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Compactifications of conic spaces in del Pezzo 3-fold

Compactifications d'espaces coniques dans la variété de del Pezzo de dimension 3

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A B S T R A C T

Let V_5 be the del Pezzo 3-fold defined by the 6-dimensional linear section of the Grassmannian variety $\mathrm{Gr}(2,5)$ under the Plücker embedding. In this paper, we present an explicit birational relation of compactifications of degree-two rational curves (i.e. conics) in V_5. By a product, we obtain the virtual Poincaré polynomial of compactified moduli spaces.

R É S U M É

Soit V_5 le del Pezzo 3 défini par la section linéaire de dimension 6 de la variété grassmannienne $\mathrm{Gr}(2,5)$ située sous l'enrobage de Plücker. Dans cet article, nous présentons une relation birationnelle explicite de compactifications de courbes rationnelles de degré deux en V_5. Au moyen d'un produit, nous obtenons le polynôme de Poincaré virtuel des espaces de modules compactifiés.

1. Introduction

We work over the complex number field $\mathbb{C}$.

1.1. Results

Let X be a smooth projective variety with a fixed embedding $i : X \hookrightarrow \mathbb{P}^r$. Let $\mathbf{R}_d(X)$ be the moduli space of all smooth rational curves of degree d in X. For $d = 1$, then $\mathbf{R}_1(X)$ (the so-called Fano scheme of lines) is compact. For $d \geq 2$, $\mathbf{R}_d(X)$ may not be compact because the degeneration of curves can be singular. There are two well-known compactifications of $\mathbf{R}_d(X)$:

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(1) **Kontsevich space**: let X be a smooth projective variety. A map $f: C \to X$ is called *stable* if C has at worst nodal singularities and $|\mathrm{Aut}(f)| < \infty$. Let $\mathcal{M}_d(X)$ be the moduli space of isomorphism classes of stable maps $f: C \to X$ with genus $g(C) = 0$ and $f_*[C] = d \in H_2(X, \mathbb{Z})$.
(2) **Hilbert scheme**: let $\mathcal{H}_d(X)$ be the Hilbert scheme of ideal sheaves I_C of X with Hilbert polynomial $\chi(\mathcal{O}_C(m)) = dm + 1$.

Let us denote by $\mathbf{M}_d(X)$ and $\mathbf{H}_d(X)$ the closure of the space $\mathbf{R}_d(X)$ in the moduli space $\mathcal{M}_d(X)$ and $\mathcal{H}_d(X)$. If X is a projective homogeneous variety, then the space $\mathbf{R}_d(X)$ is irreducible [16]. The birational relations of these compactified spaces have been studied in [15,6,4,7]. The main ingredient of the comparison consists in using the elementary modification of sheaves and variation of geometric invariant theoretical quotient ([13,22]). To apply these techniques, the fact that X is homogeneous is essential. See [4, Lemma 2.1] for the detailed conditions. In this paper, we will extend the comparison result even if X does not satisfy the conditions in [4, Lemma 2.1] (cf. Remark 2.6). Our projective variety of interest is the so-called del Pezzo 3 fold V_5, which is defined by the linear section of the Grassmannian $\mathrm{Gr}(2,5)$ under the Plücker embedding. The del Pezzo 3-fold V_5 has been known as the unique minimal compactification of $\mathbb{C}^3$ having the same topological invariant as the projective space $\mathbb{P}^3$. In this paper, we present an explicit birational relation of the compactifications: $\mathbf{M}_2(V_5)$ and $\mathbf{H}_2(V_5)$. Note that the locus of the *double lines* (Definition 2.4) in $\mathbf{H}_2(V_5)$ consists of a smooth rational quartic curve [14, Proposition 1.2.2].

Theorem 1.1 *(Proposition 3.3). There exists a smooth blow-up morphism*

$$\mathbf{M}_2(V_5) \longrightarrow \mathbf{H}_2(V_5)$$

along the double-line locus in $\mathbf{H}_2(V_5)$*. Specially, the compactification* $\mathbf{M}_2(V_5)$ *is smooth.*

The key idea of the proof of Theorem 1.1 is to use the *branchvarieties* compactification that was studied in [1]. We firstly find a flat family of conics in V_5 by using local chart computation. Secondly, we perform the normalization of the flat family followed by the base change over the blown-up space $\widetilde{\mathbf{H}_2}(V_5)$. By a local computation, one can check that the modified family provides a bijective morphism to $\mathbf{M}_2(V_5)$. Lastly, we confirm that $\mathbf{M}_2(V_5)$ is a normal variety by using a deformation theoretical argument. This implies that the blown-up space $\widetilde{\mathbf{H}_2}(V_5)$ is isomorphic to $\mathbf{M}_2(V_5)$ by Zariski's main theorem (Proposition 3.3). Using Theorem 1.1, we compute the virtual Poincaré polynomial of the Kontsevich space $\mathcal{M}_2(V_5)$ (Corollary 3.5).

