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Number theory

Parity of Schur's partition function

Parité de la fonction de partition de Schur

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ABSTRACT

Let $A(n)$ be the number of Schur's partitions of n, i.e. the number of partitions of n into distinct parts congruent to $1, 2 \pmod 3$. We prove

$$\frac{x}{(\log x)^{\frac{47}{48}}} \ll \sharp\{0 \le n \le x : A(2n+1) \text{ is odd}\} \ll \frac{x}{(\log x)^{\frac{1}{2}}}.$$

RÉSUMÉ

Soit $A(n)$ le nombre de partitions de Schur de n, c'est-à-dire le nombre de partitions de n en parts distinctes congrues à $1, 2 \pmod 3$. Nous montrons que :

$$\frac{x}{(\log x)^{\frac{47}{48}}} \ll \sharp\{0 \le n \le x : A(2n+1) \text{ impair}\} \ll \frac{x}{(\log x)^{\frac{1}{2}}}.$$

1. Introduction

The partition function $p(n)$ is the number of representations of n as nonincreasing sequence of positive integers whose sum is n. Although there has been much work on the congruence properties of $p(n)$ since Ramanujan, little is known about the parity of $p(n)$. Parkin and Shanks [22] conjectured that the partition function is even and odd equally often, i.e.

$$\sharp\{1 \le n \le x : p(n) \text{ is even (resp. odd)}\} \sim \frac{1}{2}x,\ x \to \infty. \tag{1}$$

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The best lower bound for the even case is $0.069\sqrt{x}\log\log x$ [8], and that for the odd case is $\gg \frac{\sqrt{x}}{\log\log x}$ [7], where $f(x) \gg g(x)$ means $|f(x)| \geq cg(x)$ for some constant c. We refer to [7], [20] and the references therein for more results on the parity of $p(n)$.

It seems difficult to prove Parkin and Shanks' conjecture or even improve the lower bound of (1) as $\gg x^{\frac{1}{2}+\epsilon}$. But for the Rogers–Ramanujan function $g(n)$, i.e.

$$\sum_{n=0}^{\infty} g(n)q^n := \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

we have a better lower bound for the number of odd $g(n)$. Indeed, it was shown in [12] that

$$\sharp\{0 \leq n \leq x : g(2n+1) \text{ is odd }\} \sim \frac{\pi^2}{5}\frac{x}{\log x},\ x \to \infty. \tag{2}$$

We expect to find a special partition function that satisfies the odd–even distribution like (1). In this note, we will study the parity of Schur's partition function and show that the odd values of this partition function up to x is $\gg \frac{x}{(\log x)^{\frac{47}{48}}}$. We see that this lower bound is slightly better than (2), but still far from the bound (1).

Before stating our result precisely, we recall the famous Schur's partition theorem [23]. Let $A(n)$ be the number of partitions of n into distinct parts $\equiv 1, 2 \pmod 3$, $B(n)$ be the number of partitions of n into parts $\equiv \pm 1 \pmod 6$, and $D(n)$ be the number of partitions of n of the form $n_1 + n_2 + \cdots + n_k$ such that $n_i - n_{i+1} \geq 3$ with strict inequality if $3 \mid n_i$. Schur's partition theorem states that

$$A(n) = B(n) = D(n).$$

Schur's theorem can be proved by a variety of approaches. For example, Andrews [2] gave a proof by generating functions and Bressoud [10] provided a purely combinatorial proof. For more generalizations and extensions of Schur's partition theorem, see Gleissberg [15], Andrews [4–6], Alladi and Gordon [1], to name a few.

Our main result is the following theorem.

Theorem 1.1. *We have*

$$\frac{x}{(\log x)^{\frac{47}{48}}} \ll \sharp\{0 \leq n \leq x : A(2n+1) \text{ is odd }\} \ll \frac{x}{(\log x)^{\frac{1}{2}}}.$$

Since $\sharp\{1 \leq n \leq x : A(n) \text{ is odd}\} \geq \sharp\{0 \leq n \leq \frac{x-1}{2} : A(2n+1) \text{ is odd}\}$, we have the following corollary.

