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THE TOPOLOGY OF LARGE $H$-SURFACES BOUNDED BY A CONVEX CURVE

BY BEATE SEMMLER

ABSTRACT. – We shall consider embedded compact surfaces $M$ of constant non-zero mean curvature $H$ ($H$-surfaces) in hyperbolic space $\mathbb{H}^3$. Let $L$ denote a horosphere of $\mathbb{H}^3$. Assume that $M$ is contained in the horoball bounded by $L$ and that the boundary of $M$ is a strictly convex Jordan curve $\Gamma$ in $L$. We establish the following:

(i) case $H > 1$. There is an $\delta(H)$, depending only on the geometry of $\Gamma$, such that whenever $M$ is a $H$-surface bounded by $\Gamma$, with $1 < H < \delta(H)$, then $M$ is topologically a disk.

(ii) case $H \leq 1$. Then $M$ is a graph over the domain $\Omega \subset L$ bounded by $\Gamma$ with respect to the geodesics orthogonal to $\Omega$; in particular, $M$ is topologically a disk.

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1. Introduction

Let $P$ be a plane in Euclidean space $\mathbb{R}^3$ and let $\mathbb{R}^3_+$ be one of the two halfspaces determined by $P$. Consider embedded compact surfaces $M$ of constant non-zero mean curvature $H$ ($H$-surfaces) in $\mathbb{R}^3_+$ with boundary $\partial M = \Gamma$ a convex curve in $P$. It is known that, if $H$ is sufficiently small in terms of the geometry of $\Gamma$, then a $H$-surface $M$ has genus zero. This result is established in [3] where they use a rescaling and a version of a compactness theorem to show this. Our proof of the same result will use another technique and will also work in the hyperbolic case. Indeed, in Hyperbolic space $\mathbb{H}^3$, homotheties do not exist, hence we can not apply the compactness theorem for $H$-surfaces in $\mathbb{H}^3$ proved in [2] to give a similar proof as in [3].

In this paper we shall mainly investigate the hyperbolic case to obtain a result in the same spirit as in $\mathbb{R}^3$.

Let $L$ be a horosphere in $\mathbb{H}^3$ and let $\mathcal{L}$ be the horoball of $\mathbb{H}^3$ bounded by $L$; the mean curvature of $L$ is one and the mean curvature vector of $L$ points into $\mathcal{L}$. We consider embedded compact $H$-surfaces $M$, $H$ greater than one, in $\mathcal{L}$ with boundary $\partial M = \Gamma$ a convex curve in $L$. We will
show that, if $H$ is sufficiently close to one in terms of the geometry of $\Gamma$, then $M$ has genus zero (Theorem 2). If $M$ is an embedded compact $H$-surface in $\mathcal{O}$, bounded by $\Gamma$ and $H \leq 1$, then $M$ is a geodesic graph (Theorem 4). The case for $H$ less than one and $\Gamma$ in a hyperbolic plane is treated in [2].

2. The Euclidean case

THEOREM 1 ([3, Theorem 2]). Let $P \subset P = \{x_3 = 0\}$ be a strictly convex curve. There is an $f(p)$, depending only on the geometry of $P$, such that whenever $M \subset \mathbb{R}_+^3$ is a compact embedded $H$-surface bounded by $\Gamma$, with $0 < H < f(p)$, then $M$ is topologically a disk and either $M$ is a graph over the domain $\Omega \subset P$ bounded by $\Gamma$ or $M \cap (\Omega \times [0, \infty))$ is a graph over $\Omega$ and $M \setminus (\Omega \times [0, \infty))$ is a graph over $\partial \Omega \times [0, \infty) = \Gamma \times \mathbb{R}_+$, with respect to the lines normal to $\Gamma \times \mathbb{R}_+$.

We need the following lemma which is proved in [3]:

LEMMA 2.1 ([3]). Let $r \subset P$ be a strictly convex curve. There is a $\tau > 0$, depending only upon the extreme values of the curvature of $\Gamma$, such that whenever $M \subset \mathbb{R}_+^3$ is an $H$-surface with boundary $\Gamma$, there is a $p \in \Omega$ (a depends on $M$) such that $M \cap (D(p, r) \times [0, \infty))$ is a graph over $D(p, \tau)$. (Here $D(p, \tau)$ denotes the Euclidean disk in $P$ centered at $p$, of radius $\tau$.)

Proof of Theorem 1. - Let $M$ be an $H$-surface. Let $r > 0$ and $p \in \Omega$ be given by the lemma. Let $G$ be the unique vertical catenoid meeting $P$ in the circle $C_0 = \partial D(p, r)$ where $\rho < r$ and $\rho$ is smaller than the smallest radius of curvature of $\Gamma$ (the latter condition allows us to translate $C_0$ horizontally in $\Omega$ so as to touch every point of $\Gamma$), and the angle between $G$ and $P$ along $C_0$ is $\pi/2$. Let $\Sigma = G \cap (P \times [0, 1])$ and let $C_1$ be the circle of $\Sigma$ at height one. Let $V = \{v \in P \mid C_0 + v \subset \Omega\}$ and let $D(R)$ be a sufficiently large disk in $P$ such that $C_1 + v \subset D(R) \times [1]$ for all $v \in V$.

We know that a highest point $q$ of $M$ is in $\Omega \times [0, \infty)$, and the height $d$ of $q$ is at most $2/H$. The part of $M$ over $P(d/2) = \{x_3 = d/2\}$ is a vertical graph. Also $M \setminus (\Omega \times [0, \infty))$ is a graph over $\Gamma \times \mathbb{R}_+$, with respect to the lines normal to $\Gamma \times \mathbb{R}_+$, of height at most $1/H$.

Let $\tau$ be the smallest value of the curvature of $\Gamma$ and $2\omega$ the circumscribed diameter of $\Omega$. Note by $c$ the point in $\Omega$ such that $\Gamma \subset D(c, \omega)$. As of now, we will work with $H$ sufficiently small such that $H < \tau$ and $H < 1/2\omega$.

First of all, we will prove that, if $d < 1/H$, then $M$ is a graph over $\Omega$. Let $\beta(t), 0 \leq t < \infty$, be a line segment in $P(1/2H) = \{x_3 = 1/(2H)\}$ starting at $\beta(0) = c \times \{1/(2H)\}$. We consider a straight cylinder $Z(\tau)$ of radius $1/(2H)$ and axis $\alpha$ in the horizontal plane $P(1/(2H))$ where $\alpha$ meets $\beta(t)$ orthogonally at some $t = \tau$. Let $\tilde{Z}(\tau)$ be the half-cylinder of $Z(\tau)$ by cutting $Z(\tau)$ with a vertical plane intersecting $P(1/(2H))$ along $\alpha$. We take $\tilde{Z}(\tau)$ so that $\beta(t) \cap \tilde{Z}(\tau) = \emptyset$ for $t < \tau$. For $\tau$ large, $\tilde{Z}(\tau)$ is disjoint from $M$. Now one can move $\tilde{Z}(\tau)$ towards $M$ along $\beta$. By the maximum principle, as $\partial Z \cap P$ approaches $\Gamma$ by horizontal translation, the first contact with $M$ can not be at an interior point of $M$. Therefore no accident will occur before reaching $\Gamma$. This implies that the diameter of a smallest disk centered at $c \times \{t\}$ that contains $M \cap \{x_3 = h\}$ is smaller than $2\omega + (1/H)$ for $0 \leq h \leq 1/H$. Let $S^+$ be the upper hemisphere of the sphere of mean curvature $H$ centered at $c \times \{1/H\}$. Translate $S^+$ downward, so the moving $S^+$ does not touch $M$ before it arrives at $P$, i.e., $M$ is below $S^+$ when $\partial S^+$ is on $P$. Then by the maximum principle and because $H < \tau$, one can translate $S^+$ horizontally to touch every point of $\Gamma$ and that is why $M \subset \Omega \times [0, \infty)$. Hence $M$ is a graph over $\Omega$.

