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NASH FUNCTIONS ON NONCOMPACT NASH MANIFOLDS

BY MICHEL COSTE AND MASAHIRO SHIOTA

ABSTRACT. – Several conjectures concerning Nash functions (including the conjecture that globally irreducible Nash sets are globally analytically irreducible) were proved in Coste et al. (1995) for compact affine Nash manifolds. We prove these conjectures for all affine Nash manifolds. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Plusieurs conjectures portant sur les fonctions de Nash (dont la conjecture qu'un ensemble de Nash globalement irréductible est globalement analytiquement irréductible) ont été démontrées, pour les variétés de Nash affines compactes, dans Coste et al. (1995). Nous prouvons ces conjectures pour toutes les variétés de Nash affines. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

An *affine Nash manifold* is a semialgebraic analytic submanifold of a Euclidean space. A *Nash function* on an affine Nash manifold is an analytic function with semialgebraic graph. Let M be an affine Nash manifold. Let $\mathcal{N}$ denote the sheaf of Nash functions on M (we write $\mathcal{N}_M$ if we need to emphasize M). We call a sheaf of ideals $\mathcal{I}$ of $\mathcal{N}$ *finite* if there exists a finite open semialgebraic covering $\{U_i\}$ of M such that, for each i, $\mathcal{I}|_{U_i}$ is generated by Nash functions on U_i. (See [5] and [3] for elementary properties of sheaves of $\mathcal{N}$-ideals.) Let $\mathcal{N}(M)$ (respectively $\mathcal{O}(M)$) denote the ring of Nash (respectively analytic) functions on M.

[3] showed that the following three conjectures are equivalent, and [2] gave a positive answer to the conjectures in the case where the manifold M is compact.

SEPARATION CONJECTURE. – *Let M be an affine Nash manifold. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}(M)$. Then $\mathfrak{p}\mathcal{O}(M)$ is a prime ideal of $\mathcal{O}(M)$.*

GLOBAL EQUATION CONJECTURE. – *For the same M as above, every finite sheaf $\mathcal{I}$ of $\mathcal{N}_M$-ideals is generated by global Nash functions on M.*

EXTENSION CONJECTURE. – *For the same M and $\mathcal{I}$ as above, the following natural homomorphism is surjective*:

$$H^0(M,\mathcal{N}) \longrightarrow H^0(M,\mathcal{N}/\mathcal{I}).$$

If these conjectures hold true, then the following conjecture also holds [3].

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FACTORIZATION CONJECTURE. – *Given a Nash function f on an affine Nash manifold M and an analytic factorization $f = f_1 f_2$, there exist Nash functions g_1 and g_2 on M and positive analytic functions ϕ_1 and ϕ_2 such that $\phi_1\phi_2 = 1$, $f_1 = \phi_1 g_1$ and $f_2 = \phi_2 g_2$.*

The separation conjecture means that Nash functions suffice to separate global analytic components. Note that the local version of the separation conjecture is an easy consequence of Artin's approximation theorem. The global equation and extension conjectures can be seen as substitutes for Cartan's theorems A and B, respectively, in the Nash setting. For a history of the conjectures, we refer the reader to [2,3].

In the present paper, we prove the following theorem.

THEOREM 1. – *Let $M \subset \mathbb{R}^n$ be a noncompact affine Nash manifold. Let U and V be open semialgebraic subsets of M such that $M = U \cup V$. Let $\mathcal{I}$ be a sheaf of $\mathcal{N}_M$-ideals such that $\mathcal{I}|_U$ and $\mathcal{I}|_V$ are generated by global sections on U and V, respectively. Then $\mathcal{I}$ itself is generated by global sections on M.*

This theorem obviously implies that the global equation conjecture holds true for noncompact affine Nash manifolds. By [3], we obtain a positive answer to all conjectures for every affine Nash manifold.

COROLLARY 2. – *The separation, global equation, extension and factorization conjectures hold true for every affine Nash manifold.*

The proof of the conjectures for the compact case in [2] is based on the so-called general Néron desingularization, applied to the homomorphism $\mathcal{N}(M) \to \mathcal{O}(M)$. This tool cannot be used in the noncompact case, because $\mathcal{O}(M)$ is no longer noetherian. The present proof of Theorem 1 uses the results obtained in the compact case. The main idea for the reduction to the compact case is to show that $\mathcal{I}$ can be extended to another Nash manifold in which M is relatively compact. The key for this extension is Theorem 5 on "compactification" of Nash functions.

We thank our friend Jesus M. Ruiz for his comments and suggestions which clarified several points of the proof.

2. Approximation by Nash functions

We shall use several times in the proof results on approximation by Nash functions. We recall here a few useful facts proved in [5]. Let M be an affine semialgebraic C^1 manifold. Let X_i, $i = 1, \ldots, n$, be continuous semialgebraic vector fields on M which generate the tangent vector space $T_x(M)$ at every point $x \in M$. The *semialgebraic* C^1 *topology* on the ring $\mathcal{S}^1(M)$ of semialgebraic C^1 functions $M \to \mathbb{R}$ is defined by the basis of neighborhoods of 0 consisting of all

$$U_h = \left\{ f \in \mathcal{S}^1(M);\ \forall x \in M\ |f(x)| < h(x) \text{ and } |X_i(f)(x)| < h(x),\ i = 1, \ldots, n \right\},$$

where h is a continuous positive semialgebraic function on M. If $N \subset \mathbb{R}^n$ is another semialgebraic C^1 manifold, we define the semialgebraic C^1 topology on the set $\mathcal{S}^1(M,N)$ of semialgebraic C^1 mappings $M \to N$ as the topology induced by the embedding $\mathcal{S}^1(M,N) \hookrightarrow (\mathcal{S}^1(M))^n$.

