ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4^e série, tome 33, n° 3 (2000), p. 345-359

http://www.numdam.org/item?id=ASENS_2000_4_33_3_345_0

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THE TOPOLOGY OF LARGE H -SURFACES BOUNDED BY A CONVEX CURVE

BY BEATE SEMMLER

ABSTRACT. – We shall consider embedded compact surfaces M of constant non-zero mean curvature H (H -surfaces) in hyperbolic space $\mathbb{H}^3$. Let $\mathbb{L}$ denote a horosphere of $\mathbb{H}^3$. Assume that M is contained in the horoball bounded by $\mathbb{L}$ and that the boundary of M is a strictly convex Jordan curve Γ in $\mathbb{L}$. We establish the following:

- (i) case $H > 1$. There is an $\mathfrak{H}(\Gamma)$, depending only on the geometry of Γ , such that whenever M is a H -surface bounded by Γ , with $1 < H < \mathfrak{H}(\Gamma)$, then M is topologically a disk.
- (ii) case $H \leq 1$. Then M is a graph over the domain $\Omega \subset \mathbb{L}$ bounded by Γ with respect to the geodesics orthogonal to Ω ; in particular, M is topologically a disk.

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RÉSUMÉ. – On considère une surface M plongée, compacte, à courbure moyenne constante (non nulle) dans l'espace hyperbolique $\mathbb{H}^3$. Soit $\mathbb{L}$ une horosphère de $\mathbb{H}^3$. On suppose que M est contenue dans l'horoballe bordée par $\mathbb{L}$, et que le bord de M est une courbe de Jordan strictement convexe dans $\mathbb{L}$. On établit les résultats suivants :

- (i) Cas $H > 1$. Il existe un nombre $\mathfrak{H}(\Gamma)$, qui dépende uniquement de la géométrie de Γ , tel que, quand M est une H -surface bordée par Γ , avec $1 < H < \mathfrak{H}(\Gamma)$, alors M est topologiquement un disque.
- (ii) Cas $H \leq 1$. Alors M est un graphe géodésique orthogonal au-dessus du domaine $\Omega \subset \mathbb{L}$ bordé par Γ ; en particulier, M est topologiquement un disque.

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1. Introduction

Let P be a plane in Euclidean space $\mathbb{R}^3$ and let $\mathbb{R}_+^3$ be one of the two halfspaces determined by P . Consider embedded compact surfaces M of constant non-zero mean curvature H (H -surfaces) in $\mathbb{R}_+^3$ with boundary $\partial M = \Gamma$ a convex curve in P . It is known that, if H is sufficiently small in terms of the geometry of Γ , then a H -surface M has genus zero. This result is established in [3] where they use a rescaling and a version of a compactness theorem to show this. Our proof of the same result will use another technique and will also work in the hyperbolic case. Indeed, in Hyperbolic space $\mathbb{H}^3$, homotheties do not exist, hence we can not apply the compactness theorem for H -surfaces in $\mathbb{H}^3$ proved in [2] to give a similar proof as in [3].

In this paper we shall mainly investigate the hyperbolic case to obtain a result in the same spirit as in $\mathbb{R}^3$.

Let $\mathbb{L}$ be a horosphere in $\mathbb{H}^3$ and let $\mathcal{L}$ be the horoball of $\mathbb{H}^3$ bounded by $\mathbb{L}$; the mean curvature of $\mathbb{L}$ is one and the mean curvature vector of $\mathbb{L}$ points into $\mathcal{L}$. We consider embedded compact H -surfaces M , H greater than one, in $\mathcal{L}$ with boundary $\partial M = \Gamma$ a convex curve in $\mathbb{L}$. We will

show that, if H is sufficiently close to one in terms of the geometry of Γ , then M has genus zero (Theorem 2). If M is an embedded compact H -surface in $\mathcal{L}$, bounded by Γ and $H \leq 1$, then M is a geodesic graph (Theorem 4). The case for H less than one and Γ in a hyperbolic plane is treated in [2].

2. The Euclidean case

THEOREM 1 ([3, Theorem 2]). – *Let $\Gamma \subset P = \{x_3 = 0\}$ be a strictly convex curve. There is an $\mathfrak{H}(\Gamma)$, depending only on the geometry of Γ , such that whenever $M \subset \mathbb{R}_+^3$ is a compact embedded H -surface bounded by Γ , with $0 \leq H < \mathfrak{H}(\Gamma)$, then M is topologically a disk and either M is a graph over the domain $\Omega \subset P$ bounded by Γ or $M \cap (\Omega \times [0, \infty))$ is a graph over Ω and $M \setminus (\Omega \times [0, \infty))$ is a graph over $\partial\Omega \times [0, \infty) = \Gamma \times \mathbb{R}_+$, with respect to the lines normal to $\Gamma \times \mathbb{R}_+$.*

We need the following lemma which is proved in [3]:

LEMMA 2.1 ([3]). – *Let $\Gamma \subset P$ be a strictly convex curve. There is a $r > 0$, depending only upon the extreme values of the curvature of Γ , such that whenever $M \subset \mathbb{R}_+^3$ is an H -surface with boundary Γ , there is a $p \in \Omega$ (p depends on M) such that $M \cap (D(p, r) \times [0, \infty))$ is a graph over $D(p, r)$. (Here $D(p, r)$ denotes the Euclidean disk in P centered at p , of radius r .)*

Proof of Theorem 1. – Let M be an H -surface. Let $r > 0$ and $p \in \Omega$ be given by the lemma. Let G be the unique vertical catenoid meeting P in the circle $C_0 = \partial D(p, \rho)$ where $\rho < r$ and ρ is smaller than the smallest radius of curvature of Γ (the latter condition allows us to translate C_0 horizontally in Ω so as to touch every point of Γ), and the angle between G and P along C_0 is $\pi/2$. Let $\Sigma = G \cap (P \times [0, 1])$ and let C_1 be the circle of Σ at height one. Let $V = \{v \in P \mid C_0 + v \subset \Omega\}$ and let $D(R)$ be a sufficiently large disk in P such that $C_1 + v \subset D(R) \times \{1\}$ for all $v \in V$.

