Probability theory

Kolmogorov distance between the exponential functionals of fractional Brownian motion

Distance de Kolmogorov entre les fonctionnelles exponentielles du mouvement brownien fractionnaire

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1. Introduction

Let $B^H = (B^H_t)_{t \in [0, T]}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$. We recall that fBm admits the Volterra representation

$$B^H_t = \int_0^t K_H(t, s) \, dW_s,$$

(1.1)
where \((W_t)_{t \in [0,T]}\) is a standard Brownian motion and for some normalizing constants \(c_H\) and \(c_H',\) the kernel \(K_H\) is given by
\[
K_H(t,s) = c_H \left[ \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{3}{2}}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) \int_s^t \frac{u^{H-\frac{1}{2}}}{s^{H-\frac{3}{2}}} (u-s)^{H-\frac{1}{2}} \, du \right] \quad \text{if } H < \frac{1}{2}.
\]
Given real numbers \(a\) and \(\sigma,\) we consider the exponential functional of the form
\[
F_H = \int_0^T e^{a^2 t + b^2 t} \, ds.
\]
It is known that this functional plays an important role in several domains. For example, it can be used to investigate the finite-time blowup of positive solutions to semi-linear stochastic partial differential equations \([1]\). In the special case \(H = \frac{1}{2},\) fBm reduces to a standard Brownian motion and a lot of fruitful properties of \(F_H\) can be founded in the literature, see, e.g., \([4,5,8,11]\). In particular, the distribution of \(F_{\frac{1}{2}}\) can be computed explicitly. However, to the best our knowledge, it remains a challenge to obtain the deep properties of \(F_H\) for \(H \neq \frac{1}{2}.
\]
On the other hand, because of its applications in statistical estimators, the problem of proving the continuity in law with respect to \(H\) of certain functionals has been studied by several authors. Among others, we refer the reader to \([2,3,9,10]\) and the references therein for detailed discussions and related results. Motivated by this observation, the aim of the present paper is to investigate the continuity in law of the exponential functional \(F_H.\) Intuitively, the continuity of \(F_H\) with respect to \(H\) is not surprising. However, the interesting point of Theorem 1.1 below is that we are able to give an explicit bound on Kolmogorov distance between two functionals with different Hurst indexes.

**Theorem 1.1.** For any \(H_1, H_2 \in (0, 1),\) we have
\[
\sup_{x \geq 0} |P(F_{H_1} \leq x) - P(F_{H_2} \leq x)| \leq C |H_1 - H_2|,
\]
where \(C\) is a positive constant depending on \(a, \sigma, T,\) and \(H_1, H_2.\)

## 2. Proofs

Our main tools are the techniques of Malliavin’s calculus. Hence, for the reader’s convenience, let us recall some elements of Malliavin’s calculus with respect to the Brownian motion \(W,\) where \(W\) is used to present \(B^H\) as in (1.1). We suppose that \((W_t)_{t \in [0,T]}\) is defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P}, P),\) where \(\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}\) is a natural filtration generated by the Brownian motion \(W.\) For \(h \in L^2([0,T]),\) we denote by \(W(h)\) the Wiener integral
\[
W(h) = \int_0^T h(t) \, dW_t.
\]
Let \(S\) denote the dense subset of \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) consisting of smooth random variables of the form
\[
F = f(W(h_1), \ldots, W(h_n)),
\]
where \(n \in \mathbb{N},\) \(f \in C_b^{\infty}(\mathbb{R}^n),\) \(h_1, \ldots, h_n \in L^2([0,T]).\) If \(F\) has the form (2.1), we define its Malliavin derivative as the process
\[
D_t F := D_t f(W(h_1), \ldots, W(h_n)) h_k(t).
\]
We shall denote by \(\mathbb{D}^{1,2}\) the closure of \(S\) with respect to the norm
\[
\|F\|_{1,2}^2 := \mathbb{E}|F|^2 + \mathbb{E} \left[ \int_0^T |D_u F|^2 \, du \right].
\]
An important operator in the Malliavin’s calculus theory is the divergence operator \(\delta,\) it is the adjoint of the derivative operator \(D.\) The domain of \(\delta\) is the set of all functions \(u \in L^2(\Omega, L^2([0,T]))\) such that
where $C(u)$ is some positive constant depending on $u$. In particular, if $u \in \text{Dom} \delta$, then $\delta(u)$ is characterized by the following duality relationship

$$E(DF, u)_{L^{2}[0, T]} = E[F \delta(u)]$$

for any $F \in D^{1,2}$.