FACT 1 (Diffeomorphisms form an open subset). – *If $\varphi : M \to N$ is a semialgebraic C^1 diffeomorphism between affine semialgebraic C^1 manifolds, there is a neighborhood U of φ in $\mathcal{S}^1(M,N)$ such that every $\psi \in U$ is a diffeomorphism from M onto N* [5, II.1.7].

FACT 2 (Approximation by Nash functions). – *If M and N are affine Nash manifolds, the subset $\mathcal{N}(M,N)$ of Nash mappings is dense in $\mathcal{S}^1(M,N)$* [5, II.4.1].

FACT 3 (Relative approximation). – *Moreover, if $p: M \to \mathbb{R}$ is a Nash function and $f: M \to N \subset \mathbb{R}^n$ is a semialgebraic* C^1 *mapping which is Nash in a neighborhood of $p^{-1}(0)$, then, for any neighborhood U of $0 \in \mathcal{S}^1(M,\mathbb{R}^n)$, there is a Nash mapping $\tilde{f}: M \to N \subset \mathbb{R}^n$ such that $\tilde{f} - f = pg$, where $g: M \to \mathbb{R}^n$ belongs to U and is Nash in a neighborhood of $p^{-1}(0)$* [5, II.5.1 and 2]. *By Fact* 1, *if f is a diffeomorphism and U is small enough, $\tilde{f}$ is also a diffeomorphism.*

3. Global sections of restrictions of finite sheaves

We denote by $\operatorname{clos}(M;N)$ the closure of the subset M in the space N. Recall that M is said to be relatively compact in N if $\operatorname{clos}(M;N)$ is compact.

LEMMA 3. – *Let M' be an affine Nash manifold and M a relatively compact semialgebraic open subset of M'. Let $\mathcal{I}$ be a finite sheaf of $\mathcal{N}_M$-ideals. Assume that there is a finite sheaf $\mathcal{I}'$ of $\mathcal{N}_{M'}$-ideals such that $\mathcal{I} = \mathcal{I}'|_M$. Then $\mathcal{I}$ is generated by its global sections.*

Proof. – Let $\varphi: M' \to (0,+\infty)$ be a positive proper Nash function. By assumption $K = \operatorname{clos}(M';M)$ is a compact semialgebraic subset of M'. Let $r > 0$ be such that $\varphi(K) < r$ and r is not a critical value of φ. Let

$$D = \{(x,t) \in M' \times \mathbb{R}\colon t^2 = r - \varphi(x)\},$$

and define $p: D \to M'$ by $p(x,t) = x$. Then D is a compact Nash manifold and p is a Nash mapping which induces a diffeomorphism from

$$M_1 = \{(x,t) \in M \times \mathbb{R}\colon t = \sqrt{r - \varphi(x)}\}$$

onto M. Let $\sigma: M \to M_1$ be the inverse Nash diffeomorphism. Since the global equation conjecture holds for the compact affine Nash manifold D, the finite sheaf $\mathcal{J} = p^*(\mathcal{I}')$ of $\mathcal{N}_D$-ideals is generated by its global sections. It follows that $\mathcal{I} = \sigma^*(\mathcal{J})$ is also generated by its global sections. □

The next lemma seems near to our final goal, i.e., the proof of the global equation conjecture for noncompact Nash manifolds. However, the proof of Theorem 1 will use this lemma only for a small but seemingly indispensable point.

If $\mathcal{I}$ is a sheaf of $\mathcal{N}_M$-ideals, we denote by $\mathcal{Z}(\mathcal{I})$ the set of $x \in M$ such that $\mathcal{I}_x \neq \mathcal{N}_{M,x}$.

LEMMA 4. – *Let M be an affine Nash manifold and $\mathcal{I}$ a finite sheaf of $\mathcal{N}_M$-ideals. For every compact subset K of $\mathcal{Z}(\mathcal{I})$, there is a semialgebraic open subset U of M containing $K \cup (M \setminus \mathcal{Z}(\mathcal{I}))$ such that $\mathcal{I}|_U$ is generated by its global sections.*

Proof. – Set $Z = \mathcal{Z}(\mathcal{I})$. We choose a finite Nash stratification of M compatible with Z, satisfying Whitney's regularity conditions and whose strata of maximal dimension are the connected components of $M \setminus Z$ [1, Theorem 9.7.11]. Let $\varphi: M \to \mathbb{R}$ be a positive proper Nash function. Since the set of critical values of a Nash function is finite, for $r > 0$ large enough, we have $\varphi(K) < r$ and, for every stratum S of M, either $\varphi(S) \leqslant r$ or φ restricted to $S \cap \varphi^{-1}((r,+\infty))$ is a submersion onto $(r,+\infty)$. Then, by the semialgebraic version of Thom's first isotopy lemma [4, Theorem 1], there is a semialgebraic homeomorphism

$$\tau : \varphi^{-1}((r,+\infty)) \to \varphi^{-1}(r+1) \times (r,+\infty)$$
$$x \mapsto \big(\tau_1(x), \varphi(x)\big),$$

such that, for every stratum S, τ induces a Nash diffeomorphism from $S \cap \varphi^{-1}((r,+\infty))$ onto $(S \cap \varphi^{-1}(r+1)) \times (r,+\infty)$, and $\tau_1(x) = x$ for every x in $\varphi^{-1}(r+1)$.

