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Partial differential equations

Generalized good-λ techniques and applications to weighted Lorentz regularity for quasilinear elliptic equations

Techniques λ-bonnes généralisées et applications à la régularité de Lorentz pondérée des équations elliptiques quasi linéaires

Minh-Phuong Tran [a], Thanh-Nhan Nguyen [b]

[a] *Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[b] *Department of Mathematics, Ho Chi Minh City University of Education, Ho Chi Minh City, Vietnam*

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ABSTRACT

The aim of this paper is to give some sufficient conditions, called generalized good-λ conditions, to obtain the weighted Lorentz comparisons between two measurable functions. Moreover, we also present several applications of these results for the gradient estimates of solutions to quasilinear elliptic problems in weighted Lorentz spaces.

RÉSUMÉ

Le but de cette Note est de donner des conditions suffisantes, dites λ-bonnes généralisées, pour la comparaison de deux fonctions mesurables dans les espaces de Lorentz pondérés. De plus, nous présentons des applications de ces résultats aux estimations de gradient des solutions aux problèmes elliptiques quasi linéaires dans les espaces de Lorentz pondérés.

1. Introduction

The motivation of our work comes from the problem of gradient estimates for the following elliptic equations

$$-\mathrm{div}(A(x,\nabla u)) = \mu \text{ in } \Omega, \text{ and } u = \sigma \text{ on } \partial\Omega, \tag{1}$$

where the domain Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$; the operator A is some vector-valued function that could be linear or nonlinear; μ and σ are given data, which could be in terms of functional or measure data. The existence, uniqueness, regularities, Calderón–Zygmund theory of solutions to this problem and many interesting open questions have been studied by many mathematicians in recent years, such as L.A. Caffarelli et al. [6], G. Mingione [18], S.S. Byun et al. [2–5], F. Duzaar et al. [8,9], T. Kuusi et al. [15], T. Mengesha et al. [16,17], M. Colombo et al. [7], K. Adimurthi

E-mail addresses: tranminhphuong@tdtu.edu.vn (M.-P. Tran), nhannt@hcmue.edu.vn (T.-N. Nguyen).

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et al. [1], Q.-H. Nguyen [19,20], etc. A typical type of problem (1) is the well-known p-Laplace equation $-\Delta_p u = \mu$ when $A(x,\xi) = |\xi|^{p-2}\xi$, with $p > 1$. The gradient estimates of (1) can be applied to prove the existence of solutions to some physical problems, for instance the Kardar–Parisi–Zhang equations from the physical theory of surface growth (see [11, 14]) and a stationary version of the time-dependent viscous Hamilton–Jacobi equation (see [21]), the Riccati-type equation (see [21], [23]) or some equations from various fields of science.

The problem (1) has been considered in several types of hypotheses of the data such as (1) in different conditions of the domain Ω in $\mathbb{R}^n$ (Lipschitz domain, Reifenberg flat domain, domain satisfying p-capacity condition, etc.), (2) under different assumptions imposed on nonlinear operator A (standard uniform elliptic conditions, discontinuous coefficients, monotone and growth conditions, etc.), (3) with different forms of the data μ (divergence form data, measure data, etc.), or (4) the regularity results obtained in the setting of different functional spaces (Lebesgue spaces, Lorentz spaces, Morrey spaces, weighted Lorentz-Morrey spaces, Orlicz spaces, etc.). In relation to this problem, there are several approaches to obtain the gradient estimate results, for instance the method based on the Vitali-type covering lemma by S.S. Byun et al. [3–5], the method based on the boundedness properties of singular integral potentials by G. Mingione et al. [8,9,18], or the method based on the good-λ-type bounds that was initiated by the work of Q.-H. Nguyen [19] recently. Also, we point out some interesting results recently obtained in [22,24,25]. The main objective of our study is to give some sufficient conditions of good-λ inequalities to obtain regularity results. Moreover, we consider in this paper the general form of the good-λ inequalities for abstract functions instead of the gradient of solutions or the data of Eq. (1). With our results, one can draw a complete picture of the regularity proofs for quasilinear elliptic equations (1) using the good-λ techniques and of course make it simpler than our previous ones studied in [22,24,25]. Before and after this study, we hope that the strengths of this approach could be applied to other problems in mathematical analysis.