Corollary 1.2.

$$\sharp\{1 \leq n \leq x : A(n) \text{ is odd }\} \gg \frac{x}{(\log x)^{\frac{47}{48}}}.$$

2. Proof of Theorem 1.1

First note that the generating function for $A(n)$ is

$$\sum_{n=0}^{\infty} A(n)q^n = (-q; q^3)_\infty(-q^2; q^3)_\infty,$$

where $(a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$. The odd–even dissection of this series [11, Theorem 2] is given by

$$\sum_{n=0}^{\infty} A(n)q^n = \frac{(q^4;q^4)_\infty(q^{16};q^{16})_\infty(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty(q^8;q^8)_\infty(q^{12};q^{12})_\infty(q^{48};q^{48})_\infty} + q\frac{(q^8;q^8)_\infty^2(q^{48};q^{48})_\infty}{(q^2;q^2)_\infty(q^{16};q^{16})_\infty(q^{24};q^{24})_\infty}.$$

Extracting odd exponents of q, we get

$$\begin{aligned}\sum_{n=0}^{\infty} A(2n+1)q^n &= \frac{(q^4;q^4)_\infty^2(q^{24};q^{24})_\infty}{(q;q)_\infty(q^8;q^8)_\infty(q^{12};q^{12})_\infty} \\ &\equiv \frac{(q^3;q^3)_\infty^4}{(q;q)_\infty} \pmod 2 \\ &= \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\cdot(q^3;q^3)_\infty.\end{aligned} \tag{3}$$

Expand $\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}$ as

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}=\sum_{n=0}^{\infty}a_3(n)q^n.$$

Then $a_3(n)$ is known as the number of the 3-core partitions of n. The explicit formula for $a_3(n)$ [17] is

$$a_3(n)=\sum_{\substack{d|3n+1\\ d\equiv 1 \pmod 3}}1-\sum_{\substack{d|3n+1\\ d\equiv 2 \pmod 3}}1.$$

It follows immediately that

$$a_3(n)\equiv\sum_{d|3n+1}1 \pmod 2.$$

Hence $a_3(n)$ is odd if and only if $3n+1$ is a square. Applying Euler's pentagonal theorem [3, Corollary 1.7]

$$(q;q)_\infty=\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{m(3m+1)}{2}},$$

we deduce from (3) that

$$\begin{aligned}\sum_{n=0}^{\infty}A(2n+1)q^{24n+11}&\equiv q^8\frac{(q^{72};q^{72})_\infty^3}{(q^{24};q^{24})_\infty}\cdot q^3(q^{72};q^{72})_\infty \pmod 2\\ &\equiv\sum_{m=1,3\nmid m}^{\infty}q^{8m^2}\sum_{n=-\infty}^{\infty}q^{3(6n+1)^2} \pmod 2\\ &=\sum_{\substack{m\ge 1\\ 3\nmid m}}\sum_{\substack{y\ge 1\\ y\equiv 1,5 \pmod 6}}q^{8m^2+3y^2}\\ &=\sum_{x\ge 1}\sum_{\substack{y\ge 1\\ 3\nmid y}}q^{2x^2+3y^2},\end{aligned}\tag{4}$$

where $24n+11=2x^2+3y^2$ implies that y is odd, $3\nmid x$, and that x is even by considering modulo 8, and $y\equiv 1,5 \pmod 6$ since y is odd and $3\nmid y$. For an integral binary quadratic form $ax^2+bxy+cy^2$, we denote by $R(n,ax^2+bxy+cy^2)$ the number of the representations of n by $ax^2+bxy+cy^2$ with $x,y\in\mathbb{Z}$. Then (4) is equivalent to

$$A(2n+1)\equiv\frac{1}{4}R(24n+11,2x^2+3y^2)-\frac{1}{4}R(24n+11,2x^2+27y^2) \pmod 2.\tag{5}$$

Using `BinaryQF_reduced_representatives(-24,primitive_only=True)` in software SageMath 8.1 [24], we find that the reduced primitive positive definite binary quadratic forms of discriminant -24 are $2x^2+3y^2$ and x^2+6y^2. Hence Dirichlet's theorem on binary quadratic forms [16, Theorem 1] shows that

$$R(24n+11,2x^2+3y^2)+R(24n+11,x^2+6y^2)=2\sum_{d|24n+11}\left(\frac{-6}{d}\right),$$

where $(\frac{\cdot}{\cdot})$ is the Jacobi–Kronecker symbol. Note that $24n+11$ can not be represented by x^2+6y^2 since $24n+11=x^2+6y^2$ means $2\equiv x^2 \pmod 3$, which is absurd. Therefore,

$$R(24n+11,2x^2+3y^2)=2\sum_{d|24n+11}\left(\frac{-6}{d}\right).\tag{6}$$

By SageMath 8.1, the reduced forms of discriminant -216 are given by

x^2+54y^2

$2x^2+27y^2$

$5x^2+2xy+11y^2$

$$5x^2 - 2xy + 11y^2$$
$$7x^2 - 6xy + 9y^2$$
$$7x^2 + 6xy + 9y^2.$$

It is easy to see that $24n+11$ can not be represented by x^2+54y^2 and $7x^2 \pm 6xy + 9y^2$ by considering modulo 3. Since

$$R(24n+11, 5x^2+2xy+11y^2) = R(24n+11, 5x^2-2xy+11y^2),$$

Dirichlet's theorem gives again

$$R(24n+11, 2x^2+27y^2) + 2R(24n+11, 5x^2+2xy+11y^2)$$
$$= 2\sum_{d|24n+11}\left(\frac{-216}{d}\right) = 2\sum_{d|24n+11}\left(\frac{-6}{d}\right). \tag{7}$$

