Differential geometry/Lie algebras

On left-invariant Einstein metrics that are not geodesic orbit

Sur les métriques d'Einstein invariante à gauche, qui ne sont pas à orbites géodésiques

Na Xu, Ju Tan

School of Mathematics and Physics, Anhui University of Technology, Maanshan, 243032, People's Republic of China

Abstract

In this article, we prove that compact simple Lie groups $SO(n)$ $(n > 12)$ admit at least two left-invariant Einstein metrics that are not geodesic orbit, which gives a positive answer to a problem recently posed by Nikonorov.

© 2019 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Dans cette Note, nous démontrons que les groupes de Lie simples, compacts, $SO(n)$ $(n > 12)$ admettent au moins deux métriques d'Einstein invariantes à gauche, dont des géodésiques maximales ne sont pas des orbites de sous-groupes à un paramètre du groupe d'isométries complet. Ceci répond par l'affirmative à une question récemment posée par Nikonorov.

© 2019 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Recall that a Riemannian metric on a connected manifold $M$ is said to be a geodesic orbit metric if any maximal geodesic of the metric is the orbit of a one-parameter subgroup of the full group of isometries (in this case, the Riemannian manifold is called a geodesic orbit space). It is well known that any naturally reductive metric must be geodesic orbit, but the converse is not true.

In [1], A. Arvanitoyeorgos, K. Mori, and Y. Sakane constructed non-naturally reductive Einstein metrics on compact Lie groups $SO(n)$ $(n \geq 11)$, $Sp(n)$ $(n \geq 3)$, $E_6$, $E_7$, and $E_8$. In [3], Z. Chen and K. Liang found three naturally reductive and one non-naturally reductive Einstein metric on the compact Lie group $F_4$, and I. Chrysikos and Y. Sakane obtained lots of non-naturally reductive Einstein metrics on exceptional Lie groups [4]. Moreover, based on the classification of standard homogeneous Einstein manifolds, Z. Yan and S. Deng found many non-naturally reductive Einstein metrics on compact simple Lie groups [8]. Besides, the authors constructed non-naturally reductive Einstein–Randers metrics on $Sp(n)$ [7].
However, there are only few examples of left-invariant Einstein metrics that are not geodesic orbit. In [6], Y. Nikonorov proved that there exists a left-invariant Einstein metric on compact simple Lie group $G_2$ that is not a geodesic orbit metric. The following problem is posed in [6].

**Problem 1.1.** Is there any other compact simple Lie group admitting a left-invariant Einstein metric that is not geodesic orbit?

In [2], H. Chen, Z. Chen and S. Deng obtained some left-invariant and not geodesic-orbit Einstein metrics on compact simple Lie groups that are arising from three locally symmetric spaces. They proved that the compact simple Lie groups $SU(n)$ for $n \geq 6$, $SO(n)$ for $n \geq 7$, $Sp(n)$ for $n \geq 3$, $E_6$, $E_7$, $E_8$, and $F_4$ admit left-invariant Einstein metrics that are not geodesic orbit.

In this short article, we construct new metrics that are distinct from the metrics with the same property obtained in [2], and we prove the following.

**Theorem 1.1.** The compact simple Lie groups $SO(n)$ ($n > 12$) admits at least two left-invariant Einstein metrics, which are not geodesic orbit.

## 2. Preliminaries

In this section, we will recall some basic facts and the Ricci tensor for reductive homogeneous spaces.

**Lemma 2.1.** ([5]) Let $M$ be a homogeneous Riemannian manifold and $G$ the identity component of the full group of isometries. Write $M = G/H$, where $H$ is the isotropic subgroup of $G$ at $x \in M$, and suppose the Lie algebra of $G$ has a reductive decomposition $g = h + m$, where $g = \text{Lie}(G)$, $h = \text{Lie}(H)$, and $m$ is the orthogonal complement subspace of $h$ in $g$ with respect to an $\text{Ad}(H)$-invariant inner product on $g$. Then $M$ is a geodesic orbit space if and only if, for any $X \in m$, there exists $Z \in h$ such that $([X + Z, Y]_m, X) = 0$ for all $Y \in m$.

