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Stochastic aspects of the unitary dual group

Aspects stochastiques du groupe dual unitaire

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ABSTRACT

In this note, we study the asymptotic properties of the $*$-distribution of traces of some matrices, with respect to the free Haar trace on the unitary dual group. The considered matrices are powers of the unitary matrix generating the Brown algebra. We proceed in two steps, first computing the free cumulants of any R-cyclic family, then characterizing the asymptotic $*$-distributions of the traces of powers of the generating matrix, thanks to these free cumulants. In particular, we obtain that these traces are asymptotic $*$-free circular variables.

RÉSUMÉ

Dans cette note, nous étudions la loi asymptotique de la trace de certaines matrices par rapport à la trace de Haar libre sur le groupe dual unitaire. Ces matrices sont les puissances de la matrice unitaire qui engendre l'algèbre de Brown. Nous procédons en deux étapes. Tout d'abord, nous calculons les cumulants joints d'une famille de matrices R-cyclique. Nous caractérisons ensuite la $*$-distributions asymptotique des traces considérées, à l'aide des cumulants libres. En particulier, nous obtenons que ces traces sont des variables asymptotiquement circulaires et $*$-libres.

Version française abrégée

Dans cette note, nous étudions le comportement asymptotique des $\chi(u^p)$, traces des puissances de la matrice unitaire u définissant l'algèbre de Brown U_n^{nc}, par rapport à la trace de Haar libre h déterminée par Cébron et Ulrich [2] et McClanahan [5].

Pour cela, nous remarquons tout d'abord que la propriété de h rappelée en Proposition 2.4 entraîne que $\{u, u^*\}$ est une famille de matrices R-cyclique, c'est-à-dire que les cumulants libres κ_r de coefficients $(u^e)_{ij}$ s'annulent dès lors que les indices ne sont pas cycliques.

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Définition 0.1 *([8]).* Pour une algèbre $\mathcal{A}$ fixée, une famille de matrices $\{A_l = (a_{ij}^{(l)})_{1\le i,j\le n}\}_{1\le l\le s}$ de $\mathcal{M}_n(\mathcal{A})$ est appelée *R-cyclique* si $\kappa_r(a_{i_1j_1}^{(l_1)}, \dots, a_{i_rj_r}^{(l_r)}) = 0$ lorsqu'il n'est pas vrai que $j_1 = i_2$, $j_2 = i_3$, ..., $j_{r-1} = i_r$ et $j_r = i_1$.

Nous pouvons donc nous ramener au calcul du cumulant libre des traces des puissances d'une famille de matrices R-cyclique, dont les cumulants considérés dans la définition ci-dessus ne dépendent pas des indices $i_1, \dots, i_r$. L'application de ce résultat dans notre cas nous permet de démontrer le théorème ci-dessous.

Théorème 0.2. *La famille de variables aléatoires $\left(\chi(u^p)\right)_{p\ge1}$ est asymptotiquement, lorsque la dimension tend vers l'infini, une famille $*$-libre de variables circulaires, d'espérance* 0 *et de covariance* 1.

1. Introduction

Diaconis, Shahshahani, and Evans [4,3] show that the traces of powers of a matrix chosen at random in the unitary (respectively orthogonal) group behave asymptotically like independent complex (resp. real) Gaussian random variables. Later, Banica, Curran, and Speicher investigate the case of easy quantum groups in [1], and obtain similar results in the context of free probability for free orthogonal groups.

In [2], Cébron and Ulrich study the Haar states according to the five notions of convolution (free, tensor, Boolean, monotone, and anti-monotone) of the unitary dual group $U\langle n\rangle$. They prove in particular that there is no Haar state for each of the five notions of convolution, and even no Haar trace for the Boolean, monotone or anti-monotone convolution, for $n \ge 2$. They also define a faithful Haar trace on $U\langle n\rangle$ for the free convolution, denoted h, which is in fact equal to the state given by McClanahan in [5].