Henceforth we assume that $d \geq 1/H$. The part of $M$ over $P(d/2)$ is a vertical graph.
(i) If an $H$-graph $M'$ over a domain $\Omega$ in the plane $P(t)$ where $\partial M' \subset P(t)$ has height $h$, then the radius of the smallest disk in $P(t)$ containing strictly $\Omega$ is at least $\lambda(h; H) = \sqrt{2h/H} - h^2$. To see this, suppose, on the contrary, that the domain $\Omega$ is contained in a disk $D(c, \tilde{r}) \subset P(t)$ where $\tilde{r} < \lambda(h; H)$. Let $S$ be the $H$-sphere centered at $c \times \{t\}$ and denoted by $S(h)$ the part of $S$ over the plane $P(t + (1/H) - h)$. $M'$ is contained in the vertical cylinder over $\Omega$ and the radius of $\partial S(h)$ is strictly greater than $\tilde{r}$, so by moving $S(h)$ towards $M'$ the first contact with $M'$ must occur at an interior point of $S(h)$ with a boundary point of $M'$. This means that the height of $M'$ is less than $h$ which gives a contradiction.

(ii) Let $\Omega(t)$ be the domain in $P(t)$ bounded by $M \cap P(t)$ for $t \in [d/2, d]$, and let $D_t(r)$ be the disk in $P(t)$ centered at $c_t = c \times \{t\}$, of radius $r$.

Let $r_{\max} = \inf \{ \Omega(t) \subseteq D_t(r) \}$ and $r_{\min} = \sup \{ \Omega(t) \supset D_t(r) \}$.

We want to prove: If $r_{\max} > 2\omega$ then $r_{\min} > r_{\max} - 2\omega$.

We know that $M \setminus (D(c, \omega) \times [0, \infty))$ is a graph over $\partial D(c, \omega) \times [0, \infty)$, with respect to the lines normal to $\partial D(c, \omega) \times \mathbb{R}_+$. So, for some point $\mu$ in $\partial D_t(r_{\max}) \cap M$, we consider all reflection by vertical planes and looking at the set of images of $\mu$ in $P(t)$. This set is contained in the interior of the domain in $\mathbb{R}^3$ bounded by $M \cup \Omega$; in particular in $\Omega(t)$ since each vertical plane is orthogonal to $P(t)$ and so the symmetry with respect to vertical planes leaves $P(t)$ invariant.

Doing elementary calculations, we see that the set of images of $\mu$ contains the disk $D_t(r_{\max} - 2\omega)$. In more detail, denoted by $\beta_0$ the half geodesic in $P(t)$ starting at $c_t$ and passing through $\mu$, and by $\beta_\phi$ the half geodesic in $P(t)$ where the angle between $\beta_0$ and $\beta_\phi$ at $\mu$ is $\phi$, $[\phi] \in [0, \arccos(\omega/r_{\max})]$. Consider the family of vertical planes $V_\phi(s)$ orthogonal to $\beta_\phi$ at $\beta_\phi(s + \omega)$ (Fig. 1). Apply the Alexandrov reflection technique to $M$ with the planes $V_\phi(s)$; by decreasing $s$ from $\infty$, no accident will occur up till $\partial D_t(\omega)$; i.e. $s = 0$. When $\mu'$ denotes the reflected image of $\mu$ with respect to the plane $V_\phi(0)$, the line segment $l$ joining $\mu$ to $\mu'$ is contained in $\Omega(t)$.

Now we have $\cos \phi = \frac{\omega + \text{dist}(\mu, V_\phi(0))}{r_{\max}}$ and $\sin \phi = \text{dist}(c_t, l)/r_{\max}$. The distance from $\mu'$ to $c_t$ is equal to

$$x(\phi) = \text{dist}(\mu', c_t) = \sqrt{\left(\text{dist}(\mu, V_\phi(0)) - \omega\right)^2 + \text{dist}^2(c_t, l)},$$

\begin{figure}
\centering
\includegraphics{fig1.png}
\caption{Fig. 1.}
\end{figure}
therefore \( x(\phi) = \sqrt{r_{\text{max}}^2 + 4\omega^2 - 4\omega r_{\text{max}} \cos \phi}, \) thus \( x(\phi) \geq r_{\text{max}} - 2\omega \) and this implies \( r_{\text{min}} > r_{\text{max}} - 2\omega \).

(iii) Now we are able to show that, if \( H \) is sufficiently small, then \( M \cap \{D(R) \times [1, d - 1]\} = \emptyset \) and \( M \cap \{P \times [d - 1, d]\} \) is a graph over \( P(d - 1) \). For \( h = 1 \), we get from (i) that \( r_{\text{max}} \) is at least \( \sqrt{(2/H) - 1} \) on \( P(d - 1) \); to apply (i) we need \( d/2 > 1 \) so we work with \( H \) such that \( 1/(2H) > 1 \). From (ii), by assuming \( 1/H > (1/2) + 2\omega^2 \), it follows that \( r_{\text{min}} > \sqrt{(2/H) - 1} \). Therefore, if \( 1/H > (1/2)((R + 4\omega)^2 + 1) \) then \( r_{\text{min}} > R + 2\omega \).

Set
\[
\eta = \min \left( 1/2, \left( \frac{1}{2} \{(R + 4\omega)^2 + 1\} \right)^{-1} \right).
\]

The end of the proof is the same as in [3].

3. The hyperbolic case

We work in the upper half-space model of hyperbolic space, that is,
\[
\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}
\]
with the hyperbolic metric, i.e. the Euclidean metric divided by \( x_3 \). In the following, we will represent by dist the hyperbolic geodesic distance in \( \mathbb{H}^3 \); \( \tau \) will be the hyperbolic parameter of arc length (in general used for geodesics and planes) and \( t \) the Euclidean parameter associated with the model (used for horospheres).

Let \( L(t) \) denote the horizontal horosphere \( \{x_3 = t\} \) and let \( \mathcal{L} \) be the non compact component of \( \mathbb{H}^3 \) bounded by \( L(1) \) such that the mean curvature vector of \( L(1) \) points towards \( \mathcal{L} \).

**Theorem 2.** Let \( \Gamma \subset L(1) \) be a strictly convex curve. There is an \( \mathcal{S}(\Gamma) \), depending only on the geometry of \( \Gamma \), such that whenever \( M \subset \mathcal{L} \) is a compact embedded \( H \)-surface bounded by \( \Gamma \), with \( 1 < H < \mathcal{S}(\Gamma) \), then \( M \) is topologically a disk and either \( M \) is a graph over the domain \( \Omega \subset L(1) \) bounded by \( \Gamma \) with respect to the geodesics orthogonal to \( \Omega \) or \( M \setminus (\Omega \times [1, \infty)) \) is a geodesic graph over \( \Omega \) and \( M \setminus (\Omega \times [1, \infty)) \) is a graph over \( \partial \Omega \times [1, \infty) = \Gamma \times [1, \infty) \), with respect to the geodesics orthogonal to \( \Gamma \times [1, \infty) \).