2. Preliminary

2.1. Non-free lines in V_5

We need some algebro-geometric properties of the lines to describe the blow-up center of $\mathbf{H}_2(V_5)$.

Proposition 2.1 *([10, Section 1]). The normal bundle of a line L in V_5 is isomorphic to*

$$N_{L/V_5} \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1) \ \text{or}\ \mathcal{O}_L \oplus \mathcal{O}_L.$$

Definition 2.2. The line of the first (resp. second) type in Proposition 2.1 is called a *non-free* (resp. *free*) line.

Lemma 2.3 *([10, Section 2]). The locus C_0 of the non-free lines in the Hilbert scheme* $\mathbf{H}_1(V_5)(\cong \mathbb{P}^2)$ *is a smooth conic.*

2.2. Results in [4]

In [4], as a generalization of the case $X = \mathbb{P}^r$ ([15, Section 4] and [6]), the authors compared the compactifications of rational curves such that a projective variety X is convex ([4, Lemma 2.1]). That is,

$$\mathrm{H}^1(\mathbb{P}^1, f^* T_X) = 0$$

for any morphism $f: \mathbb{P}^1 \to X$. For example, the Grassmannian variety $\mathrm{Gr}(k,n)$ is convex because the tangent bundle $T_{\mathrm{Gr}(k,n)}$ is globally generated.

Definition 2.4. On the other hand, for a line L in X, let us define the *double line* by a non-split extension sheaf F fitting into the short exact sequence

$$0 \to \mathcal{O}_L(-1) \to F \to \mathcal{O}_L \to 0$$

where $F \cong \mathcal{O}_{L^2}$, so that L^2 is a non-reduced conic.

The authors in [4] proved that compactifications of degree-two rational curves are related by explicit blow-ups/downs.

Theorem 2.5 *([4, Theorem 3.7 and Remark 3.8]). For a projective convex variety X, $\mathbf{M}_2(X)$ and $\mathbf{H}_2(X)$ are related by blow-ups as follows:*

$$\begin{array}{ccc} & \widetilde{\mathbf{M}}_2(X) & \\ {\scriptstyle \Gamma^1(X)} \swarrow & & \searrow {\scriptstyle \Gamma_2^1(X)} \\ \mathbf{M}_2(X) & & \mathbf{H}_2(X). \end{array}$$

Here $\Gamma(X)$ is the blowing up center such that

(1) $\Gamma^1(X)$ *is the locus of stable maps parameterizing the double covering of lines and*
(2) $\Gamma_2^1(X)$ *is the locus of the double lines in X.*

The comparison result of the case $X = \mathbb{P}^r$ ([15, Section 4]) was generalized in Theorem 2.5. The key point of the proof is to show that the blow-up center $\Gamma^1(\mathbb{P}^r)$ cleanly intersects with $\mathbf{M}_2(X)$ for any convex variety $X \subset \mathbb{P}^r$.

Remark 2.6. The del Pezzo variety V_5 is not convex by Proposition 2.1. In fact, let $f : \mathbb{P}^1 \to L \subset V_5$ be the degree-2 covering map where L is a non-free line. From the tangent bundle sequence, $0 \to T_L \to T_{V_5}|_L \to N_{L/V_5} \to 0$ and $f_*\mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$, we see that

$$\mathrm{H}^1(\mathbb{P}^1, f^*T_{V_5}) \twoheadrightarrow \mathrm{H}^1(\mathbb{P}^1, f^*N_{L/V_5}) \cong \mathrm{H}^1(L, N_{L/V_5} \otimes f_*\mathcal{O}_C) = \mathbb{C}$$

and thus $\mathrm{H}^1(\mathbb{P}^1, f^*T_{V_5}) \neq 0$.

Remark 2.7. Let L be a line in V_5. From the isomorphism $\mathrm{Ext}^1(\mathcal{O}_L, \mathcal{O}_L(-1)) \cong \mathrm{H}^0(N_{L/V_5}(-1))$, the supporting line L of the double line $\mathcal{O}_{L^2}$ must be non-free by Proposition 2.1.