By Lemma 3, the sheaf $\mathcal{I}$ restricted to $V = \varphi^{-1}((0,r+1))$ is generated by its global sections on V. Set $U = (M \setminus Z) \cup V$. We shall construct a Nash diffeomorphism $\tilde{\theta} : V \to U$ such that $\tilde{\theta}^*(\mathcal{I}|_U) = \mathcal{I}|_V$. This will prove that $\mathcal{I}|_U = (\tilde{\theta}^{-1})^*(\mathcal{I}|_V)$ is generated by its global sections on U.

Let $\delta : \varphi^{-1}(r+1) \to \mathbb{R}$ be a nonnegative Nash function such that $\delta < 1$ and $\delta^{-1}(0) = Z \cap \varphi^{-1}(r+1)$ (note that δ exists since the global equation conjecture holds for the compact affine Nash manifold $\varphi^{-1}(r+1)$). We extend δ to $\varphi^{-1}((r,+\infty))$ by setting $\delta(x) = \delta(\tau_1(x))$. Observe that $\delta^{-1}(0) = Z \cap \varphi^{-1}((r,+\infty))$ since, for every $x \in \varphi^{-1}((r,+\infty))$, $x \in Z$ if and only if $\tau_1(x) \in Z$. Choose a semialgebraic C^1 diffeomorphism

$$\alpha : (r,r+1) \times [0,1) \to \big((r,+\infty) \times (0,1)\big) \cup \big((r,r+1) \times \{0\}\big)$$
$$(u,v) \mapsto \big(\alpha_1(u,v), v\big),$$

such that α is the identity on the union of $(r, r+\frac{1}{2}) \times [0,1)$ and a neighborhood of $(r,r+1) \times \{0\}$. We define a semialgebraic mapping $\theta : V \to U$ by setting:

$$\theta(x) = \begin{cases} x & \text{if } \varphi(x) \leqslant r, \\ \tau^{-1}(\tau_1(x), \alpha_1(\varphi(x), \delta(x))) & \text{if } \varphi(x) > r. \end{cases}$$

The properties of τ, δ and α imply that θ is bijective. Observe that θ is the identity on a neighborhood of $Z \cap U = Z \cap V$ in V. It follows easily that θ is a C^1 diffeomorphism from V onto U. Let $p \in \mathcal{N}(V)$ be the sum of squares of a finite system of global sections of $\mathcal{I}|_V$ generating $\mathcal{I}|_V$. We have $p^{-1}(0) = Z \cap V$. Applying the relative approximation theorem (Fact 3) to θ, we obtain a Nash diffeomorphism $\tilde{\theta} : V \to U$ such that, for every $a \in Z \cap V$ and every $f_a \in \mathcal{N}_{M,a}$, the difference between the germs $(f \circ \tilde{\theta})_a$ and $(f \circ \theta)_a = f_a$ lies in $p\mathcal{N}_a$. Since $p\mathcal{N}_a \subset \mathcal{I}_a$, this proves that $\tilde{\theta}^*(\mathcal{I}|_U) = \mathcal{I}|_V$. □

4. Compactification of Nash functions

The meaning of "compactification of Nash functions" is made clear by the next theorem.

THEOREM 5. – *Let M be an affine Nash manifold, F a closed semialgebraic subset of M, and f a Nash mapping from an open semialgebraic neighborhood U of F into $\mathbb{R}^p$, such that $f|_F$ is bounded. There exists an open Nash embedding $\iota : M \to N$ into an affine Nash manifold N, such that $\iota(F)$ is relatively compact in N, and a Nash mapping $\overline{f}$ from an open semialgebraic neighborhood V of* $\mathrm{clos}(\iota(F); N)$ *into $\mathbb{R}^p$, such that $V \cap \iota(M) = \iota(U)$ and $\overline{f} \circ \iota = f$.*

Note that we can assume $N = V \cup \iota(M \setminus F)$ in the theorem. Note also that, if f is the restriction to U of a Nash mapping (still denoted by f) defined on an open subset W of M containing U, then $\overline{f}$ can be extended to $V \cup \iota(W)$, by setting $\overline{f} \circ \iota|_W = f$.

The first step in the proof of the theorem is the case $F = M$.

LEMMA 6. – *Let M be an affine Nash manifold and $f : M \to \mathbb{R}^p$ a bounded Nash mapping.*

(i) *There exist an open Nash embedding $\iota : M \to N$ into an affine Nash manifold N, such that $\iota(M)$ is relatively compact in N, and a Nash mapping $\overline{f} : N \to \mathbb{R}^p$, such that $\overline{f} \circ \iota = f$.*

(ii) *Moreover, we can assume that there is a Nash function* $\delta : N \to \mathbb{R}$ *such that* $\delta(\iota(M)) > 0$ *and* $\mathrm{clos}(\iota(M); N) \setminus \iota(M) \subset \delta^{-1}(0)$.

Proof. – (i) We can assume that M is bounded in $\mathbb{R}^n$. The graph of f is bounded in $\mathbb{R}^{n+p}$. Let X be the normalization of the Zariski closure of the graph of f. We have an open Nash embedding $\psi : M \to \mathrm{Reg}(X)$ into the set of nonsingular points of X, and $\psi(M)$ is relatively compact in X. Moreover, there is a regular mapping $g : X \to \mathbb{R}^p$ such that $g \circ \psi = f$. (This is nothing but the construction of Artin and Mazur, cf. [1, 8.4.4] or [5, I.5.1].) Now let $\pi : N \to X$ be a desingularization of X, such that N is a nonsingular real affine algebraic variety, π a proper birational mapping which induces a Nash diffeomorphism $\pi^{-1}(\mathrm{Reg}(X)) \to \mathrm{Reg}(X)$. Let $\iota : M \to N$ be the Nash embedding such that $\pi \circ \iota = \psi$, and set $\overline{f} = g \circ \pi$. Then $\iota(M)$ is relatively compact in N and $\overline{f} \circ \iota = f$.