3.1. Properties of compact surfaces in a horoball

Before proving Theorem 2, we give a representative example of hyperbolic calculations, we establish some basic properties of an \( H \)-surface as in Theorem 2 and we state a lemma whose proof we will give later.

**Notation and Example.** Let \( q \in L(1) \) be the point \((0,0,1)\) and let \( \gamma(\tau) \subset \mathcal{L} \) denote the vertical geodesic through \( q \) orthogonal to \( L(1) \) parametrized such that \( \tau = \text{dist}(\gamma(\tau), L(1)) \). Consider the family \( P_{\gamma}(\tau) \subset \mathbb{H}^3 \) of planes orthogonal to \( \gamma \) at \( \gamma(\tau) \). Let \( p \notin \gamma(\tau) \) be a point in some \( L(t), t > 1 \), denoted by \( R \) the geodesic distance from \( p \) to \( \gamma(\tau) \) and by \( \alpha \) the angle between the \( x_3 \)-axis and the Euclidean line joining \((0,0,0)\) to \( p \); (Fig. 2).

\( L(t) \) intersects \( \gamma(\tau) \) at \( \tau = \ln t \). \( R \) is related to \( \alpha \) by \( \tan \alpha = \sinh R \); since the hyperbolic metric on \( L(t) \) is the Euclidean metric divided by \( t \), the hyperbolic length from \( p \) to \( \gamma(\ln t) \) in \( L(t) \) is equal to \( \sinh R \) (notice that the geodesic distance from \( p \) to \( \gamma(\ln t) \) which is \( 2\text{arcsinh}(\sinh R/2) \), is naturally smaller than the former). The geodesic passing through \( p \) and realizing the distance \( R \) from \( p \) to \( \gamma(\tau) \) is lying on \( P_{\gamma}(\ln(t \cosh R)) \) and this implies that the length of the segment of \( \gamma \) joining \( L(t) \) to this plane is equal to \( \ln \cosh R \). The intersection between \( L(t) \) and \( P_{\gamma}(\ln(t \cosh R)) \)
is a hyperbolic circle $C$ with hyperbolic center at $\gamma(\ln(t \cosh R))$, of hyperbolic radius $R$. By hyperbolic reflection with respect to $P_{\gamma}(\ln(t \cosh R))$, the image of $L(t)$ is a horosphere, denoted by $O(t \cosh^2 R)$, containing also $C$ and intersecting $\gamma(\tau)$ at $\tau = \ln(t \cosh^2 R)$; so the distance between both horospheres on $\gamma(\tau)$ is $2 \ln \cosh R$.

**Basic properties.** Let $M$ be defined as in Theorem 2. Let $B$ be the compact component of $\Sigma$ bounded by $M$ and the domain $\Omega \subset L(1)$ such that $\partial \Omega = \partial M$. Let $H$ be the mean curvature vector of $M$; we orient $M$ by $H$. Then:

(i) $H$ points towards $\Omega = M \cup (\Omega \times (0,1))$.

(ii) Each point $q \in M$ at maximal distance from $L(1)$ is contained in the solid vertical geodesic cylinder over $\Omega$ denoted by $C$.

(iii) Let $\gamma$ be any geodesic orthogonal to $L(1)$ passing through a point of $\partial \Omega$; if $M$ is contained in the solid Killing cylinder over $\Omega$ with respect to $\gamma$ (i.e. the integral curves of the Killing vector field associated to the hyperbolic translation along $\gamma$) then $M$ is a Killing graph over $\Omega$ with respect to $\gamma$.

(iv) $M \setminus (\Omega \times [1, \infty))$ is a graph over $\Gamma \times [1, \infty)$, with respect to the geodesics orthogonal to $\Gamma \times [1, \infty)$; this part of $M$ outside $C$ is also a graph over $\Gamma \times [1, \infty)$ with respect to the horocycles in $L(t), t \in [1, \infty)$, normal to $\Gamma \times [1, \infty)$.

(v) Let $q \in M$ be a point at maximal distance $d$ from $L(1)$ and let $\gamma(\tau) \subset \Sigma$ be the geodesic through $q$ orthogonal to $L(1)$ parametrized such that $\tau = \text{dist}(\gamma(\tau), L(1))$. Consider the family $P_{\gamma}(\tau) \subset \mathbb{H}^3$ of planes orthogonal to $\gamma$ at $\gamma(\tau)$. Let $R = \max_{p \in \Gamma} \text{dist}(p, \gamma)$. Then the part of $M$ lying above $P_{\gamma}((d/2) + \ln \cosh R)$ is a Killing graph with respect to $\gamma$.

**Proof.** (i) Consider the family of horospheres $L(t) = \{x_3 = t\}$; if $t$ is large enough, then $L(t) \cap M = \emptyset$; decrease $t$ and consider the first horosphere that touches $M$. At this point of contact, the mean curvature vector of $L(t)$ points upward and since the mean curvature of $L(t)$
(which is equal to one) is smaller than the mean curvature of $M$, the maximum principle implies that $H$ points towards $\Omega$, hence the same is true at each point of $M$.

(ii) Let $d = \text{dist}(q, L(1))$ and let $\gamma$ be the geodesic through $q$ orthogonal to $L(1)$. Suppose, on the contrary, that $q \notin \mathcal{C}$ so $q_0 = \gamma \cap L(1)$ is not in $\mathcal{C}$. Let $\mathfrak{h}$ be a half horocycle in $L(1)$ starting at a point $p_0$ of $\Gamma$ passing through $q_0$ and such that $\text{dist}(p_0, q_0) = \inf_{p \in \Gamma} \text{dist}(p, q_0)$. Now consider the unique geodesic plane $E \subset \mathbb{H}^3 \setminus \mathcal{C}$ tangent to $L(1)$ at $p_0$; note by $\beta$ the half geodesic in $E$, starting at $p_0$, which is contained in the vertical half plane determined by $\mathfrak{h}$ and $q$. Let $P(\tau)$ be the family of planes in $\mathbb{H}^3$, $0 \leq \tau < \infty$, such that for each point $b$ of $\beta$, there exists one $P(\tau)$ intersecting $\beta$ orthogonally at $b$. Parametrize so that $P(0)$ contains the initial point $p_0$ of $\beta$ (Fig. 3).

Apply the Alexandrov reflection technique to $M$ with the planes $P(\tau)$ (cf. [5]). For $\tau$ large, $P(\tau)$ is disjoint from $M$. Now, if we approach $M$ by $P(\tau)$, there will be a first contact point of some $P(\tau)$ with $M$. One continues to decrease $\tau$ and considers the symmetry of the part of $M$ swept out by $P(\tau)$ with respect to $P(\tau)$. These symmetries of $M$ are in $\mathcal{B}$. Notice that the symmetry through $P(\tau)$ of the relevant part of $L(1)$ is contained in $\mathcal{C}$. (Here relevant part means the part of $L(1)$ lying on the same side of $P(\tau)$ as the part of $M$ swept out by $P(\tau)$.) So, by the maximum principle, no accident can occur until $P(0)$ and the part of $M$ in question is a Killing graph over $P(0)$ with respect to the geodesic $\beta$ (the integral curves of the Killing vector field associated to the hyperbolic translation along $\beta$ are invariant by reflection with respect to $P(\tau)$). But the Killing segment joining $q$ to $P(0)$ and its symmetry through $P(0)$ are lying above $L(e^{\Delta})$ whereas $\mathcal{B}$ is below this horosphere which gives a contradiction and therefore $q$ must be in $\mathcal{C}$.

