C. R. Acad. Sci. Paris, Ser. I 357 (2019) 655–663

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Group theory/Number theory

Coset diagrams of the modular group and continued fractions ☆

Diagrammes de classes du groupe modulaire et fractions continues

Ayesha Rafiq [a], Qaiser Mushtaq [b]

[a] *Institute of Space Technology, Islamabad, Pakistan*
[b] *The Islamia University of Bahawalpur, Bahawalpur, Pakistan*

ARTICLE INFO

Article history:
Received 14 September 2018
Accepted after revision 24 June 2019
Available online 3 September 2019

Presented by the Editorial Board

ABSTRACT

The coset diagram for each orbit under the action of the modular group on $\mathbb{Q}(\sqrt{n})^* = \mathbb{Q}(\sqrt{n}) \cup \{\infty\}$ contains a circuit C_i. For any $\alpha \in \mathbb{Q}(\sqrt{n})$, the path leading to the circuit C_i and the circuit itself are obtained through continued fractions in this paper. We show that the structure of the continued fractions of a reduced quadratic irrational element is weaved with the structure or type of the circuit. The three types of circuits of the action of V_4 on $\mathbb{Q}(\sqrt{n})^*$ are also interconnected with the structure of continued fractions. The action of the modular group on $\mathbb{Q}(\sqrt{5})^*$ is chosen specifically because a circuit of it is related to the ratio of the Fibonacci numbers being the solution to the continued fractions of the golden ratio.

RÉSUMÉ

Le diagramme des classes de chaque orbite de l'action du groupe modulaire sur $\mathbb{Q}(\sqrt{n})^* = \mathbb{Q}(\sqrt{n}) \cup \{\infty\}$ contient un circuit C_i. Dans cette Note, pour tout $\alpha \in \mathbb{Q}(\sqrt{n})$, le chemin menant au circuit C_i et le circuit lui-même sont décrits en termes de fractions continues. Nous montrons que la structure des fractions continues des nombres quadratiques irrationnels réduits est liée à la structure ou au type du circuit. Les trois types de circuits de l'action de V_4 sur $\mathbb{Q}(\sqrt{n})^*$ sont également reliés à la structure des fractions continues. L'action du groupe modulaire sur $\mathbb{Q}(\sqrt{5})^*$ est choisie précisément, car un de ses circuits est lié au fait que les rapports des nombres de Fibonacci sont les convergents de la fraction continue du nombre d'or.

☆ This work was presented at the 13th International Pure Mathematics Conference 2012, Islamabad, Pakistan.

E-mail addresses: ayesha.rafiq@ist.edu.pk, ayesharafiq@math.qau.edu.pk (A. Rafiq), pir_qmushtaq@yahoo.com (Q. Mushtaq).

https://doi.org/10.1016/j.crma.2019.07.002
1631-073X/

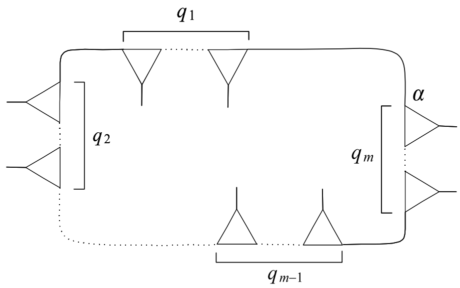

Fig. 1. A circuit of the action of the modular group on $\mathbb{Q}(\sqrt{n})^*$.

1. Introduction

If the upper half-plane model H of hyperbolic plane geometry is considered, which is a model of Lobachevsky plane $\{z = x + iy : x, y \in \mathbb{R} \text{ and } y > 0\}$ and the motions in it preserve the orientation, then the group of all orientation-preserving isometries of H consists of all *Möbius transformations* of the form $z \mapsto \dfrac{az+b}{cz+d}$, where a, b, c, and d are real numbers, and $ad - bc = \pm 1$.

From another point of view, the group $PSL(2,\mathbb{R})$ acts on the upper half-plane H according to the faithful (left-) action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}, \tag{1.1}$$

forming a group of isometries of the hyperbolic plane H. The group, which comprises linear fractional transformations of H with integer coefficients, is a discrete group of motions and forms an important subgroup of $PSL(2,\mathbb{R})$. This group of transformations is isomorphic to the projective special linear group $PSL(2,\mathbb{Z})$, which is the quotient of the 2-dimensional special linear group over the integers by its centre $\{I, -I\}$. In other words, $PSL(2,\mathbb{Z})$ consists of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, and d are integers, and $ad - bc = 1$, and the pairs of matrices A and $-A$ are considered to be identical and the group operation is the usual multiplication of matrices.

The modular group $PSL(2,\mathbb{Z})$ has the finite presentation $\langle x, y : x^2 = y^3 = 1\rangle$, where x and y correspond to the linear fractional transformations $z \mapsto \frac{-1}{z}$ and $z \mapsto \frac{z-1}{z}$. A proof of this, using coset diagrams, is given in [6].

The concept of graphs was first introduced in 1878 by A. Cayley. A number of group theorists used Cayley's diagrams to prove many important results on finitely generated groups. O. Schreier generalised the Cayley's diagrams by introducing a graph whose vertices represent the cosets of any given subgroup.