(ii) Let $\varphi : M \to (0, +\infty)$ be a positive proper Nash function. Define $\beta : M \to \mathbb{R}$ by $\beta(x) = 1/\varphi(x)$. Applying part (i) of the lemma to $(f, \beta) : M \to \mathbb{R}^{p+1}$, we obtain an open Nash embedding $\iota : M \to N$ and a Nash mapping $(\overline{f}, \delta) : N \to \mathbb{R}^{p+1}$ such that $\iota(M)$ is relatively compact in N and $(\overline{f}, \delta) \circ \iota = (f, \beta)$. It follows that $\delta(\iota(M)) > 0$. Let $y \in \mathrm{clos}(\iota(M); N) \setminus \iota(M)$. Let $\gamma : [0,1] \to N$ be a continuous semialgebraic path such that $\gamma(0) = y$ and $\gamma((0,1]) \subset \iota(M)$. We have $\lim_{t \to 0_+} \beta(\iota^{-1}(\gamma(t))) = 0$, since otherwise $\iota^{-1}(\gamma(t))$ would have a limit x in M as $t \to 0_+$, from which $y = \iota(x) \in \iota(M)$ would follow. Hence, $\delta(y) = 0$. □

Now we prove the general case of Theorem 5.

Proof of Theorem 5. –

Step 1. Shrinking U, we can assume that f is bounded on U. By Mostowski's separation theorem [1, 2.7.7], there is a Nash function $h : M \to \mathbb{R}$ such that $h(F) > 0$ and $h(M \setminus U) < 0$. Let $\mu : M \to (0, +\infty)$ be a positive Nash function such that $\mu > 1/|h|$ on $F \cup (M \setminus U)$ (cf. [1, 2.6.2]). Define $\sigma : M \to \mathbb{R}$ by $\sigma = \mu h / \sqrt{(1 + \mu^2 h^2)/2}$. Then σ is a bounded Nash function such that $\sigma(F) > 1$ and $\sigma(M \setminus U) < -1$.

We can assume that M is open and relatively compact in another affine Nash manifold $M_1 \subset \mathbb{R}^n$ (see Lemma 6).

Applying Lemma 6 to the bounded Nash mapping $x \mapsto (x, f(x), \sigma(x))$ from U to $\mathbb{R}^{n+p+1}$, we obtain an open Nash embedding $\psi : U \to L$ into an affine Nash manifold L and Nash mappings $\eta : L \to \mathbb{R}^n$, $g : L \to \mathbb{R}^p$ and $\overline{\sigma} : L \to \mathbb{R}$ such that $\psi(U)$ is relatively compact in L, $\eta \circ \psi = \mathrm{Id}_U$, $g \circ \psi = f$ and $\overline{\sigma} \circ \psi = \sigma$. Since $\eta(\psi(U)) = U$ is relatively compact in M_1, we can assume that η takes its values in M_1. Moreover, we can assume that there is a Nash function $\delta : L \to \mathbb{R}$ such that $\delta(\psi(U)) > 0$ and $\mathrm{clos}(\psi(U); L) \setminus \psi(U) \subset \delta^{-1}(0)$. We can also assume that L is a closed Nash submanifold of $\mathbb{R}^k$.

Let L' be the semialgebraic open subset of those $y \in L$ such that $\overline{\sigma}(y) > 0$. If we glue M and L' along ψ, we obtain an abstract Nash manifold satisfying the properties of the theorem. The difficulty is to obtain an *affine* Nash manifold. We shall first glue M and L' to obtain a semialgebraic C^2 submanifold N' of $\mathbb{R}^{n+1+k}$, such that the image of L' in N' is a Nash submanifold of $\mathbb{R}^{n+1+k}$.

Choose a semialgebraic C^2 function $\lambda : \mathbb{R} \to [0,1]$ such that $\lambda^{-1}(1) = (-\infty, -1]$ and $\lambda^{-1}(0) = [0, +\infty)$. Let $\alpha : M \to \mathbb{R}^{n+1+k}$ be the semialgebraic C^2 embedding defined by

$$\alpha(x) = \begin{cases} (x, 1, 0) & \text{if } x \in M \setminus U, \\ (\lambda(\sigma(x))x, \lambda(\sigma(x)), (1 - \lambda(\sigma(x)))\psi(x)) & \text{if } x \in U. \end{cases}$$

We set $M' = \alpha(M)$. For convenience, we identify L' with $\{(0,0)\} \times L' \subset \mathbb{R}^{n+1+k}$ and L with $\{(0,0)\} \times L$. Set $N' = M' \cup L' \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$, and let $N'_{>0}$ (respectively $N'_{<1}$) be the open subset of N' consisting of those (x,t,y) such that $t>0$ (respectively $t<1$). If $(x,t,y)\in N'_{<1}$, then $(1-t)^{-1}y\in L$. The sets M' and L' are both open in N', since

$$M' = N'_{>0} \cup \{(x,t,y)\in N'_{<1}\colon (1-t)^{-1}y\in\psi(U)\}$$

and

$$L' = \{(x,t,y)\in N'_{<1}\colon \overline{\sigma}((1-t)^{-1}y)>0\}.$$

Since M' is a C^2 semialgebraic submanifold and L' is a Nash submanifold of $\mathbb{R}^{n+1+k}$, N' is a C^2 semialgebraic submanifold of $\mathbb{R}^{n+1+k}$. Note that $M'\cap L'\subset\alpha(U)$. We have $g\circ\alpha=f$ on $\alpha^{-1}(L')$, and $\alpha(F)$ is relatively compact in L'.