The study that will be carried out in our paper consists in proving three theorems that contain the main idea of the good-λ techniques. The first theorem provides a basic good-λ condition corresponding to a constant $a \in (0,1)$ to obtain the Lorentz estimate in $L^{q,s}(\Omega)$ with $0 < q < a^{-1}$. In Theorem 3.2, we extend the results in weighted Lorentz estimate $L^{q,s}_{\omega}(\Omega)$ with $0 < q < \nu a^{-1}$ under the same assumptions, where ν depends on the Muckenhoupt weight ω. We finally obtain the more general estimates under a stronger good-λ condition in Theorem 3.3. More precisely, one first proves that the weighted Lorentz estimates in $L^{q,s}_{\omega}(\Omega)$ even hold for the full range $0 < q < \infty$. In the sequel, we establish a general result associated with a moderate growth function. As a consequence of Theorem 3.3, we also prove the pointwise estimates via Riesz potentials. In conclusion, we also give some applications to the gradient estimate problem for the quasilinear elliptic equations developed in weighted Lorentz spaces.

This paper is organized as follows. Section 2 is dedicated to some notations and definitions that we will use throughout the paper. The main results will be obtained in Section 3, in which we will prove three abstract theorems that we can somehow call the generalized good-λ techniques. Finally, Section 4 concentrates on some applications to regularity of solutions to quasilinear elliptic equations.

2. Notations and definitions

Throughout this study, the notation $B_r(x)$ stands for an open ball in $\mathbb{R}^n$ with radius r and centered at x. For the convenience of the reader, we use the symbol $\mathcal{L}^n(E)$ for the n-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$. In what follows, we will denote by $\fint_{B_r(x)} h(y)\,\mathrm{d}y$ the integral average of h on the variable y over the ball $B_r(x)$. We also write $\omega(E) := \int_E \omega(x)\,\mathrm{d}x$ for a weight ω that is a non-negative measurable and locally integrable function on $\mathbb{R}^n$. In this work, we consider the class of Muckenhoupt weights that are defined as follows.

Definition 2.1 *(Muckenhoupt weights).* For $1 \le p \le \infty$, we say that a weight $\omega \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\mathcal{A}_1$ if one has

$$[\omega]_{\mathcal{A}_p} = \sup_{B_r(x)\subset\mathbb{R}^n} \left(\fint_{B_r(x)} \omega(y)\,\mathrm{d}y\right)\left(\fint_{B_r(x)} \omega(y)^{-\frac{1}{p-1}}\,\mathrm{d}y\right)^{p-1} < \infty, \text{ when } 1 < p < \infty,$$

$$[\omega]_{\mathcal{A}_1} = \sup_{B_r(x)\subset\mathbb{R}^n} \left(\fint_{B_r(x)} \omega(y)\,\mathrm{d}y\right) \sup_{y\in B_r(x)} \frac{1}{\omega(y)} < \infty, \text{ when } p = 1,$$

and there are two positive constants C and ν such that

$$\omega(E) \le C\left(\frac{|E|}{|B|}\right)^{\nu} \omega(B), \text{ when } p = \infty,$$

for all ball $B = B_r(x)$ in $\mathbb{R}^n$ and all measurable subset E of B. In this case, we denote $[\omega]_{\mathcal{A}_\infty} = (C,\nu)$.

It is well known that $\mathcal{A}_1 \subset \mathcal{A}_p \subset \mathcal{A}_\infty$ for all $1 \le p \le \infty$ and $\mathcal{A}_\infty = \bigcup_{p<\infty} \mathcal{A}_p$.

