Combinatorics/Number theory

On two congruence conjectures

Sur deux conjectures de congruence

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\textbf{A B S T R A C T}

In this paper, we mainly prove a congruence conjecture of M. Apagodu [3] and a supercongruence conjecture of Z.-W. Sun [25].

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\textbf{R É S U M É}


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1. Introduction

In the past few years, a lot of researchers worked on congruences for sums of binomial coefficients (see, for instance, [7,12–15,20,26,27]). In 2011, Sun and Tauraso [27] proved that, for any prime $p > 3$,

$$
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv p^2 \binom{p}{3},
$$

(1.1)

where ($\cdot$) denotes the Legendre symbol.

Pan and Sun [17] proved that, for any odd prime $p$,

$$
\sum_{n=0}^{p-1} (3n+1) \binom{2n}{n} \equiv_p \binom{p}{3}.
$$

Then Apagodu [3] gave the following conjecture.
Conjecture 1.1. For any odd prime \( p \), we have:
\[
\sum_{n=0}^{p-1} (5n + 1) \binom{4n}{2n} \equiv_p -\left( \frac{p}{3} \right).
\]

In this paper, we first prove the above conjecture and give another congruence.

Theorem 1.1. Conjecture 1.1 is true. And we prove that, for each odd prime \( p \),
\[
\sum_{n=0}^{p-1} (3n + 1) \binom{4n}{2n} \equiv_p -\frac{1}{5} \left( \frac{p}{5} \right).
\]

Recall that the Euler numbers and the Bernoulli numbers are given by
\[
E_0 = 1, \quad E_n = -\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} E_{n-2k} \quad \text{for} \quad n \in \mathbb{Z}^+ = \{1, 2, \ldots\},
\]
\[
B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n = 2, 3, \ldots).
\]

The well-known Catalan–Larcombe–French numbers \( P_0, P_1, P_2, \ldots \) (cf. [8]) are given by
\[
P_n = \sum_{k=0}^{n} \frac{(2k)!^2 (2n-k)!^2}{(n-k)! (k)!^2},
\]
which arose from the theory of elliptic integrals (see [11]). It is known that \((n + 1)P_{n+1} = (24n(n + 1) + 8)P_n - 128n^2 P_{n-1}\) for all \( n \in \mathbb{Z}^+ \). The sequence \((P_n)_{n \geq 0}\) is also related to the theory of modular forms. See D. Zagier [29].

Many researchers worked on the Catalan–Larcombe–French numbers, (see [9,8,13]). For instance, in 2017, the author proved that
\[
\sum_{k=0}^{p-1} \frac{P_k}{8^k} \equiv 1 + 2(-1)^{(p-1)/2} p^2 E_{p-3} \pmod{p^3},
\]
\[
\sum_{k=0}^{p-1} \frac{P_k}{16^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}.
\]

In [23], Sun proved that
\[
\sum_{k=0}^{(p-1)/2} \frac{(2k)!^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3},
\]
which plays an important role for proving the above two supercongruences involving \( P_n \).

The famous Domb numbers are defined by
\[
D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}.
\]

This ubiquitous sequence (see A002895 of Sloane [19]) not only arises in the theory of third-order Apéry-like differential equations [2], odd moments of Bessel functions in quantum field theory [4], uniform random walks in the plane [5], new series for \( 1/\pi \) [6], interacting systems on crystal lattices [29] and the enumeration of abelian squares of length \( 2n \) over an alphabet with 4 letters [18], but if
\[
F(z) = \frac{\eta^4(z)\eta^4(3z)}{\eta^2(2z)\eta^2(6z)} \quad \text{and} \quad t(z) = \left( \frac{\eta(6z)\eta(2z)}{\eta(z)\eta(3z)} \right)^6,
\]
then (see [6])
There are also many researchers working on the Domb numbers (see, [16,21]). For example, the author and Wang [16] confirmed a conjecture of Sun [22]: for any prime \( p > 3 \), we have:

\[
D(p - 1) \equiv 64^{p-1} - \frac{p^3}{6} B_{p-3} \pmod{p^4}.
\]