In order to be able to prove Theorem 1.1, we need two technical lemmas.

**Lemma 2.1.** For any $H \in (0, 1)$, we have $F_{H} \in D^{1,2}$ and

$$\left( \int_{0}^{T} \left| D_{r}F_{H} \right|^{2} dr \right)^{-1} \in L^{p}(\Omega), \forall p \geq 1.$$

**Proof.** By the representation (1.1), we have $D_{r}B_{s}^{H} = K_{H}(s, r)$ for $0 \leq r < s \leq T$. Hence, $F_{H} \in D^{1,2}$ and its derivative is given by

$$D_{r}F_{H} = \int_{s}^{T} \sigma K_{H}(s, r) e^{\lambda_{s}+\sigma B_{s}^{H}} dr, \ 0 \leq r \leq T.$$ So we can deduce

$$D_{r}F_{H} \geq e^{-2|\alpha|T+\sigma \min_{0 \leq s \leq T} B_{s}^{H}} \int_{s}^{T} \sigma K_{H}(s, r) dr, \ 0 \leq r \leq T.$$ As a consequence,

$$\int_{0}^{T} \left| D_{r}F_{H} \right|^{2} dr \geq e^{-2|\alpha|T+2\sigma \min_{0 \leq s \leq T} B_{s}^{H}} \int_{0}^{T} \left( \int_{s}^{T} \sigma K_{H}(s, r) dr \right)^{2} dr$$

$$= \sigma^{2} e^{-2|\alpha|T+2\sigma \min_{0 \leq s \leq T} B_{s}^{H}} \int_{0}^{T} \left( \int_{s}^{T} K_{H}(s, r) dr \right) \left( \int_{s}^{T} K_{H}(t, r) dt \right) dr$$

$$= \sigma^{2} e^{-2|\alpha|T+2\sigma \min_{0 \leq s \leq T} B_{s}^{H}} \int_{0}^{T} \int_{0}^{T} K_{H}(s, r) K_{H}(t, r) dt dr$$

$$= \sigma^{2} e^{-2|\alpha|T+2\sigma \min_{0 \leq s \leq T} B_{s}^{H}} \int_{0}^{T} \left[ E[B_{s}^{H} B_{s}^{H}] \right] ds dr$$

$$= \frac{T^{2H+2}}{2H+2} \sigma^{2} e^{-2|\alpha|T+2\sigma \min_{0 \leq s \leq T} B_{s}^{H}}.$$ In the last equality we used the fact that $E[B_{s}^{H} B_{s}^{H}] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. We therefore obtain

$$\left( \int_{0}^{T} \left| D_{r}F_{H} \right|^{2} dr \right)^{-1} \leq \frac{2H+2}{T^{H+2}2\sigma^{2}} e^{-2|\alpha|T+2\sigma \max_{0 \leq s \leq T} (-B_{s}^{H})}.$$ By Fernique’s theorem, we have $e^{2\sigma \max_{0 \leq s \leq T} (-B_{s}^{H})} \in L^{p}(\Omega)$ for any $p \geq 1$. This completes the proof. 

**Lemma 2.2.** For any $H_{1}, H_{2} \in (0, 1)$, we have

$$E|F_{H_{1}} - F_{H_{2}}|^{2} \leq C |H_{1} - H_{2}|^{2}, \quad (2.2)$$

$$\int_{0}^{T} E|D_{r}F_{H_{1}} - D_{r}F_{H_{2}}|^{2} dr \leq C |H_{1} - H_{2}|^{2}. \quad (2.3)$$
where \( C \) is a positive constant depending on \( a, \sigma, T, \) and \( H_1, H_2. \)

**Proof.** By the Hölder inequality, we have

\[
E|F_{H_1} - F_{H_2}|^2 = E \left| \int_0^T (e^{as + \sigma B_t^H} - e^{as + \sigma B_t^H}) \, dt \right|^2
\]

\[
\leq T \int_0^T \left| e^{as + \sigma B_t^H} - e^{as + \sigma B_t^H} \right|^2 \, dt.
\]

