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Algebra

On complexity of representations of quivers

Sur la complexité des représentations de carquois

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ABSTRACT

It is shown that, given a representation of a quiver over a finite field, one can check in polynomial time whether it is absolutely indecomposable.

RÉSUMÉ

Nous montrons qu'étant donné une représentation de carquois sur un corps fini, on peut vérifier en temps polynomial si elle est absolument indécomposable.

1. Some results on absolutely indecomposable representations of quivers

Let Γ be a finite graph without self-loops (but several edges connecting two vertices are allowed), and let $\mathcal{V}$ denote the set of its vertices. The graph Γ with an orientation Ω of its edges is called a *quiver*. A *representation* of the quiver (Γ, Ω) over a field $\mathbb{F}$ is a collection of finite-dimensional vector spaces $\{U_v\}_{v\in\mathcal{V}}$ over $\mathbb{F}$ and linear maps $\{U_v \to U_w\}$ for each oriented edge $v \to w$. Homomorphisms and isomorphisms of two representations are defined in the obvious way. The *direct sum* of two representations $(\{U_v\}, \{U_v \to U_w\})$ and $(\{U'_v\}, \{U'_v \to U'_w\})$ is the representation

$$(\{U_v \oplus U'_v\}, \{U_v \oplus U'_v \to U_w \oplus U'_w\}),$$

where maps are the direct sums of maps. A representation π is called *indecomposable* if it is not isomorphic to a direct sum of two non-zero representations; π is called *absolutely indecomposable* if it is indecomposable over the algebraic closure $\bar{\mathbb{F}}$ of the field $\mathbb{F}$.

Let $r = \#\mathcal{V}$ and let $Q = \bigoplus_{v\in\mathcal{V}} \mathbb{Z}\alpha_v$ be a free abelian group of rank r with a fixed basis $\{\alpha_v\}_{v\in\mathcal{V}}$. Let $Q_+ = \bigoplus_v \mathbb{Z}_{\geq 0}\alpha_v \subset Q$. The *dimension* of a representation $\pi = \{U_v\}_{v\in\mathcal{V}}$ is the element

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$$\dim \pi = \sum_{v \in \mathcal{V}} (\dim U_v) \alpha_v \in Q_+.$$

The *Cartan matrix* of the graph Γ is the symmetric matrix $A = (a_{uv})_{u,v \in \mathcal{V}}$, where $a_{vv} = 2$ and $-a_{uv}$ is the number of edges, connecting u and v if $u \neq v$. Define a $\frac{1}{2}\mathbb{Z}$-valued symmetric bilinear form on Q, such that $(\alpha|\alpha) \in \mathbb{Z}$, by

$$(\alpha_u|\alpha_v) = \frac{1}{2} a_{uv}, \; u, v \in \mathcal{V},$$

and the following (involutive) automorphisms r_v, $v \in \mathcal{V}$, of the free abelian group Q

$$r_v(\alpha_u) = \alpha_u - a_{uv}\alpha_v, \; u \in \mathcal{V}.$$

The group $W \subset \operatorname{Aut} Q$, generated by all r_v, $v \in \mathcal{V}$, is called the *Weyl group* of the graph Γ. It is immediate to see that the bilinear form $(.|.)$ is invariant with respect to all r_v, $v \in \mathcal{V}$, hence with respect to the Weyl group W.

It is well known that the group W is finite if and only if the Cartan matrix A is positive definite, which happens if and only if all connected components of Γ are Dynkin diagrams of simple finite-dimensional Lie algebra of type A_r, D_r, E_6, E_7, E_8 (see e.g. [10]). Gabriel's theorem [4] states that for a quiver (Γ, Ω) the number of indecomposable representations, up to isomorphism, is finite if and only if the group W is finite. Moreover, in this case the map $\pi \mapsto \dim \pi$ establishes a bijective correspondence between isomorphism classes of indecomposable representations of (Γ, Ω) and the set of positive roots $\Delta_+ \subset Q_+$ of the semisimple Lie algebra with Dynkin diagram Γ, where

$$\Delta_+ = \bigcup_{v \in \mathcal{V}} ((W \cdot \alpha_v) \cap Q_+). \tag{1}$$