Let $G$ be a compact simple Lie group, consider the following inner product on the Lie algebra $g$.

$$
\langle \cdot, \cdot \rangle = u_1(-B)|_{p_1} + u_2(-B)|_{p_2} + \cdots + u_s(-B)|_{p_s}. \tag{2.1}
$$

where $B$ is the Killing form of $g$, $u_1, \ldots, u_s$ are pairwise distinct, and $u_j > 0$, $j = 1, 2, \ldots, s$. A Lie subalgebra $t$ of $g$ is called adapted for (2.1), if $t$ is the direct sum of its ideals $t \cap p_i$, $i = 1, 2, \ldots, s$, (some of these ideals could be trivial) and the $B$-orthogonal complement to $t \cap p_i$ in $p_i$ is $\text{ad}(t)$-invariant for every $i = 1, 2, \ldots, s$. It is clear that there is a maximal by-inclusion-adapted subalgebra among all subalgebras adapted for (2.1).

Now, we recall a sufficient and necessary condition for a left-invariant Riemannian metric on a compact simple Lie group to be a geodesic orbit metric.

**Theorem 2.1.** ([6]) The inner product (2.1) generates a geodesic orbit left-invariant Riemannian metric on compact simple Lie group $G$ if and only if there is a maximal by-inclusion-adapted Lie subalgebra $t$ such that, for any $X \in g$, there exists $W \in t$ such that, for any $Y \in g$, the equality $([X + W, Y], X) = 0$ holds or, equivalently, $[A(X), X + W] = 0$, where $A : g \rightarrow g$ is a metric endomorphism.

The following theorem will be useful in the proof of our main theorem.

**Theorem 2.2.** ([6]) Suppose that the inner product (2.1) generates a geodesic orbit left-invariant Riemannian metric on compact simple Lie group $G$, $t_i = t \cap p_i$, and that $n_i$ is the $B$-orthogonal complement to $t_i$ in $p_i$. Then there is a maximal by-inclusion-adapted Lie subalgebra $t$ such that one of the following assertions holds:

1. there is no more than one index $i$ such that $t_i \neq p_i$; in this case (2.1) generates a naturally reductive left-invariant Riemannian metric on $G$;
2. $\text{rank}(t) \geq 2$, and $[n_i, n_j] \subset n_i \oplus n_j$ for $i \neq j$;
3. there is only one non-zero $t_i = t \cap p_i$, hence, $t_i = t$; moreover, $\text{rank}(t) = 1$ and either $[n_i, n_j] \subset n_i$ or $[n_i, n_j] \subset n_j$ for $i \neq j$.

Next, we recall some definitions and fundamental results for a $G$-invariant Riemannian metric on a reductive homogeneous space, whose isometry representation is decomposed into the sum of non-equivalent irreducible summands. Let $G$ be a compact semisimple Lie group, $K$ a connected closed subgroup of $G$, and let $g$ and $t$ be the corresponding Lie algebras. The Killing form $B$ of $g$ is negative definite, so we can define an $\text{Ad}(G)$-invariant inner product $B$ on $g$. Let $g = t \oplus m$ be a reductive decomposition of $g$ with respect to $B$, such that $[t, m] \subset m$ and $m \cong T_0(G/K)$. We assume that $m$ admits a decomposition into mutually non-equivalent irreducible $\text{Ad}(K)$-modules as follows:
\[ m = m_1 \oplus \cdots \oplus m_q. \] (2.2)

Then any \( G \)-invariant metric on \( G/K \) can be expressed as
\[ \langle , \rangle = x_1(-B)|m_1 + \cdots + x_q(-B)|m_q, \] (2.3)
for positive real numbers \( (x_1, \ldots, x_q) \in \mathbb{R}_+^q \).