Since L is a closed Nash submanifold of $\mathbb{R}^{n+1+k}$, there is a nonnegative Nash function $\gamma\colon\mathbb{R}^{n+1+k}\to\mathbb{R}$ such that $L=\gamma^{-1}(0)$, and $\delta\colon L\to\mathbb{R}$ extends to a Nash function on $\mathbb{R}^{n+1+k}$ which we still denote by δ [5, II.5.4 and II.5.5]. Let $\varphi\colon\mathbb{R}^{n+1+k}\to\mathbb{R}$ be the Nash function defined by $\varphi=\gamma+\delta^2$. Observe that φ is positive on M', $\mathrm{clos}(M';N')\setminus M'\subset\varphi^{-1}(0)$ and $\varphi^{-1}(0)\cap N'=\varphi^{-1}(0)\cap L'$ is open in $\varphi^{-1}(0)$. See Fig. 1.

Step 2. The next step is to approximate N' by an affine Nash manifold, keeping fixed $\varphi^{-1}(0)\cap N'$ and the Nash structure of L'. This is essentially [5, III.1.3]. However, we repeat the proof because we need extra information.

Set $d=n+1+k-\dim(N')$ and let G be the grassmannian of d-dimensional vector subspaces of $\mathbb{R}^{n+1+k}$. Let E be the universal vector bundle of rank d on G. Let $\nu\colon N'\to G$ be the semialgebraic C^1 mapping which sends $z\in N'$ to the normal space $\nu(z)$ to N' at z. Observe that ν is Nash on L'. Let (z,ξ) be a point of the induced bundle $\nu^*(E)$, that is, $\xi\in\nu(z)$. The mapping $(z,\xi)\mapsto z+\xi$ induces a semialgebraic C^1 diffeomorphism θ from an open semialgebraic neighborhood Ω of the zero section of $\nu^*(E)$ onto a neighborhood W of N' in $\mathbb{R}^{n+1+k}$. We can assume that $\varphi^{-1}(0)\cap W=\varphi^{-1}(0)\cap N'$. Let $\rho\colon W\to N'$ be the semialgebraic C^1 mapping defined by $(\rho(z),z-\rho(z))=\theta^{-1}(z)$ for $z\in W$. The mapping ρ is a retraction of W on N', and (W,ρ) is a semialgebraic C^1 tubular neighborhood of N'. Define $\chi\colon W\to E$ by $\chi(z)=(\nu(\rho(z)),z-\rho(z))$. Observe that ρ and χ are Nash on $\rho^{-1}(L')$, χ is transverse to the zero

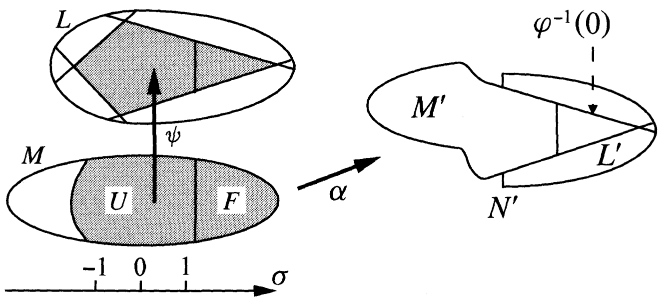
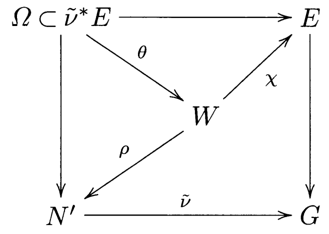


Fig. 1.

section $G \times \{0\}$ of E and $\chi^{-1}(G \times \{0\}) = N'$.



Approximate χ (in the semialgebraic C^1 topology) by a Nash mapping $\tilde{\chi}: W \to E$ such that χ and $\tilde{\chi}$ are equal on $\varphi^{-1}(0) \cap W$ (relative approximation, Fact 3). Then $\tilde{\chi}$ is transverse to $G \times \{0\}$, and $N'' = \tilde{\chi}^{-1}(G \times \{0\})$ is a Nash manifold. We have $N'' \cap \varphi^{-1}(0) = N' \cap \varphi^{-1}(0)$. If the approximation is strong enough, the retraction ρ induces a C^1 diffeomorphism $\kappa: N'' \to N'$ such that $(\kappa^{-1})|_{L'}$ is Nash and κ^{-1} is the identity on $\varphi^{-1}(0) \cap N'$. We set $L'' = \kappa^{-1}(L')$ and $M'' = \kappa^{-1}(M')$.

Step 3. We still have the problem that the diffeomorphism $\kappa^{-1} \circ \alpha: M \to M''$ is only C^1. The final step will be to approximate $\kappa^{-1} \circ \alpha$ by a Nash diffeomorphism $\iota: M \to M''$, such that $\iota \circ \alpha^{-1} \circ \kappa|_{M'' \cap L''}$ extends to a Nash diffeomorphism τ on a neighborhood of $B'' = \operatorname{clos}(M''; N'') \setminus M''$.