We know that a highest point q of M is in $\Omega \times [0, \infty)$, and the height d of q is at most $2/H$. The part of M over $P(d/2) = \{x_3 = d/2\}$ is a vertical graph. Also $M \setminus (\Omega \times [0, \infty))$ is a graph over $\Gamma \times \mathbb{R}_+$, with respect to the lines normal to $\Gamma \times \mathbb{R}_+$, of height at most $1/H$.

Let $\mathfrak{k}$ be the smallest value of the curvature of Γ and 2ω the circumscribed diameter of Ω . Note by $\mathfrak{c}$ the point in Ω such that $\Gamma \subset D(\mathfrak{c}, \omega)$. As of now, we will work with H sufficiently small such that $H < \mathfrak{k}$ and $H < 1/2\omega$.

First of all, we will prove that, if $d < 1/H$, then M is a graph over Ω . Let $\beta(t)$, $0 \leq t < \infty$, be a line segment in $P(1/2H) = \{x_3 = 1/(2H)\}$ starting at $\beta(0) = \mathfrak{c} \times \{1/(2H)\}$. We consider a straight cylinder $Z(\tau)$ of radius $1/(2H)$ and axis α in the horizontal plane $P(1/(2H))$ where α meets $\beta(t)$ orthogonally at some $t = \tau$. Let $\tilde{Z}(\tau)$ be the half-cylinder of $Z(\tau)$ by cutting $Z(\tau)$ with a vertical plane intersecting $P(1/(2H))$ along α . We take $\tilde{Z}(\tau)$ so that $\beta(t) \cap \tilde{Z}(\tau) = \emptyset$ for $t < \tau$. For τ large, $\tilde{Z}(\tau)$ is disjoint from M . Now one can move $\tilde{Z}(\tau)$ towards M along β . By the maximum principle, as $\partial\tilde{Z} \cap P$ approaches Γ by horizontal translation, the first contact with M can not be at an interior point of M . Therefore no accident will occur before reaching Γ . This implies that the diameter of a smallest disk centered at $\mathfrak{c} \times \{t\}$ that contains $M \cap \{x_3 = h\}$ is smaller than $2\omega + (1/H)$ for $0 \leq h \leq 1/H$. Let S^+ be the upper hemisphere of the sphere of mean curvature H centered at $\mathfrak{c} \times \{1/H\}$. Translate S^+ downward, so the moving S^+ does not touch M before it arrives at P , i.e., M is below S^+ when ∂S^+ is on P . Then by the maximum principle and because $H < \mathfrak{k}$, one can translate S^+ horizontally to touch every point of Γ and that is why $M \subset \Omega \times [0, \infty)$. Hence M is a graph over Ω .

Henceforth we assume that $d \geq 1/H$. The part of M over $P(d/2)$ is a vertical graph.

(i) If an H -graph M' over a domain $\mathfrak{D}$ in the plane $P(t)$ where $\partial M' \subset P(t)$ has height h , then the radius of the smallest disk in $P(t)$ containing strictly $\mathfrak{D}$ is at least $\lambda(h; H) = \sqrt{(2h/H) - h^2}$. To see this, suppose, on the contrary, that the domain $\mathfrak{D}$ is contained in a disk $D(c, \tilde{r}) \subset P(t)$ where $\tilde{r} < \lambda(h; H)$. Let S be the H -sphere centered at $c \times \{t\}$ and denoted by $S(h)$ the part of S over the plane $P(t + (1/H) - h)$. M' is contained in the vertical cylinder over $\mathfrak{D}$ and the radius of $\partial S(h)$ is strictly greater than $\tilde{r}$, so by moving $S(h)$ towards M' the first contact with M' must occur at an interior point of $S(h)$ with a boundary point of M' . This means that the height of M' is less than h which gives a contradiction.

(ii) Let $\Omega(t)$ be the domain in $P(t)$ bounded by $M \cap P(t)$ for $t \in [d/2, d]$, and let $D_t(r)$ be the disk in $P(t)$ centered at $c_t = c \times \{t\}$, of radius r .

Let $r_{\max} = \inf_r \{ \Omega(t) \subseteq D_t(r) \}$ and $r_{\min} = \sup_r \{ \Omega(t) \supset D_t(r) \}$.

We want to prove: If $r_{\max} > 2\omega$ then $r_{\min} > r_{\max} - 2\omega$.

We know that $M \setminus (D(c, \omega) \times [0, \infty))$ is a graph over $\partial(D(c, \omega) \times [0, \infty))$, with respect to the lines normal to $\partial D(c, \omega) \times \mathbb{R}_+$. So, for some point μ in $\partial D_t(r_{\max}) \cap M$, we consider all reflection by vertical planes and looking at the set of images of μ in $P(t)$. This set is contained in the interior of the domain in $\mathbb{R}_+^3$ bounded by $M \cup \Omega$; in particular in $\Omega(t)$ since each vertical plane is orthogonal to $P(t)$ and so the symmetry with respect to vertical planes leaves $P(t)$ invariant.

Doing elementary calculations, we see that the set of images of μ contains the disk $D_t(r_{\max} - 2\omega)$. In more detail, denoted by β_0 the half geodesic in $P(t)$ starting at c_t and passing through μ , and by β_ϕ the half geodesic in $P(t)$ where the angle between β_0 and β_ϕ at μ is ϕ , $|\phi| \in [0, \arccos(\omega/r_{\max})]$. Consider the family of vertical planes $V_\phi(s)$ orthogonal to β_ϕ at $\beta_\phi(s + \omega)$ (Fig. 1). Apply the Alexandrov reflection technique to M with the planes $V_\phi(s)$; by decreasing s from ∞ , no accident will occur up till $\partial D_t(\omega)$; i.e. $s = 0$. When μ' denotes the reflected image of μ with respect to the plane $V_\phi(0)$, the line segment l joining μ to μ' is contained in $\Omega(t)$.

Now we have $\cos \phi = (\omega + \text{dist}(\mu, V_\phi(0)))/r_{\max}$ and $\sin \phi = \text{dist}(c_t, l)/r_{\max}$.

The distance from μ' to c_t is equal to

$$x(\phi) = \text{dist}(\mu', c_t) = \sqrt{(\text{dist}(\mu, V_\phi(0)) - \omega)^2 + \text{dist}^2(c_t, l)},$$

Fig. 1.

therefore $x(\phi) = \sqrt{r_{\max}^2 + 4\omega^2 - 4\omega r_{\max} \cos \phi}$, thus $x(\phi) \geq r_{\max} - 2\omega$ and this implies $r_{\min} > r_{\max} - 2\omega$.