Definition 2.2 *(Weighted Lorentz spaces).* Let $\omega \in \mathcal{A}_\infty$, $0 < q < \infty$ and $0 < s \leq \infty$. We define the weighted Lorentz space $L^{q,s}_\omega(\Omega)$ by the set of all Lebesgue measurable functions h on Ω such that

$$\|h\|_{L^{q,s}_\omega(\Omega)} := \begin{cases} \left[q \int_0^\infty \lambda^s \omega(\{x \in \Omega : |h(x)| > \lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{s}} < +\infty, & \text{if } s < \infty, \\ \sup_{\lambda>0} \lambda \omega(\{x \in \Omega : |h(x)| > \lambda\})^{\frac{1}{q}} < +\infty, & \text{if } s = \infty. \end{cases}$$

If $\omega = 1$, the weighted Lorentz space $L^{q,s}_\omega(\Omega)$ becomes the Lorentz space $L^{q,s}(\Omega)$. Moreover, when $q = s$, the weighted Lorentz space $L^{q,s}_\omega(\Omega)$ coincides with the weighted Lebesgue space $L^q_\omega(\Omega)$, which is defined by the set of measure function h such that

$$\|h\|_{L^q_\omega(\Omega)} := \left(\int_\Omega |h(x)|^p \omega(x)\, dx\right)^{\frac{1}{q}} < +\infty.$$

Moreover, in applications the operator A may be assumed to satisfy a (δ, R_0)-BMO condition, which is described as follows.

Definition 2.3 *((δ, R_0)-BMO condition).* Let $\delta > 0$ and $R_0 > 0$. We say that the operator A satisfies a (δ, R_0)-BMO condition with exponent $s > 0$ if

$$[A]^{R_0}_s = \sup_{y\in\mathbb{R}^n,\ 0<r\leq R_0} \left(\fint_{B_r(y)} \left(\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|A(x,\xi) - \overline{A}_{B_r(y)}(\xi)|}{|\xi|^{p-1}}\right)^s dx\right)^{\frac{1}{s}} \leq \delta,$$

where $\overline{A}_{B_r(y)}(\xi)$ denotes the average of $A(\cdot, \xi)$ over the ball $B_r(y)$.

We now recall the definition of the fractional maximal function in relation to [12,13]. We remark that for the case $\alpha = 0$, the fractional maximal function $\mathbf{M}_\alpha$ becomes the Hardy–Littlewood maximal function $\mathbf{M}$. The classical properties of the maximal function $\mathbf{M}$ can be found in [10].

Definition 2.4 *(Fractional maximal function).* Let $0 \leq \alpha \leq n$, the fractional maximal function $\mathbf{M}_\alpha$ of a locally integrable function $h \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R})$ is defined by:

$$\mathbf{M}_\alpha h(x) = \sup_{r>0} r^\alpha \fint_{B_r(x)} |h(y)|\, dy, \qquad x \in \mathbb{R}^n.$$

3. Generalized good-λ techniques

In this study, we consider two types of good-λ condition. Let us introduce the first one. We say that f and g satisfy a good-λ condition (H1) if there exist $a \in (0,1)$, $b > 0$, and $\varepsilon_0 > 0$ such that

$$\mathcal{L}^n(\{x \in \Omega : |f(x)| > \varepsilon^{-a}\lambda;\ |g(x)| \leq \varepsilon^b\lambda\}) \leq C\varepsilon\mathcal{L}^n(\{x \in \Omega : |f(x)| > \lambda\}) \tag{H1}$$

holds for all $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$. The following theorem presents the basic version of the good-λ technique.

Theorem 3.1. *Assume that f and g satisfy a good-λ condition (H1). Then, for $q \in (0, a^{-1})$ and $0 < s \leq \infty$, if $g \in L^{q,s}(\Omega)$, then $f \in L^{q,s}(\Omega)$ and there exists a positive constant C^* depending on a, b, s, q and ε_0 such that*

$$\|f\|_{L^{q,s}(\Omega)} \leq C^*\|g\|_{L^{q,s}(\Omega)}. \tag{2}$$

Proof. By changing variable from λ to $\varepsilon^{-a}\lambda$ in the definition of Lorentz space, one has:

$$\begin{aligned} \|f\|^s_{L^{q,s}(\Omega)} &= q\int_0^\infty \lambda^s \mathcal{L}^n(\{x \in \Omega : |f(x)| > \lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} \\ &= \varepsilon^{-as} q \int_0^\infty \lambda^s \mathcal{L}^n(\{x \in \Omega : |f(x)| > \varepsilon^{-a}\lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda}. \end{aligned} \tag{3}$$