Motivated by the above work, we will work on the sequence of numbers defined by

\[
C_n = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n}{n-k}^2,
\]

which are the coefficients of the solutions to the Calabi-Yau equations. We know that the Calabi-Yau-type equation \( D_y = 0 \), with

\[
D = \theta^4 - 2^4 z(2\theta + 1)^2 (2\theta^2 + 2\theta + 1) + 2^{10} z^2 (\theta + 1)^2 (2\theta + 1)(2\theta + 3),
\]

has the solution [1, Appendix A, case #3∗]:

\[
y_0 = \sum_{n=0}^{\infty} z^n \frac{2n}{n} C_n.
\]

Sun [25] proved the following congruence involving \( C_n \)

\[
\sum_{n=0}^{p-1} \frac{n}{32^n} C_n \equiv 0 \pmod{p^3}.
\]

He gave the following conjecture:

**Conjecture 1.2.** Let \( p \) be an odd prime. Then

\[
\sum_{n=0}^{p-1} \frac{n}{32^n} C_n \equiv -2p^3 E_{p-3} \pmod{p^4}.
\]

In this paper, we confirm this conjecture.

**Theorem 1.2.** Conjecture 1.2 is true.

We end this introduction by giving the organization of this paper. We shall prove Theorem 1.1 in Section 2, and Section 3 is devoted to prove Theorem 1.2.

2. **Proof of Theorem 1.1**

**Lemma 2.1.** Let \( p \) be an odd prime. Then

\[
\binom{2k}{k} \equiv_p \binom{(p - 1)/2}{k} (-4)^k.
\]

**Proof.** It is easy to see that

\[
\binom{(p - 1)/2}{k} (-4)^k = \frac{(p^2 - 1) \cdots (p^2 - k + 1)}{k!} (-4)^k = \frac{(1 - p)(3 - p) \cdots (2k - 1 - p) 2^k}{k!} \equiv_p \frac{1 \cdots (2k - 1)}{k!} 2^k = \binom{2k}{k}.
\]

Now we finish the proof of Lemma 2.1. \( \square \)
We shall separate the left-hand side of Conjecture 1.1 into two parts, one is \( \vartheta_1 = \sum_{n=0}^{p-1} \left( \frac{4n}{2n} \right) \), the other is \( \vartheta_2 = \sum_{n=0}^{p-1} \left( \frac{4n}{2n} \right) \).

We only consider \( p > 5 \), the cases \( p = 3 \) and \( p = 5 \) can be checked directly. We calculate \( \vartheta_1 \) first. It is easy to see the identity as follows:

\[
\vartheta_1 = \frac{1}{2} \sum_{k=0}^{2p-1} \binom{2k}{k} + \sum_{k=0}^{2p-1} \binom{2k}{k} - \binom{2k}{k}.
\]

By the Lucas congruence and (1.1), we have:

\[
\sum_{k=0}^{2p-1} \binom{2k}{k} = \sum_{k=0}^{p-1} \binom{2k}{k} + \sum_{k=p}^{2p-1} \binom{2k}{k} = \sum_{k=0}^{p-1} \binom{2k}{k} + \sum_{k=0}^{p-1} \left( \frac{2p + 2k}{p + k} \right) = \equiv_p \sum_{k=0}^{p-1} \binom{2k}{k} + \sum_{k=0}^{p-1} \left( \frac{2k}{1} \right) = 3 \sum_{k=0}^{p-1} \binom{2k}{k} = \equiv_p \frac{3}{3}.
\]

and

\[
\sum_{k=0}^{2p-1} (-1)^k \binom{2k}{k} = \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k} + \sum_{k=p}^{2p-1} (-1)^k \binom{2k}{k} = \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k} + \sum_{k=0}^{p-1} (-1)^{p+k} \binom{p + 2k}{p + k} = \equiv_p \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k} - \sum_{k=0}^{p-1} (-1)^{p+k} \binom{2k}{1} = - \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}.
\]