(iii) Now we are able to show that, if H is sufficiently small, then $M \cap \{D(R) \times [1, d-1]\} = \emptyset$ and $M \cap \{P \times [d-1, d]\}$ is a graph over $P(d-1)$. For $h = 1$, we get from (i) that $r_{\max}$ is at least $\sqrt{(2/H)-1}$ on $P(d-1)$; to apply (i) we need $d/2 > 1$ so we work with H such that $1/(2H) > 1$. From (ii), by assuming $1/H > (1/2) + 2\omega^2$, it follows that $r_{\min} + 2\omega \geq \sqrt{(2/H)-1}$. Therefore, if $1/H > (1/2)\{(R+4\omega)^2 + 1\}$ then $r_{\min} > R + 2\omega$.

Set

$$\mathfrak{h} = \min\left(\mathfrak{k}; \frac{1}{2}; \left(\frac{1}{2}\{(R+4\omega)^2 + 1\}\right)^{-1}\right).$$

The end of the proof is the same as in [3]. $\square$

3. The hyperbolic case

We work in the upper half-space model of hyperbolic space, that is,

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$$

with the hyperbolic metric, i.e. the Euclidean metric divided by x_3 . In the following, we will represent by dist the hyperbolic geodesic distance in $\mathbb{H}^3$; τ will be the hyperbolic parameter of arc length (in general used for geodesics and planes) and t the euclidean parameter associated with the model (used for horospheres).

Let $\mathbb{L}(t)$ denote the horizontal horosphere $\{x_3 = t\}$ and let $\mathfrak{L}$ be the non compact component of $\mathbb{H}^3$ bounded by $\mathbb{L}(1)$ such that the mean curvature vector of $\mathbb{L}(1)$ points towards $\mathfrak{L}$.

THEOREM 2. – *Let $\Gamma \subset \mathbb{L}(1)$ be a strictly convex curve. There is an $\mathfrak{H}(\Gamma)$, depending only on the geometry of Γ , such that whenever $M \subset \mathfrak{L}$ is a compact embedded H -surface bounded by Γ , with $1 < H < \mathfrak{H}(\Gamma)$, then M is topologically a disk and either M is a graph over the domain $\Omega \subset \mathbb{L}(1)$ bounded by Γ with respect to the geodesics orthogonal to Ω or $M \cap (\Omega \times [1, \infty))$ is a geodesic graph over Ω and $M \setminus (\Omega \times [1, \infty))$ is a graph over $\partial\Omega \times [1, \infty) = \Gamma \times [1, \infty)$, with respect to the geodesics orthogonal to $\Gamma \times [1, \infty)$.*

3.1. Properties of compact surfaces in a horoball

Before proving Theorem 2, we give a representative example of hyperbolic calculations, we establish some basic properties of an H -surface as in Theorem 2 and we state a lemma whose proof we will give later.

Notation and Example. Let $q \in \mathbb{L}(1)$ be the point $(0, 0, 1)$ and let $\gamma(\tau) \subset \mathfrak{L}$ denote the vertical geodesic through q orthogonal to $\mathbb{L}(1)$ parametrized such that $\tau = \text{dist}(\gamma(\tau), \mathbb{L}(1))$. Consider the family $P_\gamma(\tau) \subset \mathbb{H}^3$ of planes orthogonal to γ at $\gamma(\tau)$. Let $p \notin \gamma(\tau)$ be a point in some $\mathbb{L}(t)$, $t > 1$, denoted by R the geodesic distance from p to $\gamma(\tau)$ and by α the angle between the x_3 -axis and the euclidean line joining $(0, 0, 0)$ to p ; (Fig. 2).

$\mathbb{L}(t)$ intersects $\gamma(\tau)$ at $\tau = \ln t$. R is related to α by $\tan \alpha = \sinh R$; since the hyperbolic metric on $\mathbb{L}(t)$ is the euclidean metric divided by t , the hyperbolic length from p to $\gamma(\ln t)$ in $\mathbb{L}(t)$ is equal to $\sinh R$ (notice that the geodesic distance from p to $\gamma(\ln t)$ which is $2\text{arcsinh}(\sinh R/2)$, is naturally smaller than the former). The geodesic passing through p and realizing the distance R from p to $\gamma(\tau)$ is lying on $P_\gamma(\ln(t \cosh R))$ and this implies that the length of the segment of γ joining $\mathbb{L}(t)$ to this plane is equal to $\ln \cosh R$. The intersection between $\mathbb{L}(t)$ and $P_\gamma(\ln(t \cosh R))$

Fig. 2.

is a hyperbolic circle C with hyperbolic center at $\gamma(\ln(t \cosh R))$, of hyperbolic radius R . By hyperbolic reflection with respect to $P_\gamma(\ln(t \cosh R))$, the image of $L(t)$ is a horosphere, denoted by $O(t \cosh^2 R)$, containing C and intersecting $\gamma(\tau)$ at $\tau = \ln(t \cosh^2 R)$; so the distance between both horospheres on $\gamma(\tau)$ is $2 \ln \cosh R$.

Basic properties. Let M be defined as in Theorem 2. Let $\mathfrak{B}$ be the compact component of $\mathfrak{L}$ bounded by M and the domain $\Omega \subset L(1)$ such that $\partial\Omega = \partial M$. Let $\mathbf{H}$ be the mean curvature vector of M ; we orient M by $\mathbf{H}$. Then:

- (i) $\mathbf{H}$ points towards $\mathfrak{U} = \mathfrak{B} \cup (\Omega \times (0, 1])$.
- (ii) Each point $q \in M$ at maximal distance from $L(1)$ is contained in the solid vertical geodesic cylinder over Ω denoted by $\mathfrak{C}$.
- (iii) Let γ be any geodesic orthogonal to $L(1)$ passing through a point of Ω ; if M is contained in the solid Killing cylinder over Ω with respect to γ (i.e. the integral curves of the Killing vector field associated to the hyperbolic translation along γ) then M is a Killing graph over Ω with respect to γ .
- (iv) $M \setminus (\Omega \times [1, \infty))$ is a graph over $\Gamma \times [1, \infty)$, with respect to the geodesics orthogonal to $\Gamma \times [1, \infty)$; this part of M outside $\mathfrak{C}$ is also a graph over $\Gamma \times [1, \infty)$ with respect to the horospheres in $L(t)$, $t \in [1, \infty)$, normal to $\Gamma \times [1, \infty)$.
- (v) Let $q \in M$ be a point at maximal distance d from $L(1)$ and let $\gamma(\tau) \subset \mathfrak{L}$ be the geodesic through q orthogonal to $L(1)$ parametrized such that $\tau = \text{dist}(\gamma(\tau), L(1))$. Consider the family $P_\gamma(\tau) \subset \mathbb{H}^3$ of planes orthogonal to γ at $\gamma(\tau)$. Let $R = \max_{p \in \Gamma} \text{dist}(p, \gamma)$. Then the part of M lying above $P_\gamma((d/2) + \ln \cosh R)$ is a Killing graph with respect to γ .

