T^k/\alpha' - \beta' \) converge to \( T \), \( T^{(k)} \) has trivial edge stabilizers, and its quotient graph of groups \( T^{(k)}/F_n \) has a non-terminal vertex with non-trivial group (in particular, \( T^{G} \)).

The proof is similar to the proof in the trivial-edge stabilizer case. With this lemma for folding sub-paths for edge stabilizers at hand, we just have to repeat the argument used when \( T \) had trivial edge stabilizers to find some paths \( \alpha \) and \( \beta \) satisfying its hypotheses. The only additional case is when \( Q' \setminus e_1 \) is a tree. This means that \( Q \) is a tree, and by minimality, \( I \) doesn't contain any terminal point of \( Q \). We then consider two endpoints \( \bar{\alpha}_0 \) and \( \bar{\beta}_0 \) lying in two distinct components of \( Q' \setminus \bar{v}_1 \) corresponding to non-adjacent components of \( Q \setminus I \). We take \( \bar{\alpha} \) and \( \bar{\beta} \) to be the simple paths joining \( \bar{v}_1 \) to \( \bar{\alpha}_0 \) and \( \bar{\beta}_0 \) respectively. If their lengths are distinct, we can apply the lemma for folding sub-paths for edge stabilizers and we are done since we thus get an approximation of \( T \) lying in \( F_n \) with trivial edge stabilizer to which we can apply Proposition 4.8 (already proved in the case of trivial edge stabilizers). If \( \bar{\alpha} \) and \( \bar{\beta} \) have the same length, we approximate \( T \) by changing some edge lengths slightly (keeping them rational) so that the lengths of \( \bar{\alpha} \) and \( \bar{\beta} \) become different. After subdivision, we can once again apply the lemma for folding sub-paths for edge stabilizers to conclude. This concludes the proof of Proposition 4.8.

Third step: approximation by a “special curve”

**Proposition 4.9.** - For \( n \geq 3 \), any action \( T \in F_n \) with trivial edge stabilizers and whose quotient graph is a tree may be approximated by a “special curve”.

Thanks to Proposition 4.8, this proposition will conclude the proof of Proposition 4.3 and therefore of Theorem 2.

**Proof.** - Recall that a “special curve” is a very small simplicial action such that there exists a basis \( \{a_1, \ldots, a_n\} \) of \( F_n \) in which every edge stabilizer is non-trivial and generated by a conjugate of a positive word in \( \{a_1, \ldots, a_n\} \).

As above, up to approximation, subdivision and rescaling, we can assume that every edge in \( T \) has length 1. We are going to perform folding operations on \( T \) to create non-trivial edge stabilizers, and we will argue by induction on the number of orbits of edges with trivial stabilizer. As above, we denote by \( Q = T/F_n \) and by \( \text{triv}(Q) \) the set of edges in \( Q \) with trivial group. We also denote by \( \overline{\text{triv}(Q)} \) the union of \( \text{triv}(Q) \) together with the vertices of \( Q \) adjacent to an edge of \( \text{triv}(Q) \).

The induction hypothesis is the following: we assume that we know how to prove the proposition for every action \( T' \in F_n \) whose quotient graph \( Q' \) is a tree such that \#\( \text{triv}(Q') \) < \#\( \text{triv}(Q) \) and

1. \( \overline{\text{triv}(Q')} \) is connected,
2. \( \overline{\text{triv}(Q')} \) is empty, or contains a vertex whose group is not cyclic, or a vertex with non-trivial group which is not terminal in \( \overline{\text{triv}(Q')} \),
3. there exist a free basis \( \{a_1, \ldots, a_n\} \) of \( F_n \), a lift \( \bar{Q}' \) of \( Q' \) and for every connected component \( C \) of \( \bar{Q}' \setminus \text{triv}(\bar{Q}') \), a possibly empty subset \( B_C \subset \{a_1, \ldots, a_n\} \) such that
   a. for all \( a_j \in B_C \), \( a_j \) fixes a point in \( C \),
   b. the stabilizer of every edge or vertex in \( C \) has a basis composed of positive words in \( B_C \).

Note that condition 3(a) implies that two sets \( B_C \) and \( B_{C'} \) are disjoint for \( C \neq C' \), and condition 3(b) shows that the union of the sets \( B_C \) equals \( \{a_1, \ldots, a_n\} \). The action \( T \) we are starting with satisfies the induction hypothesis: choose any lift \( \bar{Q} \) of \( Q \) in \( T \), and for every
component $C$ of $\widetilde{Q} \setminus \operatorname{triv}(\widetilde{Q})$ (which is a single vertex) consider a free basis $B_C$ of $\operatorname{Stab} C$ and take $\langle a_1, \ldots, a_n \rangle$ to be the union of the $B_C$. Moreover, if an action $T$ satisfies the induction hypothesis and $\operatorname{triv}(\widetilde{Q})$ is empty, then $T$ is a "special curve" so there is nothing to prove.