In 1978, G. Higman proposed coset diagrams for the modular and extended modular group. These are called coset diagrams because here the vertices are identifiable with the right cosets in a permutation group G of the stabiliser N of any point of the G-space Ω, so that an edge x_i joins the coset Ng to the coset Ngx_i for each element g of G. Since $PSL(2,\mathbb{Z})$ has two generators, the edges associated with the involution x are represented by small edges without any orientation attached to them. In the case of y, which has order 3, there is a need to distinguish y from y^{-1}. The 3-cycles of y are therefore represented by small triangles, with the convention that y permutes their vertices counter-clockwise. The fixed points of x and y, if they exist, are denoted by heavy dots. More details can be found in [11].

A quadratic irrational field is a field extension of degree 2 over $\mathbb{Q}$ denoted by $\mathbb{Q}(\sqrt{n})$. If an element $\alpha \in \mathbb{Q}(\sqrt{n})$, then $\alpha = a + b\sqrt{n}$, where $a, b \in \mathbb{Q}$. The algebraic conjugate of α is $\bar{\alpha} = a - b\sqrt{n}$. The trace and norm of α are $Tr(\alpha) = \alpha + \bar{\alpha}$ and $N(\alpha) = \alpha\bar{\alpha}$, respectively. Every $\alpha \in \mathbb{Q}(\sqrt{n})$ is the root of a monic polynomial of degree 2 with rational coefficients $(x-\alpha)(x-\bar{\alpha}) = x^2 - Tr(\alpha)x + N(\alpha)$, so an element is an integer of $\mathbb{Q}(\sqrt{n})$ if $Tr(\alpha)$ and $N(\alpha)$ belong to $\mathbb{Z}$. When $n > 0$, $\mathbb{Q}(\sqrt{n})$ is called a real, and when $n < 0$ an imaginary, quadratic field [5], [14].

Every real quadratic irrational number α can be uniquely written as $(a + \sqrt{n})/c$, where n is a non-square positive integer, and a, $(a^2 - n)/c$, and c are relatively prime integers. If α and its algebraic conjugate $\bar{\alpha}$ have positive signs, then α is called a totally positive number; if they have negative signs, then α is called a totally negative number, and if they have different signs, then α is called an *ambiguous number*. A formula for obtaining the ambiguous numbers is provided in [7], but this approach does not seem to have any connection with the continued fraction of the element.

Let the modular group $G = PSL(2,\mathbb{Z}) = \langle x, y : x^2 = y^3 = 1\rangle$ act on the *extended real quadratic irrational field* $\mathbb{Q}(\sqrt{n})^* = \mathbb{Q}(\sqrt{n}) \cup \{\infty\}$ [10]. The ambiguous numbers in the coset diagram of the orbit αG form a single closed path called a circuit. A circuit of type $(q_1, q_2, \ldots, q_m)$ means a sequence of positive integers q_i representing alternatively q_j triangles with one vertex inside the circuit and q_{j+1} triangles with one vertex outside the circuit for all $j = 1, 2, \ldots, m$. Algebraically the circuit corresponds to the word $w = (xy)^{q_1}(xy^2)^{q_2} \ldots (xy)^{q_{m-1}}(xy^2)^{q_m}$, which fixes the element α. This is represented in Fig. 1.

The coset diagram of an orbit of the action of G on $\mathbb{Q}(\sqrt{n})^*$ has only one circuit; let C_i be the circuit and Γ_{C_i} be its coset diagram. If the set

$$C(n) = \left\{C_i \mid C_i \text{ is a circuit in an orbit of the action of } G \text{ on } \mathbb{Q}(\sqrt{n})^*\right\}$$

is the collection of all circuits, then $\Gamma_{C(n)}$ denotes the complete coset diagram of the action of G on $\mathbb{Q}(\sqrt{n})^*$.

By L. Euler, every real number has a continued fraction $\alpha = [q_1; q_2, ...]$ that is finite for rational numbers and infinite for irrational numbers. The irrationals whose continued fractions repeat after a certain stage such that $\alpha = [q_1; q_2, ..., q_m, \overline{q_{m+1}, q_{m+2}, ..., q_{m+d}}]$ are the quadratic irrational numbers. Associated with these quadratic irrationals are the matrices $\begin{bmatrix} A_l & A_{l-1} \\ B_l & B_{l-1} \end{bmatrix}$, where $A_l = [q_1, q_2, ..., q_l]$ and $B_l = [q_2, q_3, ..., q_l]$ are continuants of the lth convergent $\frac{A_l}{B_l}$ (see [2]).

The conjunction of modular surfaces and continued fractions is not new (see [15]); their connection has been exploited in several directions by [1], [3] and [9], to name a few. In this study, the connection of coset diagrams of the action of G on $\mathbb{Q}(\sqrt{n})^*$, as defined by the second author in [10], to continued fractions is explored. Some authors [8] have obtained two proper G-subsets of $\mathbb{Q}(\sqrt{n})^*$ corresponding to each odd prime divisor of n. R. Qureshi and T. Nakahara also presented the process to reach an ambiguous number from a totally negative or totally positive number through continued fractions in [13].

We show that the structure of circuits in the orbits of the action G on $\mathbb{Q}(\sqrt{n})^*$ and the structure of the continued fractions of the $\mathbb{Q}(\sqrt{n})$ are intertwined. Thus, several results of continued fractions give information about the circuits and ambiguous numbers, and vice versa.