Proof. – (i) Consider the family of horospheres $L(t) = \{x_3 = t\}$; if t is large enough, then $L(t) \cap M = \emptyset$; decrease t and consider the first horosphere that touches M . At this point of contact, the mean curvature vector of $L(t)$ points upward and since the mean curvature of $L(t)$

Fig. 3.

(which is equal to one) is smaller than the mean curvature of M , the maximum principle implies that $\mathbf{H}$ points towards $\mathcal{U}$, hence the same is true at each point of M .

(ii) Let $d = \text{dist}(q, \mathbb{L}(1))$ and let γ be the geodesic through q orthogonal to $\mathbb{L}(1)$. Suppose, on the contrary, that $q \notin \mathcal{C}$ so $q_0 = \gamma \cap \mathbb{L}(1)$ is not in Ω . Let h be a half horocycle in $\mathbb{L}(1)$ starting at a point p_0 of Γ passing through q_0 and such that $\text{dist}(p_0, q_0) = \inf_{p \in \Gamma} \text{dist}(p, q_0)$. Now consider the unique geodesic plane $E \subset \mathbb{H}^3 \setminus \mathcal{L}$ tangent to $\mathbb{L}(1)$ at p_0 ; note by β the half geodesic in E , starting at p_0 , which is contained in the vertical half plane determined by h and q . Let $P(\tau)$ be the family of planes in $\mathbb{H}^3$, $0 \leq \tau < \infty$, such that for each point b of β , there exists one $P(\tau)$ intersecting β orthogonally at b . Parametrize so that $P(0)$ contains the initial point p_0 of β (Fig. 3).

Apply the Alexandrov reflection technique to M with the planes $P(\tau)$ (cf. [5]). For τ large, $P(\tau)$ is disjoint from M . Now, if we approach M by $P(\tau)$, there will be a first contact point of some $P(\tau)$ with M . One continues to decrease τ and considers the symmetry of the part of M swept out by $P(\tau)$ with respect to $P(\tau)$. These symmetries of M are in $\mathfrak{B}$. Notice that the symmetry through $P(\tau)$ of the relevant part of $\mathbb{L}(1)$ is contained in $\mathcal{L}$. (Here relevant part means the part of $\mathbb{L}(1)$ lying on the same side of $P(\tau)$ as the part of M swept out by $P(\tau)$.) So, by the maximum principle, no accident can occur until $P(0)$ and the part of M in question is a Killing graph over $P(0)$ with respect to the geodesic β (the integral curves of the Killing vector field associated to the hyperbolic translation along β are invariant by reflection with respect to $P(\tau)$). But the Killing segment joining q to $P(0)$ and its symmetry through $P(0)$ are lying above $\mathbb{L}(e^d)$ whereas $\mathfrak{B}$ is below this horosphere which gives a contradiction and therefore q must be in $\mathcal{C}$.

(iii) In this case we can do Alexandrov reflection with the family of planes orthogonal to γ until a plane below $\mathbb{L}(1)$ without any accident, so M is a Killing graph over $\mathbb{L}(1)$.

(iv) Let γ_p be a geodesic through a point p of Γ orthogonal to $\mathbb{L}(1)$ at p and let T_p be the vertical plane tangent to Γ at p . Consider any half geodesic β_g orthogonal to T_p at some point g of γ_p where Γ and β_g are on opposite sides of T_p . As in (ii), we apply the Alexandrov reflection technique to M with the family $P(\tau)$ of planes orthogonal to β_g . Therefore the relevant part of M is a Killing graph over $P(0) = T_p$ with respect to the geodesic β_g .

We can do this for each point g of γ_p and each such half geodesic β_g ; and also for γ_p associated to each point p of Γ ; this means that on each geodesic orthogonal to $\Gamma \times [1, \infty)$ there is only one point of M , hence $M \setminus (\Omega \times [1, \infty))$ is a geodesic graph and the first assertion of (iv) follows.

Let p be a point of Γ and let T_p be as above. Note by $h(s)$ the half horocycle in $\mathbb{L}(1)$ starting at $p = h(0)$ and orthogonal to $T_p \cup \mathbb{L}(1)$ at p where Γ and $h(s)$ are on opposite sides of T_p . Let $T(s)$ be a family of vertical planes such that $T(s)$ intersects h orthogonally at $h(s)$ and

$T(0) = T_p$. Apply the Alexandrov reflection process to M and the planes $T(s)$. Notice that hyperbolic symmetry through each $T(s)$ leaves $\mathbb{L}(1)$ and all horocycles orthogonal to $T(s)$ in all $\mathbb{L}(t)$ invariant. One can translate $T(s)$ along $\mathfrak{h}$ until $\partial\Omega = \Gamma$ and the part of M swept out by $T(0) = T_p$ is a graph over T_p with respect to the horocycle orthogonal to T_p .

(v) From (ii) we know that q lies in the solid geodesic cylinder $\mathfrak{C}$ over Ω . We apply the Alexandrov reflection technique to M with the planes $P_\gamma(\tau)$; the first accident occurs when the image of an interior point p_i of M touches Γ . This point p_i is situated on an integral curve of the Killing vector field with respect to $\gamma(\tau)$ over Γ ; the Killing coordinate of such point p_i is at most $d + \ln \cosh R$ where $R = \max_{p \in \Gamma} \text{dist}(p, \gamma)$ (see *Notation and Example*). Therefore the result follows. $\square$

We will prove the following Lemma 3 after the proof of Theorem 4.