2.3. Deformation theory of stable maps

The local structure of the space $\mathcal{M}_d(Y)$ was well studied in [17, Proposition 1.4, 1.5]. The deformation theory of the maps will be used for studying the normality of irreducible components of $\mathcal{M}_2(V_5)$ (Proposition 3.4).

Proposition 2.8. *Let $[f : C \to Y] \in \mathcal{M}_d(Y)$. Then, the tangent space (resp. the obstruction space) of $\mathcal{M}_d(Y)$ at $[f]$ is given by*

$$\mathrm{Ext}^1([f^*\Omega_Y \to \Omega_C], \mathcal{O}_C) \qquad (\text{resp. } \mathrm{Ext}^2([f^*\Omega_Y \to \Omega_C], \mathcal{O}_C)),$$

where $[f^*\Omega_Y \to \Omega_C]$ is thought of as a complex of sheaves concentrated on the interval $[-1, 0]$.

Lemma 2.9. *Let Y be a locally complete intersection of a smooth projective variety X. Let $f : C \to Y \subset X$ be a stable map that factors through Y. Then there exists an exact sequence:*

$$\begin{aligned} 0 \to & \mathrm{Ext}^1([f^*\Omega_Y \to \Omega_C], \mathcal{O}_C) \to \mathrm{Ext}^1([f^*\Omega_X \to \Omega_C], \mathcal{O}_C) \to \mathrm{H}^0(f^*N_{Y/X}) \\ \to & \mathrm{Ext}^2([f^*\Omega_Y \to \Omega_C], \mathcal{O}_C) \to \mathrm{Ext}^2([f^*\Omega_X \to \Omega_C], \mathcal{O}_C) \to \mathrm{H}^1(f^*N_{Y/X}) \to 0 \end{aligned}$$

where $N_{Y/X}$ is the normal bundle of Y in X.

Proof. For the proof, see [3, Lemma 2.10]. □

3. Comparison of compactifications

In this section, our main goal is to prove Theorem 1.1. To do this, we find a flat family of conics over $\mathbf{H}_2(V_5)$ and modify the family by using the normalization along the fiber. Furthermore, we prove that the irreducible components of $\mathcal{M}_2(V_5)$ are normal by using the *graph space* and deformation theory. In the last subsection, we obtain the virtual Poincaré polynomial of $\mathcal{M}_2(V_5)$ by using the comparison result.

3.1. Conics in V_5

It was proved that $\mathbf{H}_2(V_5)$ is isomorphic to $\mathbb{P}^4$ in [8] (cf. [21, Proposition 2.32]). Let us recall the correspondence between $\mathbb{P}^4$ and $\mathbf{H}_2(V_5)$. Let $\mathcal{U} := \mathcal{U}|_{V_5}$ be the restriction on V_5 of the universal rank-two subbundle on $\mathrm{Gr}(2,5)$. Note that $c_1(\mathcal{U}) = -1$ and $c_2(\mathcal{U}) = 2$. Then, by [8, Lemma 3.3], we know that

$$\mathrm{Hom}(\mathcal{U}, \mathcal{O}_{V_5}) \cong \mathrm{H}^0(\mathcal{U}^*) = \mathbb{C}^5.$$

Let

$$\phi : \mathcal{U} \to \mathcal{O}_{V_5} \tag{1}$$

be a non-zero homomorphism. Let $\mathrm{im}(\phi) \cong I_{C_\phi}$ for some subscheme C_ϕ in V_5. By the stability of $\mathcal{U}$, we have $\mathrm{codim}(C_\phi) \geq 2$. Hence, $c_1(I_{C_\phi}) = 0$. Also the kernel of ϕ in (1) is a reflexive sheaf of rank one and thus it is a line bundle on V_5 ([12, Proposition 1.1, 1.9]). Therefore, we obtain $\ker(\phi) = \mathcal{O}_{V_5}(-1)$. That is, we have

$$0 \to \mathcal{O}_{V_5}(-1) \to \mathcal{U} \to I_{C_\phi} \to 0. \tag{2}$$

By considering the Chern classes, one can check that the curve C_ϕ is a conic. Hence, we obtain a morphism

$$\Psi : \mathbb{P}(\mathrm{Hom}(\mathcal{U}, \mathcal{O}_{V_5})) = \mathbb{P}^4 \longrightarrow \mathbf{H}_2(V_5),\ \ \Psi([\phi]) = [I_{C_\phi}]. \tag{3}$$

From its construction, one can check that the morphism Ψ is injective. Since $\mathbf{H}_2(V_5)$ is irreducible and smooth ([5, Theorem 1.2, Proposition 7.2]), the map Ψ is an isomorphism by Zariski's main theorem.