(iii) In this case we can do Alexandrov reflection with the family of planes orthogonal to $\gamma$ until a plane below $L(1)$ without any accident, so $M$ is a Killing graph over $L(1)$.

(iv) Let $\gamma_p$ be a geodesic through a point $p$ of $\Gamma$ orthogonal to $L(1)$ at $p$ and let $T_p$ be the vertical plane tangent to $\Gamma$ at $p$. Consider any half geodesic $\beta_g$ orthogonal to $T_p$ at some point $g$ of $\gamma_p$ where $\Gamma$ and $\beta_g$ are on opposite sides of $T_p$. As in (ii), we apply the Alexandrov reflection technique to $M$ with the family $P(\tau)$ of planes orthogonal to $\beta_g$. Therefore the relevant part of $M$ is a Killing graph over $P(0) = T_p$ with respect to the geodesic $\beta_g$.

We can do this for each point $g$ of $\gamma_p$ and each such half geodesic $\beta_g$; and also for $\gamma_p$ associated to each point $p$ of $\Gamma$; this means that on each geodesic orthogonal to $\Gamma \times [1, \infty)$ there is only one point of $M$, hence $M \setminus (\Omega \times [1, \infty))$ is a geodesic graph and the first assertion of (iv) follows.

Let $p$ be a point of $\Gamma$ and let $T_p$ be as above. Note by $\mathfrak{h}(s)$ the half horocycle in $L(1)$ starting at $p = \mathfrak{h}(0)$ and orthogonal to $T_p \cup L(1)$ at $p$ where $\Gamma$ and $\mathfrak{h}(s)$ are on opposite sides of $T_p$. Let $T(s)$ be a family of vertical planes such that $T(s)$ intersects $\mathfrak{h}$ orthogonally at $\mathfrak{h}(s)$ and
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Apply the Alexandrov reflection process to $M$ and the planes $T(s)$. Notice that hyperbolic symmetry through each $T(s)$ leaves $L(1)$ and all horocycles orthogonal to $T(s)$ in all $L(t)$ invariant. One can translate $T(s)$ along $h$ until $\partial \Omega = \Gamma$ and the part of $M$ swept out by $T(0) = T_p$ is a graph over $T_p$ with respect to the horocycle orthogonal to $T_p$.

(v) From (ii) we know that $q$ lies in the solid geodesic cylinder $\mathcal{C}$ over $\Omega$. We apply the Alexandrov reflection technique to $M$ with the planes $P_q(\tau)$; the first accident occurs when the image of an interior point $p_i$ of $M$ touches $\Gamma$. This point $p_i$ is situated on an integral curve of the Killing vector field with respect to $\gamma(\tau)$ over $\Gamma$; the Killing coordinate of such point $p_i$ is at most $d + \ln \cosh R$ where $R = \max_{p \in \Gamma} \text{dist}(p, \gamma)$ (see Notation and Example). Therefore the result follows. \square

We will prove the following Lemma 3 after the proof of Theorem 4.

**Lemma 3.** – Let $\Gamma \subset L(1)$ be a strictly convex curve. There is a $r > 0$, depending only upon the extreme values of the curvature of $\Gamma$, such that whenever $M \subset L$ is an $H$-surface, $H > 1$, with boundary $\Gamma$, there is a $p \in \Omega$ ($p$ depends on $M$) such that the part of $M$ in the solid Killing cylinder over $D(p, \sinh r) \subset L(1)$ with respect to the vertical geodesic $\gamma_p$ passing through $p$ is a Killing graph over $D(p, \sinh r)$ with respect to $\gamma_p$.

(Here $D(p, \sinh r)$ denotes the disk in $L(1)$ centered at $p$ such that $\partial D(p, \sinh r)$ is the hyperbolic circle centered at $\gamma_p(\tau = \ln \cosh r)$ of hyperbolic radius $r$.)

### 3.2. Proof of the main result

**Proof of Theorem 2.** – Let $M$ be an $H$-surface as in Theorem 2. Let $\omega$ be the hyperbolic radius of a smallest hyperbolic circle such that the domain in $L(1)$ bounded by this circle contains $\Omega$. Note by $c$ the point in $\Omega$ such that $\Gamma \subset D(c, \sinh \omega)$, and by $\gamma_c(\tau)$ the vertical geodesic passing through $c$; parametrized such that $\tau = \text{dist}(\gamma_c(\tau), L(1))$.

By property (ii) we know that the points at maximal distance $d$ from $L(1)$ are contained in the solid vertical geodesic cylinder $\mathcal{C}$ over $\Omega$. In the proof of property (iv), we saw that, if $T_p$ is a vertical plane tangent to $\Gamma$ at a point $p \in \Gamma$, then the part of $M$ in the half space determined by $T_p$ which does not contain $\Gamma$ is a Killing graph over $T_p$ with respect to any geodesic orthogonal to $T_p$ at some point of $T_p \cap \mathcal{C}$.

The same is still true if we choose some point $p'$ in $\partial D(c, \sinh \omega) \subset L(1)$, $T_{p'}$ the vertical plane tangent to $\partial D(c, \sinh \omega)$ at $p'$ and $\beta_{p'}$ the geodesic orthogonal to $T_{p'}$ at $p'$. Now consider the point $g$ in $L(1)$ where the Killing segment $k$ (with respect to $\beta_{p'}$) that joins the point $T_{p'} \cap \{D(c, \sinh \omega) \times [1, \infty)\} \cap L(e^d) = p' \times \{e^d\}$, intersects $L(1)$ (Fig. 4).

We want to evaluate the hyperbolic distance $\kappa$ between $g$ and the geodesic $\gamma_{p'} = T_{p'} \cap \{D(c, \sinh \omega) \times [1, \infty)\}$. Recall that sinh of $\kappa$ is the hyperbolic length in $L(1)$ from $g$ to $\gamma_{p'} \cap L(1) = p'$ and this value is also equal to the euclidean distance between $g$ and $p'$ in $L(1)$. Let $a$ be the euclidean center of the Killing segment $k$ (this makes sense since $k$ looks like a part of a circle) and let $b$ be the euclidean radius of $k$. We have $b^2 = 1 + (e^d - b)^2$ and sinh$^2 \kappa = b^2 - (e^d - b - 1)^2$. Therefore sinh $\kappa$ is equal to $\sqrt{2} \sinh d$.

Since the part of $M$ outside the vertical geodesic cylinder over $D(c, \sinh \omega) \subset L(1)$ is a Killing graph with respect to $\beta_{p'}$ and $T_{p'}$ for each point $p' \in \partial D(c, \sinh \omega)$, no point of $M$ in the vertical half plane containing $\beta_{p'}$ with boundary $\gamma_{p'}$ can be a distance greater than $\kappa$ from $\gamma_{p'}$, for each $p'$. This implies that $M$ is contained in the solid Killing cylinder over $D(c, \sqrt{2} \sinh d + \sinh \omega) \subset L(1)$ with respect to $\gamma_c$.

Now we will distinguish two cases.
(1) **Small case**

Suppose that \( \sqrt{2 \sinh d + \sinh \omega} \) is strictly smaller than \( \sinh r_H \) where \( r_H \) is the radius of the sphere of mean curvature \( H \).