Using the fundamental inequality \(|e^x - e^y| \leq \frac{1}{2}|x - y|(e^x + e^y)|\) for all \( x, y, \) we deduce

\[
E|F_{H_1} - F_{H_2}|^2 \leq \frac{T}{4} \int_0^T \left( E|\sigma B_t^H - \sigma B_t^H|^4 \right)^{\frac{1}{2}} \left( E|e^{as + \sigma B_t^H} + e^{as + \sigma B_t^H}|^{\frac{1}{2}} \right) \, dt
\]

\[
\leq \frac{T \sigma^2}{4} \int_0^T \left( E|\sigma B_t^H - \sigma B_t^H|^4 \right)^{\frac{1}{2}} \left( 8E|e^{4as + 4\sigma B_t^H}| + 8E|e^{4as + 4\sigma B_t^H}| \right) \, dt
\]

\[
= \frac{T \sigma^2}{4} \int_0^T \left( E|\sigma B_t^H - \sigma B_t^H|^4 \right)^{\frac{1}{2}} \left( 8e^{4as + 8\sigma^2 s^2 H_1} + 8e^{4as + 8\sigma^2 s^2 H_2} \right) \, dt
\]

\[
\leq \frac{T \sigma^2}{4} \int_0^T \left( E|\sigma B_t^H - \sigma B_t^H|^4 \right)^{\frac{1}{2}} \left( 8e^{4as + 8\sigma^2 s^2 H_1} + 8e^{4as + 8\sigma^2 s^2 H_2} \right) \, dt.
\]

It is known from the proof of Theorem 4 in [7] that there exists a positive constant \( C \) such that

\[
\sup_{0 \leq s \leq T} E|B_t^H - B_t^H|^2 \leq C|H_1 - H_2|^2.
\]

(2.4)

On the other hand, we have \( E|B_t^H - B_t^H|^4 = 3E|B_t^H - B_t^H|^2 \) because \( B_t^H - B_t^H \) is a Gaussian random variable for every \( s \in [0, T]. \) So we can conclude that there exists a positive constant \( C \) such that

\[
E|F_{H_1} - F_{H_2}|^2 \leq C|H_1 - H_2|^2.
\]

To finish the proof, let us verify (2.3). By the Hölder and triangle inequalities, we obtain

\[
E|D_tF_{H_1} - D_tF_{H_2}|^2 \leq \sigma^2 T \int_0^T E|K_{H_1}(s, r)e^{as + \sigma B_t^H} - K_{H_2}(s, r)e^{as + \sigma B_t^H}|^2 \, dr
\]

\[
\leq 2\sigma^2 T \int_0^T |K_{H_1}(s, r) - K_{H_2}(s, r)|^2 E|e^{2as + 2\sigma B_t^H}|^2 + K_{H_2}^2(s, r)E|e^{2as + 2\sigma B_t^H} - e^{as + \sigma B_t^H}|^2 \, dr,
\]

and hence,

\[
\int_0^T E|D_tF_{H_1} - D_tF_{H_2}|^2 \, dr \leq 2\sigma^2 T \int_0^T E|e^{2as + 2\sigma B_t^H}|^2 \int_0^s |K_{H_1}(s, r) - K_{H_2}(s, r)|^2 \, dr \, ds
\]

\[
+ 2\sigma^2 T \int_0^T E|e^{2as + 2\sigma B_t^H} - e^{as + \sigma B_t^H}|^2 \int_0^s K_{H_2}^2(s, r) \, dr \, ds
\]
\[ = 2\sigma^2 T \int_0^T e^{2as + 2\sigma^2 s^2 H_1} E|B_s^{H_1} - B_s^{H_2}|^2 \, ds \]
\[ + 2\sigma^2 T \int_0^T E|e^{as + \sigma^2 s^2 H_1} - e^{as + \sigma^2 s^2 H_2}|^2 s^2 \, ds. \]

Notice that \( \int_0^1 k_{H_2}^2(s, r) \, dr = E|B_s^{H_2}|^2 = s^{2H_2}. \) Thus the estimate (2.3) follows from (2.2) and (2.4). \( \square \)

**Proof of Theorem 1.1.** For simplicity, we write \( (\ldots) \) instead of \( (\ldots)_{L^2[0, T]} \). Borrowing the arguments used in the proof of Proposition 2.1.1 in [6], we let \( \psi \) be a nonnegative smooth function with compact support, and set \( \varphi(y) = \int_{-\infty}^{y} \psi(z) \, dz \). Given \( Z \in \mathbb{D}^{1.2} \), we know that \( \varphi(Z) \) belongs to \( \mathbb{D}^{1.2} \) and, making the scalar product of its derivative with \( DF_{H_2} \), we obtain:
\[ \langle D\varphi(Z), DF_{H_2} \rangle = \varphi(Z) \langle DZ, DF_{H_2} \rangle. \]