Under the condition (H1), we deduce from (3) that

$$\|f\|^s_{L^{q,s}(\Omega)} \leq 2^{\frac{s}{q}} C^{\frac{s}{q}} \varepsilon^{-as+\frac{s}{q}} q \int_0^\infty \lambda^s \mathcal{L}^n(\{x \in \Omega : |f(x)| > \lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda} + 2^{\frac{s}{q}} \varepsilon^{-as} q \int_0^\infty \lambda^s \mathcal{L}^n(\{x \in \Omega : |g(x)| > \varepsilon^b \lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda}. \quad (4)$$

Performing a change of variables in the second integral on the right-hand side of (4) yields that

$$\|f\|^s_{L^{q,s}(\Omega)} \leq 2^{\frac{s}{q}} C^{\frac{s}{q}} \varepsilon^{-as+\frac{s}{q}} \|f\|^s_{L^{q,s}(\Omega)} + 2^{\frac{s}{q}} \varepsilon^{-as-bs} \|g\|^s_{L^{q,s}(\Omega)}. \quad (5)$$

For any $0 < q < a^{-1}$, we may choose $\varepsilon \in (0, \varepsilon_0)$ in (5) sufficiently small that $2^{\frac{s}{q}} C^{\frac{s}{q}} \varepsilon^{-as+\frac{s}{q}} \leq 1/2$, which implies (2). □

We next extend Theorem 3.1 in the weighted Lorentz spaces with a Muckenhoupt weight $\omega \in \mathcal{A}_\infty$ under the same condition (H1). If $\omega \equiv 1$, Theorem 3.2 becomes that of Theorem 3.1.

Theorem 3.2. *Let $\omega \in \mathcal{A}_\infty$ and denote $(C, \nu) = [\omega]_{\mathcal{A}_\infty}$. Assume that f and g satisfy a good-λ condition (H1). Then for $q \in (0, \nu a^{-1})$ and $0 < s \leq \infty$, if $g \in L^{q,s}_\omega(\Omega)$ then $f \in L^{q,s}_\omega(\Omega)$ and*

$$\|f\|_{L^{q,s}_\omega(\Omega)} \leq C^* \|g\|_{L^{q,s}_\omega(\Omega)} \quad (6)$$

holds for a constant C^* depending on a, b, s, q, ε_0, and $[\omega]_{\mathcal{A}_\infty}$.

Proof. Let us rephrase the definition of the norm in weighted Lorentz space and change variables from λ to $\varepsilon^{-a}\lambda$, one has:

$$\|f\|^s_{L^{q,s}_\omega(\Omega)} = \varepsilon^{-as} q \int_0^\infty \lambda^s \omega(\{x \in \Omega : |f(x)| > \varepsilon^{-a}\lambda\})^{\frac{s}{q}} \frac{d\lambda}{\lambda}. \quad (7)$$

For simplicity, let us denote $U = \{x \in \Omega : |f(x)| > \varepsilon^{-a}\lambda\}$, $V = \{x \in \Omega : |g(x)| \leq \varepsilon^b \lambda\}$ and $W = \{x \in \Omega : |f(x)| > \lambda\}$. It is easy to see that

$$\omega(U) \leq \omega(U \cap V) + \omega(\Omega \setminus V),$$

which implies, from $\omega \in \mathcal{A}_\infty$ and inequality (H1), that

$$\omega(U) \leq C_0 (C\varepsilon)^\nu \omega(W) + \omega(\Omega \setminus V).$$

Combining this estimate and (7), it follows that

$$\|f\|^s_{L^{q,s}_\omega(\Omega)} \leq 2^{\frac{s}{q}} (C_0 C^\nu)^{\frac{s}{q}} \varepsilon^{-as+\frac{\nu s}{q}} q \int_0^\infty \lambda^s \omega(U)^{\frac{s}{q}} \frac{d\lambda}{\lambda} + 2^{\frac{s}{q}} \varepsilon^{-as} q \int_0^\infty \lambda^s \omega(\Omega \setminus V)^{\frac{s}{q}} \frac{d\lambda}{\lambda}. \quad (8)$$

