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Homotopic equivalence of rational proper holomorphic discs of bounded symmetric domains of type I

Équivalence homotopique de disques rationnels propres holomorphiques de domaines bornés symétriques de type I

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ABSTRACT

We characterize homotopy classes of rational proper holomorphic Shilov maps from the unit disc to bounded symmetric domains of type I through rational proper holomorphic Shilov discs. This characterization generalizes results of D'Angelo–Huo–Xiao and D'Angelo–Lebl, where the codomains are the unit balls.

RÉSUMÉ

Nous caractérisons les classes homotopiques de fonctions de Shilov rationnelles propres holomorphiques du disque unité à valeurs dans les domaines bornés symétriques à l'aide de disques de Shilov rationnels propres holomorphiques. Cette caractérisation généralise des résultats de D'Angelo–Huo–Xiao et de D'Angelo–Lebl, où les codomaines sont les boules unité.

1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc. Let $\Omega_{r,s}$ with $r \leq s$ be the irreducible bounded symmetric domain of type I defined by

$$\Omega_{r,s} = \{Z \in M^{\mathbb{C}}_{r,s} : I_r - ZZ^* > 0\} \tag{1.1}$$

where $M^{\mathbb{C}}_{r,s}$ denotes the set of complex-valued $r \times s$ matrices. Here $Z^* = \overline{Z}^t$. In this paper, we study rational proper holomorphic maps from the unit disc to bounded symmetric domains of type I. We will call such maps rational proper holomorphic discs of $\Omega_{r,s}$. Here we say that a map $f : \Delta \to \Omega_{r,s}$ is *proper* if $f^{-1}(K) \subset \Delta$ is compact for any compact subset $K \subset \Omega_{r,s}$. In

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the case where $r = 1$ and $s = 1$, any proper holomorphic self-map of Δ has the form of the Blaschke product [1]. However, if $r \neq 1$ or $s \neq 1$, there is no fixed form of the proper holomorphic discs. For instance, if $r \geq 2$, we may observe that any map given by $z \mapsto \begin{pmatrix} z & 0 \\ 0 & h(z) \end{pmatrix}$ with a holomorphic map $h\colon \Delta \to \Omega_{r-1,s-1}$ is proper. However, if we consider homotopy classes of proper holomorphic discs through proper holomorphic discs, one has interesting results, as D'Angelo–Huo–Xiao ([2]) and D'Angelo–Lebl [3] gave proofs for rational proper holomorphic discs of the higher-dimensional balls. These authors established the following results.

Theorem 1.1 *(Theorem 5.1 in [2] and Proposition 2.1 in [3]). Let $f\colon \Delta \to \mathbb{B}^s$ be a proper holomorphic disc.*

(1) If $s = 1$, then there exists a unique positive integer m such that f is homotopic to the map $z \mapsto z^m$ through rational proper holomorphic discs.
(2) If $s \geq 2$ and f is rational, then f is homotopic to $z \mapsto (z, 0)$ through rational proper holomorphic discs.

Let $\Omega_1 \subset \mathbb{C}^{n_1}$, $\Omega_2 \subset \mathbb{C}^{n_2}$ be bounded domains and S_1, S_2 be their Shilov boundaries. We will say that a holomorphic map $f\colon \Omega_1 \to \Omega_2$ that is holomorphic near S_1 is a *Shilov map* if f maps S_1 to S_2. The Shilov boundary of $\Omega_{r,s}$ is given by

$$S_{r,s} = \{Z \in M^{\mathbb{C}}_{r,s} : I_r - ZZ^* = 0\}. \tag{1.2}$$

Any automorphism of $\Omega_{r,s}$ extends holomorphically over the boundary and preserves $S_{r,s}$. Note that the Shilov boundaries of the balls coincide with their topological boundaries. Define a rational proper holomorphic Shilov map $D_{m_1,\ldots,m_r}\colon \Delta \to \Omega_{r,s}$ by

$$z \mapsto \left(\begin{array}{c|ccc} & z^{m_1} & & 0 \\ 0 & & \ddots & \\ & 0 & & z^{m_r} \end{array}\right) \tag{1.3}$$

for some $m_1, \ldots, m_r \in \mathbb{N}$. Note that if $r = s$, there is no zero entries on the left side of (1.3). The aim of this article is to prove the following theorem.