Fixed \( x \in \mathbb{R}_+ \), by an approximation argument, the above equation holds for \( \varphi(z) = 1_{[0, x]}(z) \). Choosing \( Z = F_{H_1} \) and \( Z = F_{H_2} \), we obtain
\[ \langle D\varphi(Z), DF_{H_2} \rangle = \varphi(Z) \langle DF_{H_1}, DF_{H_2} \rangle. \]

Hence, we can get
\[ \langle D \int_{F_{H_1}}^F \mathbbm{1}_{[0, x]}(z) \, dz, DF_{H_2} \rangle = \mathbbm{1}_{[0, x]}(F_{H_1}) \langle DF_{H_1}, DF_{H_2} \rangle - \mathbbm{1}_{[0, x]}(F_{H_2}) \langle DF_{H_2}, DF_{H_2} \rangle \]
\[ = \mathbbm{1}_{[0, x]}(F_{H_1}) \mathbbm{1}_{[0, x]}(F_{H_2}) \langle DF_{H_2}, DF_{H_2} \rangle + \mathbbm{1}_{[0, x]}(F_{H_1}) \langle DF_{H_2} - DF_{H_2}, DF_{H_2} \rangle. \]

This, together with the fact that \( \|DF_{H_2}\|^2 := \langle DF_{H_2}, DF_{H_2} \rangle > 0 \) a.s. gives us
\[ \mathbbm{1}_{[0, x]}(F_{H_1}) - \mathbbm{1}_{[0, x]}(F_{H_2}) = \frac{\langle D \int_{F_{H_1}}^F \mathbbm{1}_{[0, x]}(z) \, dz, DF_{H_2} \rangle}{\|DF_{H_2}\|^2} - \frac{\mathbbm{1}_{[0, x]}(F_{H_1}) \langle DF_{H_2} - DF_{H_2}, DF_{H_2} \rangle}{\|DF_{H_2}\|^2}. \]

Taking the expectation yields
\[ P(F_{H_1} \leq x) - P(F_{H_2} \leq x) = E[\mathbbm{1}_{[0, x]}(F_{H_1}) - \mathbbm{1}_{[0, x]}(F_{H_2})] \]
\[ = E \left[ \int_{F_{H_1}}^F \mathbbm{1}_{[0, x]}(z) \, dz \delta \left( \frac{DF_{H_2}}{\|DF_{H_2}\|^2} \right) \right] - E \left[ \int_{F_{H_2}}^F \mathbbm{1}_{[0, x]}(z) \, dz \delta \left( \frac{DF_{H_2}}{\|DF_{H_2}\|^2} \right) \right]. \]

By the Hölder inequality,
\[ \sup_{x \geq 0} |P(F_{H_1} \leq x) - P(F_{H_2} \leq x)| \leq E |F_{H_1} - F_{H_2}| \delta \left( \frac{DF_{H_2}}{\|DF_{H_2}\|^2} \right) \left( \frac{1}{\|DF_{H_2}\|^2} \right) \]
\[ \leq \left( E |F_{H_1} - F_{H_2}| \right)^{1/2} \left( E \delta \left( \frac{DF_{H_2}}{\|DF_{H_2}\|^2} \right) \right)^{1/2} \left( E \|DF_{H_1} - DF_{H_2}\| \right) \left( \frac{1}{\|DF_{H_2}\|^2} \right) \]
\[ \leq \left( E |F_{H_1} - F_{H_2}| \right)^{1/2} \left( E \delta \left( \frac{DF_{H_2}}{\|DF_{H_2}\|^2} \right) \right)^{1/2} \left( E \|DF_{H_1} - DF_{H_2}\| \right) \left( \frac{1}{\|DF_{H_2}\|^2} \right). \]

Recalling Lemma 2.2, we obtain
\[
\sup_{x \geq 0} |P(F_{H_1} \leq x) - P(F_{H_2} \leq x)| \leq C|H_1 - H_2| \left( E \left( \frac{DF_{H_2}}{\|DF_{H_2}\|^2} \right)^2 \right)^{\frac{1}{2}} + \left( E \left[ \frac{1}{\|DF_{H_2}\|^2} \right] \right)^{\frac{1}{2}}.
\]