**First case:** $\operatorname{triv}(\widetilde{Q})$ contains a vertex $x$ which is terminal in $\operatorname{triv}(\widetilde{Q})$ and has non-cyclic group (see Fig. 9 where bold edges correspond to edges with non-trivial stabilizer). Let $x$ be the lift of $x$ in $\widetilde{Q}$, let $s_\alpha$ and $s_\beta$ be two terminal vertices of $\operatorname{triv}(\widetilde{Q})$ distinct from $x$. Note that $\operatorname{Stab} s_\alpha$ and $\operatorname{Stab} s_\beta$ must be non-trivial since either $s_\alpha$ is terminal in $\widetilde{Q}$ or it is the endpoint of an edge of $\widetilde{Q}$ with non-trivial stabilizer. Let $\alpha$ and $\beta$ be the paths joining $x$ to $s_\alpha$ and $s_\beta$ respectively. Since we want $\operatorname{triv}(\widetilde{Q})$ to remain connected after folding, we choose $s_\alpha$ and $s_\beta$ so that $|\alpha| = |\beta|$ is minimal, which means that $\alpha$ and $\beta$ bifurcate as soon as they meet a branch point $p$ of $\operatorname{triv}(\widetilde{Q})$. In particular, we take $s_\alpha = s_\beta$ only if $\operatorname{triv}(\widetilde{Q})$ is a segment. If $\alpha$ and $\beta$ do not have the same length, we shorten the longest one so that this condition is satisfied.

To apply the folding to approximate lemma (Section 2.3) to $\alpha$ and $\beta$, we just have to choose a sequence of distinct elements $w_k \in \operatorname{Stab} x$. Let $C$ be the component of $\widetilde{Q} \setminus \operatorname{triv}(\widetilde{Q})$ containing $x$ and let $\{g_1, \ldots, g_p\}$ be a basis of $\operatorname{Stab} x$ consisting of positive words in $B_C$. Since $\operatorname{Stab} x$ is not cyclic, we can choose a sequence $w_k$ of positive words in $\{g_1, \ldots, g_p\}$ which are not proper powers in $F_n$ and which are not conjugate to elements of $F_n$ that already fix an edge in $T$.

The hypotheses of the folding to approximate lemma are clearly satisfied, so $T^{(k)} = T/\alpha \sim w_k, \beta$ converges to $T$. Hence, we just have to prove that $T^{(k)}$ satisfy the induction hypothesis. Recall that $[x, p]$ denotes $\alpha \cap \beta$ and that $q^{(k)}: T \rightarrow T^{(k)}$ is the folding map. The stabilizer of $q^{(k)}([x, p])$ is generated by $w_k$, and $Q^{(k)} = T^{(k)}/F_n$ may be obtained from $Q$ by gluing $\alpha \setminus [x, p]$ on $\beta \setminus [x, p]$. Therefore, $Q^{(k)}$ is a tree, and $\operatorname{triv}(Q^{(k)})$ is connected since $[x, p]$ is a terminal segment of $\operatorname{triv}(Q)$. Moreover, $T^{(k)}$ is very small since $w_k$ is not a proper power and $\operatorname{Fix} q^{(k)} = q^{(k)}([x, p])$ contains no triod. Condition 2 is also satisfied by $T^{(k)}$: if $s_\alpha = s_\beta$, $\operatorname{triv}(Q^{(k)})$ is empty; if $s_\alpha \neq s_\beta$ and if $d_T(x, s_\alpha) = d_T(x, s_\beta)$ then $q^{(k)}(s_\alpha)$ has non-cyclic stabilizer and its projection to $Q^{(k)}$ lies in $\operatorname{triv}(Q^{(k)})$; if $s_\alpha \neq s_\beta$ and if $d_T(x, s_\alpha) < d_T(x, s_\beta)$ (without loss of generality), then $q^{(k)}(s_\alpha)$ has non-trivial stabilizer and is not terminal in $\operatorname{triv}(Q^{(k)})$.

To see that condition 3 is satisfied, we consider the component $S_\beta$ of $\widetilde{Q} \setminus \{p\}$ containing $s_\beta$. We obtain a lift $\widetilde{Q}^{(k)}$ of $Q^{(k)}$ by taking $q^{(k)}(\widetilde{Q} \setminus S_\beta) \cup q^{(k)}(w_k, S_\beta)$. We change the basis $\langle a_1, \ldots, a_n \rangle$ by conjugating by $w_k$ the elements of $B_C$ for each component $C$ of $\widetilde{Q} \setminus \operatorname{triv}(\widetilde{Q})$ contained in $S_\beta$. It is a free basis because $w_k$ may be written in the basis $\langle a_1, \ldots, a_n \rangle$ without using the letters of $B_C$ for $C \subset S_\beta$. It is now clear that $T^{(k)}$ satisfies the induction hypothesis.

**Second case:** $\operatorname{triv}(\widetilde{Q})$ contains a vertex $x$ with non-trivial stabilizer and which is not terminal in $\operatorname{triv}(\widetilde{Q})$. Let $x$ be the lift of $x$ in $\widetilde{Q}$ and let $s_\alpha$ and $s_\beta$ be two terminal points of $\widetilde{Q}$ lying in distinct components of $\widetilde{Q} \setminus \{x\}$. Note that $\operatorname{Stab} s_\alpha$ and $\operatorname{Stab} s_\beta$ must be non-trivial. Let $\alpha$ and $\beta$ be the simple paths joining $x$ to $s_\alpha$ and $s_\beta$ respectively. If $\alpha$ and $\beta$ do not have the same length, we
shorten the longest one so that this condition is satisfied. We take any sequence \( w_k \) of distinct elements in Stab \( x \) and we consider the folded actions \( T^{(k)} = T/\sim w_k, \beta \). It is an approximation of \( T \) when \( k \) is large enough thanks to the folding to approximate lemma.

The quotient graph \( Q^{(k)} = T^{(k)}/F_n \) is obtained from \( Q \) by identifying \( \alpha \) and \( \beta \), and the stabilizers of the edges contained in \( q^{(k)}(\alpha) = q^{(k)}(\beta) \) are trivial. Therefore, \( \text{triv}(Q^{(k)}) \) is connected, \( T^{(k)} \) is very small, and \( Q^{(k)} \) is a tree. Since the stabilizers of \( s_\alpha \) and \( s_\beta \) are non-trivial, \( \text{triv}(Q^{(k)}) \) contains a non-terminal point with non-trivial stabilizer when \( d(x, s_\alpha) \neq d(x, s_\beta) \) and a point with non-cyclic stabilizer when \( d(x, s_\alpha) = d(x, s_\beta) \).