Remark 3.1. The isomorphism Ψ in (3) can be described in the following geometric way. Let V_4 be a 4-dimensional subvector space in $\mathbb{C}^5$. Then the class $[\mathrm{Gr}(2, V_4)]$ is the Schubert cycle of type $\sigma_{1,1}$ in $\mathrm{Gr}(2,5)$. Since $\mathrm{Gr}(2, V_4)$ is a degree-two hypersurface in $\mathbb{P}(\wedge^2 V_4)$, and thus the intersection with $\mathbb{P}^6$ must be a conic:

$$C = \mathrm{Gr}(2, V_4) \cap \mathbb{P}^6 \subset \mathrm{Gr}(2,5) \cap \mathbb{P}^6 \subset \mathbb{P}^9.$$

3.2. Universal family of $\mathbf{H}_2(V_5)$

Recall that the non-free lines in V_5 consist of a conic in $\mathbf{H}_1(V_5)$ (Lemma 2.3). Also, the double structure on the non-free line L is unique because $\mathrm{H}^0(N_{L/V_5}(-1)) = \mathbb{C}$. It was proved that the locus of the double lines in $\mathbf{H}_2(V_5)$ is a smooth rational quartic curve [14, Proposition 1.2.2]. Following this argument, one can describe the universal family of conics in V_5. For details, let us consider the flag variety $\mathrm{Gr}(2,4,5)$ of type $(2,4,5)$:

$$\begin{array}{ccc} \mathrm{Gr}(2,4,5) & \hookrightarrow & \mathrm{Gr}(2,5) \times \mathrm{Gr}(4,5) \\ & & \downarrow \\ & & \mathrm{Gr}(4,5) \end{array}$$

where the vertical map is the projection onto the second component. Let

$$\mathcal{C} := \mathrm{Gr}(2,4,5)|_{V_5}$$

be the restriction of the flag variety $\mathrm{Gr}(2,4,5)$ on $V_5 = \mathrm{Gr}(2,5) \cap \mathbb{P}^6$. From the geometric construction of conics in V_5 (Remark 3.1), we have a flat family of conics in V_5 over $\mathrm{Gr}(4,5)$:

$$\begin{array}{ccccc} \mathcal{C} & \hookrightarrow & V_5 \times \mathrm{Gr}(4,5) & \longrightarrow & V_5 \\ & & \downarrow & & \\ & & \mathrm{Gr}(4,5). & & \end{array}$$

Let us find the defining equation of $\mathcal{C}$ around a double line. Let $\{x_{ij}\}$, $0 \leq i < j \leq 4$ be the Plücker coordinate of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ and the linear section $\mathbb{P}^6$ be $I_{\mathbb{P}^6} = \langle x_{12} - x_{03}, x_{13} - x_{24}, x_{14} - x_{02} \rangle$. For the standard basis $\{e_j \mid j = 0,1,2,3,4\}$ of $\mathbb{C}^5$, let us define

$$\mathbb{C}^4 = \mathrm{span}\langle t_1 e_0 + e_1,\ t_2 e_0 + e_2,\ t_3 e_0 + e_3,\ t_4 e_0 + e_4 \rangle$$

for $[1:t_1:t_2:t_3:t_4] \in \mathrm{Gr}(4,5)$. In this ordered basis of $\mathbb{C}^4$, the affine chart $\begin{bmatrix} 1 & 0 & s_1 & s_2 \\ 0 & 1 & s_3 & s_4 \end{bmatrix} \in \mathrm{Gr}(2,4)$ parameterizes the 2-dimensional subvector spaces in $\mathbb{C}^5$, which is in the following form:

$$\begin{bmatrix} t_1+t_2+s_1t_3+s_2t_4 & 1 & 1 & s_1 & s_2 \\ t_1+t_2+s_3t_3+s_4t_4 & 1 & 1 & s_3 & s_4 \end{bmatrix}.$$