For an arbitrary graph Γ denote by Δ_+^{re} the RHS of (1); note that $(\alpha|\alpha) = 1$ for all $\alpha \in \Delta_+^{\mathrm{re}}$. Furthermore, let

$$\mathcal{C} = \{\alpha \in Q_+ \backslash \{0\} \,|\, (\alpha|\alpha_v) \leq 0, \, v \in \mathcal{V}, \text{ and } \operatorname{supp}\alpha \text{ is connected}\}, \tag{2}$$

where for $\alpha = \sum_{v \in \mathcal{V}} n_v \alpha_v$, we let $\operatorname{supp}\alpha = \{v |\, n_v \neq 0\}$. We let

$$\Delta_+^{\mathrm{im}} = W \cdot \mathcal{C}, \quad \Delta_+ = \Delta_+^{\mathrm{re}} \cup \Delta_+^{\mathrm{im}}.$$

It is easy to see that $\Delta_+^{\mathrm{im}} \subset Q_+$ and that $(\alpha|\alpha) \in \mathbb{Z}_{\leq 0}$ for $\alpha \in \Delta_+^{\mathrm{im}}$. The set $\Delta_+ \subset Q_+$ is the set of *positive roots* of the Kac-Moody algebra $\mathfrak{g}(A)$, associated with the Cartan matrix A, and Δ_+^{im} is empty if and only if the matrix A is positive definite [7], [10].

Theorem 1. *Let $\mathbb{F} = \mathbb{F}_q$ be a field of q elements.*

(a) *The number of absolutely indecomposable representations over $\mathbb{F}_q$ of dimension $\alpha \in Q_+$ of a quiver (Γ, Ω) is independent of the orientation Ω. It is zero if $\alpha \notin \Delta_+$, and it is given by a monic polynomial $P_{\Gamma,\alpha}(q)$ of degree $1 - (\alpha|\alpha)$ with integer coefficients. In particular, $P_{\Gamma,\alpha}(q) = 1$ if $\alpha \in \Delta_+^{\mathrm{re}}$.*
(b) *The constant term $P_{\Gamma,\alpha}(0)$ equals to the multiplicity of the root α in $\mathfrak{g}(A)$.*
(c) *All coefficients of $P_{\Gamma,\alpha}(q)$ are non-negative.*
(d) *Consequently, for any quiver (Γ, Ω) and any $\alpha \in \Delta_+$ there exists an absolutely indecomposable representation over $\mathbb{F}_q$ of dimension α.*

Claim (a) was proved in [7] and [9]; claims (b) and (c) were conjectured in [7], [9], and proved in [5] and [6] respectively. For indivisible $\alpha \in \Delta_+$ both claims (b) and (c) were proved earlier in [2].

2. Quasi-nilpotent subalgebras of $\mathrm{End}_{\mathbb{F}}\, U$

Consider a finite-dimensional vector space U over a field $\mathbb{F}$. An endomorphism a of U is called *quasi-nilpotent* if all its eigenvalues are equal; denote these eigenvalues by $\operatorname{eig}(a)$. They are elements of the algebraic closure $\bar{\mathbb{F}}$ of the field $\mathbb{F}$. An associative subalgebra A of $\operatorname{End}_{\mathbb{F}} U$ is called *quasi-nilpotent* if it consists of quasi-nilpotent elements. For an associative algebra A we denote by A_- the Lie algebra obtained from A by taking the bracket $[a, b] = ab - ba$. We also let $\overline{A} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} A$, $\overline{U} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} U$.

Lemma 1. *Let A be a subalgebra of the associative algebra* $\operatorname{End}_{\mathbb{F}} U$.