We will first see that \( M \) stays inside in the Killing cylinder with respect to \( \gamma_c(\tau) \) over the domain in \( L(1) \) bounded by \( D(c, \sinh \omega) \). Let \( S^+ \) be the upper hemisphere of the \( H \)-sphere centered at \( \gamma_c(\theta + \ln \cosh(r_H)) \). Translate \( S^+ \) downward, so the moving \( S^+ \) does not touch \( M \) before it arrives at \( L(1) \), i.e., \( M \) is below \( S^+ \) when \( \partial S^+ \) is on \( L(1) \) (Fig. 5).

Next consider the family \( S^+(\tau) \) of upper half spheres with center at \( \gamma_c(\tau) \) for \( \tau \in [\tau_0, \tau_1] = [\ln \cosh \omega, \ln \cosh r_H] \) and \( \partial S^+(\tau) \) on \( L(1) \). This continuous family consists of surfaces in \( \mathcal{S} \) where the mean curvature starts from \( \coth \omega \), decreases to \( \coth r_H = H \) and where \( \partial S^+(\tau) \) is a foliation of the compact region in \( L(1) \) bounded by \( \partial D(c, \sinh r_H) \cup \partial D(c, \sinh \omega) \). So, for each \( \tau \), \( \Gamma \) is contained in the domain of \( L(1) \) bounded by \( \partial S^+(\tau) \). Since \( M \) is below \( S^+ = S^+(\tau_1) \) and when we decrease \( \tau \) from \( \tau_1 \) to \( \tau_0 \), the maximum principle implies that \( M \) is still below...
THE TOPOLOGY OF LARGE $H$-SURFACES BOUNDED BY A CONVEX CURVE

Our aim is now to show that, if $H$ is sufficiently small in terms of $\omega$ then $M$ is even contained in the Killing cylinder over $\Omega$ with respect to $\gamma_c$ and so we can conclude by property (iii) that $M$ is a Killing graph over $\Omega$.

Let $\delta = \sup \{ \Omega \supset D(\epsilon, \sinh \tau) \}$. To establish the result we consider the family $Z(\tau)$ of Killing cylinders over $\partial D(\epsilon, \sinh \tau) \subset L(1)$ with respect to $\gamma_c$ for $\tau \in [\delta, \omega)$. The mean curvature vector of $Z(\tau)$ points into the component of $\mathbb{H}^3$ bounded by $Z(\tau)$ which contains $\gamma_c$ and the mean curvature varies continuously in $\tau$ from $\coth(2\delta)$ decreasing to $\coth(2\omega)$ (Fig. 6).

Now, suppose on the contrary, that $M$ is not in the solid Killing cylinder over $\Omega$. Since $M$ is lying in $Z(\omega)$ and by decreasing $\tau$ from $\omega$ to $\delta$ there will be some $\tau_0$ where $Z(\tau_0)$ touches $M$ for the first time at an interior point of $M$ such that $Z(\tau_0)$ is tangent to $M$ and the mean curvature vector of both surfaces points in the same direction. However, if $H < \coth 2\tau_0$, this is impossible by the maximum principle. Therefore $M$ is contained in the Killing cylinder over $\Omega$ with respect to $\gamma_c$ for $H$ smaller than $\coth(2\omega)$.

Thus $M$ is a Killing graph over $\Omega$ with respect to $\gamma_c$.

To finish our investigation for small $H$-surfaces, we will show that $M$ is even a geodesic graph over $\Omega$ with respect to the geodesics orthogonal to $\Omega$. Let $p$ be a point in $\Omega$ and let $\gamma_p$ be the vertical geodesic passing through $p$. Since $M$ is below the upper half sphere of radius $\omega$ centered on $\gamma_c$ with boundary $D(\epsilon, \sinh \omega) \subset L(1)$, $M$ is also below the upper half sphere of radius $2\omega$ centered on $\gamma_p$ with boundary in $L(1)$. We consider Killing cylinders with axes $\gamma_p$ and conclude by the same argument as before that $M$ is a Killing graph over $\Omega$ with respect to $\gamma_p$. We can repeat this for each point $p$ in $\Omega$; this means that on each vertical geodesic there is only one point of $M$, hence $M$ is a geodesic graph over $\Omega$; in particular $M$ is topologically a disk.

(2) Large case

Henceforth we assume that $d$ is bounded from below in terms of $H$, i.e.

$$\sqrt{2 \sinh d + \sinh \omega} \geq \sinh r_H = \frac{1}{\sqrt{H^2 - 1}}.$$ 

Let $r > 0$ and $p \in \Omega$ be given by Lemma 3. Let $G$ be the unique vertical catenoid cousin meeting $L(1)$ in the circle $C_0 = \partial D(p, \sinh \rho)$ where $\rho < r$ and $\sinh \rho$ is smaller than the smallest
radius of curvature of \( \Gamma \) in \( L(1) \) (the latter condition allows us to translate \( C_0 \) horizontally in \( \Omega \) so as to touch every point of \( \Gamma \)), and \( G \) has its waist at \( L(1) \) (see [4] for catenoid cousins).

Let \( \Sigma = G \cap (L(1) \times [1, x_3 = e \cdot \cosh^3 \omega]) \) and let \( C_1 \) be the circle of \( \Sigma \) at euclidean height \( [x_3 = e \cdot \cosh^3 \omega] \). (The hyperbolic height of \( \Sigma \) is equal to \( 1 + 3 \ln \cosh \omega \)). \( \Sigma \) is a Killing graph with respect to \( \gamma_p \) over the non compact component of \( L(1) \cap C_0 \). Let \( V = \{ v \in L(1) \mid C_0 + v \subset \Omega \} \) and let \( D(c, \sinh R) \) be a sufficiently large disk in \( L(1) \) centered at \( c \) such that \( C_0 + v \subset D(c, \sinh R) \times [x_3 = e \cdot \cosh^3 \omega] \) for all \( v \in V \) (here, we translate the hyperbolic objects \( v \), respectively \( D(c, \sinh R) \), from \( L(1) \) to \( L(e \cdot \cosh^3 \omega) \) with respect to the vertical geodesic \( \gamma_p \), respectively \( \gamma_c \), and note them by \( \tilde{v} \), respectively \( \tilde{D}(c, \sinh R) \)).

As of now, we choose \( H \) such that \( d/2 > 1 + 3 \ln \cosh \omega \).

Let \( O(t) \) be the family of horospheres in \( \mathbb{H}^3 \) such that \( O(t) \) is tangent to the horosphere \( L(t) \) at \( L(t) \cap \gamma_c \); \( O(t) \neq L(t) \).

First we will show that, if \( H \) is sufficiently small, then \( M \cap \{ \text{the region in the solid Killing cylinder over } D(c, \sinh R) \text{ with respect to } \gamma_c \} \) bounded below by \( L(e \cdot \cosh^3 \omega) \) and from above by \( O(e^{d-1}/\cosh \omega) \) is empty. To establish this result we adapt our strategy from the proof in the euclidean case; we work this out in three steps in the same spirit as in (i)-(iii) Theorem 1.

By property (v), the part of \( M \) lying above \( P_{\gamma_c}((d/2) + \ln \cosh \omega) \) is a Killing graph with respect to \( \gamma_c \). \( M \) is below \( L(e^d) \). Note by \( E \) the domain in \( L(e^d) \) bounded by \( P_{\gamma_c}((d/2) + \ln \cosh \omega) \) and \( L(e^d) \). The hyperbolic distance between this plane and this horosphere is realized on \( \gamma_c \) and equal to \( (d/2) - \ln \cosh \omega \).