By changing variable in the second term on the right-hand side of (8), we obtain that

$$\|f\|^s_{L^{q,s}_\omega(\Omega)} \leq 2^{\frac{s}{q}} (C_0 C^\nu)^{\frac{s}{q}} \varepsilon^{-as+\frac{\nu s}{q}} \|f\|^s_{L^{q,s}_\omega(\Omega)} + 2^{\frac{s}{q}} \varepsilon^{-as-bs} \|g\|^s_{L^{q,s}_\omega(\Omega)}. \quad (9)$$

For $q \in (0, \nu a^{-1})$, we may conclude (6) by taking $\varepsilon \in (0, \varepsilon_0)$ in (9) such that

$$2^{\frac{s}{q}} (C_0 C^\nu)^{\frac{s}{q}} \varepsilon^{-as+\frac{\nu s}{q}} \leq 1/2.$$

The proof is complete. □

In the above theorem, we can see that the weighted Lorentz estimate only holds for $q \in (0, \nu a^{-1})$. Furthermore, in order to achieve results for a large range of q, we need to introduce a better good-λ condition. Let us describe the second type of this condition as follows. We say that f and g satisfy a good-λ condition (H2) if there exist $\vartheta > 0$, $\kappa > 0$, and $\varepsilon_0 > 0$ such that

$$\mathcal{L}^n(\{x \in \Omega : |f(x)| > \vartheta\lambda;\ |g(x)| \leq \kappa\lambda\}) \leq C\varepsilon \mathcal{L}^n(\{x \in \Omega : |f(x)| > \lambda\}) \quad \text{(H2)}$$

holds for all $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$. Under condition (H2), we are considering the more general proof related to a moderate growth function $\mathcal{H}$ as follows.

Theorem 3.3. *Assume that f and g satisfy a good-λ condition (H2). For any $\omega \in \mathcal{A}_\infty$, $0 < q < \infty$ and $0 < s \leq \infty$, if $g \in L^{q,s}_\omega(\Omega)$, then $f \in L^{q,s}_\omega(\Omega)$ and (6) holds.*

Moreover, if $\mathcal{H} : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $\mathcal{H}(0) = 0$, $\lim_{z\to\infty} \mathcal{H}(z) = \infty$ and $\mathcal{H}(2z) \leq c\mathcal{H}(z)$ for all $z \geq 0$ with a constant $c > 0$, then

$$\int_\Omega \mathcal{H}(|f(x)|)\omega(x)\,\mathrm{d}x \leq C^* \int_\Omega \mathcal{H}(|g(x)|)\omega(x)\,\mathrm{d}x, \tag{10}$$

where the constant C^* depends on ϑ, κ, c, and $[\omega]_{\mathcal{A}_\infty}$.

Proof. The proof of (6) is similar to that in Theorem 3.2 by replacing ε^{-a} by ϑ, so we only sketch it here. It is noticeable that, for ε satisfying $2^{\frac{s}{q}}(C_0C^\nu)^{\frac{s}{q}}\vartheta^s\varepsilon^{\frac{\nu s}{q}} \leq 1/2$, the norm estimate (6) holds for all $0 < q < \infty$. It remains to prove (10). For all $\lambda > 0$, by $\omega \in \mathcal{A}_\infty$ and controlled condition (H2), it is easily seen that

$$\omega(\{x \in \Omega : |f(x)| > \vartheta\lambda\}) \leq \omega(\{x \in \Omega : |g(x)| > \kappa\lambda\}) + C_0(C\varepsilon)^\nu \omega(\{x \in \Omega : |f(x)| > \lambda\}). \tag{11}$$

