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Partial differential equations

Symmetry and classification of solutions to an integral equation of the Choquard type

Symétrie et classification des solutions d'une équation intégrale de type Choquard

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A B S T R A C T

We study the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^q(z)}{|y-z|^{n-\beta}} \, dz \, dy, \quad x \in \mathbb{R}^n,$$

where $0 < \alpha, \beta < n$ and $p + q = \frac{n+\alpha+2\beta}{n-\alpha}$. We prove that all positive $L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ solutions to the equation are radially symmetric and monotone decreasing about some point, and we classify all such solutions when $p + 1 = q = \frac{n+\beta}{n-\alpha}$. As a consequence, we derive similar results for positive $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ solutions to the higher-fractional-order Choquard-type equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \frac{1}{R_{n,\alpha}} \left(\frac{1}{|x|^{n-\beta}} * u^q \right) u^p \quad \text{in } \mathbb{R}^n.$$

R É S U M É

Nous étudions l'équation intégrale

$$u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^q(z)}{|y-z|^{n-\beta}} \, dz \, dy, \quad x \in \mathbb{R}^n,$$

où $0 < \alpha, \beta < n$ et $p + q = \frac{n+\alpha+2\beta}{n-\alpha}$. Nous démontrons que toute solution positive $L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ de l'équation est à symétrie radiale et monotone décroissante autour d'un point. Nous classifions toutes les solutions telles que $p + 1 = q = \frac{n+\beta}{n-\alpha}$. Nous en déduisons des résultats

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similaires pour les solutions positives $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ de l'équation de type Choquard d'ordre fractionnaire supérieur

$$(-\Delta)^{\frac{\alpha}{2}}u = \frac{1}{R_{n,\alpha}}\left(\frac{1}{|x|^{n-\beta}} * u^q\right)u^p \quad \text{in } \mathbb{R}^n.$$

1. Introduction

This paper is concerned with the symmetry and classification of positive solutions to the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{u^q(z)}{|y-z|^{n-\beta}}\,\mathrm{d}z\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $0 < \alpha, \beta < n$ and $p + q = \frac{n+\alpha+2\beta}{n-\alpha}$. This equation is closely related to the following higher-fractional-order equation

$$(-\Delta)^{\frac{\alpha}{2}}u = \frac{1}{R_{n,\alpha}}\left(\frac{1}{|x|^{n-\beta}} * u^q\right)u^p \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $R_{n,\alpha} = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}}2^{\alpha}\Gamma(\frac{\alpha}{2})}$ is the Riesz potential's constant (see [28]).

When $p = q - 1$, this kind of equation is usually called the Choquard-type equation since, in 1976, P. Choquard used a similar equation to describe an electron trapped in its own hole, in a certain approximation to the Hartree–Fock theory of one-component plasma (see [18]). Such an equation with $\alpha = \beta = 2$ also arises in the Hartree–Fock theory of the nonlinear Schrodinger equations (see [20]) and is helpful in understanding the blow-up or the global existence and scattering of the solutions to the dynamic Hartree equation in the focusing case (see [15]). Recently, Choquard-type equations were widely used in the study of boson stars and of other physical phenomena. It also appears as a continuous-limit model for mesoscopic molecular structures in chemistry. More related mathematical and physical background can be found in [10,12, 23,25,26] and the references therein.

We say that $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is a positive weak solution to (1.2) if $u > 0$ and

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{4}}u(-\Delta)^{\frac{\alpha}{4}}\phi\,\mathrm{d}x = \frac{1}{R_{n,\alpha}}\int_{\mathbb{R}^n}\left(\frac{1}{|x|^{n-\beta}} * u^q\right)u^p\phi\,\mathrm{d}x \tag{1.3}$$

for any $\phi \in C_0^{\infty}(\mathbb{R}^n)$. Here the fractional Laplacian is defined by the ideas in [28]. More precisely, for $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$,

$$(-\Delta)^{\frac{\alpha}{4}}u = \mathcal{F}^{-1}\left(|\xi|^{\frac{\alpha}{2}}\mathcal{F}u\right),$$

where, as usual,

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^n} u(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x$$

is the Fourier transform of u and $\mathcal{F}^{-1}u$ is the inverse Fourier transform of u.

Let us also remind that $H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is the inhomogeneous Sobolev space with the norm

$$\|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \|u\|_{\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^n)},$$

where the homogeneous Sobolev norm

$$\|u\|_{\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \|(-\Delta)^{\frac{\alpha}{4}}u\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\xi|^{\alpha}|\mathcal{F}u|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

Equation (1.2) is $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^n)$-critical in the sense that (1.2) and the $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^n)$ norm are invariant under the scaling $u_\rho(x) = \rho^{\frac{n-\alpha}{2}}u(\rho x)$.

Since (1.1) and (1.2) have convolution terms, it is not easy to investigate the existence of solutions directly. By setting $v(y) = \int_{\mathbb{R}^n} \frac{u^q(z)}{|y-z|^{n-\beta}}\,\mathrm{d}z$, one can observe that Eq. (1.1) is equivalent to the integral system

$$\begin{cases} u(x) = \int\limits_{\mathbb{R}^n} \frac{u^p(y)v(y)}{|x-y|^{n-\alpha}} \, dy, & x \in \mathbb{R}^n, \\ v(x) = \int\limits_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\beta}} \, dy, & x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

The idea of considering the equivalent systems of integral equations like this was initially used by Ma and Zhao [23]. Much effort has been devoted to study the symmetry of positive solutions to Eq. (1.1) and its equivalent integral system (1.4) when $p+1=q=\frac{n+\beta}{n-\alpha}$ in recent years. These symmetry results usually lead to classification results by using the techniques in [5,19].