The Ricci tensor \( r \) of a \( G \)-invariant Riemannian metric on \( G/K \) is of the same form as (2.3), that is
\[ r = y_1(-B)|m_1 + \cdots + y_q(-B)|m_q, \] (2.4)
for some real numbers \( y_1, \ldots, y_q \).

Let \( e_\alpha \) be a \((-B)\)-orthonormal basis adapted to the decomposition of \( m \), i.e. \( e_\alpha \in m_i \) for some \( i \), and \( \alpha < \beta \) if \( i < j \). We put
\[ A_{\alpha\beta} = B([e_\alpha, e_\beta]_m) = \sum \gamma A_{\alpha\beta\gamma} e_\gamma, \]
and set
\[ \begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A_{\alpha\beta\gamma})^2, \]
where the sum is taken over all indices \( \alpha, \beta, \gamma \) with \( e_\alpha \in m_i, e_\beta \in m_j, e_\gamma \in m_k \), and \([,]_m\) denotes the m-component. Then the positive numbers \( [k]_{ij} \) are independent of the \( B \)-orthonormal bases chosen for \( m_i, m_j, m_k \), and
\[ [k]_{ij} = \begin{bmatrix} k \\ jk \end{bmatrix}, \]
because of the operation law of bracket and Killing form.

### 3. Non-geodesic orbit Einstein metrics on the compact lie groups \( SO(n) \)

For \( G = SO(k_1 + k_2 + k_3 + k_4) \), \( K = \text{diag}(SO(k_1) \times SO(k_2) \times SO(k_3) \times SO(k_4)) \), we take into account the diffeomorphism:
\[ G/e \cong (G \times SO(k_1) \times SO(k_2) \times SO(k_3) \times SO(k_4))/\text{diag}(SO(k_1) \times SO(k_2) \times SO(k_3) \times SO(k_4)), \]
where \( G \times K \) acts on \( G \) by \((g, k)y = gyk^{-1}\). We denote \( so(k_1) \) as \( m_1 \), \( so(k_2) \) as \( m_2 \), \( so(k_3) \) as \( m_3 \), \( so(k_4) \) as \( m_4 \). We denote by \( M(p, q) \) the set of all \( p \times q \) matrices,
\[
\begin{align*}
m_{12} &= \begin{pmatrix}
0 & A_{12} & 0 & 0 \\
 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad |A_{12} \in M(k_1, k_2)|, \\
n_{13} &= \begin{pmatrix}
0 & 0 & A_{13} & 0 \\
 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad |A_{13} \in M(k_1, k_3)|, \\
n_{14} &= \begin{pmatrix}
0 & 0 & 0 & A_{14} \\
 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad |A_{14} \in M(k_1, k_4)|, \\
n_{23} &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
 0 & 0 & A_{23} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad |A_{23} \in M(k_2, k_3)|, \\
n_{24} &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
 0 & 0 & 0 & A_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad |A_{24} \in M(k_2, k_4)|, \\
n_{34} &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
0 & 0 & A_{34} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad |A_{34} \in M(k_3, k_4)|,
\end{align*}
\]
where \( A_{ij} \) denotes the transposed matrix of the matrix \( A_{ij}, 1 \leq i, j \leq 4 \). Note that the action of \( Ad(k) \) \((k \in K)\) on \( m \) is given by
\[ Ad(k) \begin{pmatrix}
0 & A_{12} & A_{13} & A_{14} \\
 0 & 0 & A_{23} & A_{24} \\
0 & 0 & 0 & A_{34} \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \begin{pmatrix}
h_1 A_{12} h_2 & h_1 A_{13} h_3 & h_1 A_{14} h_4 \\
 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \]
where \( \begin{pmatrix}
h_1 & 0 & 0 & 0 \\
0 & h_2 & 0 & 0 \\
0 & 0 & h_3 & 0 \\
0 & 0 & 0 & h_4
\end{pmatrix} \in K \), hence the subspaces \( m_{12}, m_{13}, m_{23}, m_{24}, m_{34} \) are irreducible \( Ad(K) \)-submodules.