To see that condition 3 of the induction hypothesis is satisfied, we consider the component \( S_\beta \) of \( Q \setminus \{ q \} \) containing \( s_\beta \). As above, we consider the lift \( \tilde{Q}^{(k)} \) of \( Q^{(k)} \) defined by

\[
\tilde{Q}^{(k)} = q^{(k)}(\tilde{Q} \setminus S_\beta) \cup q^{(k)}(w_k, S_\beta).
\]

We change the basis \( \langle a_1, \ldots, a_n \rangle \) by conjugating by \( w_k \) the elements of \( B_C \) for each component \( C \) of \( \tilde{Q} \setminus \text{triv}(\tilde{Q}) \) contained in \( S_\beta \). We get that \( T^{(k)} \) satisfies the induction hypothesis which ends the proof of Proposition 4.9 and hence of Theorem 2.

5. Does Out(\( F_n \)) act with dense orbits on \( F_n \)?

We still don’t know whether \( \text{Out}(F_n) \) acts with dense orbits on \( F_n \). This question is equivalent to asking whether \( M_n = F_n \). To prove this equality, it would be sufficient to approximate every non-simplicial action by a simplicial action lying in \( F_n \). In [4], Bestvina and Feighn show how to approximate a very small action \( T \) by a simplicial very small action \( T' \). Their argument shows that if \( T \) has a non-trivial arc stabilizer, then \( T' \) may be assumed to have a non-trivial edge stabilizer so \( T' \in F_n \) and \( T \in M_n \). They also prove that if a geometric approximation of \( T \) has an orientable surface component, then \( T \) can be approximated by a very small simplicial action with a non-trivial edge stabilizer and hence lies in \( M_n \).

If \( T \) is a very small action of \( F_n \), Gaboriau and Levitt show in [14] that \( T \) has only finitely many orbits of branch points. Therefore, we can apply [19] to conclude that \( T \) may be seen as the action corresponding to a graph of actions on \( \mathbb{R} \)-trees whose vertex actions have dense orbits. Therefore, proving that \( M_n = F_n \) reduces to showing that any very small action with dense orbits lies in \( M_n \). The following theorem partially answers this question (see Section 5.1 for definitions):

**Theorem.** Let \( n \geq 3 \) and let \( T \) be a very small action with dense orbits. If the Lebesgue measure on \( T \) is the sum of at most \( n-1 \) ergodic measures, then \( T \in M_n \).

We will see in Section 5.1 that because of the topological dimension of \( \overline{CV}_n \), the Lebesgue measure is always a sum of at most \( 3n-4 \) ergodic measures.

**Remark.** Let \( \alpha \) be an irreducible automorphism of \( F_n \) with irreducible powers. This means that no power of \( \alpha \) fixes a free factor of \( F_n \) up to conjugation. Then Lustig has proved that \( \alpha \) has exactly two fixed points in \( \overline{CV}_n \) and no other periodic point [22]. This implies that those fixed points are uniquely ergodic. As a matter of fact, there is a natural way to associate to an action \( T \in \overline{CV}_n \) a simplex \( \sigma(T) \subset \overline{CV}_n \) built on its set of invariant measures. If \( T \) is not uniquely ergodic, this simplex is not reduced to one point and some power of \( \alpha \) fixes this simplex pointwise which is impossible. Therefore, Theorem 3 implies that the fixed points of an irreducible automorphism with irreducible powers must lie in \( M_n \).
5.1. Measures on $\mathbb{R}$-trees

Length measures and uniquely ergodic actions

The classical measure theory is not adapted to $\mathbb{R}$-trees because they are not locally compact. In [26] is proposed an alternative called length measure. For shortness's sake, we will sometimes use the shortcut measure to mean a length measure.

**Definition.** A length measure $\mu$ on an $\mathbb{R}$-tree $T$ consists of a finite Borel measure $\mu_I$ for every compact interval $I$ of $T$ such that if $J \subset I$, $\mu_J = (\mu_I)_| J$.

If $T$ is endowed with an action of a group $\Gamma$, we say that a length measure is invariant if $\mu_{gJ} = (g^*\mu)_J$. The Lebesgue measure of an $\mathbb{R}$-tree is the collection of the Lebesgue measures of the intervals of $T$. If $\Gamma$ acts by isometries on $T$, then the Lebesgue measure is invariant. If $\mu$ is a length measure on an $\mathbb{R}$-tree $T$, we write $\mu(I)$ for $\mu_I(I)$. We say that $\mu$ is non-atomic or positive if every $\mu_I$ is non-atomic or positive.

**Remark.** It may happen that an action with dense orbits has an invariant measure with atoms, but this is impossible if every orbit is dense in the segments.

Let $f : T \to T'$ be a map such that every segment $I$ in $T$ may be subdivided into finitely many intervals on which $f$ preserves alignment (this is the case when $f$ is a morphism of $\mathbb{R}$-trees or a map preserving alignment). Any non-atomic measure $\mu'$ on $T'$ may be carried to a measure $\mu = f^*\mu'$ in the following way: let $J$ be a segment in $T$ and subdivide $I$ into finitely many subsegments $I_p$ on which $f$ preserves alignment. Then take $\mu_I$ to be the only (non-atomic) measure on $I$ such that for every interval $J$ inside some $I_p$, $\mu(I) = \mu'_I(f(J))$.