(i) Let \( O(t) \) be the domain in \( O(t) \) bounded by \( M \cap O(t) \) for \( t \in [e^{d/2} \cosh \omega, e^d] \). The part of \( M \) above \( O(t) \) is also a Killing graph with respect to \( \gamma_c \). Our aim is now to show that the radius of the smallest disk in \( O(t) \) containing \( O(t) \) and centered on \( \gamma_c \) can not be too small in terms of \( h = d - \ln t \) and \( H \). Let \( M' \) be a \( H \)-Killing graph over \( O(e^{d-h}) \) with respect to \( \gamma_c \), \( \partial M' \subset O(e^{d-h}) \) and with a highest point on \( L(e^d) \) (here highest means the \( x_3 \) coordinate). Then the hyperbolic radius of the smallest disk in \( O(e^{d-h}) \) centered on \( \gamma_c \) containing strictly \( O(e^{d-h}) \) is at least

\[
\lambda(h; H) = \arccosh \left( \sqrt{\frac{H + 1}{H - 1} \left( e^{-h} - e^{-2h} \right) + e^{-h}} \right).
\]

To see this, suppose, on the contrary, that \( \partial M' \) is contained in a disk of \( O(e^{d-h}) \) of radius smaller than \( \lambda(h; H) \). Therefore \( M' \) must lie in the Killing cylinder over the disk of radius \( \lambda(h; H) \) with respect to \( \gamma_c \). Next consider the \( H \)-sphere \( S \) with center at \( \gamma_c(d - \arcoth H) \), tangent to \( L(e^d) \) at \( \gamma_c(d) = \gamma_c \cap L(e^d) \) and denoted by \( S(h) \) the part of \( S \) over \( O(e^{d-h}) \) (Fig. 7).

We will show that the hyperbolic radius of the hyperbolic circle \( \partial S(h) = S(h) \cap O(e^{d-h}) \) is exactly \( \lambda(h; H) \). Let \( a \in \gamma_c \) be the hyperbolic center of \( S \) and \( b \in \gamma_c \) the hyperbolic center of \( \partial S(h) \). When \( q \) is some point in \( \partial S(h) \), consider the geodesic triangle \( \Delta a, b, q \). The angle at \( b \) is \( \pi/2 \); by using hyperbolic trigonometry formulas [1] we obtain that

\[
cosh r_H = \cosh \text{dist}(a, b) \cdot \cosh \text{dist}(b, q)
\]

(here \( r_H \) is the radius of the \( H \)-sphere). On the other hand, the distance from \( b \) to \( O(e^{d-h}) \) is \( \ln \cosh \text{dist}(b, q) \) (see in Notation and Example above) and so \( \text{dist}(a, b) = d - h - \ln \cosh \text{dist}(b, q) \).

It is straightforward to check that

\[
\cosh^2 \text{dist}(b, q) = (2 \cosh r_H - \cosh(r_H - h) - \sinh(r_H - h))(\cosh(r_H - h) - \sinh(r_H - h))
\]

\[= e^{2r_H} \left( e^{-h} - e^{-2h} \right) + e^{-h};\]
so, by taking $H = \coth r_H$ into account, we find out that

$$\text{dist}(b, q) = \lambda(h; H).$$

(Notice that by construction $\partial S(h)$ is also the intersection $O(e^{d-h}) \cap P(d - h - \ln \cosh \lambda)$ or $O(e^{d-h}) \cap L(e^{d-h} / \cosh^2 \lambda)$.)

We continue the proof of (i). Translate $S(h)$ along $\gamma_c$ upward to be disjoint from $M'$. Now come back down; the moving $S(h)$ does not touch $M'$ before it arrives at $O(e^{d-h})$ again; one continues displacement of $S(h)$ along $\gamma_c$ and the first contact with $M'$ must occur at an interior point of $S(h)$ with a boundary point of $M'$. This means that no point of $M'$ is on $L(e^d)$ which gives a contradiction.

In the following, when we desire to use this result, there is some obstacle (quite different from the euclidean case): how one can ensure that, for fixed $h$, the part of $M$ over $O(e^{d-h})$ in $E$ has its boundary on $O(e^{d-h})$? However what we need in (iii) below, is only to find a disk in $O(e^{d-h})$ of hyperbolic radius at least $\arcsinh(\sinh R + 2 \sinh \omega)$. Since, for $h$ fixed, the largest radius on $O(e^{d-h})$ in $E$ is equal to $r(h) = \arcsinh(e^{(d/2) - h / \cosh^2 r});$ we assume up to now that $d$ is big enough (or $H$ is small enough) such that $\sqrt{\sinh r(h)} > (\sinh R + 2 \sinh \omega)$. (To evaluate $r(h)$ we apply again Notation and Example: the distance on $\gamma_c$ between $O(e^{d-h})$ and $P_c((d/2) + \ln \cosh \omega)$ which is $(d/2) - h - \ln \cosh \omega$, must be equal to $\ln \cosh r(h)$.)

The assumption above implies that if each point of $M$ in $O(e^{d-h})$ is at most a distance

$$\arcsinh((\sinh R + 2 \sinh \omega)^2)$$

from $\gamma_c$ then $M$ is a Killing graph over $O(e^{d-h})$ with boundary on $O(e^{d-h})$.

(ii) Let $\tilde{\Omega}(t)$ be the domain in $L(t)$ bounded by $M \cap L(t)$ for $t \in [e^{d/2} \cosh \omega, e^d]$ and let $D_t(\sinh r)$ be the disk in $L(t)$ centered on $\gamma_c$ of hyperbolic radius $r$. Let $r_{\max} = \inf_t \{\tilde{\Omega}(t) \subseteq D_t(\sinh r)\}$ and $r_{\min} = \sup_t \{\tilde{\Omega}(t) \supset D_t(\sinh r)\}$. We want to prove: If $\sinh r_{\max} > (2/t) \sinh \omega$ then $\sinh r_{\min} > \sinh r_{\max} - (2/t) \sinh \omega$.

By property (iv) we know that $M \setminus \{D(c, \sinh \omega) \times [1, \infty)\}$ is a graph over $\partial}\{D(c, \sinh \omega) \times [1, \infty)\}$ with respect to the horocycles in $L(t)$, $t \in [1, \infty)$, normal to $\partial \{D(c, \sinh \omega) \times [1, \infty)\}$.