When $\beta=\alpha\in(1,n)$ and $p+1=q=\frac{n+\alpha}{n-\alpha}$, Xu and Lei [30] and Lei [13] classified all positive $L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ solutions to (1.4). Their proof rely on a previous classification result by Chen and Li [3] for a system that is more general than (1.4), but with the restriction $\alpha=\beta$. Later, Wang and Tian [29] obtained a classification result for positive $H^\alpha(\mathbb{R}^n)$ solutions to (1.2) when $\beta=\alpha\in(0,\frac{n}{2})$ and $p+1=q=\frac{n+\alpha}{n-\alpha}$.

The case $\beta=n-2\alpha$ also got the attention from some authors. Liu proved in [22] a classification result for positive $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ solutions of integral system (1.4) when $\alpha=2$, $\beta=n-4$ and $p+1=q=2$. Liu's result was lately extended to the case where $\alpha\in(0,\frac{n}{2})$, $\beta=n-2\alpha$ and $p+1=q=2$ by Dai et al. [8]. Recently, a symmetry result for Eq. (1.2) in the case $\alpha\in(0,2)$ and $p+1=q$ was studied by some authors [7,11,24] using a direct method of moving planes developed in [4].

To the best of our knowledge, Eq. (1.1) in the case $q\neq p+1$ or $\beta\in(0,n)\setminus\{\alpha,n-2\alpha\}$, where $\alpha\in(0,n)$, has not been fully studied in the literature. Our main purpose in writing this paper is to establish the radial symmetry of positive solutions to (1.1) in that general case. Our first result therefore extends and unifies previously mentioned symmetry results.

Theorem 1.1. *Assume that $0<\alpha,\beta<n$ and $p+q=\frac{n+\alpha+2\beta}{n-\alpha}$, where*

$$\frac{2(\alpha+\beta-n)}{n-\alpha} < p < \frac{2n}{n-\alpha}, \quad p \geq 1, \tag{1.5}$$

$$\max\left\{\frac{2\beta}{n-\alpha}, 1\right\} < q < \frac{2n}{n-\alpha}. \tag{1.6}$$

Then every positive solution $u\in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ to (1.1) is radially symmetric and monotone decreasing about some point.

We will employ the method of moving planes in integral forms by Chen, Li and Ou [5] to prove Theorem 1.1. The methods of moving planes was founded by Alexanderoff in the early 1950s. Later, it was further developed by Serrin [27], Gidas, Ni and Nirenberg [9], Caffarelli, Gidas and Spruck [1], Chen and Li [2], Li and Zhu [16], Li [14], Lin [21], Chen, Li and Ou [5], Chen, Li and Li [4], and many others.

The method of moving planes in integral forms requires the use of the Hardy–Littlewood–Sobolev inequality. Technical assumptions (1.5) and (1.6) are required to apply the Hardy–Littlewood–Sobolev inequality and Hölder's inequality in our proof. These assumptions automatically hold in the original Choquard model (where $p+1=q$), as we can see in our next theorem.

Theorem 1.2. *Assume that $0<\alpha,\beta<n$ and $p+1=q=\frac{n+\beta}{n-\alpha}\geq 2$. Then every positive solution $u\in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ to (1.1) must assume the form*

$$u(x)=\mu^{\frac{n-\alpha}{2}}Q(\mu(x-x^0)) \quad \text{for some } \mu>0 \text{ and } x^0\in\mathbb{R}^n,$$

where $Q(x)=\left[I\left(\frac{n-\alpha}{2}\right)I\left(\frac{n-\beta}{2}\right)\right]^{-\frac{n-\alpha}{2(\alpha+\beta)}}\left(\frac{1}{1+|x|^2}\right)^{\frac{n-\alpha}{2}}$, *with* $I(s)=\frac{\pi^{\frac{n}{2}}\Gamma(\frac{n-2s}{2})}{\Gamma(n-s)}$ *for* $0<s<\frac{n}{2}$.

In this paper, we also establish the equivalence between the integral equation (1.1) and the Choquard-type equation (1.2).

Theorem 1.3. *Assume that $0<\alpha,\beta<n$ and $p,q\geq 0$. If $u\in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is a positive weak solution to Eq. (1.2), then it satisfies the integral equation (1.1), and vice versa.*

Combining Theorem 1.1 and 1.2 for the integral equation (1.1) with Theorem 1.3, we have the following corollary for the Choquard-type equation (1.2) immediately.