Note that \( \binom{2k}{k} \equiv 0 \pmod{p} \) for each \( k \) such that \( p/2 < k < p \). So, by Lemma 2.1, we have:

\[
\sum_{k=0}^{2p-1} (-1)^k \binom{2k}{k} = \equiv_p - \sum_{k=0}^{p-1} \binom{2k}{k} - 1 = - \left( 2n + 1 \right) \equiv_p \frac{5}{p} = - \left( \frac{5}{p} \right).
\]

Hence,

\[
\vartheta_1 = \frac{1}{2} \left( 3 \left( \frac{p}{3} \right) - \left( \frac{5}{p} \right) \right).
\]

(2.1)

Now we turn to calculate \( \vartheta_2 \); like the identity of \( \vartheta_1 \), we have the following identity:

\[
\vartheta_2 = \frac{1}{4} \sum_{k=0}^{2p-1} \binom{2k}{k} + \sum_{k=0}^{2p-1} (-1)^k \binom{2k}{k}.
\]

It is easy to see that

\[
\sum_{k=0}^{2p-1} \binom{2k}{k} = \sum_{k=0}^{p-1} \binom{2k}{k} + \sum_{k=p}^{2p-1} \binom{2k}{k} = \sum_{k=0}^{p-1} \binom{2k}{k} + \sum_{k=0}^{p-1} \binom{p + 2k}{p + k} \]

and

\[
\sum_{k=0}^{2p-1} (-1)^k \binom{2k}{k} = \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k} + \sum_{k=p}^{2p-1} (-1)^k \binom{2k}{k} = \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k} + \sum_{k=0}^{p-1} \binom{p + k}{p + k} \binom{2p + 2k}{p + k}.
\]
Then, by Lucas congruence, we have:

\[
\sum_{k=0}^{2p-1} k \binom{2k}{k} \equiv p \sum_{k=0}^{p-1} k \binom{2k}{k} + (p-1)/2 \sum_{k=0}^{p-1} k \binom{2k}{1} \binom{2k}{k} = 3 \sum_{k=0}^{p-1} k \binom{2k}{k} 
\]

\[
\equiv p 3 \sum_{k=0}^{(p-1)/2} k \binom{(p-1)/2}{k} (-4)^k 
\]

\[
\equiv p 6 \sum_{k=0}^{(p-3)/2} \binom{(p-3)/2}{k} (-4)^k \equiv p -2 \binom{3}{p} = -2 \binom{3}{p}
\]

and

\[
\sum_{k=0}^{2p-1} (-1)^k k \binom{2k}{k} \equiv p \sum_{k=0}^{p-1} (-1)^k k \binom{2k}{k} - \sum_{k=0}^{p-1} (-1)^k \binom{2k}{1} \binom{2k}{k} 
\]

\[
= - \sum_{k=0}^{p-1} (-1)^k k \binom{2k}{k} \equiv p - \sum_{k=0}^{(p-1)/2} k \binom{(p-1)/2}{k} 4^k 
\]

\[
\equiv p 2 \cdot 5^{(p-3)/2} \equiv p \frac{2}{5} \left( \frac{p}{5} \right).
\]

Therefore,

\[
\vartheta_2 \equiv p \frac{1}{4} \left( -2 \binom{3}{p} + \frac{2}{5} \binom{5}{p} \right) = - \frac{1}{2} \binom{3}{p} + \frac{1}{10} \binom{5}{p}. \tag{2.2}
\]

Combining (2.1) and (2.2), we immediately obtain that

\[
\sum_{n=0}^{p-1} (5n+1) \binom{4n}{2n} = \vartheta_1 + 5 \vartheta_2 \equiv p - \binom{3}{p}
\]

and

\[
\sum_{n=0}^{p-1} (3n+1) \binom{4n}{2n} \equiv p - \frac{1}{5} \binom{5}{p}.
\]

So the proof of Theorem 1.1 is complete. \(\square\)