**Measures and maps preserving alignment**

From now on, we only consider positive invariant measures.

Let $T$ be an $\mathbb{R}$-tree with an isometric action of $\Gamma$. If $q : T \to T'$ is an equivariant 1-Lipschitz map preserving alignment, by carrying to $T$ the Lebesgue measure of $T'$, we obtain an invariant positive measure whose density with respect to the Lebesgue measure is at most 1. Conversely, given a invariant positive measure $\mu$ on $T$ whose density with respect to the Lebesgue measure is at most 1, we consider the pseudo-metric on $T$ given by $d_\mu(x, y) = \mu([x, y])$. One easily checks that making this pseudo-metric Hausdorff gives an $\mathbb{R}$-tree $T'_\mu$. This tree is naturally endowed with an isometric action of $\Gamma$ and the quotient map $q : T \to T'_\mu$ preserves alignment. Note that if $\mu$ is obtained by pulling back $\mu'$ under $f : T \to T'$ then $T'_\mu$ is isometric to $T''_{\mu'}$.

Here are some simple properties of maps preserving alignment:

**Lemma 5.1.** Let $T$ and $T'$ be $\mathbb{R}$-trees endowed with an isometric action of a group $\Gamma$ and let $q : T \to T'$ be an equivariant map preserving alignment.

Then the preimage of a convex set is convex. For every $\gamma \in \Gamma$, $\text{Char}_{T'} \gamma = q(\text{Char}_T \gamma)$. Moreover, if $\gamma$ is hyperbolic in $T$ and elliptic in $T'$ then $\gamma$ has only one fixed point $a = q(\text{Axis}_T \gamma)$ in $T'$.

**Proof.** Let $K'$ be a convex set in $T'$ and let $a, b \in K = q^{-1}(K')$. Every $x \in [a, b]$ is sent by $q$ to a point in $[q(a), q(b)]$ so $K$ is convex. Now, $q(\text{Char}_T \gamma) \subset \text{Char}_{T'} \gamma$ because a point $a$ lies in the characteristic set of $\gamma$ if and only if $a \in [\gamma^{-1}a, \gamma a]$. If $\gamma$ is hyperbolic in $T'$, then $q(\text{Char}_T \gamma)$ is connected and $\gamma$-invariant, so it must contain the axis of $\gamma$ in $T'$. If $\gamma$ is elliptic in $T'$, then the preimage of a fixed point of $\gamma$ in $T'$ is connected and $\gamma$-invariant. Hence it must intersect the characteristic set of $\gamma$ in $T$. Therefore, $\text{Char}_{T'} \gamma = q(\text{Char}_T \gamma)$ and $\gamma$ fixes at most one point in $T'$ when it is hyperbolic in $T$. □

**Corollary.** Let $T$ and $T'$ be two minimal $F_n$ actions and $q : T \to T'$ be a map preserving alignment. If $T$ is very small then so is $T'$. 

4° SERIE - TOME 33 - 2000 - N° 4
Proof. – From the previous lemma, $T'$ is small because any element fixing the non-degenerate arc $[x, y]$ fixes the arc joining the subtrees $q^{-1}(x)$ and $q^{-1}(y)$. The previous lemma allows one to deduce that $\text{Fix}_{T'} \gamma = \text{Fix}_{T'} \gamma^k$ from $\text{Fix}_T \gamma = \text{Fix}_T \gamma^k$. Finally, $\text{Fix}_{T'} \gamma = q(\text{Fix}_T \gamma)$ shows that $\gamma$ may not fix any triod in $T'$. □

**Ergodic measures**

A homothety of a length measure $\mu$ is the multiplication of every $\mu_1$ by a same positive real number.

**Definition.** – We say that an $\mathbb{R}$-tree endowed with an isometric action of a group $\Gamma$ is uniquely ergodic if the Lebesgue measure is the only non-zero positive invariant measure on $T$ up to homothety.

In particular, if $T$ is uniquely ergodic, and if $q : T \to T'$ is equivariant and preserves alignment, then $q$ is a homothety.

If $T$ is an action of a group $\Gamma$, we denote by $M(T)$ the convex cone of invariant positive measures on $T$.

A subset $E \subset T$ is said to be measurable if each intersection of $E$ with an arc of $T$ is measurable. Thus, a function $f : T \to \mathbb{R}$ is measurable if its restriction to every interval of $T$ is measurable. We say that a measurable subset $E \subset T$ has $\mu$-measure 0 if for every arc $I$ of $T$, $\mu(E \cap I) = 0$, and $E$ has $\mu$-full measure if $T \setminus E$ has measure 0. A function $f : T \to \mathbb{R}$ is constant $\mu$-almost everywhere if there exists $c \in \mathbb{R}$ such that $f^{-1}(c)$ has full $\mu$-measure.

**Definition.** – A measure $\mu \in M(T) \setminus \{0\}$ is said to be ergodic if the following equivalent conditions hold:

1. every $\Gamma$-invariant measurable function is constant $\mu$-almost everywhere;
2. every measure $\nu \in M(t)$ with density at most one with respect to $\mu$ is homothetic to $\mu$;
3. $\mu$ is extremal in $M(T)$, i.e. if $\mu = \mu_1 + \mu_2$ with $\mu_1, \mu_2 \in M(T)$, then $\mu_1$ and $\mu_2$ are homothetic to $\mu$;
4. every measurable invariant subset of $T$ either has full or 0 measure with respect to $\mu$.