LEMMA 3. – *Let $\Gamma \subset \mathbb{L}(1)$ be a strictly convex curve. There is a $r > 0$, depending only upon the extreme values of the curvature of Γ , such that whenever $M \subset \mathfrak{L}$ is an H -surface, $H > 1$, with boundary Γ , there is a $p \in \Omega$ (p depends on M) such that the part of M in the solid Killing cylinder over $D(p, \sinh r) \subset \mathbb{L}(1)$ with respect to the vertical geodesic γ_p passing through p is a Killing graph over $D(p, \sinh r)$ with respect to γ_p .*

(Here $D(p, \sinh r)$ denotes the disk in $\mathbb{L}(1)$ centered at p such that $\partial D(p, \sinh r)$ is the hyperbolic circle centered at $\gamma_p(\tau = \ln \cosh r)$ of hyperbolic radius r .)

3.2. Proof of the main result

Proof of Theorem 2. – Let M be an H -surface as in Theorem 2. Let ω be the hyperbolic radius of a smallest hyperbolic circle such that the domain in $\mathbb{L}(1)$ bounded by this circle contains Ω . Note by $\mathfrak{c}$ the point in Ω such that $\Gamma \subset D(\mathfrak{c}, \sinh \omega)$, and by $\gamma_{\mathfrak{c}}(\tau)$ the vertical geodesic passing through $\mathfrak{c}$; parametrized such that $\tau = \text{dist}(\gamma_{\mathfrak{c}}(\tau), \mathbb{L}(1))$.

By property (ii) we know that the points at maximal distance d from $\mathbb{L}(1)$ are contained in the solid vertical geodesic cylinder $\mathfrak{C}$ over Ω . In the proof of property (iv), we saw that, if T_p is a vertical plane tangent to Γ at a point $p \in \Gamma$, then the part of M in the half space determined by T_p which does not contain Γ is a Killing graph over T_p with respect to any geodesic orthogonal to T_p at some point of $T_p \cap \mathfrak{C}$.

The same is still true if we choose some point p' in $\partial D(\mathfrak{c}, \sinh \omega) \subset \mathbb{L}(1)$, $T_{p'}$ the vertical plane tangent to $\partial D(\mathfrak{c}, \sinh \omega)$ at p' and $\beta_{p'}$ the geodesic orthogonal to $T_{p'}$ at p' . Now consider the point g in $\mathbb{L}(1)$ where the Killing segment k (with respect to $\beta_{p'}$) that joins the point $T_{p'} \cap \{D(\mathfrak{c}, \sinh \omega) \times [1, \infty)\} \cap \mathbb{L}(e^d) = p' \times \{e^d\}$, intersects $\mathbb{L}(1)$ (Fig. 4).

We want to evaluate the hyperbolic distance κ between g and the geodesic $\gamma_{p'} = T_{p'} \cap \{D(\mathfrak{c}, \sinh \omega) \times [1, \infty)\}$. Recall that $\sinh$ of κ is the hyperbolic length in $\mathbb{L}(1)$ from g to $\gamma_{p'} \cap \mathbb{L}(1) = p'$ and this value is also equal to the euclidean distance between g and p' in $\mathbb{L}(1)$. Let a be the euclidean center of the Killing segment k (this makes sense since k looks like a part of a circle) and let b be the euclidean radius of k . We have $b^2 = 1 + (e^d - b)^2$ and $\sinh^2 \kappa = b^2 - (e^d - b - 1)^2$. Therefore $\sinh \kappa$ is equal to $\sqrt{2 \sinh d}$.

Since the part of M outside the vertical geodesic cylinder over $D(\mathfrak{c}, \sinh \omega) \subset \mathbb{L}(1)$ is a Killing graph with respect to $\beta_{p'}$ and $T_{p'}$ for each point $p' \in \partial D(\mathfrak{c}, \sinh \omega)$, no point of M in the vertical half plane containing $\beta_{p'}$ with boundary $\gamma_{p'}$ can be a distance greater than κ from $\gamma_{p'}$, for each p' . This implies that M is contained in the solid Killing cylinder over $D(\mathfrak{c}, \sqrt{2 \sinh d} + \sinh \omega) \subset \mathbb{L}(1)$ with respect to $\gamma_{\mathfrak{c}}$.

Now we will distinguish two cases.

Fig. 4.

Fig. 5.

(1) Small case

Suppose that $\sqrt{2 \sinh d} + \sinh \omega$ is strictly smaller than $\sinh r_H$ where r_H is the radius of the sphere of mean curvature H .

We will first see that M stays inside in the Killing cylinder with respect to $\gamma_c(\tau)$ over the domain in $L(1)$ bounded by $D(c, \sinh \omega)$. Let S^+ be the upper hemisphere of the H -sphere centered at $\gamma_c(d + \ln \cosh(r_H))$. Translate S^+ downward, so the moving S^+ does not touch M before it arrives at $L(1)$, i.e., M is below S^+ when ∂S^+ is on $L(1)$ (Fig. 5).

Next consider the family $S^+(\tau)$ of upper half spheres with center at $\gamma_c(\tau)$ for $\tau \in [\tau_0, \tau_1] = [\ln \cosh \omega, \ln \cosh r_H]$ and $\partial S^+(\tau)$ on $L(1)$. This continuous family consists of surfaces in $\mathcal{L}$ where the mean curvature starts from $\coth \omega$, decreases to $\coth r_H = H$ and where $\partial S^+(\tau)$ is a foliation of the compact region in $L(1)$ bounded by $\partial D(c, \sinh r_H) \cup \partial D(c, \sinh \omega)$. So, for each τ , Γ is contained in the domain of $L(1)$ bounded by $\partial S^+(\tau)$. Since M is below $S^+ = S^+(\tau_1)$ and when we decrease τ from τ_1 to τ_0 , the maximum principle implies that M is still below

Fig. 6.

$S^+(\tau_0)$, the upper half sphere of radius ω . Therefore M is contained in the Killing cylinder with respect to γ_c over $\partial D(c, \sinh \omega) \subset \mathbb{L}(1)$.

Our aim is now to show that, if H is sufficiently small in terms of ω then M is even contained in the Killing cylinder over Ω with respect to γ_c and so we can conclude by property (iii) that M is a Killing graph over Ω .