Proposition 1.4. *The same conclusions of Theorem 1.1 and 1.2 hold for every positive solution $u\in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ to (1.2).*

The classification result in Proposition 1.4 would provide the best constant for the corresponding Hardy–Littlewood–Sobolev inequality. We define the norm

$$\|u\|_{L^{\alpha,\beta}} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^{\frac{n+\beta}{n-\alpha}}|u(y)|^{\frac{n+\beta}{n-\alpha}}}{|x-y|^{n-\beta}}\,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{n-\alpha}{2(n+\beta)}}.$$

For any $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$, we have the following Hardy–Littlewood–Sobolev inequality (see [19,28])

$$\|u\|_{\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^n)} \geq S_{n,\alpha,\beta}\|u\|_{L^{\alpha,\beta}}, \tag{1.7}$$

where the best constant $S_{n,\alpha,\beta}$ in (1.7) is given by

$$S_{n,\alpha,\beta} = \inf_{u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)\setminus\{0\}} \frac{\|u\|_{\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^n)}}{\|u\|_{L^{\alpha,\beta}}}. \tag{1.8}$$

It is known that $S_{n,\alpha,\beta}$ is achieved by the extremal functions $u(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{n-\alpha}{2}}$ (see [19,28]). However, the explicit formula for $S_{n,\alpha,\beta}$ has not been derived in previous works, except for the case $\beta = n - 2\alpha$ (see [8, Corollary 2]). Using Proposition 1.4, we are able to compute $S_{n,\alpha,\beta}$ explicitly in terms of the gamma function as follows.

Proposition 1.5. *The best constant $S_{n,\alpha,\beta}$ in the Hardy–Littlewood–Sobolev inequality (1.7) is given explicitly by*

$$2^{\frac{\alpha}{2}}\pi^{\frac{n(2\alpha+\beta-n)}{4(n+\beta)}} \left(\frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha}{2})}\right)^{\frac{\beta(n-\alpha)}{2\alpha(n+\beta)}} \left(\frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\right)^{\frac{1}{2}} \left(\frac{\Gamma\left(\frac{n+\beta}{2}\right)}{\Gamma(\beta)}\right)^{\frac{n-\alpha}{2(n+\beta)}} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{\frac{\alpha+\beta}{2(n+\beta)}}.$$

This paper is organized as follows. In Section 2, we prove the symmetry of positive solutions to (1.1), namely, Theorem 1.1. This result allows us to prove Theorem 1.2 on the classification of positive solutions in Section 3. The last section is devoted to the proofs of Theorem 1.3 and Proposition 1.5.

Unless specified, C denotes the generic positive constant whose concrete value may vary from line to line or even in the same line, depending on the situation. We also denote by $|\Omega|$ the Lebesgue measure of $\Omega \subset \mathbb{R}^n$ and by $B_r(x)$ the ball of radius $r > 0$ and center $x \in \mathbb{R}^n$.

2. Symmetry of the positive solutions

To prove Theorem 1.1, we carry out the method of moving planes in integral forms (see [5]) to the integral equation (1.1) in the x_1 direction. For any $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}$$

be the moving plane,

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

be the half-space to the left of the plane and

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$$

be the reflection of the point $x = (x_1, x_2, \dots, x_n)$ about the plane T_λ. We also define

$$u_\lambda(x) := u(x^\lambda), \quad v(x) := \int_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\beta}}\,\mathrm{d}y, \quad v_\lambda(x) := v(x^\lambda)$$

and

$$w_\lambda(x) := u_\lambda(x) - u(x).$$

Let us recall a version of the Hardy–Littlewood–Sobolev inequality that will be used in the method of moving planes in integral forms.

Lemma 2.1 *(Hardy–Littlewood–Sobolev inequality [19,28]). Let $0 < \alpha < n$ and $1 < p < q$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then, we have*

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \right\|_{L^q(\mathbb{R}^n)} \leq C_{n,\alpha,p,q}\|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$.

Proof of Theorem 1.1. One can observe that, for any $x \in \Sigma_\lambda$,

$$u_\lambda(x) - u(x) = \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) \left[u_\lambda^p(y) v_\lambda(y) - u^p(y) v(y) \right] \mathrm{d}y$$
$$= \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) \left[u_\lambda^p(y) - u^p(y) \right] v(y) \, \mathrm{d}y$$
$$+ \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) u_\lambda^p(y) \left[v_\lambda(y) - v(y) \right] \mathrm{d}y.$$

Similarly,

$$v_\lambda(x) - v(x) = \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-z|^{n-\beta}} - \frac{1}{|x^\lambda - z|^{n-\beta}} \right) \left[u_\lambda^q(z) - u^q(z) \right] \mathrm{d}z.$$

Combining the above two formulas, we obtain

$$\begin{aligned} w_\lambda(x) = & \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) \left[u_\lambda^p(y) - u^p(y) \right] v(y) \, \mathrm{d}y \\ & + \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) u_\lambda^p(y) \int\limits_{\Sigma_\lambda} \left(\frac{1}{|y-z|^{n-\beta}} - \frac{1}{|y^\lambda - z|^{n-\beta}} \right) \left[u_\lambda^q(z) - u^q(z) \right] \mathrm{d}z \, \mathrm{d}y. \end{aligned} \tag{2.1}$$