Theorem 1.2. *Let $\Omega_{r,s}$ be an irreducible bounded symmetric domain of type I. Then all nonconstant rational proper holomorphic Shilov maps from Δ to $\Omega_{r,s}$ are homotopic, through rational proper holomorphic Shilov maps, to the following:*

(1) $D_{1,\ldots,1}$ if $r < s$,
(2) $D_{m_1,\ldots,m_r}$ for some $m_1, \ldots, m_r \in \mathbb{N}$ if $r = s$. Furthermore, $D_{m_1,\ldots,m_r}$ and $D_{l_1,\ldots,l_r}$ are homotopically equivalent through rational proper holomorphic Shilov maps if and only if

$$m_1 + \cdots + m_r = l_1 + \cdots + l_r. \tag{1.4}$$

At this point, it is worth mentioning the dimension of the codomain, a subtle aspect addressed in [2,3]. For a proper holomorphic disc $f\colon \Delta \to \mathbb{B}^s$, if one identifies f with $(f, 0)$ as a proper holomorphic disc into $\mathbb{B}^{s+1}$, f is always homotopically equivalent to the map $z \mapsto (0, z)$ through the homotopy $H_t\colon \Delta \to \mathbb{B}^{s+1}$ defined by $z \mapsto \left(\sqrt{1-t}\, f(z), \sqrt{t}z\right)$ for $t \in [0, 1]$. Moreover, if the dimension of the codomain is bigger than two, then, by Theorem 1.1, all rational proper holomorphic discs are homotopically equivalent. Similar to the situation of the ball, when f is a rational proper holomorphic Shilov disc into $\Omega_{r,s}$, we may identify f with $(f|0)$, which is also a proper Shilov map into $\Omega_{r,s+1}$. But, if $r \geq 2$, we do not have a simple homotopy that induces the homotopy equivalence to one specific rational proper holomorphic Shilov disc. However, by Theorem 1.2, we see that f is always homotopically equivalent to $D_{1,\ldots,1}$ in $\Omega_{r,s+1}$. Furthermore, all rational proper holomorphic Shilov discs into $\Omega_{r,s}$ with $r < s$ are homotopically equivalent.

2. Proof of Theorem 1.2

Lemma 2.1. *Let $f\colon \Delta \to \Omega_{r,s}$ be a rational proper holomorphic Shilov map. For any $\phi \in SU(r,s)$, f and $\phi \circ f$ are homotopically equivalent through rational proper holomorphic Shilov maps.*

Proof. Since $SU(r,s)$ is connected, we can take a path $\gamma\colon [0,1] \to SU(r,s)$ such that $\gamma(0)$ is the identity map and $\gamma(1) = \phi$. Since any automorphism of $\Omega_{r,s}$ preserves the Shilov boundary, $z \mapsto \gamma(t) \circ f$ gives a homotopy between f and $\phi \circ f$, which is what we want. $\square$

Proposition 2.2 *(cf. Proposition 2.2 in [2]). Let F be a rational proper holomorphic disc of $\Omega_{r,s}$ such that $F(z_j) \to S_{r,s}$ whenever $z_j \to \partial\Delta$. Then F extends holomorphically over $\partial\Delta$.*

Proof. The proof given here is the same as that of the ball case given in [2]. We write it for the reader's convenience. Denote $F = p/q$, where p is a matrix-valued polynomial and q is a scalar-valued polynomial. We may assume that F is reduced to the lowest terms. Suppose $q(z_0) = 0$ for some $z_0 \in \partial\Delta$. Since q is a polynomial, it is divisible by $(z - z_0)$. Since $F(z_j) \to S_{r,s}$ whenever $z_j \to \partial\Delta$, one has $p(z_j)p(z_j)^* \to |q(z_0)|^2 I_r = 0$ as $j \to \infty$. This implies $p(z_0) = 0$ and hence each component of p is divisible by $(z - z_0)$. Therefore, f is not reduced to lowest terms. Hence q is not zero on the circle, and F extends holomorphically past the circle. □

Remark 2.3. In general proper holomorphic discs of $\Omega_{r,s}$ cannot extend holomorphically over the circle. Indeed let $f : \Delta \to \Omega^I_{3,3}$ be a proper holomorphic disc given by $z \mapsto \begin{pmatrix} z & 0 \\ 0 & h(z) \end{pmatrix}$ for a holomorphic map $h : \Delta \to \Omega^I_{2,2}$. If we choose h, which cannot be extended holomorphically over the circle, f also cannot extend over the circle.