(a) *If A is a quasi-nilpotent subalgebra, then in some basis of $\overline{U}$, all endomorphisms $a \in A$ have upper triangular matrices with $\operatorname{eig}(a)$ on the diagonal. In particular, $\operatorname{eig}(a + b) = \operatorname{eig}(a) + \operatorname{eig}(b)$ for $a, b \in A$, and A_- is a nilpotent Lie algebra.*
(b) *If A_- is a nilpotent Lie algebra and A has a basis, consisting of quasi-nilpotent endomorphisms, then A is a quasi-nilpotent subalgebra.*

Proof. Burnside's theorem says that any subalgebra of the $\overline{\mathbb{F}}$-algebra $\mathrm{End}_{\overline{\mathbb{F}}}\overline{U}$, where $\overline{U}$ is a finite-dimensional vector space over $\overline{\mathbb{F}}$, which acts irreducibly on $\overline{U}$, coincides with $\mathrm{End}\,\overline{U}$. Hence, in some basis of $\overline{U}$ the algebra $\overline{A}$ consists of upper triangular block matrices with blocks $\mathrm{End}_{\overline{\mathbb{F}}}\overline{\mathbb{F}}^{m_i}$ on the diagonal, where $m_i \geq 1$, $\sum_i m_i = \dim \bar{U}$.

If A is a quasi-nilpotent subalgebra, then so is $\overline{A}$, and, in particular, $\mathrm{End}_{\overline{\mathbb{F}}}\overline{\mathbb{F}}^{m_i}$ for all i. This implies that all $m_i = 1$. Hence $\overline{A}$ consists of upper triangular quasi-nilpotent matrices. This proves (a).

In order to prove (b), note that if A_- is a nilpotent Lie algebra, then so is $\bar{A}_-$, and, in particular so are all $(\mathrm{End}_{\bar{\mathbb{F}}}\overline{\mathbb{F}}^{m_i})_-$. It follows that all $m_i = 1$, so that $\overline{A}_-$ consists of upper triangular matrices in some basis of $\overline{U}$. Since A has a basis, consisting of quasi-nilpotent elements, the subalgebra A is quasi-nilpotent. This proves (b). □

Corollary 1. *A subalgebra A of the associative algebra $\mathrm{End}_{\mathbb{F}} U$ is quasi-nilpotent if and only if the Lie algebra A_- is nilpotent and A has a basis, consisting of quasi-nilpotent endomorphisms.* □

3. Criterion of absolute indecomposability

Let $\pi = (\{U_v\}, \{U_v \to U_w\})$ be a representation of a quiver (Γ, Ω) over a field $\mathbb{F}$, of dimension $\alpha = \sum_{v\in\mathcal{V}} n_v\ \alpha_v$. Let $U = \bigoplus_{v\in\mathcal{V}} U_v$. Then the space $\mathrm{Hom}_{\mathbb{F}}(U_v, U_w)$ is naturally identified with a subspace of $\mathrm{End}_{\mathbb{F}} U$, so that the representation π is identified with a collection of endomorphisms for each oriented edge $v \to w$ of the quiver (Γ, Ω): $\{\pi_{v,w} : U_v \to U_w\} \subset \mathrm{End}_{\mathbb{F}} U$. An endomorphism a of π decomposes as $a = \sum_{v\in\mathcal{V}} a_v$, where $a_v \in \mathrm{End}_{\mathbb{F}} U_v \subset \mathrm{End}_{\mathbb{F}} U$, and the condition that $a \in \mathrm{End}\,\pi$, the algebra of endomorphisms of π, means that

$$a_w \pi_{v,w} = \pi_{v,w} a_v \text{ for all oriented edges } v \to w. \tag{3}$$

This simply means that the block diagonal endomorphism a commutes with all endomorphisms $\pi_{v,w}$ in the algebra $\mathrm{End}_{\mathbb{F}} U$. Note that (3) has an obvious solution $a_v = cI_{U_v}$, $v \in \mathcal{V}$, where $c \in \mathbb{F}$, hence $\dim \mathrm{End}\,\pi \geq 1$. In the case of equality, α lies in Δ_+, and it is called a *Schur vector*; in this and only in this case a generic representation of dimension α is absolutely indecomposable [8].