Let $\zeta: N'' \to M_1 \subset \mathbb{R}^n$ be the semialgebraic C^1 mapping defined by $\zeta(z) = \alpha^{-1}(\kappa(z))$ if $z \in M''$ and $\zeta(z) = \eta(\kappa(z))$ if $z \in L''$. The restriction $\zeta|_{L''}$ is Nash, and the restriction $\zeta|_{M''}$ is a semialgebraic C^1 diffeomorphism onto M. The construction of the Nash diffeomorphism τ will use Tougeron's implicit function theorem applied to an equation involving ζ. For this, we need to have a good control on the sheaf $\mathcal{J}$ of jacobian ideals of ζ. We can take finitely many Nash charts on L'' and M_1, such that the image by ζ of the domain of any chart of L'' is contained in the domain of some chart of M_1. Let $\mathcal{J} \subset \mathcal{N}_{N''}$ be the finite sheaf of ideals which is generated by 1 on M'' and by $\det(\partial\zeta_i/\partial z_j)$ on every chart of L'', where the z_j are the coordinates in this chart and the ζ_i are the coordinates of ζ in the corresponding chart of M_1. By Lemma 4, we can assume that $\mathcal{J}$ is generated by its global sections on N''; indeed, this becomes true if we remove from N'' some closed subset of $\mathcal{Z}(\mathcal{J})$, disjoint from the compact subset $\operatorname{clos}(\kappa^{-1}(\alpha(F)); N'')$. Let $J \in \mathcal{N}_{N''}$ be the sum of squares of a finite system of global sections generating $\mathcal{J}$. Note that $J^{-1}(0) \subset L''$.

We approximate ζ (in the semialgebraic C^1 topology) by a Nash mapping $\tilde{\zeta}: N'' \to M_1 \subset \mathbb{R}^n$, such that $\tilde{\zeta} - \zeta = \varphi^\ell J \varepsilon$, where $\varepsilon: N'' \to \mathbb{R}^n$ is a semialgebraic C^1 mapping close to zero, Nash in a neighborhood of $\varphi^{-1}(0) \cap N''$ (cf. Fact 3).

We claim that, if ℓ is large enough and ε sufficiently close to 0, $\tilde{\zeta}|_{M''}$ is a close approximation of $\zeta|_{M''}$ with respect to the semialgebraic C^1 topology on the space of semialgebraic C^1 mappings $M'' \to M_1 \subset \mathbb{R}^n$. Indeed, let X be a continuous semialgebraic vector field on N''. We have

$$X(\tilde{\zeta} - \zeta) = \varphi^{\ell-1}\big(\ell X(\varphi)J + \varphi X(J)\big)\varepsilon + \varphi^\ell J X(\varepsilon).$$

Let $\mu: M'' \to \mathbb{R}$ be a positive continuous semialgebraic function. Since M'' is a union of connected components of $N'' \setminus \varphi^{-1}(0)$, there is a positive integer m such that φ^m/μ can be continuously extended by 0 on $N'' \setminus M''$ [1, 2.6.4]. Hence, taking $\ell = m + 1$ and ε close enough to 0, we obtain $\|X(\tilde{\zeta} - \zeta)\| < \mu$ on M''. This proves the claim. It follows from the claim that we can assume that $\tilde{\zeta}|_{M''}$ is a Nash diffeomorphism from M'' onto M. Let $\iota: M \to M''$ be the inverse Nash diffeomorphism.

We now claim that there are open semialgebraic neighborhoods C and D of B'' in L'' and a Nash diffeomorphism $\tau: C \to D$ such that $\tau|_{C\cap M''} = \iota \circ \zeta$. By uniqueness of continuation of Nash mappings, the problem is local on B''. Let $z_0 \in B''$. Using Nash charts in a neighborhood of z_0 (respectively $\zeta(z_0) \in M_1$), we can assume that $N'' = M_1 = \mathbb{R}^m$ and $z_0 = 0$. Consider the equation $\Phi(z,y) = \zeta(z+y) - \tilde{\zeta}(z) = 0$, where $z \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$. We know that $\Phi(z,0)$ is divisible by $\varphi^\ell(z)J(z)$, and J is in the square of the jacobian ideal $\mathcal{J}_{z_0}$ generated by $\det(\partial\Phi_i/\partial y_j)(z,0)$. Hence, by Tougeron's implicit function theorem [6, Chapter 3, Theorem 3.2], there is a Nash germ $y(z) \in \varphi^\ell \mathcal{J}_{z_0}$, such that $\zeta(z + y(z)) = \tilde{\zeta}(z)$. The germ $z + y(z)$ is a germ of diffeomorphism (we assume $\ell \geqslant 2$). Its inverse τ_{z_0} is a germ of diffeomorphism $(N'', z_0) \to (N'', z_0)$ such that $\tau_{z_0} = \iota \circ \zeta$ on M''. Therefore the claim is proved.