Remark 2.4. Let Ω and Ω' be irreducible bounded symmetric domains such that Ω is a characteristic subspace of Ω'; see [4] for the definition. All proper holomorphic discs of Ω are homotopically equivalent through proper holomorphic discs in Ω' for the following reason. Let f be a proper holomorphic disc in Ω. Since Ω is a characteristic subspace of Ω', there exists a minimal disc Δ_α such that $\Delta_\alpha \times \Omega$ can be totally geodesically embedded into Ω'. Take $H_t(z) = (2tz, f(z)) \in \Delta_\alpha \times \Omega \subset \Omega'$ for $t \in [0, 1/2]$ and $H_t(z) = (z, (2 - 2t)f(z)) \in \Delta_\alpha \times \Omega \subset \Omega'$ for $t \in [1/2, 1]$.

Lemma 2.5 *(D'Angelo–Huo–Xiao [2]).*

(1) Let $f : \mathbb{C} \to \mathbb{C}^n$ be a rational map of degree d. Denote f by p/q, where $p(z) = \sum_{j=0}^d P_j z^j$ with $P_j \in \mathbb{C}^n$ and $q(z) = \sum_{j=0}^d q_j z^j$ with $q_j \in \mathbb{C}$. Then $f|_\Delta$ is a proper map from Δ to $\mathbb{B}^n$ if and only if $\{P_0, \dots, P_d\}$ and $\{q_1, \dots, q_d\}$ satisfy the following:

$$\sum_{k=0}^{d-l} q_{k+l}\bar{q}_k = \sum_{k=0}^{d-l} \langle P_{k+l}, P_k \rangle \quad \text{for} \quad l = 0, 1, \dots, d \tag{2.1}$$

where $\langle \, , \, \rangle$ denotes the standard inner product of $\mathbb{C}^n$.

(2) Let $f = \frac{p}{q} : \Delta \to \Omega_{r,s}$ be a rational proper holomorphic map with $f(0) = 0$. Then $\deg(p) > \deg(q)$.

Remark 2.6. Lemma 2.5 also holds for a rational proper holomorphic Shilov disc of bounded symmetric domains of type I.

(1) Let $f : \mathbb{C} \to M^{\mathbb{C}}_{r,s}$ be a rational map of degree d. Denote f by p/q, with

$$p(z) = \sum_{j=0}^{d} P_j z^j \text{ with } P_j \in M^{\mathbb{C}}_{r,s} \quad \text{and} \quad q(z) = \sum_{j=0}^{d} q_j z^j \text{ with } q_j \in \mathbb{C}. \tag{2.2}$$

If f is a Shilov map, then $\{P_0, \dots, P_d\}$ and $\{q_1, \dots, q_d\}$ satisfy the following:

$$\left(\sum_{k=0}^{d-l} q_{k+l}\bar{q}_k \right) I_r = \sum_{k=0}^{d-l} P_{k+l} P_k^* \text{ for } l = 0, 1, \dots, d. \tag{2.3}$$

(2) Let $f = \frac{p}{q}$ be a rational proper holomorphic Shilov map from Δ to $\Omega_{r,s}$ with $f(0) = 0$. Then $\deg(p) > \deg(q)$.

Lemma 2.7.

(1) Let f be a proper holomorphic Shilov disc of $\Omega_{r,s}$. Then the map $z \mapsto zf(z)$ is also a proper holomorphic Shilov disc of $\Omega_{r,s}$.

(2) Let $g(z) = zf(z)$ be a proper holomorphic Shilov disc of $\Omega_{r,s}$ with nonconstant holomorphic disc f. Then f is also a proper holomorphic Shilov disc of $\Omega_{r,s}$.

From now on, for any given map $g : \Delta \to M^{\mathbb{C}}_{r,s}$, we denote $g = \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}$, where g_j with $1 \le j \le r$ are $1 \times s$ matrix-valued mappings from Δ.

Proof of Theorem 1.2. The argument involves three steps.