Under the Plücker embedding $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, eliminating the variables $\{s_1, s_2, s_3, s_4\}$ by using the Macaulay2 computer program ([11]), one can see that the defining equation of $\mathcal{C}$ is given by (cf. [14, Section 2.5.4])

$$\begin{cases} x_{12}-x_{03}=x_{13}-x_{24}=x_{14}-x_{02}=0, \\ x_{01}=-t_2x_{12}-t_3x_{13}-t_4x_{14}, \\ x_{02}=t_1x_{12}-t_3x_{23}-t_4x_{24}, \\ x_{03}=t_1x_{13}+t_2x_{23}-t_4x_{34}, \\ x_{04}=t_1x_{14}+t_2x_{24}+t_3x_{34}, \\ -t_3x_{23}^2-t_4x_{23}x_{24}-x_{24}^2+t_2x_{23}x_{34}+t_1x_{24}x_{34}-t_4x_{34}^2+t_1^2x_{23}x_{24}+t_1t_2x_{23}^2-t_1t_4x_{23}x_{34}=0. \end{cases} \tag{4}$$

Note that the fiber $\mathcal{C}|_{(0,0,0,0)}$ at the origin $(t_1,t_2,t_3,t_4)=(0,0,0,0)$ defines the double line

$$I_{L^2} = \langle x_{01}, x_{02}, x_{03}, x_{04}, x_{12}, x_{14}, x_{13}-x_{24}, x_{24}^2\rangle$$

in V_5.

Corollary 3.2 *([14, Proposition 1.2.2]). Under the above notation, the locus of the double lines in $\mathbf{H}_2(V_5)=\mathbb{P}^4$ is defined by*

$$\{(t_1, -\frac{t_1^3}{8}, \frac{t_1^4}{64}, \frac{t_1^2}{4}) \in \mathbb{C}^4_{(t_1,t_2,t_3,t_4)}\}.$$

Proof. The double-line locus around the origin $(0,0,0,0) \in \mathbb{C}^4_{(t_1,t_2,t_3,t_4)} \subset \mathbf{H}_2(V_5)$ can be directly computed. The condition that the last equation in (4) should be a square is exactly the rank-one condition of the symmetric matrix

$$\mathrm{rk}\begin{bmatrix} t_1t_2-t_3 & \frac{t_1^2-t_4}{2} & \frac{t_2-t_1t_4}{2} \\ \frac{t_1^2-t_4}{2} & -1 & \frac{t_1}{2} \\ \frac{t_2-t_1t_4}{2} & \frac{t_1}{2} & -t_4 \end{bmatrix} \le 1.$$

Using the Macaulay2 computer program ([11]) again, one can check that this is equivalent to

$$\langle t_4^2-4t_3, t_1t_4+2t_2, 2t_1t_3+t_2t_4, t_2^2-4t_3t_4, t_1t_2+2t_4^2, t_1^2-4t_4\rangle,$$

which is the defining ideal of the rational normal quartic curve in $\mathbf{H}_2(V_5)$. □

3.3. Birational relation between $\mathbf{M}_2(V_5)$ and $\mathbf{H}_2(V_5)$

By modifying the presentation (4) of the universal family $\mathcal{C}$ in V_5, we have the following Proposition.

Proposition 3.3. *There exists a smooth blow-up*

$$\mathbf{M}_2(V_5) \longrightarrow \mathbf{H}_2(V_5)$$

along the double-line locus $C_0(\cong \mathbb{P}^1)$ in $\mathbf{H}_2(V_5)$. Especially, the compactification $\mathbf{M}_2(V_5)$ is smooth.

Proof. Let

$$p : \mathrm{bl}_{C_0}\mathbf{H}_2(V_5) \longrightarrow \mathbf{H}_2(V_5)$$

be the blow-up space of $\mathbf{H}_2(V_5)$ along the double line locus C_0. Let $\mathcal{C}' := (p \times \mathrm{id})^*\mathcal{C}$ be the pull-back of the flat family $\mathcal{C}$ by the map $p \times \mathrm{id}$. Let

$$p_2 : \widetilde{\mathrm{bl}_{C_0}\mathbf{H}_2(V_5)} \longrightarrow \mathrm{bl}_{C_0}\mathbf{H}_2(V_5)$$

be the two-fold covering map ramified along the exceptional divisor $p^{-1}(C_0)$ and $\mathcal{C}'' := (p_2 \times \mathrm{id})^*\mathcal{C}'$. Let

$$q:\widetilde{\mathcal{C}''}\longrightarrow\mathcal{C}''$$

be the normalization of $\mathcal{C}''$ in the (general) fiber over $\widetilde{\mathrm{bl}_{C_0}\mathbf{H}_2(V_5)}\setminus p^{-1}(C_0)$. Then, we have a flat family of stable maps over $\mathrm{bl}_{C_0}\mathbf{H}_2(V_5)$ ([1, Theorem 2.5])

$$\begin{array}{ccc}\widetilde{\mathcal{C}''} & \xrightarrow{\ \mathrm{ev}\ } & V_5\\ \downarrow{\scriptstyle\pi} & & \\ \mathrm{bl}_{C_0}\mathbf{H}_2(V_5). & & \end{array}$$