We know that \( g \) admits a decomposition into mutually non-equivalent irreducible \( Ad(K) \)-modules as follows:
\[ g = m_1 + m_2 + m_3 + m_4 + m_{12} + m_{13} + m_{14} + m_{23} + m_{24} + m_{34} \] (3.1)
and consider left-invariant metrics on \( G \) that are determined by the \( Ad(SO(k_1) \times SO(k_2) \times SO(k_3) \times SO(k_4)) \)-invariant scalar products on \( so(k_1 + k_2 + k_3 + k_4) \) given by
\[ \langle , \rangle = x_1(-B)|_{so(5)} + x_2(-B)|_{so(5)} + x_3(-B)|_{so(5)} + x_4(-B)|_{so(5)} + x_{12}(-B)|_{m_{12}} \\
+ x_{13}(-B)|_{m_{13}} + x_{14}(-B)|_{m_{14}} + x_{23}(-B)|_{m_{23}} + x_{24}(-B)|_{m_{24}} + x_{34}(-B)|_{m_{34}}. \]

(3.2)

**Proposition 3.1.** The submodules in the decomposition (3.1) satisfy the following bracket relations:

\[
\begin{align*}
[m_1, m_1] &= m_1, & [m_2, m_2] &= m_2, & [m_3, m_3] &= m_3, & [m_4, m_4] &= m_4. \\
[m_1, m_{12}] &= m_{12}, & [m_2, m_{12}] &= m_{12}, & [m_3, m_{13}] &= m_{13}, & [m_4, m_{14}] &= m_{14}. \\
[m_1, m_{13}] &= m_{13}, & [m_2, m_{23}] &= m_{23}, & [m_3, m_{23}] &= m_{23}, & [m_4, m_{24}] &= m_{24}. \\
[m_1, m_{14}] &= m_{14}, & [m_2, m_{24}] &= m_{24}, & [m_3, m_{34}] &= m_{34}, & [m_4, m_{34}] &= m_{34}. \\
[m_{12}, m_{23}] &\subset m_{13}, & [m_{12}, m_{24}] &\subset m_{14}, & [m_{13}, m_{34}] &\subset m_{14}, & [m_{13}, m_{23}] &\subset m_{12}, \\
[m_{14}, m_{24}] &\subset m_{12}, & [m_{14}, m_{34}] &\subset m_{13}, & [m_{12}, m_{13}] &\subset m_{23}, & [m_{23}, m_{34}] &\subset m_{24}. \\
[m_{23}, m_{24}] &\subset m_{34}, & [m_{23}, m_{34}] &\subset m_{23}, & [m_{12}, m_{12}] &\subset m_1 + m_2, & [m_{13}, m_{13}] &\subset m_1 + m_3. \\
[m_{14}, m_{14}] &\subset m_1 + m_4, & [m_{23}, m_{23}] &\subset m_2 + m_3, & [m_{24}, m_{24}] &\subset m_2 + m_4, & [m_{34}, m_{34}] &\subset m_3 + m_4.
\end{align*}
\]

and all the other pairs of subspaces not appearing in the above list are all multiply commutative.

From [9], we know that the compact simple Lie group \( SO(n) \) \((n > 12)\) admits at least two left-invariant non-naturally reductive Einstein metrics \( \rho_i, i = 1, 2 \), which both correspond to the coefficients of the metric (3.2) satisfying the conditions

\[ x_{12} = x_{13} = x_{14} = 1, \quad x_{34} = x_{23}, \quad x_2 = x_3 = x_4, \quad x_2 \neq x_3, \quad x_3 \neq 1. \]

Moreover, it is easy to see that \( x_1, x_2, x_{23}, x_{12} \) are pairwise distinct ([9]).