Note that $\sum_{d|24n+11}\left(\frac{-6}{d}\right)$ is even because $24n+11 \equiv 2 \pmod 3$ implies that there exists a prime $p \equiv 2 \pmod 3$ such that the exponent of p in the prime factorization of $24n+11$ is odd, hence

$$\sum_{d|24n+11}\left(\frac{-6}{d}\right) \equiv \sum_{d|24n+11} 1 \equiv 0 \pmod 2.$$

Putting (5), (6) and (7) together, we obtain

$$A(2n+1) \equiv \frac{1}{2}R(24n+11, 5x^2+2xy+11y^2) \pmod 2. \tag{8}$$

Let $\mathcal{S}$ be a subset of primes defined as

$$\mathcal{S} = \{p : p \equiv 11 \pmod{24}, p = 5x^2+2xy+11y^2\}.$$

For convenience, we write

$$f = 5x^2+2xy+11y^2.$$

We claim that for any $2t-1$ distinct primes $p_1, p_2, \cdots, p_{2t-1} \in \mathcal{S}$,

$$R(p_1p_2\cdots p_{2t-1}, f) \equiv 2 \pmod 4. \tag{9}$$

We prove the claim by induction on t. If $t=1$, then $R(p,f)=2$ for any $p \in \mathcal{S}$ because the opposite form of f is $f^{-1} = 5x^2-2xy+11y^2$ and is improperly equivalent to f [13, pp. 24–25], thereby the classes of forms equivalent to f and f^{-1} are not equal, and we have $R(p,f)=2$ by [21,Theorem 4]. Assume that (9) holds for $t=k-1$, i.e.

$$R(p_1\cdots p_{2k-3}, f) \equiv 2 \pmod 4. \tag{10}$$

Let f, g be any primitive positive binary quadratic forms of the same negative discriminant d and p a prime not dividing d and represented by g. Pall [21] showed that for every positive integer n,

$$R(pn, f) + R\left(\frac{n}{p}, f\right) = R(n, f\circ g) + R(n, f\circ g^{-1}), \tag{11}$$

where $f\circ g$ is the Dirichlet composition of f and g, g^{-1} is the opposite form of g (see [13, p. 49] for definitions). Taking

$$f = g = 5x^2+2xy+11y^2$$

and applying (11) twice, we find for $2k-1$ distinct primes $p_1, p_2, \cdots, p_{2k-1} \in \mathcal{S}$

$$\begin{aligned} R(p_1p_2\cdots p_{2k-1}, f) &= R(p_1\cdots p_{2k-2}, f\circ f) + R(p_1\cdots p_{2k-2}, f\circ f^{-1}) \\ &= R(p_1\cdots p_{2k-3}, f\circ f\circ f) + R(p_1\cdots p_{2k-3}, f) \\ &\quad + R(p_1\cdots p_{2k-3}, f) + R(p_1\cdots p_{2k-3}, f^{-1}) \\ &= R(p_1\cdots p_{2k-3}, f\circ f\circ f) + 3R(p_1\cdots p_{2k-3}, f), \end{aligned} \tag{12}$$

where $R(p_1p_2\cdots p_{2k-3}, f) = R(p_1p_2\cdots p_{2k-3}, f^{-1})$ follows the fact that a solution (x_0, y_0) to $p_1p_2\cdots p_{2k-3} = f = 5x^2+2xy+11y^2$ corresponds to a solution $(x_0, -y_0)$ to $p_1p_2\cdots p_{2k-3} = f^{-1} = 5x^2-2xy+11y^2$. We compute $f\circ f\circ f$ explicitly and find

$$f \circ f \circ f = 125x^2 + 222xy + 99y^2.$$

Moreover, its reduce form is $2x^2 + 27y^2$. Since equivalent forms represent the same numbers ([13, Ex.2.2]), it follows that

$$R(p_1 \cdots p_{2k-3}, f \circ f \circ f) = R(p_1 \cdots p_{2k-3}, 2x^2 + 27y^2).$$

If n is coprime to the discriminant -216, then

$$R(n, 2x^2 + 27y^2) \equiv 0 \pmod 4$$

because $n = 2x^2 + 27y^2$ means $n = 2(\pm x)^2 + 27(\pm y)^2$. Therefore,

$$R(p_1 \cdots p_{2k-3}, f \circ f \circ f) \equiv 0 \pmod 4. \tag{13}$$

Inserting (10) and (13) into (12), we find (9) holds for $t = k$. This proves the claim.