This can be checked by a local computation. Let $(0,a\epsilon,b\epsilon,c\epsilon)$ be the arbitrary normal curve in $\mathbb{C}^4_{(t_1,t_2,t_3,t_4)}$ at the double line (the origin). Then the universal curve $\mathcal{C}$ in (4) becomes

$$x_{13}-x_{24}=x_{12}=x_{14}=x_{01}=x_{02}=x_{03}=x_{04}=0 \pmod{\epsilon}$$

and

$$pf(\epsilon):=pf(0,a\epsilon,b\epsilon,c\epsilon)=-b\epsilon x^2-c\epsilon xy-y^2+a\epsilon xz-c\epsilon z^2=0,$$

where $x=x_{23},\ y=x_{24},\ z=x_{34}$. Let us perform the double covering $\epsilon=t^2$ along the divisor. Then

$$pf(t^2)=-bt^2x^2-ct^2xy-y^2+at^2xz-ct^2z^2=0.$$

After normalization along the general fiber (i.e. $\bar{y}=\frac{y}{t}$ and dividing by t^2), we have a flat family of degree-two curves

$$-bx^2-ctx\bar{y}-\bar{y}^2+axz-cz^2=0.$$

Now, the central fiber at $t=0$ becomes

$$\widetilde{\mathcal{C}''}|_0=-bx^2-\bar{y}^2+axz-cz^2=0.$$

This is obviously a reduced curve of degree two (i.e. smooth conic or pair of lines) in the plane $\mathbb{P}^2_{[x:\bar{y}:z]}$ whenever $(a,b,c)\neq 0$. Also, this defines a double covering map $\pi:\widetilde{\mathcal{C}''}|_0\subset\mathbb{P}^2_{[x:\bar{y}:z]}\to V(\bar{y}=0)=\mathbb{P}^1$ given by the projection from a point $[0:1:0]$. Note that the covering map π is bijectively determined by the homogeneous coordinates $\mathbb{P}^2=\mathbb{P}(\mathbb{C}^3_{(a,b,c)})$, because on the line $\bar{y}=0$, two ramification points are uniquely defined by the equation $-bx^2+axz-cz^2=0$.

After all, we have a bijective morphism $\mathrm{bl}_{C_0}\mathbf{H}_2(V_5)\to\mathbf{M}_2(V_5)$ by the functoriality of the moduli space of stable maps ([9, Theorem 1]). From the normality of $\mathbf{M}_2(V_5)$ (Proposition 3.4 below), we conclude that the morphism is an isomorphism by Zariski's main theorem. □

3.4. Normality of irreducible components of $\mathcal{M}_2(V_5)$

As it has been done in [3, Proposition 4.1], one can see that the Kontsevich space $\mathcal{M}_2(V_5)$ has two irreducible components. That is,

$$\mathcal{M}_2(V_5)=\mathbf{M}_2(V_5)\cup\mathbf{L}_2(V_5),$$

where $\mathbf{M}_2(V_5)$ is the irreducible component containing the smooth conic space $\mathbf{R}_2(V_5)$ and $\mathbf{L}_2(V_5)$ is the locus of the double covering of a line in V_5. Also, the intersection part parameterizes double-covering maps of a non-free line in V_5. Note that $\dim\mathbf{M}_2(V_5)=\dim\mathbf{L}_2(V_5)=4$ and $\dim\mathbf{M}_2(V_5)\cap\mathbf{L}_2(V_5)=3$. In this subsection, we finish the proof of Proposition 3.3 by proving the following thing.

Proposition 3.4. *The two irreducible components $\mathbf{M}_2(V_5)$ and $\mathbf{L}_2(V_5)$ are normal.*

Proof. It is straightforward to check that the obstruction space of the map in the complement $\mathcal{M}_2(V_5)\setminus(\mathbf{M}_2(V_5)\cap\mathbf{L}_2(V_5))$ vanishes. Therefore, the moduli space has at most finite group quotient singularity, which implies the normality on the complement.