2. Continued fractions and coset diagrams of $PSL(2, \mathbb{Z})$

We start by giving a few simple equalities.

Lemma 1. Let x, $y \in G$, with $x: z \to \frac{-1}{z}$, $y: z \to \frac{z-1}{z}$ and $\alpha \in \mathbb{Q}(\sqrt{n})$, then the following equalities hold:

(1) $\left[(\alpha)\left(xy^2\right)^{q_i}\right]^{-1} = \left(\alpha^{-1}\right)(xy)^{q_i}$.

(2) $\left[(\alpha)(xy)^{q_i}\right]^{-1} = \left(\alpha^{-1}\right)\left(xy^2\right)^{q_i}$.

(3) $\left[(\alpha)\left(xy^2\right)^{q_i}(xy)^{q_j}\right]^{-1} = \left(\alpha^{-1}\right)(xy)^{q_i}\left(xy^2\right)^{q_j}$.

Proof. 1. As $(\alpha)\left(xy^2\right)^{q_i} = \frac{\alpha}{q_i\alpha+1}$, so $\left[(\alpha)\left(xy^2\right)^{q_i}\right]^{-1} = \left[\frac{\alpha}{q_i\alpha+1}\right]^{-1} = \frac{q_i\alpha+1}{\alpha} = q_i + \alpha^{-1} = \left(\alpha^{-1}\right)(xy)^{q_i}$.

2. As $(\alpha)(xy)^{q_i} = \alpha + q_i$, so $\left[(\alpha)(xy)^{q_i}\right]^{-1} = [\alpha + q_i]^{-1} = \frac{1}{\alpha+q_i} = \frac{\alpha^{-1}}{q_i\alpha^{-1}+1} = \left(\alpha^{-1}\right)\left(xy^2\right)^{q_i}$.

3. By the previous two equalities, $\left[(\alpha)\left(xy^2\right)^{q_i}(xy)^{q_j}\right]^{-1} = \left((\alpha)\left(xy^2\right)^{q_i}\right)^{-1}\left(xy^2\right)^{q_j} = \left(\left(\alpha^{-1}\right)(xy)^{q_i}\right)\left(xy^2\right)^{q_j} = \left(\alpha^{-1}\right)(xy)^{q_i}\left(xy^2\right)^{q_j}$. □

A Möbius transformation of the form $(z)\, w = \dfrac{az+b}{cz+d}$ may be represented in a matrix notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} = \frac{az+b}{cz+d}.$$

So, x and y have matrix representations of the form:

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

The following theorem connects the structure of continued fraction of an element $\alpha \in \mathbb{Q}\left(\sqrt{n}\right)$ to the structure of words with xy and xy^2 of G, which will later be used for the coset diagrams.

Theorem 1. *The continued fraction $[q_1; q_2, ..., q_m]$ of $\alpha \in \mathbb{Q}\left(\sqrt{n}\right)$ gives a path $(xy)^{q_1}\left(xy^2\right)^{q_2} \ldots \left(xy^2\right)^{q_m}$ from α to another α' in the coset diagram of the action of G on $\mathbb{Q}\left(\sqrt{n}\right)^*$.*

Proof. Let x, $y \in PSL(2, \mathbb{Z})$ with matrix representations:

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then,

$$(YX)^{q_i}=\begin{bmatrix}1 & q_i\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}^{q_i},\ \left(Y^2X\right)^{q_j}=\begin{bmatrix}1 & 0\\ q_j & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^{q_j}$$

are the respective matrices of the transformations $(z)\,(xy)^{q_i}=z+q_i$ and $(z)\left(xy^2\right)^{q_j}=\frac{z}{q_jz+1}$.

By [16], which goes back at least to [4], a result of [17] states that if

$$\begin{bmatrix}q_1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}q_2 & 1\\ 1 & 0\end{bmatrix}\cdots\begin{bmatrix}q_m & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}A_m & A_{m-1}\\ B_m & B_{m-1}\end{bmatrix},\ m=1,2,3,\ldots \tag{2.1}$$

then $\frac{A_m}{B_m}=[q_1;q_2,...,q_m]$, where $\frac{A_m}{B_m}$ is the mth convergent of α and $\begin{vmatrix}A_m & A_{m-1}\\ B_m & B_{m-1}\end{vmatrix}=(-1)^m$.

It is observed that

$$\begin{bmatrix}q_i & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}q_j & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}q_iq_j+1 & q_i\\ q_j & 1\end{bmatrix}=\begin{bmatrix}1 & q_i\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ q_j & 1\end{bmatrix}. \tag{2.2}$$

By equations (2.2) and (2.1), we get:

$$\begin{bmatrix}1 & q_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ q_2 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & 0\\ q_m & 1\end{bmatrix}=\begin{bmatrix}A_m & A_{m-1}\\ B_m & B_{m-1}\end{bmatrix}. \tag{2.3}$$

Since these matrices are powers of linear fractional transformations of xy and xy^2, we have

$$\begin{bmatrix}1 & q_i\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ q_j & 1\end{bmatrix}=(YX)^{q_i}\left(Y^2X\right)^{q_j}$$

which implies that Eq. (2.3) can be written as:

$$(xy)^{q_1}\left(xy^2\right)^{q_2}\ldots\left(xy^2\right)^{q_m}=\begin{bmatrix}A_m & A_{m-1}\\ B_m & B_{m-1}\end{bmatrix}. \tag{2.4}$$

This sets up a correspondence between the words of powers of xy and xy^2 of G and continued fractions. □

We state the following important theorem.