As the hyperbolic metric on $L(t)$ is the Euclidean metric divided by $t$, and the hyperbolic symmetries in vertical planes induce euclidean symmetries in $L(t)$, the euclidean calculation in (ii), Proof of Theorem 1, yields (ii) here.
(iii) Now we apply (i) for \( h = \ln \cosh \omega \); so we need that
\[
\sinh r(h) = \sqrt{\frac{e^d}{\cosh^4 \omega} - 1} > (\sinh R + 2 \sinh \omega)^2
\]
and because \( \sqrt{2 \sinh d} + \sinh \omega \geq 1/\sqrt{H^2 - 1} \) we work with \( H \) such that
\[
\frac{1}{\sqrt{H^2 - 1}} > \sinh \omega + \cosh^2 \omega \sqrt{1 + (\sinh R + 2 \sinh \omega)^4}.
\]

Let \( q \) be a point in \( O(e^{d-h}) \cap M \) at maximal distance from \( \gamma_c \); (i) implies that this hyperbolic distance from \( q \) to \( \gamma_c \) is greater than
\[
r_0 = \min(\arcsinh((\sinh R + 2 \sinh \omega)^2), \lambda(\ln \cosh \omega; H)).
\]

The point \( q \) is also lying on some horizontal horosphere \( L(t_0) \) for \( t_0 \) smaller than \( e^{d-h}/\cosh^2 r_0 \) and the hyperbolic length, denoted by \( \sinh r_1 \), from \( q \) to \( \gamma_c \) in \( L(t_0) \) is greater than \( \sinh r_0 \). Now (ii) implies that there is a disk in \( L(t_0) \) centered at \( \gamma_c \cap L(t_0) \) of radius (the hyperbolic length in \( L(t_0) \)) greater than \( \sinh r_1 - (2/t_0) \sinh \omega \) and this disk is contained in the interior of the domain in \( \mathbb{H}^3 \) bounded by \( M \cup \Omega \) (Fig. 8).

Next consider the horosphere \( O(t') \) which intersects \( L(t_0) \) in the hyperbolic circle centered on \( \gamma_c \) of radius \( r_2 = \arcsinh(\sinh r_1 - (2/t_0) \sinh \omega) \) and denoted by \( O^+ \) the part of \( O(t') \) above \( L(t_0) \). When \( \mathcal{B}' \) is the domain in \( \mathbb{H}^3 \) bounded by \( L(t_0) \) and the part of \( M \) above \( L(t_0) \); we observe that \( O^+ \) is contained in \( \mathcal{B}' \). To see this we move \( O^+ \) downward to be disjoint from \( \mathcal{B}' \); then come back upward; by the maximum principle the moving \( O^+ \) can not touch \( M \) before it arrives again at its starting position.

Remark that the distance between \( O(t') \) and \( O(e^{d-h}) \) on \( \gamma_c \) is equal to \( 2 \ln(\cosh r_1/\cosh r_2) \). Since \( \sinh r_2 = \sinh r_1 - (2/t_0) \sinh \omega; t_0 = e^d/(\cosh \omega \cosh^2 r_1) \) and if we assume furthermore that \( r_1 \) is sufficiently large in terms of \( H \) and \( \omega \) then \( t' = e^d \cosh^2 r_2/(\cosh \omega \cosh^2 r_1) > e^{d-h-1} \).
Therefore the part of $M$ lying in the Killing cylinder with respect to $\gamma_c$ over the hyperbolic disk in $P_{\gamma_c}((d/2)+\ln \cosh \omega)$ of radius $r_2$ is contained in the slice between $L(e^d)$ and $O(e^{d-h-1})$.

(Since the highest points of $M$ are in the vertical geodesic cylinder over $\Omega$, $M$ has points in this domain for $r$ large enough.)

Let $P_{\gamma_c}(\tau) \subset \mathbb{H}^3$ be the family of planes orthogonal to $\gamma_c$ at $\gamma_c(\tau)$; we can apply the Alexandrov reflection technique to $M$ with $P_{\gamma_c}(\tau)$; by decreasing $r$ from $\infty$ until $\tau_0 = (d/2) + \ln \cosh \omega$ no accident will occur. The symmetry of $O(e^{d-h-1})$ through $P_{\gamma_c}(\tau_0)$ is exactly $L(e \cdot \cosh^3 \omega)$ and this implies that the intersection between $M$ and the part of the solid Killing cylinder over $D(r, \sinh r_2) \subset L(1)$ bounded below by $L(e \cdot \cosh \omega)$ and from above by $O(e^{d-1}/\cosh \omega)$ is empty.

To finish our investigation, we will choose $H$ such that $\sinh r_2 > \sinh R$. We know that

$$\sinh r_2 = \sinh r_1 - \frac{2}{t_0} \sinh \omega > \sinh r_1 - 2 \sinh \omega > \sinh r_0 - 2 \sinh \omega$$

hence we take $H$ such that $\lambda(\ln \cosh \omega; H) > \arcsinh(\sinh R + 2 \sinh \omega)$, i.e.,

$$\frac{H + 1}{H - 1} > \cosh \omega + \frac{\cosh^2 \omega}{\cosh \omega - 1} (\sinh R + 2 \sinh \omega)^2.$$

Now, in the second part of the proof, we will show that $\Omega \times [1, e \cdot \cosh^3 \omega] \subset \mathcal{B}$.

Recall that by Lemma 3 the family $C_0(t)$ of disks obtained by translating $C_0$ with respect to the vertical geodesic $\gamma_P$, (i.e. $C_0(t) = \partial D(p(t), \sinh \rho) \subset L(t)$, for $t \in [t_1, t_2] = [1, e \cdot \cosh^3 \omega]$), is contained in $\mathcal{B}$. Let $\Sigma(t)$ denote the family of the translated $\Sigma$ where $\partial \Sigma(t) \cap L(t) = C_0(t)$. Our result above implies that the upper boundary of $\Sigma(t)$ for all $t \in [t_1, t_2]$ and $\Sigma(t_2)$ are contained in $\mathcal{B}$ and therefore $\Sigma(t)$ must also lie in $\mathcal{B}$. Otherwise when one translates $\Sigma(t_2)$ down to $\Sigma(t_1)$, there would be a first point of contact of some $\Sigma(t)$ with $M$. This contact point occurs on the inner side of $M$; the mean curvature vector of both surfaces points in the same direction. This is impossible since the point of contact is an interior point of both $M$ and $\Sigma(t)$ and the mean curvature of $\Sigma(t)$ (which is equal to one) is smaller than $H$.

We know that the upper boundary component of $\Sigma + v$, for $v \in V$, at height $t_2$ is contained in $\mathcal{B}$. Hence $\Sigma + v \subset \mathcal{B}$ for each $v \in V$ by similar reasoning as above: the family $\Sigma + sv$, $s \in [0, 1]$ can have no first point of interior contact with $M$ as $s$ goes from $0$ to $1$.

Our choice of $C_0$ guarantees that for each $q \in \Gamma$, there is a $v \in V$ such that $C_0 + v$ is tangent to $\Gamma$ at $q$. The angle $\theta$ between $\Sigma$ and non compact component of $L(1) \cap C_0$ along $C_0$ is equal to $\arcsin(\cosh^{-1} \rho)$. Therefore the outer angle that $\mathcal{B}$ makes with $L(1)$ at $q$ is smaller than $\theta$; in particular $M$ stays outside the solid vertical geodesic cylinder over $\Gamma$ between $L(1)$ and $L(t_2)$.

Since the horizontal translations $\Sigma + sv$, $v \in V$, $0 \leq s < 1$ are all in $\mathcal{B}$ and $D(p, \sinh r) \times [t_1, t_2] \subset \mathcal{B}$ by Lemma 3, we conclude that $\Omega \times [t_1, t_2] \subset \mathcal{B}$. Also $M$ meets the solid Killing cylinder over $D(\gamma_c(\tau = \ln t_2), \sinh R) \subset L(t_2)$ with respect to $\gamma_c$ in a Killing graph above $O(e^{d-1}/\cosh \omega)$.