Thanks to Lemma 2.1 we have
\[
E \left[ \frac{1}{\|DF_{H_2}\|^2} \right] = E \left[ \left( \int_0^T |D_t F_{H_2}|^2 \, dt \right)^{-1} \right] < \infty.
\]

Thus we can obtain (1.2) by checking the finiteness of \( E[\delta(u)^2] \), where
\[
u_r := \frac{D_r F_{H_2}}{\|DF_{H_2}\|^2}, \quad 0 \leq r \leq T.
\]

It is known from Proposition 1.3.1 in [6] that
\[
E[\delta(u)^2] \leq \int_0^T E|\nu_r|^2 \, dr + \int_0^T \int_0^T E|D_\theta \nu_r|^2 \, d\theta \, dr.
\]

We have
\[
\int_0^T E|\nu_r|^2 \, dr = E \left[ \frac{1}{\|DF_{H_2}\|^2} \right] < \infty.
\]

Furthermore, by the chain rule for Malliavin derivative, we have
\[
D_\theta \nu_r = \frac{D_\theta D_r F_{H_2}}{\|DF_{H_2}\|^2} - 2 \frac{D_r F_{H_2} (D_\theta D F_{H_2})}{\|DF_{H_2}\|^4}, \quad 0 \leq \theta \leq T.
\]

Hence, by the Hölder inequality,
\[
\int_0^T \int_0^T E|D_\theta \nu_r|^2 \, d\theta \, dr \leq 2 E \left[ \int_0^T \int_0^T \frac{|D_\theta D_r F_{H_2}|^2}{\|DF_{H_2}\|^4} \, d\theta \, dr \right] + 8 E \left[ \int_0^T \int_0^T \frac{|D_\theta D_r F_{H_2}|^2}{\|DF_{H_2}\|^4} \, d\theta \, dr \right] \leq 10 \left( E \left[ \left( \int_0^T \int_0^T |D_\theta D_r F_{H_2}|^2 \, d\theta \, dr \right)^2 \right] \right)^{\frac{1}{2}} \left( E \left[ \frac{1}{\|DF_{H_2}\|^8} \right] \right)^{\frac{1}{2}}.
\]

We now observe that
\[
D_\theta D_r F_{H_2} = \int_{r \wedge \theta}^T \sigma^2 K_{H_2}(s, r) K_{H_2}(s, \theta) e^{\sigma s+\sigma b^H_2} \, ds, \quad 0 \leq r, \theta \leq T.
\]

Hence,
\[
|D_\theta D_r F_{H_2}|^2 \leq T \sigma^4 \int_{r \wedge \theta}^T K_{H_2}^2(s, r) K_{H_2}^2(s, \theta) e^{2\sigma s+2\sigma b^H_2} \, ds, \quad 0 \leq r, \theta \leq T
\]

and we obtain
\[
\int_0^T \int_0^T |D_\theta D_r F_{H_2}|^2 \, d\theta \, dr \leq T \sigma^4 \int_0^T s^{4H_2} e^{2\sigma s+2\sigma b^H_2} \, ds,
\]

which implies that
\[
E \left[ \left( \int_0^T \int_0^T |D_\theta D_r F_{H_2}|^2 \, d\theta \, dr \right)^2 \right] \leq T^4 \sigma^8 \int_0^T s^{8H_2} e^{4\sigma s+8\sigma^2 b^H_2} \, ds < \infty.
\]
Finally, we have $E \left[ \frac{1}{\|DF_{H_2}\|^2} \right] < \infty$ due to Lemma 2.1. So we can conclude that $E[\delta(u)^2]$ is finite. This finishes the proof of Theorem 1.1. □

**Remark 2.1.** Given a bounded and continuous function $\psi$, with the exact proof of Theorem 1.1, we also have

$$|E[\psi(F_{H_1})] - E[\psi(F_{H_2})]| \leq C|H_1 - H_2|.$$

This kind of estimates has been investigated by Richard and Talay for the solution to fractional stochastic differential equations. However, Theorem 1.1 in [9] requires $H_2 = \frac{1}{2}$ and $\psi$ to be Hölder continuous of order $2 + \beta$ with $\beta > 0$.

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**References**


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