Step 1. Let $f = \frac{p}{q} : \Delta \to \Omega_{r,s}$ be a rational proper holomorphic Shilov map of degree d. Since f is a Shilov map, whenever $z \in \partial\Delta$, we have

$$I_r = f(z)f(z)^* = \begin{pmatrix} f_1(z)f_1(z)^* & \cdots & f_1(z)f_r(z)^* \\ \vdots & \ddots & \vdots \\ f_r(z)f_1(z)^* & \cdots & f_r(z)f_r(z)^* \end{pmatrix}. \tag{2.4}$$

In particular, $f_r(z)f_r(z)^* = 1$ whenever $z \in \partial\Delta$ and hence f_r is a rational proper holomorphic map from Δ to $\mathbb{B}^s$. Note that any element in $\mathrm{Aut}(\mathbb{B}^s) = U(1,s)$ extends to an automorphism of $\Omega_{r,s}$, that is, to $U(r,s)$. This embedding of $U(1,s)$ into $U(r,s)$ is given by

$$U(1,s) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \hookrightarrow \begin{pmatrix} I_{r-1} & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \in U(r,s). \tag{2.5}$$

Let ϕ be an automorphism of $\mathbb{B}^s$ and $\begin{pmatrix} I_{r-1} & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$ denotes ϕ as an element in $U(r,s)$. Note that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(r,s)$ acts on $\Omega_{r,s}$ by $Z \mapsto (A+ZC)^{-1}(B+ZD)$. Hence ϕ acts on $\Omega_{r,s}$ by

$$\begin{aligned} Z \mapsto & \left(\begin{pmatrix} I_{r-1} & 0 \\ 0 & a \end{pmatrix} + Z \begin{pmatrix} 0 & c \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} 0 \\ b \end{pmatrix} + ZD \right) \\ & = \begin{pmatrix} I_{r-1} & -\frac{Z'c}{a+Z_r c} \\ 0 & \frac{1}{a+Z_r c} \end{pmatrix} \left(\begin{pmatrix} 0 \\ b \end{pmatrix} + ZD \right). \end{aligned} \tag{2.6}$$

Here we express Z by $\begin{pmatrix} Z' \\ Z_r \end{pmatrix}$ with $Z' \in M^{\mathbb{C}}_{r-1,s}$ and $Z_r \in M^{\mathbb{C}}_{1,s}$. Then $(\phi \circ f)_r(z)$ is given by

$$\left(\frac{b_1 + f_r(z)D^1}{a + f_r(z)c}, \dots, \frac{b_s + f_r(z)D^s}{a + f_r(z)c} \right) = \left(\frac{b_1 q(z) + p_r(z)D^1}{aq(z) + p_r(z)c}, \dots, \frac{b_s q(z) + p_r(z)D^s}{aq(z) + p_r(z)c} \right) \tag{2.7}$$

where we denote D by $(D^1, \dots, D^s)$ with columns D^j. Hence $(\phi \circ f)_r(z)$ has degree less than or equal to d and $(\phi \circ f)_r = \phi \circ f_r$.

Step 2. In this step, we will show that f is homotopic to $D_{m_1,\dots,m_r}$ for some $m_1, \dots, m_r \in \mathbb{N}$. One notices that it is enough to prove the following: any rational proper holomorphic Shilov disc is homotopic to a map

$$z \mapsto \begin{pmatrix} \hat{f} & 0 \\ 0 & z^m \end{pmatrix} \tag{2.8}$$

through rational proper holomorphic Shilov discs where $\hat{f} : \Delta \to \Omega_{r-1,s-1}$ is a rational proper holomorphic Shilov map and $m \in \mathbb{N}$; we may repeat this process to $\hat{f}$. We will use induction to prove it.

Suppose that $d = 1$. Choose $\phi \in SU(1,s) \subset SU(r,s)$ so that $\phi(f_r(0)) = 0 \in \mathbb{B}^s$. Since $(\phi \circ f)_r$ is a degree-one rational proper holomorphic map from Δ to $\mathbb{B}^s$, by Lemma 2.5 (2) $(\phi \circ f)_r$ has the form Pz where $P \in \mathbb{C}^s$. Moreover, by Lemma 2.5 (1), we have $P \in \partial\mathbb{B}^s$ and hence $(\phi \circ f)_r = (\phi \circ f_r)$ is homotopic to $z \mapsto (0, z) \in \mathbb{B}^s$ through $U(s)$. In particular, f is homotopic through $SU(r,s)$ to a rational proper holomorphic Shilov map of the form (2.8), with $m = 1$.