For all $z \geq 0$, let us apply (11) by $\lambda = \vartheta^{-1}\mathcal{H}^{-1}(z)$, one gets

$$\omega(\{x \in \Omega : |f(x)| > \mathcal{H}^{-1}(z)\}) \leq \omega(\{x \in \Omega : |g(x)| > \kappa\vartheta^{-1}\mathcal{H}^{-1}(z)\}) + C_0(C\varepsilon)^\nu \omega(\{x \in \Omega : |f(x)| > \vartheta^{-1}\mathcal{H}^{-1}(z)\}),$$

which guarantees that

$$\omega(\{x \in \Omega : \mathcal{H}(|f(x)|) > z\}) \leq \omega(\{x \in \Omega : \mathcal{H}(\vartheta\kappa^{-1}|g(x)|) > z\}) + C_0(C\varepsilon)^\nu \omega(\{x \in \Omega : \mathcal{H}(\vartheta|f(x)|) > z\}). \tag{12}$$

We note that $\mathcal{H}(2z) \leq c\mathcal{H}(z)$ for all $z \geq 0$, it follows that $\mathcal{H}(2^k z) \leq c^k\mathcal{H}(z)$ for all $z \geq 0$ and $k \in \mathbb{N}$. Moreover, since $\mathcal{H}$ is strictly increasing function, it is obvious that

$$\mathcal{H}(2^r z) \leq \mathcal{H}(2^{[r]+1}z) \leq c^{[r]+1}\mathcal{H}(z),$$

for all $z \geq 0$, $r \in \mathbb{R}$ and $[r]$ denotes the largest integer smaller or equal than r. Thanks to this fact, we can deduce from (12) that

$$\omega(\{x \in \Omega : \mathcal{H}(|f(x)|) > z\}) \leq \omega(\{x \in \Omega : \alpha_1\mathcal{H}(|g(x)|) > z\}) + C_0(C\varepsilon)^\nu \omega(\{x \in \Omega : \alpha_2\mathcal{H}(|f(x)|) > z\}), \tag{13}$$

where $\alpha_1 = c^{[\log_2(\vartheta\kappa^{-1})]+1}$ and $\alpha_2 = c^{[\log_2(\vartheta)]+1}$. Integrating the two sides of (13) over the range $[0, \infty)$ and then changing variable on the right-hand side, one has

$$\int_0^\infty \omega(\{x \in \Omega : \mathcal{H}(|f(x)|) > z\})\,\mathrm{d}z \leq \alpha_1 \int_0^\infty \omega(\{x \in \Omega : \mathcal{H}(|g(x)|) > z\})\,\mathrm{d}z$$
$$+ C_0(C\varepsilon)^\nu\alpha_2 \int_0^\infty \omega(\{x \in \Omega : \mathcal{H}(|f(x)|) > z\})\,\mathrm{d}z. \tag{14}$$

One remarks that

$$\int_\Omega \mathcal{H}(|f(x)|)\omega(x)\,\mathrm{d}x = \int_0^\infty \omega(\{x \in \Omega : \mathcal{H}(|f(x)|) > z\})\,\mathrm{d}z,$$

so we may choose $\varepsilon \in (0, \varepsilon_0)$ in (14) such that $C_0(C\varepsilon)^\nu\alpha_2 \leq 1/2$ to obtain (10). □

As a consequence of Theorem 3.3, we obtain a pointwise estimate for the Riesz potential.

Corollary 3.4. *Assume that f and g satisfy a good-λ condition (H2). For any $0 < \alpha < n$, there exists a constant $C^* > 0$ such that*

$$\mathbf{I}_\alpha f(z) \leq C^*\mathbf{I}_\alpha g(z), \quad \text{for almost everywhere } z \in \mathbb{R}^n. \tag{15}$$

Here the Riesz potential is defined by

$$\mathbf{I}_\alpha[\mu](x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}}\,\mathrm{d}\mu, \quad x \in \mathbb{R}^n.$$

Proof. Applying Theorem 3.3 with $\mathcal{H}$ is the identity function, there exists a constant C^* only depending on ϑ, κ such that

$$\int_{\Omega} |f(x)|\omega(x)\,\mathrm{d}x \le C^* \int_{\Omega} |g(x)|\omega(x)\,\mathrm{d}x, \quad \forall \omega \in \mathcal{A}_\infty. \tag{16}$$