Theorem 2. *If $\alpha\in\mathbb{Q}\left(\sqrt{n}\right)$, where n is a square-free positive integer, then for the continued fraction $[q_1;q_2,...,q_m,\overline{q_{m+1},q_{m+2},...,q_{m+d}}]$ of α, the period of the continued fraction forms a circuit in the coset diagram of the action of G on $\mathbb{Q}(\sqrt{n})^*$. The word $w_\alpha=\left(y^2x\right)^{q_1}(yx)^{q_2}\cdots\left(y^2x\right)^{q_{m-1}}(yx)^{q_m}$, where m is even, leads α to the reduced quadratic irrational number α', and the word $w_{C_i}=\left(xy^2\right)^{q_{m+d}}(xy)^{q_{m+d-1}}\ldots\left(xy^2\right)^{q_{m+2}}(xy)^{q_{m+1}}$ fixes α'.*

Proof. By Eq. (2.3), for the continued fraction $[q_1;q_2,...,q_m,\overline{q_{m+1},q_{m+2},...,q_{m+d}}]$ of α,

$$\begin{bmatrix}1 & q_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ q_2 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & q_{m-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ q_m & 1\end{bmatrix}\begin{bmatrix}\alpha'\\ 1\end{bmatrix}=\begin{bmatrix}\alpha\\ 1\end{bmatrix}$$

which leads from α' to α, that is, it is a path from α' to α in the coset diagram of the action of $PSL(2,Z)$ on $\mathbb{Q}(\sqrt{n})^*$. Denote this path by a word $w=\left(xy^2\right)^{q_m}(xy)^{q_{m-1}}\cdots\left(xy^2\right)^{q_2}(xy)^{q_1}$ such that

$$(\alpha')w=\alpha. \tag{2.5}$$

Then $\alpha'=(\alpha)\,w^{-1}=(\alpha)\,w_\alpha$, which gives the path from α to α' such that

$$\alpha'=(\alpha)\left(y^2x\right)^{q_1}(yx)^{q_2}\cdots\left(y^2x\right)^{q_{m-1}}(yx)^{q_m}.$$

The periodic part of the continued fraction gives

$$\begin{bmatrix}1 & q_{m+1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ q_{m+2} & 1\end{bmatrix}\cdots\begin{bmatrix}1 & q_{m+d-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ q_{m+d} & 1\end{bmatrix}\begin{bmatrix}\alpha'\\ 1\end{bmatrix}=\begin{bmatrix}\alpha'\\ 1\end{bmatrix}.$$

Denote this path by a word $w_{C_i}=\left(xy^2\right)^{q_{m+d}}(xy)^{q_{m+d-1}}\ldots\left(xy^2\right)^{q_{m+2}}(xy)^{q_{m+1}}$ such that

$$(\alpha')w_{C_i}=\alpha'. \tag{2.6}$$

As $w_{C_i}\in G$ and fixes α', by [10] such a word is a circuit $(q_{m+1},q_{m+2},...,q_{m+d})$ in the respective coset diagram, where x is represented by — and the three cycles of y by Δ permuted anticlockwise for the action of G on $\mathbb{Q}(\sqrt{n})^*$ with q_{m+1}

triangles inside the circuit and q_{m+2} triangles outside the circuit, alternatively. So, the period of an element of $\mathbb{Q}\left(\sqrt{n}\right)$ provides the only circuit residing in the coset diagram of the orbit to which α belongs.

For the circuit, if d is even, then word w_{C_i} is associated with the matrix $\begin{bmatrix} A_{m+d} & A_{m+d-1} \\ B_{m+d} & B_{m+d-1} \end{bmatrix}$, which has determinant 1. Hence, w_{C_i} belongs to G. But if d is odd, the determinant of the associated matrix is -1, which does not belong to G. Taking w_{C_i} twice makes it even, that is,

$$\left(\alpha'\right)\left(xy^2\right)^{q_{m+d}}(xy)^{q_{m+d-1}}\cdots\left(xy^2\right)^{q_m}(xy)^{q_{m+d}}\cdots\left(xy^2\right)^{q_{m+1}}(xy)^{q_m}=\alpha'$$

which yields a matrix with determinant 1, which thus belongs to G. □

Corollary 1. *The uniquely represented reduced quadratic irrational numbers, whose continued fractions are cyclically not equivalent, reveal the circuits in the orbits of the action of G on* $\mathbb{Q}\left(\sqrt{n}\right)^*$.

Proof. The uniquely represented reduced quadratic irrational numbers α are elements that satisfy $\gcd\left(a, \frac{a^2-n}{c}, c\right)=1$, $\alpha>1$ and $-1<\bar{\alpha}<0$. If two elements have cyclically equivalent continued fractions, either both belong in the same circuit or one of them belongs in the circuit of its algebraic conjugate. Hence, all the circuits of the action are obtained by the continued fractions of uniquely represented reduced quadratic irrational numbers, whose continued fractions are cyclically not equivalent. □

We illustrate this by considering the following example.