Set \( p_1 = m_1, \quad p_2 = m_2 + m_3 + m_4, \quad p_3 = m_{12} + m_{13} + m_{14}, \quad p_4 = m_{23} + m_{24} + m_{34} \).

Then the metric (3.2) reduces to

\[ \langle , \rangle = x_1(-B)\langle , \rangle_{p_1} + x_2(-B)\langle , \rangle_{p_2} + x_{12}(-B)\langle , \rangle_{p_3} + x_{23}(-B)\langle , \rangle_{p_4}. \]

(3.3)

Now we can give the proof of the main result of this paper.

**Proof of Theorem 1.1.** Let us consider Lie group \( SO(n) \) \((n > 12)\) supplied with two left-invariant non-naturally reductive Einstein metrics \( \rho_i, i = 1, 2 \), generated with the inner product (3.3) (see [9]), now we show that the Riemannian manifolds \( (SO(n) \ (n > 12), \rho_i) \) are not geodesic orbit.

Choose any maximal by-inclusion subalgebra \( \mathfrak{t} \) adapted to (3.3); by the definition of \( \mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{p}_i, i = 1, 2, 3, 4 \), we have \([\mathfrak{t}_1, \mathfrak{t}_1] \subset \mathfrak{t}_1\) and \([\mathfrak{t}_1, \mathfrak{n}_1] \subset \mathfrak{t}_1, \mathfrak{n}_1\), where \( \mathfrak{n}_i \) is the orthogonal complement to \( \mathfrak{t}_i \) in \( \mathfrak{p}_i \). On the other hand, \([\mathfrak{p}_3, \mathfrak{p}_3] \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_4\) and \([\mathfrak{p}_4, \mathfrak{p}_4] \subset \mathfrak{p}_2 \oplus \mathfrak{p}_4 \). So \([\mathfrak{t}_3, \mathfrak{p}_3] = 0,\) notice \([\mathfrak{t}_3, \mathfrak{t}_3] \subset \mathfrak{t}_3\); it is easy to get \( \mathfrak{t}_3 = 0 \). From \([\mathfrak{t}_4, \mathfrak{t}_4] \subset \mathfrak{t}_4\) and \([\mathfrak{t}_4, \mathfrak{n}_4] \subset \mathfrak{n}_4\), by Proposition 3.1, it is easy to get \( \mathfrak{t}_4 = 0 \). Thus, \( \mathfrak{t}_3 \) and \( \mathfrak{t}_4 \) are trivial and \( \mathfrak{t} \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2 \).

Suppose that the inner product (3.3) generates a geodesic orbit left-invariant Riemannian metric, take \( X_{13} \in m_{13} \) and \( X_{23} \in m_{23}\) such that \([X_{13}, X_{23}] \neq 0\). By Theorem 2.1, for \( X_{13} + X_{23} \in m_{13} + m_{23}\), there exists \( W \in \mathfrak{t} \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2\), such that \([A(X_{13} + X_{23}), X_{13} + X_{23} + W] = 0\). Then, we have

\[ (x_{12} - x_{23})[(X_{13}, X_{23}) + (x_{12}X_{13} + x_{23}X_{23}, W)] = 0. \]

Since \([m_{13}, m_{23}] \subset m_{12}, \) and \([m_{13}, m_{23}] \subset \text{ad}(\mathfrak{p}_1 \oplus \mathfrak{p}_2)\)-invariant submodules, we have \( x_{12} = x_{23} \), which is impossible.

Thus, the Riemannian manifolds \( (SO(n) \ (n > 12), \rho_i) \), \( i = 1, 2 \) are not geodesic orbit. This completes the proof.

**Acknowledgements**

The authors are deeply grateful to the referees for valuable comments and helpful suggestions. This research was supported by the Youth Foundation of Anhui University of Technology (Nos. QZ201818 and QZ201819), by NSFC (Nos. 11671212 and 11771331) and by the Natural Science Foundation of Anhui Province (No. 1908085QA03).

**References**