3. **Proof of Theorem 1.2**

**Lemma 3.1.** ([23, Lemma 2.1]) Let \(p\) be an odd prime. Then, for any \(k = 1, \ldots, p - 1\), we have:

\[
k \binom{2k}{k} \binom{2(p-k)}{p-k} \equiv (-1)^{2k/p} -1 \pmod{p^2}.
\]

**Lemma 3.2.** ([24, Lemma 3.1]) For any \(n = 0, 1, 2, \ldots\), we have:

\[
\sum_{k=0}^{n} k \binom{2k}{k} \binom{2n-2k}{n-k} = \sum_{k=0}^{n} k \binom{2k}{k} \binom{2n}{n-k} (-16)^{n-k}.
\]

**Lemma 3.3.** (Sun [23, (1.4)]) For any prime \(p > 3\), we have:

\[
\sum_{k=1}^{(p-1)/2} \frac{4^k}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} 4E_{p-3} \pmod{p}.
\]
Proof of Theorem 1.2. The \( p = 3 \) case is easy to check. So we just need to prove that for \( p > 3 \). First by Lemma 3.2, we have:

\[
\sum_{n=0}^{p-1} \frac{n}{32^n} C_n = \sum_{n=0}^{p-1} \frac{n}{32^n} \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{k}{n-k} (-16)^{n-k}
\]

\[
= \sum_{k=0}^{p-1} \binom{2k}{k}^3 \sum_{n=k}^{p-1} \frac{n}{32^n} \binom{k}{n-k} (-16)^{n-k}
\]

\[
= \sum_{k=0}^{p-1} \binom{2k}{k}^3 \sum_{n=0}^{p-1-k} (n+k) \binom{k}{n} \left(-\frac{1}{2}\right)^n.
\]

Then we divide the sum into two parts for \( p - 1 - k \geq k \) and \( p - 1 - k < k \). Set

\[
\theta_1 = \sum_{k=0}^{(p-1)/2} \frac{(2k)!}{32^k} \sum_{n=0}^{p-1-k} (n+k) \binom{k}{n} \left(-\frac{1}{2}\right)^n
\]

and

\[
\theta_2 = \sum_{k=(p+1)/2}^{p-1} \frac{(2k)!}{32^k} \sum_{n=0}^{p-1-k} (n+k) \binom{k}{n} \left(-\frac{1}{2}\right)^n.
\]

Thus,

\[
\sum_{n=0}^{p-1} \frac{n}{32^n} C_n = \theta_1 + \theta_2.
\]

Now we calculate \( \theta_1 \). Recall that we have \( p - 1 - k \geq k \); thus,

\[
\sum_{n=0}^{p-1-k} (n+k) \binom{k}{n} \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{k} (n+k) \binom{k}{n} \left(-\frac{1}{2}\right)^n
\]

\[
= \binom{k}{2k} + k \sum_{n=1}^{k} \binom{k-1}{n} \left(-\frac{1}{2}\right)^n
\]

\[
= \binom{k}{2k} - k \frac{1}{2k} = 0.
\]

Hence

\[
\theta_1 = 0.
\]

Then we turn to compute \( \theta_2 \); now \( p - 1 - k < k \); by Lemma 3.1, we have:

\[
\theta_2 = \sum_{k=1}^{(p-1)/2} \frac{(2p-2k)!}{32^k} \sum_{n=0}^{k-1} (n+p-k) \binom{p-k}{n} \left(-\frac{1}{2}\right)^n
\]

\[
= - \frac{p^3}{4} \sum_{k=1}^{(p-1)/2} \frac{32^k}{k^3} \sum_{n=0}^{k-1} (n-k) \binom{-k}{n} \left(-\frac{1}{2}\right)^n \pmod{p^4}.
\]