Lemma 2. *The representation π is absolutely indecomposable if and only if the algebra of its endomorphisms $\mathrm{End}\,\pi$ is quasi-nilpotent in $\mathrm{End}_{\mathbb{F}} U$.*

Proof. An endomorphism $a \in \mathrm{End}\,\pi \subset \mathrm{End}_{\mathbb{F}} U \subset \mathrm{End}_{\bar{\mathbb{F}}}\overline{U}$ decomposes in a sum of commuting endomorphisms $a = a_{(s)} + a_{(n)}$, where the endomorphism $a_{(s)}$ is diagonalizable and the endomorphone $a_{(n)}$ is nilpotent (Jordan decomposition). Condition (3) means that a commutes with $\pi_{v,w}$ for all oriented edges $v \to w$. By a well-known fact of linear algebra, it follows that the $\pi_{v,w}$ commute with $a_{(s)}$. But then the decomposition of $\overline{U}$ in a direct sum of eigenspaces of $a_{(s)}$ is a decomposition of the representation π in a direct sum of representation of the quiver (Γ, Ω). Thus, π is absolutely indecomposable if and only if $a_{(s)}$ is a scalar endomorphism of $\overline{U}$, which is equivalent to say that a is a quasi-nilpotent endomorphism of U. □

4. Main theorem

The following is the main result of the paper.

Theorem 2. *Let $\mathbb{F}_q$ be a fixed finite field. Then there exists an algorithm which, given as input a quiver (Γ, Ω) and its representation $\pi = (\{U_v\}, \{U_v \to U_w\})$ over $\mathbb{F}_q$ of dimension $\sum_{v\in\mathcal{V}} n_v\alpha_v$, can decide in polynomial in $N := \sum_v n_v$ time whether π is absolutely indecomposable or not.*

Proof. By Lemma 2 one has to check whether $\mathrm{End}\,\pi \subset \mathrm{End}_{\mathbb{F}_q} U$, where $U = \bigoplus_{v\in\mathcal{V}} U_v$, consists of quasi-nilpotent elements. By Corollary 1 one has to check two things:

(i) $\mathrm{End}\,\pi$ has a basis, consisting of quasi-nilpotent elements;
(ii) the Lie algebra $(\mathrm{End}\,\pi)_-$ is nilpotent.

For this we identify U_v with the vector space $\mathbb{F}_q^{n_v}$, so that U is identified with $\mathbb{F}_q^N$ and $\mathrm{End}_{\mathbb{F}_q} U$ with the algebra of $N \times N$-matrices over $\mathbb{F}_q$. $\mathrm{End}\,\pi$ is a subspace of $\mathrm{End}_{\mathbb{F}_q} U$, given by linear homogeneous equation (3), hence, using Gauss elimination, we can construct in polynomial in N time a basis $a_1, \ldots, a_m$ of $\mathrm{End}\,\pi$, where $m \leq N$.

First, we check that all the a_i are quasi-nilpotent. This simply means that

$$\det{}_U(\lambda I_N + a_i) = (\lambda + \gamma_i)^N, \text{ where } \gamma_i \in \overline{\mathbb{F}}_q. \tag{4}$$

The left-hand side of (4) can be computed in polynomial in N time by Gauss elimination. By the separability of $\bar{\mathbb{F}}_q$ over $\mathbb{F}_q$, (4) implies that all γ_i lie in $\mathbb{F}_q$. Hence we have to check that (4) holds for each i and some element $\gamma_i \in \mathbb{F}_q$, which can be done in polynomial in N time.

Second, we check that $(\operatorname{End}\pi)_-$ is a nilpotent Lie algebra. Recall that a Lie algebra $\mathfrak{g}$ of dimension m is nilpotent if and only if the member $\mathfrak{g}^m$ of the sequence of subspaces, defined inductively by

$$\mathfrak{g}^1 = \mathfrak{g}, \quad \mathfrak{g}^j = [\mathfrak{g}, \mathfrak{g}^{j-1}] \text{ for } j \geq 2,$$

is zero. Given a basis $\{a_i\}$ of $\mathfrak{g}$ (which we already have), the subspace $\mathfrak{g}^2$ is the span over $\mathbb{F}_q$ of all commutators $[a_i, a_j]$. Using Gauss elimination, construct a basis $\{b_i\}$ of $\mathfrak{g}^2$. Next, $\mathfrak{g}^3$ is the span of commutators $[a_i, b_j]$, and again, using Gauss elimination, choose a basis $\{c_i\}$ of $\mathfrak{g}^3$, etc. The Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}^m = 0$. □