Let us define

$$\Sigma_\lambda^- = \{ x \in \Sigma_\lambda \mid w_\lambda(x) < 0 \}.$$

Using the mean value theorem, we get from (2.1) that, for any $x \in \Sigma_\lambda^-$,

$$\begin{aligned} 0 > w_\lambda(x) \geq & \, p \int\limits_{\Sigma_\lambda^-} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) u^{p-1}(y) w_\lambda(y) v(y) \, \mathrm{d}y \\ & + q \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) u_\lambda^p(y) \int\limits_{\Sigma_\lambda^-} \left(\frac{1}{|y-z|^{n-\beta}} - \frac{1}{|y^\lambda - z|^{n-\beta}} \right) u^{q-1}(z) w_\lambda(z) \, \mathrm{d}z \, \mathrm{d}y \\ \geq & \, p \int\limits_{\Sigma_\lambda^-} \frac{u^{p-1}(y) v(y) w_\lambda(y)}{|x-y|^{n-\alpha}} \mathrm{d}y + q \int\limits_{\Sigma_\lambda} \frac{u_\lambda^p(y)}{|x-y|^{n-\alpha}} \int\limits_{\Sigma_\lambda^-} \frac{u^{q-1}(z) w_\lambda(z)}{|y-z|^{n-\beta}} \mathrm{d}z \, \mathrm{d}y. \end{aligned} \tag{2.2}$$

We choose any $r > \max\{\frac{n}{n-\alpha}, \frac{2n}{(p+2)(n-\alpha)-2\beta}\}$. If $p > \frac{2\alpha}{n-\alpha}$, then r is chosen in such a way that it also satisfies $r < \frac{2n}{p(n-\alpha)-2\alpha}$. We apply the Hardy–Littlewood–Sobolev inequality to (2.2) to obtain

$$\|w_\lambda\|_{L^r(\Sigma_\lambda^-)} \leq C \|u^{p-1} v w_\lambda\|_{L^{\frac{nr}{n+\alpha r}}(\Sigma_\lambda^-)} + C \left\| u_\lambda^p(y) \int\limits_{\Sigma_\lambda^-} \frac{u^{q-1}(z) w_\lambda(z)}{|y-z|^{n-\beta}} \mathrm{d}z \right\|_{L^{\frac{nr}{n+\alpha r}}(\Sigma_\lambda^-)}. \tag{2.3}$$

On the one hand,

$$\begin{aligned} \|u^{p-1} v w_\lambda\|_{L^{\frac{nr}{n+\alpha r}}(\Sigma_\lambda^-)} & \leq \|u^{p-1}\|_{L^{\frac{2n}{(n-\alpha)(p-1)}}(\Sigma_\lambda^-)} \|v\|_{L^{\frac{2n}{(n-\alpha)q-2\beta}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^r(\Sigma_\lambda^-)} \\ & = \|u\|^{p-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)} \left\| \int\limits_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\beta}} \mathrm{d}y \right\|_{L^{\frac{2n}{(n-\alpha)q-2\beta}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^r(\Sigma_\lambda^-)} \\ & \leq C \|u\|^{p-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)} \|u^q\|_{L^{\frac{2n}{(n-\alpha)q}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^r(\Sigma_\lambda^-)} \\ & = C \|u\|^{p+q-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^r(\Sigma_\lambda^-)}, \end{aligned} \tag{2.4}$$

where we use the convention that $L^{\frac{2n}{(n-\alpha)(p-1)}}(\Sigma_\lambda^-)=L^\infty(\Sigma_\lambda^-)$ if $p=1$.

On the other hand,

$$\begin{aligned}\left\| u_\lambda^p(y)\int_{\Sigma_\lambda^-}\frac{u^{q-1}(z)w_\lambda(z)}{|y-z|^{n-\beta}}\,dz\right\|_{L^{\frac{nr}{n+\alpha r}}(\Sigma_\lambda^-)} &\le \|u_\lambda^p\|_{L^{\frac{2n}{(n-\alpha)p}}(\Sigma_\lambda^-)}\left\|\int_{\Sigma_\lambda^-}\frac{u^{q-1}(z)w_\lambda(z)}{|y-z|^{n-\beta}}\,dz\right\|_{L^{\frac{2nr}{2n+2\alpha r-(n-\alpha)pr}}(\Sigma_\lambda^-)}\\ &\le C\|u_\lambda^p\|_{L^{\frac{2n}{(n-\alpha)p}}(\mathbb{R}^n)}\|u^{q-1}w_\lambda\|_{L^{\frac{2nr}{2n+2(\alpha+\beta)r-(n-\alpha)pr}}(\Sigma_\lambda^-)}\\ &\le C\|u^p\|_{L^{\frac{2n}{(n-\alpha)p}}(\mathbb{R}^n)}\|u^{q-1}\|_{L^{\frac{2n}{(n-\alpha)(q-1)}}(\Sigma_\lambda^-)}\|w_\lambda\|_{L^r(\Sigma_\lambda^-)}\\ &= C\|u\|^p_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)}\|u\|^{q-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}\|w_\lambda\|_{L^r(\Sigma_\lambda^-)}.\end{aligned}\tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3), we arrive at the following key estimate

$$\|w_\lambda\|_{L^r(\Sigma_\lambda^-)}\le C\left(\|u\|^{p+q-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}+\|u\|^p_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)}\|u\|^{q-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}\right)\|w_\lambda\|_{L^r(\Sigma_\lambda^-)}.\tag{2.6}$$

With the aid of (2.6), we are able to start moving the plane T_λ from near $\lambda=-\infty$ to the right until it reaches the limiting position in order to derive symmetry. This procedure contains two steps.

Step 1. We show that, for λ sufficiently negative,

$$w_\lambda\ge 0\quad\text{in } \Sigma_\lambda.\tag{2.7}$$

Indeed, since $u\in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ and $q>1$, we can choose $R_0>0$ sufficiently large that, for $\lambda\le -R_0$, we have

$$\|u\|^{p+q-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}+\|u\|^p_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)}\|u\|^{q-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}\le\frac{1}{2C},\tag{2.8}$$

where the constant C is the same as in (2.6).