Example 1. Let $\alpha=\frac{24-\sqrt{15}}{17}\in\mathbb{Q}\left(\sqrt{15}\right)$. The continued fraction of α is $\left[1;5,\overline{2,3}\right]$ which implies $(\alpha)\left(y^2x\right)(yx)^5=\alpha'$, and $\left(\alpha'\right)\left(xy^2\right)^3(xy)^2=\alpha'$, where α' is either $\frac{3+\sqrt{15}}{3}$ or its algebraic conjugate $\frac{3-\sqrt{15}}{3}$.

By the theorem (2), the continued fraction of an element of $\mathbb{Q}\left(\sqrt{n}\right)$ is related to the path that leads from that element to an ambiguous number, and also to the type of circuit of the orbit to which the element belongs in the action of G on $\mathbb{Q}\left(\sqrt{n}\right)^*$. So, the structure of the continued fraction is interwoven with the structure or type of the circuit. Thus, obtaining all the reduced quadratic irrational numbers is equivalent to obtaining all the ambiguous numbers.

Lemma 2. If $\left(\alpha'\right)\left(xy^2\right)^{q_m}(xy)^{q_{m-1}}\cdots\left(xy^2\right)^{q_2}(xy)^{q_1}=\alpha$, then, for m even,

$$(\alpha)\left(y^2x\right)^{q_1}(yx)^{q_2}\cdots\left(y^2x\right)^{q_{m-1}}(yx)^{q_m}=\alpha',$$

and, for m odd,

$$(\alpha)\left(y^2x\right)^{q_1}(yx)^{q_2}\cdots(yx)^{q_{m-1}}\left(y^2x\right)^{q_m}=\left(\alpha'\right)^{-1}.$$

Theorem 3. *For every distinct periodic part of the reduced quadratic irrational number in* $\mathbb{Q}(\sqrt{n})$*, there are two orbits of the action with the same type of circuit, and only one orbit if the circuit is of type* $(q_{m+1}, q_{m+2}, \ldots q_{m+d}, q_{m+d}, \ldots, q_{m+2}, q_{m+1})$.

Proof. Let α and $\bar{\alpha}$ be roots of the quadratic equation of the word w_2 in Eq. (2.6). This means that they belong to a circuit of type $(q_{m+1}, q_{m+2}, \ldots, q_{m+d})$. If α and $\bar{\alpha}$ belong in the same circuit, that is, the circuit is of type

$$(q_{m+1}, q_{m+2}, \ldots q_{m+d}, q_{m+d}, \ldots, q_{m+2}, q_{m+1})$$

then there is only one orbit with circuit $(q_{m+1}, q_{m+2}, \ldots, q_{m+d})$; otherwise there are two orbits with circuit

$$(q_{m+1}, q_{m+2}, \ldots, q_{m+d})$$

one orbit containing the circuit of α and the other containing the conjugates of the aforementioned orbit. □

Corollary 2. *There are* $4\sum_{i=1}^{m} q_i$ *ambiguous numbers for the circuit of type* $(q_1, q_2, \ldots, q_m, q_m, \ldots, q_2, q_1)$ *and, for any other circuit of type* $(q_1, q_2, \ldots, q_{2n})$*, there are two orbits, so that there are* $4\sum_{i=1}^{2n} q_i$ *of them.*

So, obtaining all the circuits of the action of $PSL\left(2,\mathbb{Z}\right)$ on $\mathbb{Q}\left(\sqrt{n}\right)^*$ gives all the ambiguous numbers of the action.

In the following table, we list a few relations between the periodic continued fractions of a real quadratic irrational number and the path and the circuit in the coset diagram of the action:

	Continued fraction	Path	Circuit
1.	$[\overline{a}]=\frac{a+\sqrt{a^2+4}}{2}$		$(\alpha')\left(xy^2\right)^a(xy)^a=\alpha'$
2.	$[1;\overline{a}]=\frac{2-a+\sqrt{a^2+4}}{2}$	$(\alpha)\left(y^2x\right)=\alpha'$	$(\alpha')\left(xy^2\right)^a(xy)^a=\alpha'$
3.	$\left[a;\overline{2a}\right]=\sqrt{a^2+1}$	$(\alpha)\left(y^2x\right)^a=\alpha'$	$(\alpha')\left(xy^2\right)^{2a}(xy)^{2a}=\alpha'$
4.	$\left[\overline{a;b}\right]=\frac{-ab+\sqrt{ab(ab+4)}}{2a}$		$(\alpha')\left(xy^2\right)^a(xy)^b=\alpha'$
5.	$\left[a;\overline{b_1,\ldots,b_m}\right]$ $=a+\frac{1}{\left[\overline{b_1;\ldots,b_m}\right]}$	$(\alpha)\left(y^2x\right)^a=\alpha'$	$(\alpha')\left(xy^2\right)^{b_m}(xy)^{b_{m-1}}$ $\ldots\left(xy^2\right)^{b_2}(xy)^{b_1}=\alpha'$

Finally, we state that the uniquely represented reduced quadratic irrational numbers reveal the circuits of the orbits of the action of $PSL\left(2,\mathbb{Z}\right)$ on $\mathbb{Q}\left(\sqrt{n}\right)^*$.