Proof of the equivalence of the conditions. – (1) $\Rightarrow$ (2) because if $\nu \in M(T)$ has density at most one with respect to $\mu$, on every arc $I$ we may write $\nu_I = f_I \mu_I$ for some measurable functions $f_I$ defined $\mu_I$-almost everywhere, and the $f_I$ are the restrictions $\mu$-almost everywhere of an invariant measurable function $f : T \to \mathbb{R}$. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear. If $f$ is a $\Gamma$-invariant measurable function which is not constant almost everywhere, then there exists $M \in \mathbb{R}$ such that neither $A^+ = \{x \in T \mid f(x) \geq M\}$ nor $A^- = \{x \in T \mid f(x) < M\}$ have $\mu$-measure 0. □

Note that if $T$ is uniquely ergodic, then the Lebesgue measure is ergodic. We denote by $M_0(T)$ the set of non-atomic invariant positive measures on $T$ and $M_1(T) \subset M_0(T)$ the set of invariant positive measures with density at most 1 with respect to the Lebesgue measure. Both $M_0(T)$ and $M_1(T)$ are convex.

**Lemma 5.2.** – A non-atomic measure $\mu$ is ergodic if and only if $M_1(T_\mu)$ has dimension 1.

Proof. – The measure $\mu$ is ergodic if and only if every non-zero measure whose density is at most 1 with respect to $\mu$ is homothetic to $\mu$. Now there is a natural isomorphism between the set of measures on $T$ with density at most 1 with respect to $\mu$ and the set of measures on $T_\mu$ with density at most 1 with respect to the Lebesgue measure: if $q : T \to T_\mu$ denotes the quotient map, the isomorphism is given by $\nu \in M_1(T_\mu) \mapsto q^* \nu$. □
Weak topology on sets of measures

The set $M(T)$ is naturally endowed with the weak topology (see [26]). For this topology, a sequence $\mu^{(k)}$ of measures converges to $\mu$ if and only if for every interval $I$ and every continuous function $f : I \to \mathbb{R}$,

$$\int f \, d\mu^{(k)} \xrightarrow{k \to \infty} \int f \, d\mu.$$

This topology is not projectively compact in general. One should keep in mind the following phenomenon [26]: if $I$ is an arc in $T$, if $b \in I \setminus \partial I$ is a branch point of $T$, and if $\delta_k$ is the Dirac measure at $x_k \notin I$ with $x_k \to b$, then $\delta_k$ does not converge to the Dirac measure at $b$.

If $T$ is a minimal action of a finitely generated group, then there exists a finite tree $K \subset T$ such that every arc $I$ of $T$ may be subdivided into finitely many sub-arcs which may be sent into $K$ by an element of $\Gamma$. Therefore, the set $M_0(T)$ of non-atomic length measures on $T$ is naturally identified with the set of (usual) measures $\mu$ on $K$ which are $\Gamma$-invariant i.e. such that for all $\gamma \in \Gamma$,

$$(\gamma|_{K\cap \gamma^{-1}.K})_*\mu|_{K\cap \gamma^{-1}.K} = \mu|_{K\cap \gamma.K}.$$

The topology induced on $M_0(T)$ by the weak topology coincides with the usual topology on the space of invariant measures on $K$. This implies that $M_1(T)$ is compact (but it contains the null measure). This identification may be extended to the set of measures for which no branch point of $T$ has non-zero measure, but we won’t need this fact. Note that on $M_0(T)$, the applications $\mu \mapsto \mu(I)$ are continuous for every arc $I$ (because the measures in $M_0(T)$ have no atom).

Measures and simplices

**Lemma 5.3.** - Let $T$ be a minimal action of a non-abelian finitely generated group $\Gamma$ with dense orbits. Then the map $\sigma_T$ from $M_0(T) \setminus \{0\}$ to the set of actions of $\Gamma$ on $\mathbb{R}$-trees modulo equivariant isometry defined by $\sigma_T(\mu) = T_\mu$ is one-to-one.

**Remark.** - This lemma is of course false if we don’t assume that $T$ has dense orbits. This map $\sigma_T$ is linear in the following sense:

$$l_{\sigma(t_1\mu_1 + t_2\mu_2)} = t_1 l_{\sigma(\mu_1)} + t_2 l_{\sigma(\mu_2)} \quad \text{for all } t_1, t_2 \geq 0.$$

The map $\sigma_T$ is continuous on $M_0(T)$ because $\mu \mapsto \mu(I)$ is continuous for every interval $I$.

**Proof.** - We can assume that $T$ is not a line since we know in this case that $T$ is uniquely ergodic. Since $T$ is minimal, every $T_\mu$ is minimal (the preimage of an invariant subtree is an invariant subtree). Assume that $f : T_{\mu_1} \to T_{\mu_2}$ is an equivariant isometry for some $\mu_1, \mu_2 \in M_0(T)$. We denote by $q_i : T \to T_{\mu_i}$ the quotient maps.

For $\gamma, \delta \in \Gamma$, we denote by $\text{bridge}_T(\gamma, \delta)$ the segment joining the characteristic sets of $\gamma$ and $\delta$ (if they are disjoint) or their intersection point when they meet in exactly one point. We don’t define $\text{bridge}_T(\gamma, \delta)$ if the intersection of their characteristic sets contains more than one point.