For the intersection part, we use the result of [20, Theorem 0.1] (cf. [18, Theorem 6.1.3]). By the Plücker embedding $V_5\subset\mathbb{P}^9$, one can see that $\mathcal{M}_2(V_5)$ is a SL(2)-quotient of the moduli space $\mathcal{M}_{(1,2)}(\mathbb{P}^1\times V_5)$ of stable maps $f:C\to\mathbb{P}^1\times V_5$ with bi-degree $f_*[C]=(1,2)$:

$$\pi:\mathcal{M}_{(1,2)}(\mathbb{P}^1\times V_5)\to\mathcal{M}_{(1,2)}(\mathbb{P}^1\times V_5)/\!/\mathrm{SL}(2)\cong\mathcal{M}_2(V_5).$$

Let us denote the inverse image $\pi^{-1}(\mathbf{M}_2(V_5))$ and $\pi^{-1}(\mathbf{L}_2(V_5))$ by the same notation. Let $\mathbf{Q}=\mathbf{M}_2(V_5)\cap\mathbf{L}_2(V_5)$. We prove that the two spaces $\mathbf{M}_2(V_5)$ and $\mathbf{L}_2(V_5)$ are smooth at $[f:C\to\mathbb{P}^1\times L\subset\mathbb{P}^1\times V_5]\in Q$, $L\in C_0$ (Lemma 2.3) and thus that their SL(2)-quotient spaces are normal (cf. [15, Proposition 6.2]). By the projection formula and $(p_2\circ f)_*\mathcal{O}_C\cong\mathcal{O}_L\oplus\mathcal{O}_L(-1)$, one can see that the tangent space of $[f]$ in $\mathcal{M}_{(1,2)}(\mathbb{P}^1\times V_5)$ is canonically isomorphic to

$$\mathrm{H}^0((p_2\circ f)^*T_{V_5})\cong\mathrm{H}^0(T_{V_5}\otimes(p_2\circ f)_*\mathcal{O}_C)\cong\mathrm{H}^0(T_{V_5}|_L)\oplus\mathrm{H}^0(T_{V_5}|_L(-1)), \tag{5}$$

where $p_2:\mathbb{P}^1\times V_5\to V_5$ is the projection into the second component.

Let us consider the deformation of the map $[f]$ in $\mathbf{M}_2(V_5)$. Recall that the locus of double lines in $\mathbf{H}_1(V_5)$ is a smooth conic C_0 (Lemma 2.3). Thus, the normal space $N_{C_0/\mathbf{H}_1(V_5)}$ at $[L]$ is canonically isomorphic to the quotient space $\mathrm{H}^0(N_{L/V_5})/T_{[L]}C_0$, which is the normal deformation of $\mathbf{Q}$ in $\mathbf{L}_2(V_5)$. Hence, the deformation of $[f]$ in $\mathbf{M}_2(V_5)$ is cut out by the composition map

$$\mathrm{H}^0((p_2\circ f)^*T_{V_5})\twoheadrightarrow\mathrm{H}^0(T_{V_5}|_L)\twoheadrightarrow\mathrm{H}^0(N_{L/V_5})\twoheadrightarrow\mathrm{H}^0(N_{L/V_5})/T_{[L]}C_0(\cong\mathbb{C}),$$

where the second map comes from the tangent bundle sequence $0\to T_L\to T_{V_5}|_L\to N_{L/V_5}\to 0$. Therefore $\mathbf{M}_2(V_5)$ is smooth at $[f]$.

Let us describe the space $\mathrm{H}^0(N_{L/V_5}|_L(-1))$ to find the deformation space of $[f]$ in $\mathbf{L}_2(V_5)$. From the normal bundle sequence $0\to N_{L/V_5}\to N_{L/\mathbb{P}^9}\to N_{V_5/\mathbb{P}^9}|_L\to 0$ of $L\subset V_5\subset\mathbb{P}^9$, we obtain an inclusion map

$$\mathrm{H}^0(N_{L/V_5}|_L(-1))\subset\mathrm{H}^0(N_{L/\mathbb{P}^9}|_L(-1)). \tag{6}$$