Let $\delta = \sup_r \{ \Omega \supset D(c, \sinh r) \}$. To establish the result we consider the family $Z(\tau)$ of Killing cylinders over $\partial D(c, \sinh \tau) \subset \mathbb{L}(1)$ with respect to γ_c for $\tau \in [\delta, \omega]$. The mean curvature vector of $Z(\tau)$ points into the component of $\mathbb{H}^3$ bounded by $Z(\tau)$ which contains γ_c and the mean curvature varies continuously in τ from $\coth(2\delta)$ decreasing to $\coth(2\omega)$ (Fig. 6).

Now, suppose on the contrary, that M is not in the solid Killing cylinder over Ω . Since M is lying in $Z(\omega)$ and by decreasing τ from ω to δ there will be some τ_0 where $Z(\tau_0)$ touches M for the first time at an interior point of M such that $Z(\tau_0)$ is tangent to M and the mean curvature vector of both surfaces points in the same direction. However, if $H < \coth 2\tau_0$, this is impossible by the maximum principle. Therefore M is contained in the Killing cylinder over Ω with respect to γ_c for H smaller than $\coth(2\omega)$.

Thus M is a Killing graph over Ω with respect to γ_c .

To finish our investigation for small H -surfaces, we will show that M is even a geodesic graph over Ω with respect to the geodesics orthogonal to Ω . Let p be a point in Ω and let γ_p be the vertical geodesic passing through p . Since M is below the upper half sphere of radius ω centered on γ_c with boundary $D(c, \sinh \omega) \subset \mathbb{L}(1)$, M is also below the upper half sphere of radius 2ω centered on γ_p with boundary in $\mathbb{L}(1)$. We consider Killing cylinders with axes γ_p and conclude by the same argument as before that M is a Killing graph over Ω with respect to γ_p . We can repeat this for each point p in Ω ; this means that on each vertical geodesic there is only one point of M , hence M is a geodesic graph over Ω ; in particular M is topologically a disk.

(2) Large case

Henceforth we assume that d is bounded from below in terms of H , i.e.

$$\sqrt{2 \sinh d} + \sinh \omega \geq \sinh r_H = \frac{1}{\sqrt{H^2 - 1}}.$$

Let $r > 0$ and $p \in \Omega$ be given by Lemma 3. Let G be the unique vertical catenoid cousin meeting $\mathbb{L}(1)$ in the circle $C_0 = \partial D(p, \sinh \rho)$ where $\rho < r$ and $\sinh \rho$ is smaller than the smallest

radius of curvature of Γ in $\mathbb{L}(1)$ (the latter condition allows us to translate C_0 horizontally in Ω so as to touch every point of Γ), and G has its waist at $\mathbb{L}(1)$ (see [4] for catenoid cousins).

Let $\Sigma = G \cap (\mathbb{L}(1) \times [1, x_3 = e \cdot \cosh^3 \omega])$ and let C_1 be the circle of Σ at euclidean height $\{x_3 = e \cdot \cosh^3 \omega\}$. (The hyperbolic height of Σ is equal to $1 + 3 \ln \cosh \omega$.) Σ is a Killing graph with respect to γ_p over the non compact component of $\mathbb{L}(1) \cap C_0$. Let $V = \{v \in \mathbb{L}(1) \mid C_0 + v \subset \Omega\}$ and let $D(c, \sinh R)$ be a sufficiently large disk in $\mathbb{L}(1)$ centered at c such that $C_1 + \tilde{v} \subset \tilde{D}(c, \sinh R) \times \{x_3 = e \cdot \cosh^3 \omega\}$ for all $v \in V$ (here, we translate the hyperbolic objects v , respectively $D(c, \sinh R)$, from $\mathbb{L}(1)$ to $\mathbb{L}(e \cdot \cosh^3 \omega)$ with respect to the vertical geodesic γ_p , respectively γ_c , and note them by $\tilde{v}$, respectively $\tilde{D}(c, \sinh R)$).

As of now, we choose H such that $d/2 > 1 + 3 \ln \cosh \omega$.

Let $O(t)$ be the family of horospheres in $\mathbb{H}^3$ such that $O(t)$ is tangent to the horosphere $\mathbb{L}(t)$ at $\mathbb{L}(t) \cap \gamma_c$; $O(t) \neq \mathbb{L}(t)$.

First we will show that, if H is sufficiently small, then $M \cap \{\text{the region in the solid Killing cylinder over } D(c, \sinh R) \text{ with respect to } \gamma_c \text{ bounded below by } \mathbb{L}(e \cdot \cosh^3 \omega) \text{ and from above by } O(e^{d-1}/\cosh \omega)\}$ is empty. To establish this result we adapt our strategy from the proof in the euclidean case; we work this out in three steps in the same spirit as in (i)–(iii) Theorem 1.

By property (v), the part of M lying above $P_{\gamma_c}((d/2) + \ln \cosh \omega)$ is a Killing graph with respect to γ_c . M is below $\mathbb{L}(e^d)$. Note by E the domain in $\mathcal{L}$ bounded by $P_{\gamma_c}((d/2) + \ln \cosh \omega)$ and $\mathbb{L}(e^d)$. The hyperbolic distance between this plane and this horosphere is realized on γ_c and equal to $(d/2) - \ln \cosh \omega$.

(i) Let $\Omega(t)$ be the domain in $O(t)$ bounded by $M \cap O(t)$ for $t \in [e^{d/2} \cosh \omega, e^d]$. The part of $M \cap E$ above $O(t)$ is also a Killing graph with respect to γ_c . Our aim is now to show that the radius of the smallest disk in $O(t)$ containing $\Omega(t)$ and centered on γ_c can not be *too small* in terms of $h = d - \ln t$ and H . Let M' be a H -Killing graph over $O(e^{d-h})$ with respect to γ_c , $\partial M' \subset O(e^{d-h})$ and with a highest point on $\mathbb{L}(e^d)$ (here highest means the x_3 coordinate). Then the hyperbolic radius of the smallest disk in $O(e^{d-h})$ centered on γ_c containing strictly $\Omega(e^{d-h})$ is at least

$$\lambda(h; H) = \operatorname{arcosh} \left(\sqrt{\frac{H+1}{H-1} (e^{-h} - e^{-2h})} + e^{-h} \right).$$

To see this, suppose, on the contrary, that $\partial M'$ is contained in a disk of $O(e^{d-h})$ of radius smaller than $\lambda(h; H)$. Therefore M' must lie in the Killing cylinder over the disk of radius $\lambda(h; H)$ with respect to γ_c . Next consider the H -sphere S with center at $\gamma_c(d - \operatorname{arccoth} H)$, tangent to $\mathbb{L}(e^d)$ at $\gamma_c(d) = \gamma_c \cap \mathbb{L}(e^d)$ and denoted by $S(h)$ the part of S over $O(e^{d-h})$ (Fig. 7).