![Diagram](image-url)
Let \( x, y \) be two distinct points in \( T \). We want to prove that \( \mu_1([x,y]) = \mu_2([x,y]) \). Since the orbits of \( \Gamma \) are dense in \( T \), and since \( T \) is not a line, the branch points of \( T \) are dense in every segment: if \( I \) is an arc and if \( x \in I \), we find a branch point in \( I \) close to \( x \) by projecting to \( I \) any branch point of \( T \) that is close enough to \( x \). Now, since \( T \) is non-abelian, every segment is contained in the axis of a hyperbolic element (see [8] or [25, Lemma 4.3]). This implies that for every \( \varepsilon > 0 \), we can find elements \( \gamma, \gamma', \delta, \delta' \in \Gamma \) hyperbolic in \( T \) whose axes are pairwise disjoint and

\[
x \in \text{bridge}_T(\gamma, \gamma') \quad \text{and} \quad y \in \text{bridge}_T(\delta, \delta')
\]

with

\[
\mu_i(\text{bridge}_T(\gamma, \gamma')) = \mu_i(\text{bridge}_T(\delta, \delta')) \leq \varepsilon \quad \text{for } i \in \{1, 2\}.
\]

One has \( q_i(\text{bridge}_T(\gamma, \gamma')) = \text{bridge}_{T_{\mu_i}}(\gamma, \gamma') \) and \( q_i(x) \in \text{bridge}_{T_{\mu_i}}(\gamma, \gamma') \) (and similar facts for \( y \) with \( \delta, \delta' \) instead of \( \gamma, \gamma' \)). This implies that

\[
\left| \mu_i([x,y]) - d(\text{bridge}_{T_{\mu_i}}(\gamma, \gamma'), \text{bridge}_{T_{\mu_2}}(\delta, \delta')) \right| \leq 2\varepsilon.
\]

But \( f \) sends \( \text{bridge}_{T_{\mu_1}}(\gamma, \gamma') \) and \( \text{bridge}_{T_{\mu_2}}(\delta, \delta') \) respectively to \( \text{bridge}_{T_{\mu_1}}(\gamma, \gamma') \) and \( \text{bridge}_{T_{\mu_2}}(\delta, \delta') \). We deduce that \( \mu_1([x,y]) \) is \( 4\varepsilon \)-close to \( \mu_2([x,y]) \) and this holds for every \( \varepsilon > 0 \) so that \( \mu_1 = \mu_2 \).

**Corollary 5.4.** Let \( T \) be a very small \( F_n \)-action with dense orbits. Then \( \mathcal{M}_0(T) \) is a finite dimensional convex set and \( \mathcal{M}_0(T) \) is projectively compact. Moreover, \( T \) has at most \( 3n - 4 \) non-atomic ergodic measures (up to homothety), and every measure in \( \mathcal{M}_0(T) \) is a sum of these ergodic measures. Moreover, \( \mathcal{M}_1(T) \) is compact, and \( \sigma_T \) and \( (\sigma_T)|_{\mathcal{M}_1(T) \setminus \{0\}} \) define two simplices in outer space.

**Proof.** First notice that if \( \mu_1, \ldots, \mu_p \) are ergodic measures which are mutually not homothetic, then they are linearly independent in \( \mathcal{M}(T) \). This is because there exist disjoint measurable sets \( E_1, \ldots, E_p \) that cover \( T \) such that \( E_i \) has full \( \mu_i \)-measure.

On the other hand, the set of very small actions of \( F_n \) which are not free simplicial (i.e. the non-projective boundary of outer space) has topological dimension \( 3n - 4 \) (see [14]). Since \( \sigma_T \) is linear, continuous and injective, \( \mathcal{M}_0(T) \) has dimension at most \( 3n - 4 \), and \( T \) has at most \( 3n - 4 \) non-atomic ergodic measures up to homothety.

We now prove that any measure \( \mu \) is a sum of ergodic measures. The set of measures \( \mathcal{M}_\mu(T) \) with density at most 1 with respect to \( \mu \) is compact since it is isomorphic to the set of invariant measures on a finite tree \( K \) with density at most 1 with respect to \( \mu \). The Krein–Millman theorem shows that \( \mu \) is a convex combination of extremal points of \( \mathcal{M}_\mu(T) \) [20, Theorem IV.1.5, p. 88]. Such an extremal point must be ergodic (if non-zero).

**5.2. Limits and maps preserving alignment**

The following proposition is crucial in this section:

**Proposition 5.5.** Let \( T \) be a minimal non-abelian action with dense orbits of a finitely generated group \( \Gamma \), and assume that \( T \) is not a line. Assume we are given actions \( T_p, T'_p \) and \( T' \) such that \( T_p \xrightarrow{\sim} T \) and \( T'_p \xrightarrow{\sim} T' \), and assume that we have equivariant 1-Lipschitz maps preserving alignment \( q_p : T_p \to T'_p \).

Then there exists a natural equivariant 1-Lipschitz map \( q : T \to T' \) preserving alignment.
Remark. – This proposition can easily be checked to hold under the weaker assumption that $q_p$ is 1-Lipschitz and has a backtracking constant going to 0 as $p$ tends to infinity.

Proof. – Let $K_p$ and $K'_p$ be two exhaustions of $T$ and $T'$ by finite subtrees, $F_p$ an exhaustion of $T'$ by finite subsets, and $\varepsilon_p$ a sequence of numbers decreasing towards zero. By passing to a subsequence, we may assume that

• there is an $F_p$-equivariant $\varepsilon_p$-approximation $R_p$ between $K_p \subset T$ and $H_p \subset T'$,
• there is an $F'_p$-equivariant $\varepsilon_p$-approximation $R'_p$ between $K'_p \subset T'$ and $H'_p \subset T'$.

Here is a method to construct $x' = q(x) \in T'$. Take $x \in T$ and assume that $p$ is large enough so that $x \in K_p$. Let $x_p \in H_p$ be an $R_p$-approximation point of $x$ and let $x'_p = q_p(x_p)$. Let $y'_p$ be an $R'_p$-approximation point of the projection of $x'_p$ on $H'_p$. We are going to prove that $d(x'_p, H'_p) \rightarrow 0$ and that $y'_p$ converges in $T'$ to a point which we will define to be $q(x)$.