5. A brief discussion on P vs NP

In terms of matrices over $\mathbb{F}_q$, a representation π over $\mathbb{F}_q$ of a quiver (Γ, Ω) of dimension $\alpha = \sum_{v\in\mathcal{V}} n_v\alpha_v \in Q_+$ is a collection of $n_w \times n_v$ matrices $\pi_{v,w}$ over $\mathbb{F}_q$ for each oriented edge $v \longrightarrow w$. An endomorphism of π is a collection of $n_v \times n_v$ matrices a_v over $\mathbb{F}_q$ for each vertex $v \in \mathcal{V}$, such that the linear homogeneous equations (3) hold. The representation π is absolutely indecomposable if for each endomorphism of π all matrices a_v, $v \in \mathcal{V}$, are quasi-nilpotent (equivalently, by Corollary 1, $\operatorname{End}\pi$ has a basis of elements with this property).

The following discussion was outlined to me by Mike Sipser. Given a representation π over a fixed finite field $\mathbb{F}_q$ of a quiver (Γ, Ω) of dimension $\alpha \in \Delta_+$, which is a collection of $M_\alpha := \sum_{v\to w} n_v n_w$ numbers from $\mathbb{F}_q$, the output is YES if π is absolutely indecomposable and NO otherwise. Call this problem INDEC; it is a P problem, according to Theorem 2. Define a generalization of INDEC, where some of the numbers are replaced by variables x_i, $i = 1, \ldots, M$, where M is an integer, such that $1 \leq M \leq M_\alpha$, and call this problem INDEC$[x_1, \ldots, x_M]$. Say YES for the latter problem if there exist $\gamma_1, \ldots, \gamma_M \in \mathbb{F}_q$ we can substitute for $x_1, \ldots, x_M$, such that the resulting INDEC problem is YES. Obviously INDEC is in P implies that INDEC$[x_1, \ldots, x_M]$ is in NP.

Now assume that INDEC$[x_1, \ldots, x_{M_\alpha}]$ is actually in P. We give a polynomial in M_α time procedure to output an absolutely indecomposable representation. Test INDEC$[x_1, \ldots, x_{M_\alpha}]$. The answer is YES by Theorem 1(d). Now reduce M_α by 1, by trying all possible numbers from $\mathbb{F}_q$ in place of x_{M_α} and test INDEC$[x_1, \ldots, x_{M_\alpha - 1}]$ for each of these numbers. The answer must be YES for at least one of these numbers. Repeat this procedure until we find all M_α numbers. That is our answer.

6. Conjectures and examples

Conjecture 1. *INDEC*$[x_1, \ldots, x_{M_\alpha}]$ *is not in P.*

Conjecture 2. *INDEC*$[x_1, \ldots, x_{M_\alpha}]$ *is in P for any quiver* (Γ, Ω) *if* $\alpha \in \Delta_+$ *is a Schur vector.*

Conjecture 3. *INDEC*$[x_1, \ldots, x_{M_\alpha}]$ *is in P for any quiver* (Γ, Ω) *if* $\alpha \in \mathcal{C}$ *(defined by (2)).*

Example 1. Let Γ be a Dynkin diagram of type A_r, D_r, E_6, E_7, E_8. In this case for any orientation Ω of Γ all indecomposable representations have been constructed explicitly in [4], which shows that in this case INDEC$[x_1, \ldots, x_{M_\alpha}]$ is in P.