The part of $M$ in $(\Omega \times [1, \infty))$ is even a geodesic graph over $\Omega$; we find this out by coming down with planes $P_q$ orthogonal to the vertical geodesic $\gamma_q$ passing through $q$, $q$ any point in $\Omega$; and we consider the symmetries of $M$ with respect to $P_q$ until $P_q$ is below $O(e^{d-1}/\cosh \omega) \cap (\Omega \times [1, \infty))$. For $H$ small, we are far from $\Gamma$, so no accident will occur and on any geodesic $\gamma_0$ is exactly one point of $M$.

The part of $M$ outside $\Omega \times [1, \infty)$ is of genus zero, so $M$ is topologically a disk and Theorem 2 is established. $\square$
Theorem 4. Let \( \Gamma \subset L(1) \) be a strictly convex curve. If \( M \subset L \) is a compact embedded \( H \)-surface bounded by \( \Gamma \), with \( H \leq 1 \) then \( M \) is a graph over the domain \( \Omega \subset L(1) \) bounded by \( \Gamma \) with respect to the geodesics orthogonal to \( \Omega \); in particular, \( M \) is topologically a disk.

Proof. \( M \) is compact hence there exists a compact half sphere \( S^+ \) with \( \partial S^+ \) on \( L(1) \) and \( M \) below \( S^+ \). The mean curvature of \( S^+ \) is greater than \( H \). We conclude by the same argument as in the proof of Theorem 2: we are in the situation of the Small case. \( \square \)

The following proof is quite similar to the proof in the euclidean case of Lemma 2.1 in [3].

Proof of Lemma 3. Let \( \omega \) be the hyperbolic radius of a smallest hyperbolic circle such that the domain in \( L(1) \) bounded by this circle contains \( \Omega \). Note by \( c \) the point in \( \Omega \) such that \( \Gamma \subset D(c, \sinh \omega) \), and by \( \gamma(\tau) \) the vertical geodesic passing through \( c \); parametrized such that \( \tau = \text{dist}(\gamma(\tau), L(1)) \). Consider the family \( \mathcal{P}_C(\tau) \subset \mathbb{H}^3 \) of planes orthogonal to \( \gamma \) at \( \gamma(\tau) \). For \( p \in \Omega \), let \( \eta_p(\tau) \) be the orbit through \( p \) of the hyperbolic translation along \( \gamma \), i.e. the integral curve of the Killing vector field associated to the hyperbolic translation.

Apply the Alexandrov reflection technique to \( M \) with the planes \( \mathcal{P}_C(\tau) \) by decreasing \( \tau \) from \( \infty \). If we can come down to \( \mathcal{P}_C(0) \), then \( M \) is a Killing graph above \( \Omega \) and the lemma is clear. Otherwise there is a \( \tau_0 \) where the reflected surfaces with respect to \( \mathcal{P}_C(\tau_0) \) touches \( \Gamma \) for the first time at a point \( q \in \Gamma \). So \( \eta_q \) intersects \( M \) exactly once; and the segment of \( \eta_q(\tau) \) for \( \tau \in [\ln \cosh \rho_q, 2 \cdot \tau_0 - \ln \cosh \rho_q] \) is contained in \( \text{int } \mathcal{B} \) where \( \rho_q \) denote the hyperbolic distance from \( q \) to \( \gamma \) (to find the values of \( \tau \) see in Notation and Example). Also the part of \( M \) above \( \mathcal{P}_C(\tau_0) \) is a Killing graph with respect to \( \gamma \).

Next consider Alexandrov reflection with vertical planes \( \mathcal{Q} \); let \( v \) be the normal to \( \mathcal{Q} \) in \( L(1) \), \( |v| = 1 \). Suppose one can do Alexandrov reflection of \( M \), moving the plane \( \mathcal{Q} \) slightly beyond \( q \), and denote by \( J(v) \) the segment in \( \Omega \) joining \( q \) to its reflected image by this plane \( \mathcal{Q}' \). Since the part of \( M \) swept out by \( \mathcal{Q} \) is a geodesic graph over \( Q \) with respect to the geodesics orthogonal to \( \mathcal{Q} \) (property (iv)), the vertical domain \( G(v) \) bounded by \( J(v) \), the segment \( \eta_q(\tau) \) and its reflected image through \( \mathcal{Q}' \), \( \tau \in [\ln \cosh \rho_q, 2 \cdot \tau_0 - \ln \cosh \rho_q] \), and by the segment of the geodesic orthogonal to \( Q' \) joining the point \( \eta_q(2 \cdot \tau_0 - \ln \cosh \rho_q) \) to its reflected image is contained in \( \text{int } \mathcal{B} \).
Suppose we could repeat this reasoning for a family of directions $v \in \mathcal{L}(1)$, $|v| = 1$, such that, for some $p \in \mathcal{O}$ and $r > 0$, we have $D(p, \sinh r) \subseteq \bigcup_{v} J(v)$. Note by $\gamma_p$ the vertical geodesic passing through $p$ and by $P_{\gamma_p}(\tau)$ the family of planes orthogonal to $\gamma_p$; then we would have that the domain in the solid Killing cylinder over $D(p, \sinh r)$ with respect to $\gamma_p$ between $\mathcal{L}(1)$ and $P_{\gamma_p}(\tau_1)$ is also contained in int $\mathcal{B}$ where $P_{\gamma_p}(\tau_1)$ is the plane orthogonal to $\gamma_p$ that intersects $\eta_p(\tau)$ at $\tau = 2\tau_0 - \ln \cosh \rho_q$ (i.e., the point of $M$ that touches $\Gamma'$ for the first time by applying Alexandrov technique with respect to the planes $P_{\gamma}(\tau)$ above). Hence the points of $M$ in this Killing cylinder are only above $P_{\gamma_p}(\tau_1)$. Now we can apply the Alexandrov reflection technique to $M$ with the planes $P_{\gamma_p}(\tau)$; by decreasing $\tau$ from $\infty$ until $\tau_1$ no accident will occur (the plane $P_{\gamma}(\tau_0)$ is always below $P_{\gamma_p}(\tau_1)$), so the part of $M$ in the solid Killing cylinder over $D(p, \sinh r) \subseteq \mathcal{L}(1)$ is a Killing graph with respect to $\gamma_p$ as desired (see Fig. 9). So we have to understand the horizontal directions $v$ for which Alexandrov reflection goes beyond a point $q \in \Gamma'$.

First recall, that for horizontal directions $v$, one can always do Alexandrov reflection up till $\Gamma$. Let $k$ be the minimum curvature of $\Gamma$ and let $C \subseteq \mathcal{L}(1)$ be a circle of curvature $k$. So if $C$ is tangent to $\Gamma$ at $q$, then $\Gamma$ is inside $C$. Let $\rho$ be chosen so that the tubular neighborhood of $\Gamma$ of radius $\rho$ is an embedded annulus.

Then for each horizontal $v$, we can do Alexandrov reflection with vertical planes at least a distance $\rho/2$ beyond each point of $\Gamma$ and so at least a distance $\rho/2$ beyond the first time the horizontal plane meets the circle $C$. Now consider those planes which left behind $q$. This will hold for those directions in some neighborhood $V = \{v \in \mathcal{L}(1); \ |v| = 1\}$ of the inward pointing normal to $C$ at $q$. It is clear from the geometry of the circle, that $\bigcup_{v \in V} J(v)$ contains a disk $D(p, \sinh r), r > 0$ which depends on $\rho$ and $C$ but not on $q$. This completes the proof of Lemma 3. $\square$

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ANNALES SCIENTIFIQUES DE L’ÉCOLE NORMALE SUPÉRIEURE