Now we deduce from (8) and (9) that

$$A\left(\frac{p_1 p_2 \cdots p_{2t-1} + 1}{12}\right) \equiv 1 \pmod 2$$

for any $2t - 1$ distinct primes $p_1, p_2, \cdots, p_{2t-1} \in \mathcal{S}$. Thus,

$$\sum_{\substack{0 \le n \le x \\ A(2n+1) \text{ odd}}} 1 \ge \sum_{\substack{m \le x \\ \mu(m) = -1 \\ p|m \Rightarrow p \in \mathcal{S}}} 1, \tag{14}$$

where μ is the usual Möbius function. Since the number of classes of discriminant -216 is 6, the Chebotarev density theorem [13, Theorem 9.12] shows that the Dirichlet density of the set of primes represented by $5x^2 + 2xy + 11y^2$ is $\frac{1}{6}$. Applying the orthogonality of Dirichlet character modulo 24, we see that the Dirichlet density of $\mathcal{S}$ is $\frac{1}{6} \cdot \frac{1}{\phi(24)} = \frac{1}{48}$, where ϕ is Euler's totient function. By a classical result of Wirsing [25] on multiplicative functions (see also [14, Proposition 4]), we find

$$\sum_{\substack{m \le x \\ p|m \Rightarrow p \in \mathcal{S}}} 1 \sim c \frac{x}{(\log x)^{\frac{47}{48}}},$$

where c is a constant. An elementary argument (see, for example, [19, Lemma 3.6]) shows that

$$\sum_{\substack{m \le x \\ \mu(m) = -1 \\ p|m \Rightarrow p \in \mathcal{S}}} 1 \gg \frac{x}{(\log x)^{\frac{47}{48}}}. \tag{15}$$

Hence, the lower bound of Theorem 1.1 follows from (14) and (15). On the other hand, Bernays' theorem [9] (see also [18, Theorem 2]) implies that the number of integers less than x represented integrally by $5x^2 + 2xy + 11y^2$ is

$$c_1 \frac{x}{(\log x)^{\frac{1}{2}}} \left(1 + O\left(\frac{1}{(\log x)^{c_2}}\right)\right)$$

for some constants c_1 and c_2. Therefore, from (8) we infer

$$\sum_{\substack{0 \le n \le x \\ A(2n+1) \text{ odd}}} 1 \ll \sum_{\substack{n \le 2x+1 \\ R(n, 5x^2+2xy+11y^2) \equiv 2 \pmod 4}} 1 \ll \sum_{\substack{n \le 2x+1 \\ R(n, 5x^2+2xy+11y^2) > 0}} 1 \ll \frac{x}{(\log x)^{\frac{1}{2}}}.$$

This completes the proof of Theorem 1.1.

Remark 2.1. The relation (8) implies that $A(2n + 1)$ is even if $24n + 11$ has a prime divisor ℓ satisfying $(\frac{-6}{\ell}) = -1$ and the exponent of ℓ in the prime factorization of $24n + 11$ is odd. To prove this statement, we observe that if $R(24n + 11, f) > 0$, then

$$24n + 11 = 5x^2 + 2xy + 11y^2 \equiv 0 \pmod \ell$$

for some x and y. It follows that

$$(5x + y)^2 \equiv -54y^2 \pmod \ell,$$

and so

$$\left(\frac{(5x+y)^2}{\ell}\right)=\left(\frac{-54y^2}{\ell}\right)=\left(\frac{-6}{\ell}\right)\left(\frac{9y^2}{\ell}\right)=-\left(\frac{9y^2}{\ell}\right).$$

This implies that $\ell \mid y$, hence $\ell \mid x$ and $\ell^2 \mid 24n+11$. Replacing $24n+11$ by $\frac{24n+11}{\ell^2}$ and repeating the arguments above, we find that the exponents of ℓ in the prime factorization of $24n+11$ must be even, which contradicts our assumption on ℓ. Therefore, $R(24n+11, f)=0$ and $A(2n+1)$ is even by (8).

For any prime $\ell \equiv 13, 17, 19$ and $23 \pmod{24}$, any positive integers s and m with $\ell \nmid m$, we see that $(\frac{-6}{\ell})=-1$ and $24\ell^{2s-1}m+11\ell^{2s}$ has a prime divisor ℓ with exponent $2s-1$. Therefore, we have

$$A\left(2\ell^{2s-1}m+\frac{11\ell^{2s}+1}{12}\right)\equiv 0 \pmod{2}.$$

This gives infinitely many congruences for $A(n) \pmod{2}$.

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