Therefore, (2.6) and (2.8) imply that $\|w_\lambda\|_{L^r(\Sigma_\lambda^-)}=0$ and hence $|\Sigma_\lambda^-|=0$ for $\lambda\le -R_0$. Furthermore, we can deduce from (2.1) that $w_\lambda(x)\ge 0$ for any $x\in\Sigma_\lambda$. Thus $\Sigma_\lambda^-=\emptyset$ and (2.7) holds for $\lambda\le -R_0$. This completes Step 1.

Step 2. Let

$$\lambda_0=\sup\{\lambda\in\mathbb{R}\mid w_\mu\ge 0 \text{ in } \Sigma_\mu \text{ for all } \mu\le\lambda\}.\tag{2.9}$$

By using a similar argument as in Step 1, we can also start moving the plane from near $\lambda=+\infty$ to the left, thus we must have $\lambda_0<\infty$. Now, we will show that

$$w_{\lambda_0}=0\quad\text{in } \Sigma_{\lambda_0}.\tag{2.10}$$

Suppose, on the contrary, that $w_{\lambda_0}\ge 0$, but that w_{λ_0} is not identically zero in Σ_{λ_0}. Using (2.1), we deduce that $w_{\lambda_0}>0$ in Σ_{λ_0}. We will obtain a contradiction to (2.9) by showing the existence of an $\varepsilon>0$ small enough that $w_\lambda\ge 0$ in Σ_λ for all $\lambda\in[\lambda_0,\lambda_0+\varepsilon)$.

It can be clearly seen from (2.6) that our main task is to prove the existence of $\varepsilon>0$ sufficiently small that

$$\|u\|^{p+q-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}+\|u\|^p_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)}\|u\|^{q-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^-)}\le\frac{1}{2C}\tag{2.11}$$

for all $\lambda\in[\lambda_0,\lambda_0+\varepsilon)$, where the constant C is the same as in (2.6).

Since $u\in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$, there exists $R>0$ large enough that

$$\|u\|^{p+q-1}_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n\setminus B_R(0))}+\|u\|^p_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)}\|u\|^{q-1}_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n\setminus B_R(0))}\le\frac{1}{4C}.\tag{2.12}$$

Now fix this R; in order to derive (2.11), we only need to show that

$$\lim_{\lambda\to\lambda_0^+}|\Sigma_\lambda^-\cap B_R(0)|=0.\tag{2.13}$$

To prove this, we define $E_\delta=\{x\in\Sigma_{\lambda_0}\cap B_R(0)\mid w_{\lambda_0}(x)>\delta\}$ and $F_\delta=\{x\in\Sigma_{\lambda_0}\cap B_R(0)\mid w_{\lambda_0}(x)\le\delta\}$ for any $\delta>0$, and let $D_\lambda=(\Sigma_\lambda\setminus\Sigma_{\lambda_0})\cap B_R(0)$ for any $\lambda>\lambda_0$. Then,

$$\lim_{\delta\to 0^+} |F_\delta| = 0, \quad \lim_{\lambda\to\lambda_0^+} |D_\lambda| = 0 \tag{2.14}$$

and

$$\Sigma_\lambda^- \cap B_R(0) \subset \Sigma_\lambda^- \cap (E_\delta \cup F_\delta \cup D_\lambda) \subset (\Sigma_\lambda^- \cap E_\delta) \cup F_\delta \cup D_\lambda. \tag{2.15}$$

Therefore, for an arbitrarily fixed $\eta > 0$, one can choose $\delta > 0$ small enough that $|F_\delta| \le \eta$. For this fixed δ, we will point out that

$$\lim_{\lambda\to\lambda_0^+} |\Sigma_\lambda^- \cap E_\delta| = 0. \tag{2.16}$$

Indeed, for all $x \in \Sigma_\lambda^- \cap E_\delta$, we have $u(x^{\lambda_0}) - u(x^\lambda) = w_{\lambda_0}(x) - w_\lambda(x) > \delta$. It follows that $\Sigma_\lambda^- \cap E_\delta \subset G_\delta^\lambda := \{x \in B_R(0) \mid u(x^{\lambda_0}) - u(x^\lambda) > \delta\}$. By Chebyshev's inequality, we get

$$\begin{aligned}
|G_\delta^\lambda| &\le \frac{1}{\delta^{\frac{2n}{n-\alpha}}} \int_{G_\delta^\lambda} \left|u(x^{\lambda_0}) - u(x^\lambda)\right|^{\frac{2n}{n-\alpha}} \, dx \\
&= \frac{1}{\delta^{\frac{2n}{n-\alpha}}} \int_{B_R(2\lambda_0 e_1)} |u(x) - u(x + 2(\lambda_0 - \lambda)e_1)|^{\frac{2n}{n-\alpha}} \, dx,
\end{aligned}$$

where $e_1 = (1, 0, \dots, 0)$. Hence, $\lim_{\lambda\to\lambda_0^+} |G_\delta^\lambda| = 0$, from which (2.16) follows.

Therefore, by (2.14), (2.15) and (2.16), we have:

$$\lim_{\lambda\to\lambda_0^+} |\Sigma_\lambda^- \cap B_R(0)| \le |F_\delta| \le \eta.$$

This implies (2.13), since $\eta > 0$ is arbitrarily chosen. From (2.12) and (2.13), we arrive at (2.11).