On the other hand, we can easily check that there exists a constant $K > 0$ such that

$$\mathbf{M}(\omega_0)(x) \le K\omega_0(x), \quad \forall x \in \mathbb{R}^n,$$

where $\omega_0(x) = |x|^{1-n}$, $x \in \mathbb{R}^n$. By applying Fubini's Theorem, it can be concluded that for all $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^+)$, there holds

$$\mathbf{M}(\mathbf{I}_\alpha h)(x) \le K\mathbf{I}_\alpha h(x), \quad \forall x \in \mathbb{R}^n,$$

which implies that $\mathbf{I}_\alpha h \in \mathcal{A}_1$. In addition, we recall here $\mathcal{A}_1 \subset \mathcal{A}_\infty$. Therefore, for any $z \in \mathbb{R}^n$ and $\varepsilon > 0$ small enough, we may choose $\omega = \mathbf{I}_\alpha[\chi_{B_\varepsilon(z)}] \in \mathcal{A}_1$ in (16); one has:

$$\int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \frac{\chi_{B_\varepsilon(z)}(y)}{|y-x|^{n-\alpha}}\,\mathrm{d}y\,\mathrm{d}x \le C^* \int_{\mathbb{R}^n} g(x) \int_{\mathbb{R}^n} \frac{\chi_{B_\varepsilon(z)}(y)}{|y-x|^{n-\alpha}}\,\mathrm{d}y\,\mathrm{d}x.$$

Thanks to Fubini's Theorem again, we obtain the following estimate

$$\int_{\mathbb{R}^n} \chi_{B_\varepsilon(z)}(y) \int_{\mathbb{R}^n} \frac{f(x)}{|y-x|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C^* \int_{\mathbb{R}^n} \chi_{B_\varepsilon(z)}(y) \int_{\mathbb{R}^n} \frac{g(x)}{|y-x|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y,$$

from which we deduce that

$$\fint_{B_\varepsilon(z)} \mathbf{I}_\alpha f(y)\,\mathrm{d}y \le C^* \fint_{B_\varepsilon(z)} \mathbf{I}_\alpha g(y)\,\mathrm{d}y. \tag{17}$$

Passing ε to 0 in (17), we obtain that (15) holds almost everywhere for $z \in \mathbb{R}^n$. $\square$

4. Applications to quasilinear elliptic equations

In this section, we apply the results in Section 3 to obtain the regularity estimates of solutions to the quasilinear elliptic equations (1) if the good-λ bounds (H1) or (H2) hold. Let us now consider Eqs. (1), where the nonlinear operator $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory vector valued function that satisfies the growth and monotonicity conditions for $1 < p \le n$:

$$|A(x,\xi_1)| \le \Lambda_1 |\xi_1|^{p-1},$$
$$\langle A(x,\xi_1) - A(x,\xi_2), \xi_1 - \xi_2 \rangle \ge \Lambda_2 (|\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2,$$

for all $\xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\}$ and almost everywhere, $x \in \mathbb{R}^n$, and Λ_1, Λ_2 are two positive constants. We refer the reader to several recent articles [22,24,25] for the key ideas to get these good-λ inequalities under different hypotheses of nonlinearity of A.

4.1. Problems with measure data

We study in this subsection Eq. (1) with a finite Radon measure data μ and the boundary condition $\sigma = 0$. We assume that the complement of domain Ω in $\mathbb{R}^n$ satisfies the p-capacity uniform thickness condition (see [22] for the definition) in the singular case $\frac{3n-2}{2n-1} \le p < 2 - \frac{1}{n}$. In order to get the regularity result, a good-λ type bound was established in [22] and [24] as follows

$$\mathcal{L}^n(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}} > \varepsilon^{\frac{-1}{\Theta}}\lambda; (\mathbf{M}_1(\mu))^{\frac{1}{p-1}} \le \varepsilon^{\frac{1}{(p-1)\gamma_0}}\lambda\} \cap \Omega)$$
$$\le C\varepsilon\mathcal{L}^n(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}} > \lambda\} \cap \Omega),$$