The aim of this note is to extend the study of Diaconis, Shashahani, and Evans to the framework of the unitary dual group and its free Haar trace. The paper is organized as follows. We first introduce the tools of our study. Then, in the last section, we discuss the computation of the joint cumulants of traces of powers of an R-cyclic family and determine the asymptotic $*$-distribution of the traces of powers of the generating matrix, with respect to the free Haar trace.

2. Preliminaries

We recall here some facts about the unitary dual group, free cumulants, and R-cyclicity.

2.1. The unitary dual group

Let $n \ge 1$, and U_n^{nc}, sometimes called the Brown algebra, be the noncommutative $*$-algebra generated by n^2 elements $\{u_{ij}\}_{1\le i,j\le n}$ such that the matrix $u = (u_{ij})_{1\le i,j\le n}$ is unitary. It is possible to endow this algebra with a structure of dual group in the sense of Voiculescu [10], $U\langle n\rangle = (U_n^{\mathrm{nc}}, \Delta, \delta, \Sigma)$, called the unitary dual group. This is a generalization of the notion of groups, like Hopf algebras, but using the free product instead of the tensor product.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be unital $*$-algebras. The *free product of $\mathcal{A}$ and $\mathcal{B}$* is the unique unital $*$-algebra $\mathcal{A} \sqcup \mathcal{B}$ with two $*$-homomorphisms $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \sqcup \mathcal{B}$ and $i_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \sqcup \mathcal{B}$, such that, for all $*$-homomorphisms $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$, there exists a unique $*$-homomorphism $f \sqcup g : \mathcal{A} \sqcup \mathcal{B} \to \mathcal{C}$ satisfying $f = (f \sqcup g) \circ i_{\mathcal{A}}$ and $g = (f \sqcup g) \circ i_{\mathcal{B}}$.

We sometimes refer to $\mathcal{A}$ and $\mathcal{B}$ as the left and right legs of the free product $\mathcal{A} \sqcup \mathcal{B}$. Therefore, for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we denote $i_{\mathcal{A}}(a)$ and $i_{\mathcal{B}}(b)$ by $a^{(1)}$ and $b^{(2)}$, respectively.

Let $f : \mathcal{A}_1 \to \mathcal{A}_2$ and $g : \mathcal{B}_1 \to \mathcal{B}_2$ be unital $*$-homomorphisms between the four unital $*$-algebras $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$. Then we denote by $f \underline{\sqcup} g : \mathcal{A}_1 \sqcup \mathcal{B}_1 \to \mathcal{A}_2 \sqcup \mathcal{B}_2$ the unital $*$-homomorphism given by the free product $(i_{\mathcal{A}_2} \circ f) \sqcup (i_{\mathcal{B}_2} \circ g)$.

Definition 2.2. Let $n \ge 1$. The *unitary dual group $U\langle n\rangle$* is defined by the unital $*$-algebra U_n^{nc} and three unital $*$-homomorphisms $\Delta : U_n^{\mathrm{nc}} \to U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$, $\delta : U_n^{\mathrm{nc}} \to \mathbb{C}$ and $\Sigma : U_n^{\mathrm{nc}} \to U_n^{\mathrm{nc}}$, such that

– U_n^{nc} is the Brown algebra, generated by the u_{ij}'s satisfying

$$\forall 1 \le i, j \le n,\ \sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^{n} u_{ik} u_{jk}^*,$$

– the map Δ is a coassociative coproduct, i.e. $(\mathrm{id} \underline{\sqcup} \Delta) \circ \Delta = (\Delta \underline{\sqcup} \mathrm{id}) \circ \Delta$, given on the generators by $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik}^{(1)} u_{kj}^{(2)}$,
– the map δ is a counit, i.e. $(\delta \underline{\sqcup} \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \underline{\sqcup} \delta) \circ \Delta$, given by $\delta(u_{ij}) = \delta_{ij}$,
– the map Σ is a coinverse, i.e. $(\Sigma \sqcup \mathrm{id}) \circ \Delta = \delta(\cdot) 1_{U_n^{\mathrm{nc}}} = (\mathrm{id} \sqcup \Sigma) \circ \Delta$, given by $\Sigma(u_{ij}) = u_{ji}^*$.