Now we deduce from (2.6) and (2.11) that there exists an $\varepsilon > 0$ sufficiently small that $|\Sigma_\lambda^-| = 0$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$. Furthermore, by (2.1), we must have $w_\lambda \ge 0$ in Σ_λ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$. This contradicts the definition of λ_0 in (2.9). Therefore, (2.10) must hold.

Since Eq. (1.1) is invariant under rotation, the x_1 direction can be chosen arbitrarily; we conclude that u must be radially symmetric and monotone decreasing about some point $x^0 \in \mathbb{R}^n$. □

3. Classification of positive solutions

Proposition 3.1. *Assume that $0 < \alpha, \beta < n$ and $p + 1 = q = \frac{n+\beta}{n-\alpha}$. Then Q, which is defined in Theorem 1.2, is a positive solution to (1.1).*

Proof. The following identity was obtained in [8] for any $0 < s < \frac{n}{2}$ (see (37) in [8])

$$\int_{\mathbb{R}^n} \frac{1}{|x-y|^{2s}} \left(\frac{1}{1+|y|^2}\right)^{n-s} dy = I(s) \left(\frac{1}{1+|x|^2}\right)^s. \tag{3.1}$$

Denoting

$$d = \left[I\left(\frac{n-\beta}{2}\right) I\left(\frac{n-\alpha}{2}\right)\right]^{-\frac{n-\alpha}{2(\alpha+\beta)}}$$

and using (3.1), we have:

$$\int_{\mathbb{R}^n} \frac{Q^q(z)}{|y-z|^{n-\beta}} dz = \int_{\mathbb{R}^n} \frac{d^{\frac{n+\beta}{n-\alpha}}}{|y-z|^{n-\beta}} \left(\frac{1}{1+|y|^2}\right)^{\frac{n+\beta}{2}} dz = d^{\frac{n+\beta}{n-\alpha}} I\left(\frac{n-\beta}{2}\right) \left(\frac{1}{1+|y|^2}\right)^{\frac{n-\beta}{2}}.$$

Therefore, for all $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \frac{Q^p(y)}{|x-y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{Q^q(z)}{|y-z|^{n-\beta}} dz\, dy = d^{\frac{n+\alpha+2\beta}{n-\alpha}} I\left(\frac{n-\beta}{2}\right) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n+\alpha}{2}} dy$$

$$= d^{\frac{n+\alpha+2\beta}{n-\alpha}} I\left(\frac{n-\beta}{2}\right) I\left(\frac{n-\alpha}{2}\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{n-\alpha}{2}}$$
$$= \mathrm{d}\left(\frac{1}{1+|x|^2}\right)^{\frac{n-\alpha}{2}} = Q(x).$$

Hence, Q is a solution to (1.1). □

Using the ideas from [5,8,11], we are able to prove the following uniqueness result.

Proposition 3.2. *Assume that $0<\alpha,\beta<n$, $p+1=q=\frac{n+\beta}{n-\alpha}\geq 2$ and $u\in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ is a positive solution to (1.1). Then there exist $\mu>0$ and $x^0\in\mathbb{R}^n$ such that*

$$u(x)=\mu^{\frac{n-\alpha}{2}}Q\left(\mu(x-x^0)\right) \quad \textit{for all } x\in\mathbb{R}^n,$$

where Q is defined in Theorem 1.2.

Proof. By Theorem 1.1 and the invariance of Eq. (1.1) under translations and scalings, we may assume that $u\in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ is radially symmetric and monotone decreasing about some $x^0\in\mathbb{R}^n$. Moreover, one can observe that u satisfies the following asymptotic property

$$u_\infty := \lim_{|x|\to\infty}|x|^{n-\alpha}u(x)<\infty.$$

Step 1. We claim that if $x^0=0$, then

$$u(sx+a)=\frac{1}{|x|^{n-\alpha}}u\left(\frac{sx}{|x|^2}+a\right) \tag{3.2}$$

for any $x\in\mathbb{R}^n\setminus\{0\}$ and $a\in\mathbb{R}^n$, where $s=\left(\frac{u_\infty}{u(a)}\right)^{\frac{1}{n-\alpha}}$.

We first assume $a=0$. Let $x^1\in\mathbb{R}^n\setminus\{0\}$ be any fixed point and $e=\frac{x^1}{|x^1|}$. We define

$$w(x)=\frac{1}{|x|^{n-\alpha}}u\left(s(\frac{x}{|x|^2}-e)\right), \tag{3.3}$$

then it is clear that $w(0)=s^{\alpha-n}u_\infty=u(0)=w(e)$ and $s^{\frac{n-\alpha}{2}}w$ is also a positive solution to (1.1). Therefore, w must be radially symmetric with respect to some point $\bar{x}$ that lies on the hyperplane $e^\perp+\frac{1}{2}e$ through $\frac{1}{2}e$, which is perpendicular to e. Furthermore, since u is radially symmetric about 0, for any $\frac{1}{2}<r<1$, consider two different points $y^1, y^2\in\partial B_r(0)\cap\partial B_r(e)$, we can deduce from (3.3) that

$$w(y^1)=\frac{1}{|y^1|^{n-\alpha}}u\left(s(\frac{y^1}{|y^1|^2}-e)\right)=\frac{1}{|y^2|^{n-\alpha}}u\left(s(\frac{y^2}{|y^2|^2}-e)\right)=w(y^2).$$

Therefore, $w(x)=w(|x-\frac{1}{2}e|)$ on the hyperplane $e^\perp+\frac{1}{2}e$, and hence $\bar{x}=\frac{1}{2}e$ and w is actually radially symmetric about $\frac{1}{2}e$.