3. Continued fractions of three types

By [12], in the coset diagram for the action of $PSL\left(2,\mathbb{Z}\right)$ on $\mathbb{Q}\left(\sqrt{n}\right)^*$, a point α is on a circuit if and only if it is fixed by some element $g=(xy)^{q_1}(xy^2)^{q_2}\ldots(xy^2)^{q_{2m}}$, which means that the circuits are permuted by any permutation g of $\mathbb{Q}\left(\sqrt{n}\right)^*$ that normalises the set $\{xy, xy^{-1}\}$. Two such permutations are $s: z\longrightarrow \bar{z}$ and $t: z\longrightarrow 1/z$. Since $s^2=t^2=(st)^2=1$, a 4-permutation group permutes the circuits.

It is given that for the action of $V_4=\langle s,t : s^2=t^2=(st)^2=1\rangle$, under s the circuits that contain α and its image $\bar{\alpha}$ are of type $(q_1,q_2,\ldots,q_m,q_m,\ldots,q_2,q_1)$, under t the circuits that contain α and its image $1/\alpha$ are of type $(q_1,q_2,\ldots,q_m,q_1,q_2,\ldots,q_k)$, and under st the circuits that contain α and its image $1/\bar{\alpha}$ are of type

$$(q_1,q_2,\ldots,q_m,q_{m+1},q_m,\ldots,q_2).$$

Thus, there are three types of circuits in the orbits due to the action of V_4 on $\mathbb{Q}(\sqrt{n})^*$. We explain these circuits with respect to continued fractions.

(1) For the continued fractions $[q_1;q_2,\ldots,q_m]$ of the reduced quadratic α, it is known that

$$\alpha=\frac{A_m\alpha+A_{m-1}}{B_m\alpha+B_{m-1}}, \tag{3.1}$$

and that the Eq. (2.3) and reverse of the cycle $[q_m;\ldots,q_1]$ yields

$$\beta=\frac{A_m\beta+B_m}{A_{m-1}\beta+B_{m-1}}. \tag{3.2}$$

If $\beta=-1/\alpha$, then Eq. (3.2) converts into equation (3.1), which means that $-1/\beta$ is a root of this equation. Since α and β are positive, so the other root of Eq. (3.1) is $\bar{\alpha}=-1/\alpha$.

We observe that, for the circuit of type $(q_1,q_2,\ldots,q_m,q_m,\ldots,q_1)$ in the coset diagram, the reverse circuit does not change, which means that, under the transformation s, α and $\bar{\alpha}=-1/\alpha$ lie in the same circuit.

(2) For a cycle of odd length $[q_1;q_2,\ldots,q_{2m+1}]$, the matrix $\begin{bmatrix} A_m & A_{m-1}\\ B_m & B_{m-1}\end{bmatrix}$ has determinant -1. To obtain a matrix with determinant 1, continue the series of convergents until an even length is reached. Take an odd cycle

$$(\alpha)\left(xy^2\right)^{q_{2m+1}}(xy)^{q_{2m}}\ldots(xy)^{q_2}\left(xy^2\right)^{q_1}=\alpha^{-1}, \tag{3.3}$$

by lemma (1) we get $\left(\alpha^{-1}\right)(xy)^{q_{2m+1}}\left(xy^2\right)^{q_{2m}}\ldots\left(xy^2\right)^{q_2}(xy)^{q_1}=\alpha$. Conjoining both of the equations, we get

$$(\alpha)\left(xy^2\right)^{q_{2m+1}}(xy)^{q_{2m}}\ldots\left(xy^2\right)^{q_1}(xy)^{q_{2m+1}}\left(xy^2\right)^{q_{2m}}\ldots\left(xy^2\right)^{q_2}(xy)^{q_1}=\alpha.$$

Hence if the continued fraction is of type $[q_1;q_2,\ldots,q_{2m+1},q_1,q_2,\ldots,q_{2m+1}]$ then α and α^{-1} belong to the same circuit where the position of α^{-1} is determined by equation (3.3).

(3) Combining the above two facts, if the cycle is of type $[q_1;q_2,\ldots q_m,q_{m+1},q_m,\ldots,q_2]$, then α and $1/\bar{\alpha}$ belong to the same orbit.

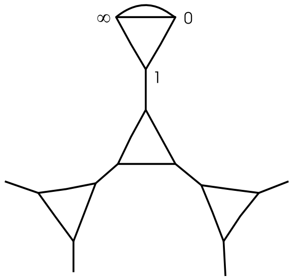

Fig. 2. An orbit with circuit of type $(1, 0)$.

4. Circuits of the action of $PSL(2, \mathbb{Z})$ on $\mathbb{Q}\left(\sqrt{5}\right)^*$

In this section, we determine all the circuits of the action of G on $\mathbb{Q}\left(\sqrt{5}\right)^*$ and investigate their structure because a circuit of this action is related to the ratio of the Fibonacci numbers that are the solutions to the continued fractions of the golden ratio.

We denote the fact $(\infty)(xy^2)^0(xy)^1 = \infty + 1 = \infty$ as a circuit of the type $(1, 0)$. This is represented in Fig. 2.