We will show that the hyperbolic radius of the hyperbolic circle $\partial S(h) = S(h) \cap O(e^{d-h})$ is exactly $\lambda(h; H)$. Let $a \in \gamma_c$ be the hyperbolic center of S and $b \in \gamma_c$ the hyperbolic center of $\partial S(h)$. When q is some point in $\partial S(h)$, consider the geodesic triangle $\triangle a, b, q$. The angle at b is $\pi/2$; by using hyperbolic trigonometry formulas [1] we obtain that

$$\cosh r_H = \cosh \operatorname{dist}(a, b) \cdot \cosh \operatorname{dist}(b, q)$$

(here r_H is the radius of the H -sphere). On the other hand, the distance from b to $O(e^{d-h})$ is $\ln \cosh \operatorname{dist}(b, q)$ (see in *Notation and Example* above) and so $\operatorname{dist}(a, b) = d - h - \ln \cosh \operatorname{dist}(b, q)$. It is straightforward to check that

$$\begin{aligned} & \cosh^2 \operatorname{dist}(b, q) \\ &= (2 \cosh r_H - \cosh(r_H - h) - \sinh(r_H - h)) (\cosh(r_H - h) - \sinh(r_H - h)) \\ &= e^{2r_H} (e^{-h} - e^{-2h}) + e^{-h}; \end{aligned}$$

(iii) Now we apply (i) for $h = \ln \cosh \omega$; so we need that

$$\sinh r(h) = \sqrt{\frac{e^d}{\cosh^4 \omega} - 1} > (\sinh R + 2 \sinh \omega)^2$$

and because $\sqrt{2 \sinh d} + \sinh \omega \geq 1/\sqrt{H^2 - 1}$ we work with H such that

$$\frac{1}{\sqrt{H^2 - 1}} > \sinh \omega + \cosh^2 \omega \sqrt{1 + (\sinh R + 2 \sinh \omega)^4}.$$

Let q be a point in $O(e^{d-h}) \cap M$ at maximal distance from γ_c ; (i) implies that this hyperbolic distance from q to γ_c is greater than

$$r_0 = \min(\operatorname{arcsinh}((\sinh R + 2 \sinh \omega)^2), \lambda(\ln \cosh \omega; H)).$$

The point q is also lying on some horizontal horosphere $L(t_0)$ for t_0 smaller than $e^{d-h}/\cosh^2 r_0$ and the hyperbolic length, denoted by $\sinh r_1$, from q to γ_c in $L(t_0)$ is greater than $\sinh r_0$. Now (ii) implies that there is a disk in $L(t_0)$ centered at $\gamma_c \cap L(t_0)$ of radius (the hyperbolic length in $L(t_0)$) greater than $\sinh r_1 - (2/t_0) \sinh \omega$ and this disk is contained in the interior of the domain in $\mathbb{H}^3$ bounded by $M \cup \Omega$ (Fig. 8).

Next consider the horosphere $O(t')$ which intersects $L(t_0)$ in the hyperbolic circle centered on γ_c of radius $r_2 = \operatorname{arcsinh}\{\sinh r_1 - (2/t_0) \sinh \omega\}$ and denoted by O^+ the part of $O(t')$ above $L(t_0)$. When $\mathfrak{B}'$ is the domain in $\mathbb{H}^3$ bounded by $L(t_0)$ and the part of M above $L(t_0)$; we observe that O^+ is contained in $\mathfrak{B}'$. To see this we move O^+ downward to be disjoint from $\mathfrak{B}'$; then come back upward; by the maximum principle the moving O^+ can not touch M before it arrives again at its starting position.

Remark that the distance between $O(t')$ and $O(e^{d-h})$ on γ_c is equal to $2 \ln(\cosh r_1 / \cosh r_2)$. Since $\sinh r_2 = \sinh r_1 - (2/t_0) \sinh \omega$; $t_0 = e^d / (\cosh \omega \cosh^2 r_1)$ and if we assume furthermore that r_1 is sufficiently large in terms of H and ω then $t' = e^d \cosh^2 r_2 / (\cosh \omega \cosh^2 r_1) > e^{d-h-1}$.

Fig. 8.

Therefore the part of M lying in the Killing cylinder with respect to γ_c over the hyperbolic disk in $P_{\gamma_c}((d/2) + \ln \cosh \omega)$ of radius r_2 is contained in the slice between $\mathbb{L}(e^d)$ and $O(e^{d-h-1})$. (Since the highest points of M are in the vertical geodesic cylinder over Ω , M has points in this domain for r_2 large enough.)

Let $P_{\gamma_c}(\tau) \subset \mathbb{H}^3$ be the family of planes orthogonal to γ_c at $\gamma_c(\tau)$; we can apply the Alexandrov reflection technique to M with $P_{\gamma_c}(\tau)$; by decreasing τ from ∞ until $\tau_0 = (d/2) + \ln \cosh \omega$ no accident will occur. The symmetry of $O(e^{d-h-1})$ through $P_{\gamma_c}(\tau_0)$ is exactly $\mathbb{L}(e \cdot \cosh^3 \omega)$ and this implies that the intersection between M and the part of the solid Killing cylinder over $D(c, \sinh r_2) \subset \mathbb{L}(1)$ bounded below by $\mathbb{L}(e \cdot \cosh^3 \omega)$ and from above by $O(e^{d-1}/\cosh \omega)$ is empty.

To finish our investigation, we will choose H such that $\sinh r_2 > \sinh R$. We know that

$$\sinh r_2 = \sinh r_1 - \frac{2}{t_0} \sinh \omega > \sinh r_1 - 2 \sinh \omega > \sinh r_0 - 2 \sinh \omega$$

hence we take H such that $\lambda(\ln \cosh \omega; H) > \operatorname{arcsinh}(\sinh R + 2 \sinh \omega)$, i.e.,

$$\frac{H+1}{H-1} > \cosh \omega + \frac{\cosh^2 \omega}{\cosh \omega - 1} (\sinh R + 2 \sinh \omega)^2.$$

Now, in the second part of the proof, we will show that $\Omega \times [1, (e \cdot \cosh^3 \omega)] \subset \mathfrak{B}$.