We choose some $\eta\in(-\frac{1}{2},\frac{1}{2})$ such that $|x^1|=\frac{\frac{1}{2}+\eta}{\frac{1}{2}-\eta}$, then, from (3.3), we have

$$\frac{1}{|\frac{1}{2}-\eta|^{n-\alpha}}u\left(s\frac{\frac{1}{2}+\eta}{\frac{1}{2}-\eta}e\right)=w\left((\frac{1}{2}-\eta)e\right)=w\left((\frac{1}{2}+\eta)e\right)=\frac{1}{|\frac{1}{2}+\eta|^{n-\alpha}}u\left(s\frac{\frac{1}{2}-\eta}{\frac{1}{2}+\eta}e\right),$$

which implies that

$$u(sx^1)=\frac{1}{|x^1|^{n-\alpha}}u\left(\frac{sx^1}{|x^1|^2}\right).$$

If $a\neq 0$, then (3.2) follows by considering $u(\cdot+a)$ instead of u itself.

Step 2. We have

$$u(x^0)u_\infty=\left[I\left(\frac{n-\alpha}{2}\right)I\left(\frac{n-\beta}{2}\right)\right]^{-\frac{n-\alpha}{\alpha+\beta}}. \tag{3.4}$$

To prove (3.4), we may use (3.2) and similar arguments as in the proof of [5, Lemma 3.2].

Step 3. We define

$$v(x)=\mu^{-\frac{n-\alpha}{2}}u(\mu^{-1}x+x^0),$$

where $\mu=\left(\frac{u(x_0)}{u_\infty}\right)^{\frac{1}{n-\alpha}}$, then v is also a positive solution to (1.1) with radial symmetry about the origin and $v(0)=v_\infty$. We show that $v=Q$ in $\mathbb{R}^n$.

To prove this, using (3.2) and (3.4), similar argument as in [5, p. 338] yields that $v\le Q$ in $\mathbb{R}^n$ or $v\ge Q$ in $\mathbb{R}^n$. Since (3.4) implies $v(0)=Q(0)$ and v, Q are solutions to Eq. (1.1), we must have $v=Q$ in $\mathbb{R}^n$. Hence,

$$u(x)=\mu^{\frac{n-\alpha}{2}}Q(\mu(x-x^0)) \quad \text{for all } x\in\mathbb{R}^n.$$

The proof is completed. □

Proof of Theorem 1.2. Theorem 1.2 is a consequence of Propositions 3.1 and 3.2. □

Remark 3.1. Instead of relying on Theorem 1.1, one may also use the method of moving spheres in integral forms (see [17, 31]) to derive the property (3.2) directly when $p+1=q=\frac{n+\beta}{n-\alpha}\ge 2$. Then the classification result follows as above. This approach would lead us to another proof of Theorem 1.2.

However, the method of moving spheres cannot be used to prove the symmetry of positive solutions to (1.1) in the case $p+1\neq q$. Therefore, we have to use the moving plane method to establish Theorem 1.1. Then we choose to exploit that symmetry result to prove Theorem 1.2 for the consistency of the presentation of this paper. It is well known that, if u satisfies (3.2), then u has the form $u(x)=c\left(\frac{t}{t^2+|x-x^0|^2}\right)^{\frac{n-\alpha}{2}}$ for some $c,t>0$ and $x^0\in\mathbb{R}^n$ (see [5,17]). The purpose of Proposition 3.1 is that it gives an explicit formula for c so that such u is a solution to (1.1).

4. Choquard-type equations and the Hardy–Littlewood–Sobolev inequality

This last section is devoted to the proofs of the equivalence between Eqs. (1.1) and (1.2). We also compute the best constant $S_{n,\alpha,\beta}$ in the Hardy–Littlewood–Sobolev inequality (1.7).

Proof of Theorem 1.3. Assume $u\in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is a positive solution to Eq. (1.2). For any $\phi\in C_0^\infty(\mathbb{R}^n)$, let

$$\psi(x)=\int_{\mathbb{R}^n}\frac{R_{n,\alpha}\phi(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y.$$

Then $(-\Delta)^{\frac{\alpha}{2}}\psi=\phi$. Consequently, $\psi\in H^\alpha(\mathbb{R}^n)\subset H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ and hence (1.3) holds for ψ

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{\alpha}{4}}u(x)(-\Delta)^{\frac{\alpha}{4}}\psi(x)\,\mathrm{d}x=\frac{1}{R_{n,\alpha}}\int_{\mathbb{R}^n}\left(\frac{1}{|x|^{n-\beta}}*u^q\right)(x)u^p(x)\psi(x)\,\mathrm{d}x.$$

Integrating by parts the left-hand side and exchanging the order of integration of the right-hand side yield that

$$\int_{\mathbb{R}^n}u(x)\phi(x)\,\mathrm{d}x=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{u^p(y)}{|x-y|^{n-\alpha}}\int_{\mathbb{R}^n}\frac{u^q(z)}{|y-z|^{n-\beta}}\,\mathrm{d}z\,\mathrm{d}y\right)\phi(x)\,\mathrm{d}x.$$

Since $\phi\in C_0^\infty(\mathbb{R}^n)$ is arbitrary, we conclude that u satisfies the integral equation (1.1).