As in the proof of Lemma 5.3, for every $\varepsilon > 0$, we can find hyperbolic elements $\gamma, \delta \in \Gamma$ such that

• Axis $(\gamma) \cap$ Axis $(\delta) = \emptyset,$
• $x \in \text{bridge}_{T_p}(\gamma, \delta)$,
• the diameter of $\text{bridge}_{T_p}(\gamma, \delta)$ is at most $\varepsilon$.

An easy argument about the Gromov topology shows that for $p$ large enough, $\gamma$ and $\delta$ are hyperbolic in $T_p$, their axes do not intersect, they are at most $2\varepsilon$-far from each other, and $x_p$ is $\varepsilon$-close to $\text{bridge}_{T_p}(\gamma, \delta)$.

Lemma 5.1 implies that the characteristic sets of $\gamma$ and $\delta$ intersect in at most one point. Moreover, $q_p[\text{bridge}_{T_p}(\gamma, \delta)] = \text{bridge}_{T'_p}(\gamma, \delta)$. Since $q_p$ is 1-Lipschitz, $d(x'_p, \text{bridge}_{T'_p}(\gamma, \delta)) \leq \varepsilon$ and the diameter of $\text{bridge}_{T'_p}(\gamma, \delta)$ is at most $2\varepsilon$. To show the first part of the claim, just notice that for $p$ large enough, $H'_p$ contains $\text{bridge}_{T'_p}(\gamma, \delta)$ because it meets $\text{Char}_{\gamma}$ and $\text{Char}_{\delta}$ (this is a simple argument about the Gromov topology).

Let $x''_p$ be the projection of $x'_p$ on $H'_p$ and let $y''_p$ be an approximation point of $x''_p$ in $T'$. The condition $\#(\text{Char } \gamma \cap \text{Char } \delta) \leq 1$ being a closed condition in the equivariant Gromov topology (see for instance [25]), $\text{Char}_{T'} : \gamma \cap \text{Char}_{T'} : \delta$ contains at most one point. Moreover, since the diameter of $\text{bridge}_{T'_p}(\gamma, \delta)$ is at most $2\varepsilon$ for every $p$, so is the diameter of $\text{bridge}_{T'_p}(\gamma, \delta)$. For sufficiently large $p$, $y''_p$ is $2\varepsilon$-close to $\text{bridge}_{T'_p}(\gamma, \delta)$, which implies that for $p, q$ large enough, $d(y''_p, y''_q) \leq 6\varepsilon$, so $y''_p$ is Cauchy. Note that $T'$ may not be complete (and in this case its completion is not minimal). But the argument above shows that if $\gamma_0, \delta_0$ are fixed hyperbolic elements of $\Gamma$ such that $x \in \text{bridge}_{T_p}(\gamma_0, \delta_0)$, $d(y''_p, \text{bridge}_{T'_p}(\gamma_0, \delta_0))$ tends to 0 as $p$ tends to infinity. Since $\text{bridge}_{T'_p}(\gamma_0, \delta_0)$ is compact, $y''_p$ converges to a point in this set which proves the claim.

The limit $q(x)$ of $y''_p$ is independent of the choices made since we may apply the claim to the sequence obtained by alternating the terms of two sequences $y^{(1)}_p$ and $y^{(2)}_p$ corresponding to different choices. The fact that $q$ is equivariant and 1-Lipshitz is clear. To prove that $q$ preserves alignment, pick $a, b, c \in T'$ aligned in this order, i.e. such that $(a|c)_b = 0$. Some approximation points $a_p, b_p, c_p$ in $T'_p$ satisfy $(a_p|c_p)_b \leq 3\varepsilon_p/2$. Since a 1-Lipschitz map preserving alignment decreases the Gromov product, $(a''_p|c''_p)_b \leq (a'_p|c'_p)_b \leq 3\varepsilon_p/2$ where $a'_p, b'_p, c'_p$ are the images through $q$ of $a_p, b_p, c_p$ and $a''_p, b''_p, c''_p$ are their projection on $H'_p$ (this projection is 1-Lipschitz and preserves alignment). We deduce that $(q(a)|q(c))_{q(b)} = 0$, and $q$ preserves alignment. \qed

5.3. Approximation of actions with few ergodic measures

Theorem 3. – Let $n \geq 3$ and let $T$ be a very small action with dense orbits. If the Lebesgue measure on $T$ is the sum of at most $n - 1$ ergodic measures, then $T \in \mathcal{M}_n$.

Remark. – There exist actions for which the Lebesgue measure is non-ergodic since Keynes–Newton and Keane have shown how to build interval exchanges for which the Lebesgue measure...
is non-ergodic [17,16]. In fact, the number of ergodic measures of an orientable measured foliation on a compact orientable surface with fundamental group $F_n$ is at most $n - 1$ and equality is reached (see [27] for instance). The number of ergodic measures of a non-orientable measured foliation on a non-orientable surface with fundamental group $F_n$ is at most $3n - 4$. More recently, Martin proved that there exist non-ergodic systems of isometries of exotic type [23]. There is a very easy way to construct non-geometric very small actions with dense orbits for which the Lebesgue measure is not ergodic: start from two non-geometric free $F_3$-actions $T_1, T_2$ with dense orbits. Given two base points *1 and *2 in $T_1, T_2$, the action $T = T_1 * T_2$ of $F_3 * F_3$ has dense orbits and is free. The Lebesgue measure of $T$ is not ergodic since one may multiply by $\lambda_1$ and $\lambda_2$ the metrics on $T_1$ and $T_2$.