Now we can conclude the proof. We set $N = M'' \cup D \subset N''$ and $V = \iota(U) \cup D$. It is clear that $\iota(F)$ is relatively compact in N and V is an open semialgebraic neighborhood of $\iota(F)$ in N. Take $z \in M'' \cap D$ and set $x = \iota^{-1}(z) \in M$. The relations

$$\alpha(x) = \kappa\big(\zeta^{-1}(x)\big) = \kappa\big(\tau^{-1}(z)\big) \in \kappa(M'' \cap L'') = M' \cap L' \subset \alpha(U)$$

imply $x \in U$ and $g(\kappa(\tau^{-1}(z))) = g(\alpha(x)) = f(x)$. Hence, $\iota(M) \cap V = M'' \cap V = \iota(U)$ and the Nash function $\overline{f}: V \to \mathbb{R}$ given by $\overline{f}(\iota(x)) = f(x)$ if $x \in U$ and $\overline{f}(z) = g(\kappa(\tau^{-1}(z)))$ if $z \in D$ is well defined. $\square$

5. Proof of Theorem 1

Let $f_1, \dots, f_k \in H^0(U, \mathcal{I}|_U)$ and $g_1, \dots, g_k \in H^0(V, \mathcal{I}|_V)$ be systems of generators of $\mathcal{I}|_U$ and $\mathcal{I}|_V$, respectively. Replacing the generators f_i and g_j with $f_i/(1+f_i^2)$ and $g_j/(1+g_j^2)$, respectively, we can assume that they are all bounded. Note that the restriction of each system of generators to $U \cap V$ generates $\mathcal{I}|_{U\cap V}$. Hence, by I.6.5 in [5], there exist Nash functions $\alpha_{i,j}$ and $\beta_{i,j}$ on $U \cap V$, $i, j = 1, \dots, k$, such that, for each i,

$$f_i = \sum_{j=1}^{k} \alpha_{i,j} g_j \quad \text{and} \quad g_i = \sum_{j=1}^{k} \beta_{i,j} f_j \quad \text{on } U \cap V.$$

The idea of the proof is to extend the sheaf $\mathcal{I}$ to a bigger Nash manifold in which M is relatively compact. For this, we extend the generators f_i and g_j and the functions $\alpha_{i,j}$ and $\beta_{j,i}$. We have to modify a little these functions to satisfy the boundedness condition in Theorem 5.

Let φ be a positive proper Nash function on M, and set $\delta = 1/\varphi$. Using Mostowski's separation theorem as in the beginning of the proof of Theorem 5, we obtain a bounded Nash

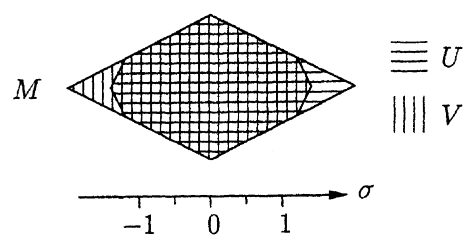


Fig. 2.

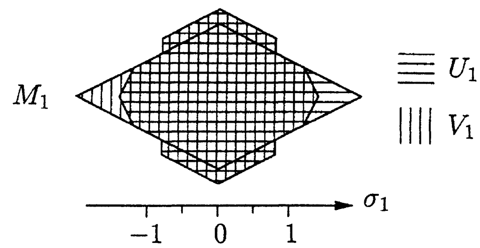


Fig. 3.

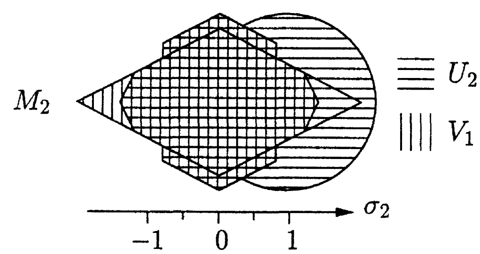


Fig. 4.

function σ on M such that $\sigma > 1$ on $M \setminus V (= U \setminus V)$ and $\sigma < -1$ on $M \setminus U (= V \setminus U)$. By Łojasiewicz inequality (cf. [1, 2.6.4]), there is a positive integer ℓ such that all $\delta^\ell \alpha_{i,j} = \alpha'_{i,j}$ and $\delta^\ell \beta_{i,j} = \beta'_{i,j}$ are bounded on $\sigma^{-1}([-\frac{1}{2}, +\frac{1}{2}])$ (Fig. 2).

By Theorem 5, we can assume that:

- M is an open semialgebraic subset of an affine Nash manifold M_1 and $\sigma^{-1}([-\frac{1}{2}, \frac{1}{2}]) \subset M$ is relatively compact in M_1.
- There are Nash functions $\delta_1 : M_1 \to \mathbb{R}$ and $\sigma_1 : M_1 \to \mathbb{R}$ such that $\delta_1|_M = \delta$, δ_1 is bounded, $\sigma_1|_M = \sigma$ and $|\sigma_1| < 1$ on $M_1 \setminus M$. We set $U_1 = U \cup \sigma_1^{-1}((-1, 1))$ and $V_1 = V \cup \sigma_1^{-1}((-1, 1))$ (Fig. 3).
- There are Nash functions $f_{i,1} : U_1 \to \mathbb{R}$, $g_{j,1} : V_1 \to \mathbb{R}$ and $\overline{\alpha}'_{i,j}, \overline{\beta}'_{j,i} : U_1 \cap V_1 \to \mathbb{R}$, for $i, j = 1, \ldots, k$, such that all $f_{i,1}$ and $g_{j,1}$ are bounded,

$$f_{i,1}|_U = f_i, \quad g_{j,1}|_V = g_j, \quad \overline{\alpha}'_{i,j}|_{U \cap V} = \alpha'_{i,j}, \quad \overline{\beta}'_{j,i}|_{U \cap V} = \beta'_{j,i}.$$