Now we assume $u\in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is a positive weak solution to Eq. (1.1). Applying the Fourier transform to both sides of (1.1) (see [28]), we get

$$\mathcal{F}u(\xi)=\frac{1}{R_{n,\alpha}(2\pi|\xi|)^\alpha}\mathcal{F}\left[\left(\frac{1}{|x|^{n-\beta}}*u^q\right)u^p\right](\xi).$$

It follows that, for any $\phi\in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n}(-\Delta)^{\frac{\alpha}{4}}u(-\Delta)^{\frac{\alpha}{4}}\phi\,\mathrm{d}x&=\frac{1}{R_{n,\alpha}}\int_{\mathbb{R}^n}\mathcal{F}\left[\left(\frac{1}{|x|^{n-\beta}}*u^q\right)u^p\right]\overline{\mathcal{F}\phi}\,\mathrm{d}\xi\\&=\frac{1}{R_{n,\alpha}}\int_{\mathbb{R}^n}\left(\frac{1}{|x|^{n-\beta}}*u^q\right)u^p\phi\,\mathrm{d}x,\end{aligned}$$

which implies that u is also a weak solution to Eq. (1.2). □

Proof of Proposition 1.5. Equation (1.2) is the corresponding Euler–Lagrange equation for the minimization problem described in (1.8). Since minimization problem (1.8) can be attained by the extremal function Q defined in Theorem 1.2, one can deduce from the definition of $S_{n,\alpha,\beta}$, Eq. (1.2) and Proposition 1.4 that

$$\|Q\|_{\dot H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = S_{n,\alpha,\beta}\|Q\|_{L^{\alpha,\beta}} \quad \text{and} \quad \|Q\|^2_{\dot H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \frac{1}{R_{n,\alpha}}\|Q\|^{\frac{2(n+\beta)}{n-\alpha}}_{L^{\alpha,\beta}}.$$

Therefore, the best constant $S_{n,\alpha,\beta}$ for the Hardy–Littlewood–Sobolev inequality (1.7) can be calculated as

$$S_{n,\alpha,\beta} = R_{n,\alpha}^{\frac{\alpha-n}{2(n+\beta)}}\|Q\|^{\frac{\alpha+\beta}{n+\beta}}_{\dot H^{\frac{\alpha}{2}}(\mathbb{R}^n)}. \tag{4.1}$$

Let $S_{n,\alpha}$ be the best constant in the Sobolev inequality

$$\|u\|_{\dot H^{\frac{\alpha}{2}}(\mathbb{R}^n)} \geq S_{n,\alpha}\|u\|_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)}$$

for any $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$. Then $S_{n,\alpha}$ is given by

$$S_{n,\alpha} = (4\pi)^{\frac{\alpha}{4}}\left(\frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}\right)^{\frac{1}{2}}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right)^{\frac{\alpha}{2n}}, \tag{4.2}$$

(see [6,19]) and $S_{n,\alpha}$ is achieved by the extremal function

$$P(x) := I\left(\frac{n-\alpha}{2}\right)^{-\frac{n-\alpha}{2\alpha}}\left(\frac{1}{1+|x|^2}\right)^{\frac{n-\alpha}{2}},$$

which solves the critical fractional Lane–Emden equation

$$(-\Delta)^{\frac{\alpha}{2}}u = \frac{1}{R_{n,\alpha}}u^{\frac{n+\alpha}{n-\alpha}} \quad \text{in } \mathbb{R}^n.$$

Hence,

$$\|P\|_{\dot H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = S_{n,\alpha}\|P\|_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)} \quad \text{and} \quad \|P\|^2_{\dot H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = \frac{1}{R_{n,\alpha}}\|P\|^{\frac{2n}{n-\alpha}}_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)}.$$

This leads to

$$\|P\|_{\dot H^{\frac{\alpha}{2}}(\mathbb{R}^n)} = R_{n,\alpha}^{\frac{n-\alpha}{2\alpha}}S_{n,\alpha}^{\frac{n}{\alpha}}. \tag{4.3}$$

From (4.1), (4.3), and the fact that

$$Q(x) = I\left(\frac{n-\alpha}{2}\right)^{\frac{\beta(n-\alpha)}{2\alpha(\alpha+\beta)}} I\left(\frac{n-\beta}{2}\right)^{-\frac{n-\alpha}{2(\alpha+\beta)}} P(x),$$

we have

$$S_{n,\alpha,\beta} = R_{n,\alpha}^{\frac{\beta(n-\alpha)}{2\alpha(n+\beta)}} I\left(\frac{n-\alpha}{2}\right)^{\frac{\beta(n-\alpha)}{2\alpha(n+\beta)}} I\left(\frac{n-\beta}{2}\right)^{-\frac{n-\alpha}{2(n+\beta)}} S_{n,\alpha}^{\frac{n(\alpha+\beta)}{\alpha(n+\beta)}}. \tag{4.4}$$

Now we may use (4.4), (4.2) and the definition of $R_{n,\alpha}$ and $I(s)$ to get the conclusion. □

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