**Proof.** - We have to approximate $T$ by simplicial actions in $\mathcal{F}_n$. We first prove the theorem when the Lebesgue measure on $T$ is ergodic, since the proof is simpler.

Take a sequence of very small (or even free) simplicial actions $T_p$ converging to $T$. Given an edge $e_p \in T_p$, we consider the action $T'_p$ obtained by collapsing to a point every edge which is not in the orbit of $e_p$. $T'_p$ may be seen as $(T_p, \mu_p)$, where $\mu_p$ is the restriction of the Lebesgue measure on $T_p$. The collapsing map $q_p : T_p \to T''_p$ is 1-Lipschitz and preserves alignment.

We show that $e_p$ may be chosen so that a subsequence of $T'_p$ converges to a very small action $T'$ (without rescaling the metric on $T'_p$): take $g \in F_n$ hyperbolic in $T$. Since $T_p$ has at most $3n - 3$ orbits of edges, there is an edge $e_p$ of $T_p$ whose orbit contributes at least $1/(3n - 3)$ to the translation length of $g$ (if $I$ is a fundamental domain for the action of $g$ on its axis, $|I \cap F_n, e_p| \geq l_{T_p}(e_p)/3n - 3$). Compactness of $C^\infty$ and the fact that $l_{T''_p}(g)$ remains bounded away from 0, implies that up to taking a subsequence, we may assume that $T'_p$ converges to a very small action $T'$.

Proposition 5.5 and ergodicity then show that $T'$ is homothetic to $T$. Moreover, since the quotient graph of $T_p$ has exactly one edge, $T'_p$ cannot lie in $\mathcal{O}_n$. $T'_p$ being simplicial, we get that $T'_p \in \mathcal{M}_n$. Therefore, $T'$ (and hence $T$) lies in $\mathcal{M}_n$.

Now let's turn to the proof of the general case. First, Bestvina and Feighn show that $T$ may be approximated by simplicial very small actions $T_p$ such that there exist equivariant morphisms of $\mathbb{R}$-trees $f_p : T_p \to T$ [4]. Let $\lambda$ be the Lebesgue measure on $T$, and let $\mu_1, \ldots, \mu_k$ be ergodic measures such that $\lambda = \mu_1 + \cdots + \mu_k$ for some $k \leq n - 1$. Denote by $\nu_i^p = f_p^* \mu_i$ the pull-back measure on $T_p$. The density of $\nu_i^p$ with respect to the Lebesgue measure on $T_p$ is at most 1. Let $T_p' = (T_p)_{\sigma^p}$ be the corresponding simplicial action.

We show that $T_p'$ converges to $T_{\mu_i}$ when $p \to \infty$. For every $g \in F_n$, $l_{T_{\mu_i}}(g) \leq l_{T_p'}(g)$ so if $g$ is elliptic in $T$ then $l_{T_p'}(g)$ converges to $l_{T_{\mu_i}}(g) = 0$. When $g$ is hyperbolic in $T$, $g$ is hyperbolic in $T_p'$ for large $p$. Let $I$ be an interval of length $l_{T_p'}(g)$ in $\text{Axis}_{T_p'}(g)$ and subdivide $I$ into sub-intervals isometrically embedded in $T$ through $f_p$. This subdivision may be refined so that there exists a finite union $E$ of the sub-intervals such that $f_p(E)$ is an interval of length $l_T(g)$ contained in $\text{Axis}_T(g)$, and such that $f_p$ is one-to-one in restriction to $E \setminus \partial E$. This implies that $f_p$ is isometric in restriction to each component of $E$, so that the Lebesgue measure of $I \setminus E$ is $l_{T_p'}(g) - l_T(g)$ and thus tends to 0 as $p$ tends to infinity. In the same way, $l_{T_p'}(g) - l_{T_{\mu_i}} = \nu_i^p(I \setminus E)$, hence tends to 0 since $\nu_i^p(I \setminus E)$ is smaller than the Lebesgue measure of $(I \setminus E)$. This shows that $T_p'$ converges to $T_{\mu_i}$ when $p \to \infty$.

The argument in the ergodic case tells us that for each $i \in \{1, \ldots, k\}$, up to taking a subsequence, we can collapse edges in $T_i$ to obtain actions $T_i'$ having exactly one orbit of edges, and which converge to some action $T_i''$ homothetic to $T_{\mu_i}$ since the Lebesgue measure of $T_{\mu_i}$ is ergodic. Let $0 < t_i \leq 1$ be such that $T_i'' = t_i T_{\mu_i}$. We denote by $\sigma_i^p$ the measure on $T_p$ corresponding to this collapse: $\sigma_i^p$ is the restriction of $\nu_i^p$ to the non-collapsed edges. With these notation, $T_p'' = (T_p)_{\sigma_i^p}$.
Now consider the measure $\sigma$ on $T_p$ defined by

$$\sigma_p = \sum_{i=1}^{k} \frac{1}{t_i} \sigma_{i, p}.$$

Then $(T_p)_{\sigma_p}$ tends to $T$ as $p$ tends to infinity. Since each $\sigma_{i, p}$ is non-zero on at most 1 orbit of edges, $(T_p)_{\sigma_p}$ has at most $k \leq n - 1$ orbits of edges. To conclude, just notice that any very small simplicial action having at most $n - 1$ orbits of edges cannot lie in $O_n$. 

REFERENCES


(Manuscript received September 28, 1998; accepted, after revision, September 10, 1999.)

Vincent GUIRARDEL
Institut Fourier, BP 74, Université Grenoble-I,
38402 Saint-Martin-d’Hères cedex, France
E-mail: vincent.guirardel@ujf-grenoble.fr