Now assume that $d \geq 2$ and the claim holds whenever the degree of the map is smaller than d. Let ϕ be a rational proper holomorphic Shilov disc of degree d. Choose $\phi \in SU(1,s) \subset SU(r,s)$ so that $\phi(f_r(0)) = 0 \in \mathbb{B}^s$. Hence, the degree of the numerator of $(\phi \circ f)_r$ is bigger than that of the denominator of $(\phi \circ f)_r$. We can express $(\phi \circ f)_r(z) = z\widetilde{(\phi \circ f)}_r(z)$. Define a map

$$\widetilde{\phi \circ f}(z) = \begin{pmatrix} \frac{1}{z}(\phi \circ f)_1(z) \\ \vdots \\ \frac{1}{z}(\phi \circ f)_{r-1}(z) \\ \widetilde{(\phi \circ f)}_r(z) \end{pmatrix}.$$

Since $\widetilde{\phi \circ f}$ is a rational proper holomorphic Shilov disc of $\Omega_{r,s}$, by the induction hypothesis $\widetilde{\phi \circ f}$ is homotopic to (2.8) for some $m \in \mathbb{N}$. This implies that f is also homotopic to (2.8) for some $m \in \mathbb{N}$ through rational proper holomorphic discs.

Step 3. Firstly suppose that $r \neq s$. Let $H_t : \Delta \to \Omega_{r,s}$ be a proper holomorphic Shilov map defined by

$$H_t(z) = \left(\begin{array}{c|c|cccc} & \sqrt{1-t^2}z & tz^{m_1} & & & 0 \\ & & & z^{m_2} & & \\ 0 & 0 & 0 & & \ddots & \\ & & & & & z^{m_r} \end{array} \right)$$

for each $t \in [0, 1]$. This map guarantees the homotopy equivalence of the map (1.3) and $D_{1,m_2,\dots,m_r}$, since H_0 is homotopically equivalent to the map $D_{1,m_2,\dots,m_r}$ through $SU(r, s)$. By similar maps to H_t above for the ith row $i = 2, \dots, r$, one obtains that the map (1.3) is homotopically equivalent to $D_{1,\dots,1}$.

Secondly suppose that $r = s$. Consider a homomorphism $h\colon \Omega_{r,r} \to \Delta$ defined by $Z \mapsto \det Z$. Then it is clear that $h \circ D_{m_1,\dots,m_r}\colon \Delta \to \Delta$ is a rational proper holomorphic self-map $z \mapsto z^{m_1+\dots+m_r}$. Hence if $D_{m_1,\dots,m_r}$ and $D_{l_1,\dots,l_r}$ are homotopically equivalent to each other, then $m_1 + \dots + m_r = l_1 + \dots + l_r$ by Theorem 1.1 (1). Now we will show that this condition is sufficient by using induction. When $r = 1$, it was proved by D'Angelo and Lebl (Proposition 2.1 in [3]). Let us consider the case where $r = 2$. Since $m_1 + m_2 = l_1 + l_2$, the following rational proper holomorphic Shilov homotopy maps H_t $(0 \le t \le 1)$ from Δ to $\Omega_{2,2}$ are well defined:

$$z \mapsto \begin{pmatrix} \sqrt{1-t^2}z^{m_1} & tz^{l_1} \\ -tz^{l_2} & \sqrt{1-t^2}z^{m_2} \end{pmatrix}. \tag{2.9}$$

Note that H_0 is D_{m_1,m_2} and H_1 is homotopically equivalent to D_{l_1,l_2} through $SU(2, 2)$. Now assume that the claim holds for all r less than R and $D_{m_1,\dots,m_R}$, $D_{l_1,\dots,l_R}$ are given of the form (1.3) provided $m_1 + \dots + m_R = l_1 + \dots + l_R$. Without loss of generality we may assume that $m_1 \le m_2 \le \dots \le m_R$, $l_1 \le l_2 \le \dots \le l_R$ and $m_1 \le l_1$. Then by applying the homotopy map (2.9) to the first 2×2 block submatrix of $D_{m_1,\dots,m_R}$, we can show that it is homotopy equivalent to $D_{l_1,m_1+m_2-l_1,m_3,\dots,m_R}$. By the induction hypothesis, we obtain that $D_{l_1,m_1+m_2-l_1,m_3,\dots,m_R}$ is homotopically equivalent to $D_{l_1,\dots,l_R}$ through rational proper holomorphic discs, and hence the theorem is proved. □

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