for some constants $\Theta > p$ and $\gamma_0 \in \left[\frac{2-p}{2}, \frac{(p-1)n}{n-1}\right)$. Here $\mathbf{M}$ and $\mathbf{M}_1$ denote the Hardy–Littlewood maximal function and the 1-fractional maximal function, respectively. Applying Theorem 3.2 with $a = \frac{1}{\Theta} < 1$, $b = \frac{1}{(p-1)\gamma_0}$, $f = (\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}$ and $g = (\mathbf{M}_1(\mu))^{\frac{1}{p-1}}$, we can obtain that, for $\omega \in \mathcal{A}_\infty$, $0 < q < \Theta\nu$, $0 < s \le \infty$ with $(C,\nu) = [\omega]_{\mathcal{A}_\infty}$, one has:

$$\|\nabla u\|_{L^{q,s}_\omega(\Omega)} \le C^* \|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{q,s}_\omega(\Omega)}. \tag{18}$$

We emphasize that, with $\omega \equiv 1$, this estimate coincides with the gradient estimate in [22], which can be obtained by applying Theorem 3.1. In fact, in [24], we obtained the global Lorentz–Morrey gradient estimate which is more complicated than (18) in Lorentz spaces. However, the two sufficient conditions to obtain the gradient estimates are the same.

On the other hand, we can even improve the regularity result with an additional hypothesis on A and Ω. In particular, when Ω is a (δ_1, R_0)-Reifenberg flat domain, then the authors in [20] have proved that

$$\mathcal{L}^n(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}} > \Lambda_0\lambda; (\mathbf{M}_1(\mu))^{\frac{1}{p-1}} \leq \delta_2\lambda\} \cap \Omega) \leq C\varepsilon\mathcal{L}^n(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}} > \lambda\} \cap \Omega), \tag{19}$$

for some constant Λ_0 and δ_2. We refer the reader to [20] for the detailed proof of inequality (19). In this case, we apply Theorem 3.3 to obtain that the weighted Lorentz estimate (18) even holds for $0 < q < \infty$. Moreover, we also get the pointwise estimate for the Riesz potential of the gradient of u by using Corollary 3.4.

4.2. Problems with divergence form data

Let us now consider the problem (1) under the non-homogeneous boundary condition $\sigma \in W^{1,p}(\Omega; \mathbb{R})$ and the divergence form data $\mu = \mathrm{div}(|F|^{p-2}F)$, with $F \in L^p(\Omega; \mathbb{R}^n)$. The complement of domain Ω also satisfies the p-capacity uniform thickness condition. In [25], the authors have showed that, for all $\lambda > 0$ and $\varepsilon \in (0, \varepsilon_0)$, the following good-$\lambda$ condition

$$\mathcal{L}^n(\{\mathbf{MM}_\alpha(|\nabla u|^p) < \varepsilon^{-a}\lambda; \mathbf{M}_\alpha(|F|^p + |\nabla\sigma|^p) \leq \varepsilon^b\lambda\} \cap \Omega) \leq C\mathcal{L}^n(\{\mathbf{MM}_\alpha(|\nabla u|^p) < \lambda\} \cap \Omega),$$

holds for some constants $a \in (0,1)$, $b > 0$ and $\varepsilon_0 > 0$. Here $\mathbf{M}_\alpha$ denotes the α-fractional maximal function with $0 \leq \alpha < n$. In this case, we can apply Theorem 3.2 for $f = \mathbf{MM}_\alpha(|\nabla u|^p)$ and $g = \mathbf{M}_\alpha(|F|^p + |\nabla\sigma|^p)$ to conclude that

$$\|\mathbf{MM}_\alpha(|\nabla u|^p)\|_{L^{q,s}_\omega(\Omega)} \leq C^*\|\mathbf{M}_\alpha(|F|^p + |\nabla\sigma|^p)\|_{L^{q,s}_\omega(\Omega)}, \tag{20}$$

for any $\omega \in \mathcal{A}_\infty$, $0 < q < \nu a^{-1}$, $0 < s \leq \infty$. It is similar to the previous subsection. We also obtain the gradient estimate (20) for $0 < q < \infty$ and the pointwise estimate under the considered Reifenberg flat domain, whereas the BMO norm of A is small enough.

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