Recall that by Lemma 3 the family $C_0(t)$ of disks obtained by translating C_0 with respect to the vertical geodesic γ_p , (i.e. $C_0(t) = \partial D(p(t), \sinh \rho) \subset \mathbb{L}(t)$, for $t \in [t_1, t_2] = [1, e \cdot \cosh^3 \omega]$), is contained in $\mathfrak{B}$. Let $\Sigma(t)$ denote the family of the translated Σ where $\partial \Sigma(t) \cap \mathbb{L}(t) = C_0(t)$. Our result above implies that the upper boundary of $\Sigma(t)$ for all $t \in [t_1, t_2]$ and $\Sigma(t_2)$ are contained in $\mathfrak{B}$ and therefore $\Sigma(t)$ must also lie in $\mathfrak{B}$. Otherwise when one translates $\Sigma(t_2)$ down to $\Sigma(t_1)$, there would be a first point of contact of some $\Sigma(t)$ with M . This contact point occurs on the inner side of M ; the mean curvature vector of both surfaces points in the same direction. This is impossible since the point of contact is an interior point of both M and $\Sigma(t)$ and the mean curvature of $\Sigma(t)$ (which is equal to one) is smaller than H .

We know that the upper boundary component of $\Sigma + v$, for $v \in V$, at height t_2 is contained in $\mathfrak{B}$. Hence $\Sigma + v \subset \mathfrak{B}$ for each $v \in V$ by similar reasoning as above: the family $\Sigma + sv$, $s \in [0, 1]$ can have no first point of interior contact with M as s goes from 0 to 1.

Our choice of C_0 guarantees that for each $q \in \Gamma$, there is a $v \in V$ such that $C_0 + v$ is tangent to Γ at q . The angle θ between Σ and non compact component of $\mathbb{L}(1) \cap C_0$ along C_0 is equal to $\arcsin(\cosh^{-1} \rho)$. Therefore the outer angle that $\mathfrak{B}$ makes with $\mathbb{L}(1)$ at q is smaller than θ ; in particular M stays outside the solid vertical geodesic cylinder over Γ between $\mathbb{L}(1)$ and $\mathbb{L}(t_2)$.

Since the horizontal translations $\Sigma + sv$, $v \in V$, $0 \leq s < 1$ are all in $\mathfrak{B}$ and $D(p, \sinh r) \times [t_1, t_2] \subset \mathfrak{B}$ by Lemma 3, we conclude that $\Omega \times [t_1, t_2] \subset \mathfrak{B}$. Also M meets the solid Killing cylinder over $D(\gamma_c(\tau = \ln t_2), \sinh R) \subset \mathbb{L}(t_2)$ with respect to γ_c in a Killing graph above $O(e^{d-1}/\cosh \omega)$.

The part of M in $(\Omega \times [1, \infty))$ is even a geodesic graph over Ω ; we find this out by coming down with planes P_q orthogonal to the vertical geodesic γ_q passing through q , q any point in Ω ; and we consider the symmetries of M with respect to P_q until P_q is below $O(e^{d-1}/\cosh \omega) \cap (\Omega \times [1, \infty))$. For H small, we are far from Γ , so no accident will occur and on any geodesic γ_q is exactly one point of M .

The part of M outside $\Omega \times [1, \infty)$ is of genus zero, so M is topologically a disk and Theorem 2 is established. $\square$

Suppose we could repeat this reasoning for a family of directions $v \in \mathbb{L}(1)$, $|v| = 1$, such that, for some $p \in \Omega$ and $r > 0$, we have $D(p, \sinh r) \subset \bigcup_v J(v)$. Note by γ_p the vertical geodesic passing through p and by $P_{\gamma_p}(\tau)$ the family of planes orthogonal to γ_p ; then we would have that the domain in the solid Killing cylinder over $D(p, \sinh r)$ with respect to γ_p between $\mathbb{L}(1)$ and $P_{\gamma_p}(\tau_1)$ is also contained in $\text{int } \mathfrak{B}$ where $P_{\gamma_p}(\tau_1)$ is the plane orthogonal to γ_p that intersects $\eta_q(\tau)$ at $\tau = 2\tau_0 - \ln \cosh \rho_q$ (i.e., the point of M that touches Γ for the first time by applying Alexandrov technique with respect to the planes $P_{\gamma_p}(\tau)$ above). Hence the points of M in this Killing cylinder are only above $P_{\gamma_p}(\tau_1)$. Now we can apply the Alexandrov reflection technique to M with the planes $P_{\gamma_p}(\tau)$; by decreasing τ from ∞ until τ_1 no accident will occur (the plane $P_{\gamma_p}(\tau_0)$ is always below $P_{\gamma_p}(\tau_1)$), so the part of M in the solid Killing cylinder over $D(p, \sinh r) \subset \mathbb{L}(1)$ is a Killing graph with respect to γ_p as desired (see Fig. 9). So we have to understand the horizontal directions v for which Alexandrov reflection goes beyond a point $q \in \Gamma$.

First recall, that for horizontal directions v , one can always do Alexandrov reflection up till Γ . Let k be the minimum curvature of Γ and let $C \subset \mathbb{L}(1)$ be a circle of curvature k . So if C is tangent to Γ at q , then Γ is inside C . Let ρ be chosen so that the tubular neighborhood of Γ of radius ρ is an embedded annulus.

Then for each horizontal v , we can do Alexandrov reflection with vertical planes at least a distance $\rho/2$ beyond each point of Γ and so at least a distance $\rho/2$ beyond the first time the horizontal plane meets the circle C . Now consider those planes which left behind q . This will hold for those directions in some neighborhood $V = \{v \in \mathbb{L}(1); |v| = 1\}$ of the inward pointing normal to C at q . It is clear from the geometry of the circle, that $\bigcup_{v \in V} J(v)$ contains a disk $D(p, \sinh r)$, $r > 0$ which depends on ρ and C but not on q . This completes the proof of Lemma 3. $\square$

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(Manuscript received September 4, 1997,
accepted, after revision, October 14, 1999.)

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