Note that \( \binom{-k}{n} = (-1)^n \binom{n+k-1}{n} \); we have:

\[
\sum_{n=0}^{k-1} (n-k) \binom{-k}{n} \left(-\frac{1}{2}\right)^n
\]

\[
= - k \sum_{n=0}^{k-1} \binom{k}{n} \left(-\frac{1}{2}\right)^n - k \sum_{n=1}^{k-1} \binom{k-1}{n} \left(-\frac{1}{2}\right)^n
\]

\[
= - k \sum_{n=0}^{k-1} \binom{n+k-1}{n} \left(-\frac{1}{2}\right)^n + k \sum_{n=0}^{k-2} \binom{n+k}{n} \frac{1}{2^n}.
\]
By taking $m = n$ and $x = 1/2$ in, e.g., [10, (1.1)], we obtain that
\begin{equation*}
\sum_{n=0}^{k} \binom{n+k}{n} \frac{1}{2^n} = 2^k.
\end{equation*}
So,
\begin{equation*}
\begin{align*}
\sum_{n=0}^{k-1} (n-k) \binom{-k}{n} \left( -\frac{1}{2} \right)^n &= -k2^{k-1} \left( \frac{k}{2} \right) - \left( \sum_{n=0}^{k} \binom{n+k}{n} \frac{1}{2^n} \right) \left( \frac{2k-1}{k-1} \right) - \left( \frac{2k}{2^k} \right) \\
&= -k2^{k-1} + \frac{k}{2} \left( 2^k - \frac{2k}{2^k} \right) = -k \frac{2k}{2^k}.
\end{align*}
\end{equation*}
Hence,
\begin{equation*}
\theta_2 \equiv \frac{p}{4} \sum_{k=1}^{(p-1)/2} \frac{\sum_{k=0}^{n} \left( \sum_{k=0}^{n} \frac{1}{n^2} \right) = 1}{\binom{n}{k}^2 (2k) \binom{2k}{k}} \equiv \frac{16k}{4} \sum_{k=1}^{(p-1)/2} \frac{\sum_{k=0}^{n} \frac{1}{n^2} \right) = 1}{\binom{n}{k}^2 (2k) \binom{2k}{k}} \binom{2k}{k} \mod p^4.
\end{equation*}
Set $n = (p-1)/2$; then we have:
\begin{equation*}
\binom{p}{2} \frac{\sum_{k=0}^{n} \frac{1}{n^2} \right) = 1}{\binom{n}{k}^2 (2k) \binom{2k}{k}} \equiv \frac{4}{4} \sum_{k=0}^{(n-1)/2} \binom{n-1}{k} \binom{2k}{k} \mod p.
\end{equation*}
We have the following identity in [28],
\begin{equation*}
\sum_{k=0}^{n-1} \frac{1}{n^2} = \frac{2n^2}{n+1} \sum_{k=0}^{n} \frac{1}{n^2} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k^2}.
\end{equation*}
So,
\begin{equation*}
\binom{p}{2} \frac{\sum_{k=0}^{n} \frac{1}{n^2} \right) = 1}{\binom{n}{k}^2 (2k) \binom{2k}{k}} \equiv 4 \sum_{k=1}^{n} \frac{1}{k^2} \binom{k}{n} \binom{k}{(n-1)} = 4n \sum_{k=1}^{n} \frac{1}{k^2} \binom{k}{n} \binom{k}{(n-1)} = 4n \sum_{k=1}^{n} \frac{1}{k^2} \binom{k}{n} \binom{k}{(n-1)} = -2(-1)^n \sum_{k=1}^{n} \frac{4k}{k^2} \binom{2k}{k} \mod p.
\end{equation*}
Therefore, with the help of Lemma 3.3, we finally obtain
\begin{equation*}
\binom{p}{2} \frac{\sum_{k=1}^{(p-1)/2} \frac{16k}{k^2 (2k) \binom{2k}{k}} \equiv -8E_{p-3} \mod p}
\end{equation*}
and hence
\begin{equation*}
\theta_2 \equiv -2p^3 E_{p-3} \mod p^4.
\end{equation*}
This, with (3.1) and (3.2), yields that
\begin{equation*}
\sum_{n=0}^{p-1} \frac{n}{32n} C_n \equiv -2p^3 E_{p-3} \mod p^4,
\end{equation*}
as desired. □

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