By Lemma 2.9, the projection formula and $g_*\mathcal{O}_C\cong\mathcal{O}_L\oplus\mathcal{O}_L(-1)$ for $g:=p_2\circ f$, we have

$$\begin{array}{ccccc}
\mathrm{Ext}^1([g^*\Omega_L\to\Omega_C],\mathcal{O}_C) & \hookrightarrow & \mathrm{Ext}^1([g^*\Omega_{\mathbb{P}^9}\to\Omega_C],\mathcal{O}_C) & \twoheadrightarrow & \mathrm{H}^0(g^*N_{L/\mathbb{P}^9}) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
T_{[g]}\mathcal{M}_2(L) & \hookrightarrow & T_{[g]}\mathcal{M}_2(\mathbb{P}^9) & \twoheadrightarrow & \mathrm{H}^0(N_{L/\mathbb{P}^9})\oplus\mathrm{H}^0(N_{L/\mathbb{P}^9}|_L(-1)),
\end{array} \tag{7}$$

where from the surjective map in the diagram (7) of the first row comes $\mathrm{Ext}^2([g^*\Omega_L\to\Omega_C],\mathcal{O}_C)=0$ because L is convex. From this, the latter space in (6) is the normal deformation space of $[g]$ along the double-covering locus in $\mathbf{M}_2(\mathbb{P}^9)$. Hence, the deformation space of $[f]$ in $\mathbf{L}_2(V_5)$ is cut out by the surjective map

$$\mathrm{H}^0((p_2\circ f)^*T_{V_5})\twoheadrightarrow\mathrm{H}^0(T_{V_5}|_L(-1))\twoheadrightarrow\mathrm{H}^0(N_{L/V_5}|_L(-1))=\mathbb{C},$$

where the first map comes from the isomorphism in (5). After all, we finish the proof of the normality of two irreducible components. □

3.5. *Virtual Poincaré polynomial of $\mathcal{M}_2(V_5)$*

In this section, we compute the virtual Poincaré polynomial of $\mathcal{M}_2(V_5)$ by Proposition 3.5. Let X be a quasi-projective variety. For the Hodge–Deligne polynomial $\mathrm{E}_c(X)(u,v)$ for compactly supported cohomology of X, let

$$\mathrm{P}(X):=\mathrm{E}_c(X)(-t,-t)$$

be the *virtual* Poincaré polynomial of X. The motivic properties of the virtual Poincaré polynomial is well studied in [19, Theorem 2.2] and [2, Lemma 3.1].

Proposition 3.5.

(1) $\mathrm{P}(\mathbb{P}^n)=\frac{t^{2n+2}-1}{t^2-1}$.
(2) $\mathrm{P}(X)=\mathrm{P}(Z)+\mathrm{P}(X\setminus Z)$ *for any closed subset* $Z\subset X$.
(3) $\mathrm{P}(X)=\mathrm{P}(F)\cdot\mathrm{P}(B)$ *for the Zariski (resp. étal) locally trivial fibration* $X\to B$ *with constant fiber* F *(resp.* $\mathrm{Gr}(k,n)$*).*

Corollary 3.6. *The virtual Poincaré polynomial of* $\mathbf{M}_2(V_5)$ *and* $\mathbf{L}_2(V_5)$ *is given by*

$$\mathrm{P}(\mathbf{M}_2(V_5))=\mathrm{P}(\mathbf{L}_2(V_5))=1+2t^2+3t^4+2t^6+t^8.$$

Hence, the virtual Poincaré polynomial of the Kontsevich space $\mathcal{M}_2(V_5)$ *is*

$$\mathrm{P}(\mathcal{M}_2(V_5))=1+2t^2+4t^4+3t^6+2t^8.$$

Proof. From Proposition 3.3 and the fact that $\mathbf{L}_2(V_5)$ is a $\mathcal{M}_2(\mathbb{P}^1)(\cong \mathbb{P}^2)$-fibration over $\mathbf{H}_1(V_5)$,

$$\mathrm{P}(\mathbf{M}_2(V_5)) = \mathrm{P}(\mathbb{P}^4) + \mathrm{P}(\mathbb{P}^1)(\mathrm{P}(\mathbb{P}^2) - 1),\ \mathrm{P}(\mathbf{L}_2(V_5)) = \mathrm{P}(\mathbb{P}^2) \cdot \mathrm{P}(\mathbf{H}_1(V_5)).$$

By the property (2) of Proposition 3.5, we have

$$\mathrm{P}(\mathcal{M}_2(V_5)) = \mathrm{P}(\mathbf{M}_2(V_5)) + \mathrm{P}(\mathbf{L}_2(V_5)) - \mathrm{P}(\mathbb{P}^2) \cdot \mathrm{P}(C_0).$$

Cooking up the above, we obtain the results. □

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