Theorem 4. *The action of G on $\mathbb{Q}\left(\sqrt{5}\right)^*$ has only three orbits with circuits $(1, 0)$, $(1, 1)$ and $(4, 4)$ up to the unique representation of the quadratic irrational numbers.*

Proof. Let $\alpha_{ij} = \frac{a_i+\sqrt{5}}{c_j}$ be a uniquely represented reduced quadratic irrational number of $\mathbb{Q}(\sqrt{5})$, with $\gcd\left(a_i, \frac{a_i^2-5}{c_j}, c_j\right) = 1$ and $a_i, c_j \in \mathbb{Z}^+$. By the properties of the reduced quadratic irrational numbers $\alpha_{ij} > 1$ and $-1 < \bar{\alpha}_{ij} < 0$, we have

$$c_j < a_i + \sqrt{5} \text{ and } \sqrt{5} - a_i < c_j.$$

The inequality $a_i < \sqrt{5}$ implies two possible values, 1 and 2, of a_i. For $a_i = 1$, c_j has two possible values 2 and 3, and for $a_i = 2$, c_j has four possible values 1, 2, 3, and 4 implying 6 reduced quadratic irrational numbers, so that:

a_i	c_j	α_{ij}	continued fractions	circuits
1	2	$\frac{1+\sqrt{5}}{2}$	$[\overline{1}]$	$(1, 1)$
1	3	$\frac{1+\sqrt{5}}{3}$	$[\overline{1; 12, 1, 2, 2, 2}]$	$(2, 2, 1, 12, 1, 2)$
2	1	$2+\sqrt{5}$	$[\overline{4}]$	$(4, 4)$
2	2	$\frac{2+\sqrt{5}}{2}$	$[\overline{2; 8}]$	$(2, 8)$
2	3	$\frac{2+\sqrt{5}}{3}$	$[\overline{1; 2, 2, 2, 1, 12}]$	$(2, 2, 1, 12, 1, 2)$
2	4	$\frac{2+\sqrt{5}}{4}$	$[\overline{1; 16}]$	$(1, 16)$

As the two reduced real quadratic irrational numbers $\frac{1+\sqrt{5}}{3}$ and $\frac{2+\sqrt{5}}{3}$ have continued fractions that are cyclically equivalent, they belong to the same orbit. Since the circuits $(1, 1)$ and $(4, 4)$ are of type $(q_{m+1}, \ldots q_{m+d}, q_{m+d}, \ldots, q_{m+1})$, by theorem (3) only one orbit of the form exists.

Recall that every real quadratic irrational number can be written uniquely as $(a_i + \sqrt{n})/c_j$, n is a non-square positive integer, and a_i, b_{ij} and c_j are relatively prime integers, where $b_{ij} = (a_i^2 - n)/c_j$:

(1) for $a = 1$, $c = 2$, and $b = \frac{1^2-5}{2} = -2$, $\gcd(a, b, c) = 1$ implies that $\frac{1+\sqrt{5}}{2}$ belongs to the circuit $(1, 1)$ of $\mathbb{Q}\left(\sqrt{5}\right)$;

(2) $\frac{1+\sqrt{5}}{3}$ (or $\frac{2+\sqrt{5}}{3}$) $\in (2, 2, 1, 12, 1, 2)$. For $a = 1$ or 2, $c = 3$ and $b = \frac{1^2-5}{3} = \frac{-4}{3}$ and $\frac{2^2-5}{3} = \frac{-1}{3}$, $\gcd(a, b, c) = \frac{1}{3}$. Converting $\frac{1+\sqrt{5}}{3}$ in $\frac{3+\sqrt{45}}{9}$, which belongs to $\mathbb{Q}\left(\sqrt{45}\right)$, one has that $\gcd(3, \frac{4^2-20}{4}, 4) = 1$;

(3) $2 + \sqrt{5}$ belongs to the circuit $(4, 4)$ of $\mathbb{Q}\left(\sqrt{5}\right)$, one has that $\gcd(2, \frac{2^2-5}{1}, 1) = 1$;

(4) $\frac{2+\sqrt{5}}{2}$ belongs to the circuit $(2, 8)$, so that $\gcd(2, \frac{2^2-5}{2}, 2) = \frac{1}{2} \neq 1$. Converting $\frac{2+\sqrt{5}}{2}$ into $\frac{4+\sqrt{20}}{4}$, which belongs to $\mathbb{Q}\left(\sqrt{20}\right)$, one has that $\gcd(4, \frac{4^2-20}{4}, 4) = 1$;

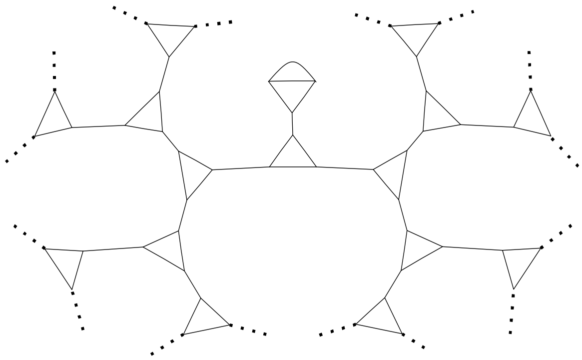

Fig. 3. Orbit with circuit $(1, 0)$.

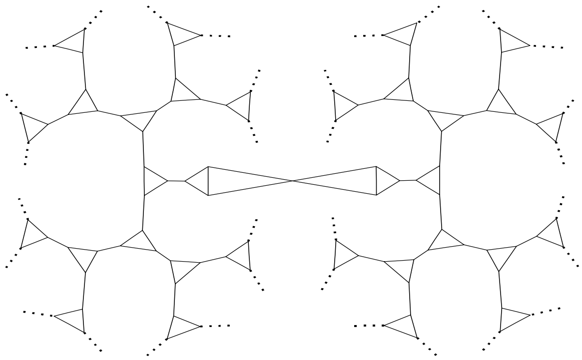

Fig. 4. Orbit with circuit $(1, 1)$.