Example 2. Let Γ be the extended (connected) Dynkin diagram, so that $\#\mathcal{V} = r + 1$ and $\det A = 0$. These are the only connected graphs, for which the Cartan matrix is positive semidefinite and singular. In this case all absolutely indecomposable representations for any orientation Ω have been constructed in [11] and in [3], which shows that in this case INDEC$[x_1, \ldots, x_{M_\alpha}]$ is in P as well. Note that in this case [7] $\Delta_+^{\mathrm{im}} = \mathbb{Z}_{\geq 1}\delta$, where $A\delta = 0$ and $(\delta|\delta) = 0$, and one can show that $P_{\Gamma, n\delta}(q) = q + r$ for $n \in \mathbb{Z}_{\geq 1}$.

Example 3. Let Γ_m be the quiver with two vertices v_1 and v_2, and m arrows from v_1 to v_2. For $m = 1$ and 2 this is a quiver from Examples 1 and 2 respectively. For $m \geq 3$ the explicit expressions for the polynomials $P_{\Gamma_m,\alpha}(q)$ for an arbitrary $\alpha \in \Delta_+^{\mathrm{im}}$ are unknown. Note that in this case $\Delta_+^{\mathrm{re\,(resp.\,im)}} = \{\alpha = n_1\alpha_1 + n_2\alpha_2 \,|\, n_i \in \mathbb{Z}_{\geq 0} \text{ and } n_1^2 + n_2^2 - mn_1n_2 = 1 \text{ (resp.} < 0)\}$.

Now, let (Γ, Ω) be a quiver, and let v be a vertex, which is a source or a sink. In [1] an explicitly computable reflection functor R_v was constructed, which sends a representation π of dimension $\alpha \neq v$ of (Γ, Ω) to a representation $R_v(\pi)$ of the reflected quiver $(\Gamma, R_v(\Omega))$ of dimension $r_v(\alpha)$, preserving indecomposability, see also [7]. It follows that if the problem INDEC$[x_1, \ldots, x_{M_\alpha}]$ is in P for the quiver (Γ, Ω) and dimension $\alpha \neq v$, and v is a source or a sink of (Γ, Ω), then it is in P for the quiver $(\Gamma, R_v(\Omega))$ and dimension $r_v(\alpha)$.

Remark 1. If v is a source or a sink of the quiver (Γ, Ω) and $\alpha \in \Delta_+ \backslash \{v\}$ is a Schur vector, then $r_v(\alpha)$ is a Schur vector for $(\Gamma, R_v(\Omega))$. Also, if α is such that INDEC$[x_1, \ldots, x_{M_\alpha}]$ is in P, then the same holds for $r_v(\alpha)$.

Remark 2. For an arbitrary quiver (Γ, Ω) the set $\mathcal{C}$ consists of Schur vectors, except for the vectors with $(\alpha|\alpha) = 0$ [7], in which case, $\mathrm{supp}\,\alpha$ is a graph from Example 2. Hence Conjecture 2 implies Conjecture 3.

Remark 3. Let Γ_m be a quiver from Example 3. Then, using the reflection functors, we see that for all $\alpha \in \Delta_+^{\mathrm{re}}$, $\mathrm{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P. Since for this quiver $(\alpha|\alpha) < 0$ for all $\alpha \in \mathcal{C}$, we see that all $\alpha \in \Delta_+^{\mathrm{im}}$ are Schur vectors [7], and it follows from Remark 1 and Conjecture 2 that for all $\alpha \in \Delta_+^{\mathrm{im}}$, $\mathrm{INDEC}[x_1, \dots, x_{M_\alpha}]$ is in P as well.

However, in general, $\alpha \in \Delta_+$ is not a Schur vector, so that a generic representation of a quiver (Γ, Ω) of dimension $\alpha \in \Delta_+$ is not absolutely indecomposable. In this case $\mathrm{INDEC}[x_1, \dots, x_{M_\alpha}]$ becomes a problem of finding a needle in a haystack, which leads me to (naively) believe in Conjecture 1.

In fact, I believe that for any connected quiver, different from those in Examples 1, 2, and 3, there exists $\alpha \in \Delta_+$, for which $\mathrm{INDEC}[x_1, \dots, x_{M_\alpha}]$ is not in P.

Remark 4. As explained in [9], claim (a) of Theorem 1 extends to the case of Γ with self-loops. Claim (c) is proved in [6] in this generality. Theorem 2 holds in this generality as well.

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