- Every connected component of $U_1 \cap V_1$ meets $U \cap V$ (otherwise, we can remove from M_1 the closure of these components). Therefore,

$$(*) \qquad \delta_1^\ell f_{i,1} = \sum_{j=1}^{k} \overline{\alpha}'_{i,j} g_{j,1} \quad \text{and} \quad \delta_1^\ell g_{j,1} = \sum_{i=1}^{k} \overline{\beta}'_{j,i} f_{i,1} \quad \text{on } U_1 \cap V_1.$$

Again by Theorem 5, we can assume that:

- M_1 is an open semialgebraic subset of an affine Nash manifold M_2 and $\sigma_1^{-1}([\frac{1}{2}, +\infty)) \subset M_1$ is relatively compact in M_2.
- There are Nash functions $\delta_2 : M_2 \to \mathbb{R}$ and $\sigma_2 : M_2 \to \mathbb{R}$ such that $\delta_2|_{M_1} = \delta_1$, $\sigma_2|_{M_1} = \sigma_1$ and $\sigma_2 > 0$ on $M_2 \setminus M_1$. We set $U_2 = U_1 \cup \sigma_2^{-1}((0, +\infty))$ (Fig. 4).
- There are Nash functions $\overline{f}_i : U_2 \to \mathbb{R}$ such that $\overline{f}_i|_{U_1} = f_{i,1}$.

By a last application of Theorem 5, we can assume that:

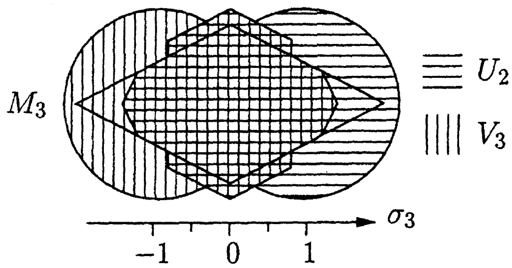


Fig. 5.

- M_2 is an open semialgebraic subset of an affine Nash manifold M_3 and $\sigma_2^{-1}((-\infty,-\frac{1}{2}]) \subset M_2$ is relatively compact in M_3.
- There are Nash functions $\overline{\delta}: M_3 \to \mathbb{R}$ and $\sigma_3 : M_3 \to \mathbb{R}$ such that $\overline{\delta}|_{M_2} = \delta_2$, $\sigma_3|_{M_2} = \sigma_2$ and $\sigma_3 < 0$ on $M_3 \setminus M_2$. We set $V_3 = V_1 \cup \sigma_3^{-1}((-\infty,0))$ (Fig. 5).
- There are Nash functions $\overline{g}_j : V_3 \to \mathbb{R}$ such that $\overline{g}_j|_{V_1} = g_{j,1}$.

Observe that

$$M \subset \sigma^{-1}\big(\big[-\tfrac{1}{2},\tfrac{1}{2}\big]\big) \cup \sigma_1^{-1}\big(\big[\tfrac{1}{2},+\infty\big)\big) \cup \sigma_2^{-1}\big(\big(-\infty,-\tfrac{1}{2}\big]\big)$$

is relatively compact in M_3.

We define the ideals

$$\begin{aligned}
&I = (\overline{f}_1,\dots,\overline{f}_k) \subset \mathcal{N}(U_2),\\
&I' = \big\{h \in \mathcal{N}(U_2)\colon\ \overline{\delta}^{\nu} h \in I \text{ for some } \nu\big\} = \bigcup_{\nu} \big(I : (\overline{\delta}^{\nu}|_{U_2})\big) \subset \mathcal{N}(U_2),\\
&J = (\overline{g}_1,\dots,\overline{g}_k) \subset \mathcal{N}(V_3),\\
&J' = \big\{h \in \mathcal{N}(V_3)\colon\ \overline{\delta}^{\nu} h \in J \text{ for some } \nu\big\} = \bigcup_{\nu} \big(J : (\overline{\delta}^{\nu}|_{V_3})\big) \subset \mathcal{N}(V_3).
\end{aligned}$$

We claim that the ideals I' and J' generate the same sheaf of ideals over $U_2 \cap V_3 = U_1 \cap V_1$. Let $x \in U_1 \cap V_1$. Since $\mathcal{N}_x$ is flat over $\mathcal{N}(U_2)$ and $\mathcal{N}(V_3)$, we have

$$I'\mathcal{N}_x = \bigcup_{\nu} \big(I\mathcal{N}_x : (\overline{\delta}_x^{\nu})\big) \quad \text{and} \quad J'\mathcal{N}_x = \bigcup_{\nu} \big(J\mathcal{N}_x : (\overline{\delta}_x^{\nu})\big).$$

The above relations $(*)$ imply that $J\mathcal{N}_x \subset (I\mathcal{N}_x : (\overline{\delta}_x^{\ell}))$ and $I\mathcal{N}_x \subset (J\mathcal{N}_x : (\overline{\delta}_x^{\ell}))$. Hence, $I'\mathcal{N}_x = J'\mathcal{N}_x$ and the claim is proved.

It follows that we can define a finite sheaf of ideals $\overline{\mathcal{I}}$ over M_3 by setting $\overline{\mathcal{I}}_x = I'\mathcal{N}_x$ if $x \in U_2$ and $\overline{\mathcal{I}}_x = J'\mathcal{N}_x$ if $x \in V_3$. If $x \in M$, $\overline{\delta}_x = \delta_x$ is invertible and, therefore, $\overline{\mathcal{I}}_x = \mathcal{I}_x$. By Lemma 3, since M is relatively compact in M_3 and $\mathcal{I} = \overline{\mathcal{I}}|_M$, the sheaf $\mathcal{I}$ is generated by its global sections on M.

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