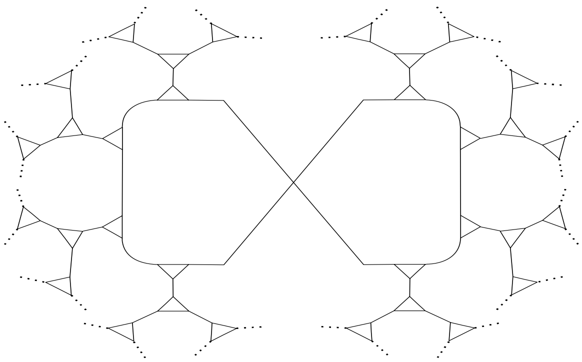

Fig. 5. Orbit with circuit $(4, 4)$.

(5) $\frac{2+\sqrt{5}}{4} \in (1, 16)$, so that $\gcd(2, \frac{2^2-5}{4}, 4) = \frac{1}{4} \neq 1$. Converting $\frac{2+\sqrt{5}}{4}$ into $\frac{8+\sqrt{80}}{16}$, which belongs to $\mathbb{Q}\left(\sqrt{80}\right)$, one has that $\gcd(8, \frac{8^2-80}{16}, 16) = 1$.

Out of six circuits, only two satisfy the condition of unique representation. Hence, the action has only three orbits containing the circuit $(1, 0)$ (Fig. 3), the circuit $(1, 1)$ (Fig. 4) and the circuit $(4, 4)$ (Fig. 5) unique up to the representation of the quadratic irrational numbers, and gives 20 ambiguous numbers of the action, whereas the rest of the circuits belong to the orbits of the action of G on $\mathbb{Q}\left(s\sqrt{5}\right)$ for the positive integers $s = 2, 3, 4$. $\square$

Conclusion. *By theorem (2), the structure of the continued fraction of an element of $\mathbb{Q}\left(\sqrt{n}\right)$ is interwoven with the structure or the type of the circuit. Thus, obtaining all the reduced quadratic irrational numbers is equivalent to obtaining all the circuits and thus all the ambiguous numbers. The continued fraction of an element α_{ij} of $\mathbb{Q}\left(\sqrt{n}\right)$ gives the path that leads to the ambiguous number and the* *period of α_{ij} gives the circuit of the orbit to which α_{ij} belongs. Hence, the uniquely represented reduced quadratic irrational numbers reveal the circuits of the orbits of the action of $PSL\left(2,\mathbb{Z}\right)$ on $\mathbb{Q}\left(\sqrt{n}\right)^*$.*

References

[1] A. Ash, L. Rudolph, The modular symbol and continued fractions in higher dimensions, Invent. Math. 55 (3) (1979) 241–250.
[2] H. Davenport, The Higher Arithmetic: An Introduction to the Theory of Numbers, Cambridge University Press, Cambridge, UK, 1999.
[3] D. Rosen, A class of continued fractions associated with certain properly discontinuous groups, Duke Math. J. 21 (3) (1954) 549–563.
[4] J.S. Frame, Continued fractions and matrices, Amer. Math. Mon. 56 (1949) 98–103.
[5] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1938.
[6] G. Higman, Q. Mushtaq, Coset diagrams and relations for $PSL(2,\mathbb{Z})$, Arab Gulf J. Sci. Res. 1 (1) (1983) 159–164.
[7] S.M. Husnine, M.A. Malik, A. Majeed, On ambiguous numbers of an invariant subset $\mathbb{Q}^*(k\sqrt{m})$ of $\mathbb{Q}(\sqrt{m})$ under the action of the modular group $PSL(2,Z)$, Studia Sci. Math. Hung. 42 (4) (2005) 401–412.
[8] M.A. Malik, M.A. Zafar, Real quadratic irrational numbers and modular group action, Southeast Asian Bull. Math. 35 (2011) 439–445.
[9] Y.I. Manin, M. Marcolli, Continued fractions, modular symbols, and noncommutative geometry, Sel. Math. New Ser. 8 (3) (2002) 475–521.
[10] Q. Mushtaq, Modular group acting on real quadratic fields, Bull. Aust. Math. Soc. 37 (1988) 303–309.
[11] Q. Mushtaq, G.C. Rota, Alternating groups as quotients of two generators groups, Adv. Math. 96 (1) (1992) 113–121.
[12] Q. Mushtaq, On word structure of the modular group over finite and real quadratic fields, Discrete Math. 178 (1998) 155–164.
[13] R. Qureshi, T. Nakahara, Behavior of the ring class numbers of a real quadratic field, Ars Comb. 113 (2014) 257–271.
[14] I. Samuel, Algebraic Theory of Numbers, first edition, Éditions Hermann, Paris, France, 1970.
[15] C. Series, The modular surface and continued fractions, J. Lond. Math. Soc. 31 (2) (1985) 69–80.
[16] H.M. Stark, An Introduction to Number Theory, MIT Press, 1978.
[17] A.J. van der Poorten, An introduction to continued fractions, in: Diophantine Analysis, in: LMS Lecture Notes in Math., vol. 109, Cambridge University Press, 1986, pp. 99–138.