2.2. Free cumulants

Note that the free cumulants characterize random variables and free independence. We will use them to study the asymptotic law of $\chi(u^p)$, as Banica, Curran, and Speicher in [1]. Let us introduce some notations:

- $[s]$ denotes the set $\{1, 2, \ldots, s\}$ for each $s \geq 1$,
- the set of noncrossing partitions of $[s]$ is denoted $NC(s)$,
- $NC_2(s)$ corresponds to the noncrossing pairings.

The family of free cumulants $(\kappa_r)_{r\in\mathbb{N}}$ is uniquely characterized by the corresponding multiplicative family of functionals satisfying the moment-cumulant formula [9, equation (11.7)]:

$$\forall s \in \mathbb{N}, \forall \{a_i\}_{1\leq i\leq s} \subset U_n^{\mathrm{nc}},\ h(a_1 \ldots a_s) = \sum_{\pi\in \mathrm{NC}(s)} \kappa_\pi[a_1, \ldots, a_s]$$

where $\kappa_\pi[a_1, \ldots, a_s] = \prod_{V\in\pi} \kappa_V[a_1, \ldots, a_s] := \prod_{\substack{V\in\pi \\ V=\{v_1<\ldots<v_l\}}} \kappa_l(a_{v_1}, \ldots, a_{v_l})$.

In particular, they satisfy the following property:

Proposition 2.3 *([9, equation (11.11)]). Let $s, r \in \mathbb{N}$ and $\underline{p} = (p_1, \ldots, p_r) \in [n]^r$ be given such that the sum of the p_i's is s. Then, for any $\{a_i\}_{1\leq i\leq s} \subset U_n^{\mathrm{nc}}$,*

$$\kappa_r\left(a_1 \ldots a_{p_1}, a_{p_1+1} \ldots a_{p_1+p_2}, \ldots, a_{s-p_r+1} \ldots a_s\right) = \sum_{\substack{\pi\in NC(s) \\ \pi\vee\gamma_{\underline{p}}=1_s}} \kappa_\pi[a_1, \ldots, a_s]$$

where $\gamma_{\underline{p}}$ denotes the noncrossing partition associated with $\underline{p}$, i.e. $\gamma_{\underline{p}} = \{\{1, \ldots, p_1\}, \ldots, \{s - p_r + 1, \ldots, s\}\}$, and $1_s = \{\{1, \ldots, s\}\}$.

Cébron and Ulrich [2] compute the free cumulants associated with the free Haar trace h of the generators of the Brown algebra u_{ij} and their adjoints $(u^*)_{ij} = u^*_{ji}$.

Proposition 2.4 *([2]). The free cumulants of $(u_{ij})_{1\leq i,j\leq n}$ and $((u^*)_{ij})_{1\leq i,j\leq n}$ in the noncommutative probability space (U_n^{nc}, h) are given as follows.*

Let $1 \leq i_1, j_1, \ldots, i_r, j_r \leq n$ and $\epsilon_1, \ldots, \epsilon_r$ be either $\emptyset$ or $*$. If the indices are cyclic (i.e. if $j_{l-1} = i_l$ for $2 \leq l \leq r$ and $i_1 = j_r$), r is even and the ϵ_i are alternating we have

$$\kappa_r\left((u^{\epsilon_1})_{i_1 j_1}, \ldots, (u^{\epsilon_r})_{i_r j_r}\right) = n^{1-r}(-1)^{\frac{r}{2}-1} C_{\frac{r}{2}-1}$$

where $C_i = \frac{(2i)!}{(i+1)!i!}$ designate the Catalan numbers.

Otherwise, the left-hand side term is equal to zero.

2.3. R-cyclicity

Definition 2.5. For an algebra $\mathcal{A}$, a family of matrices $\{A_l = (a^{(l)}_{ij})_{1\leq i,j\leq n}\}_{1\leq l\leq s}$ in $\mathcal{M}_n(\mathcal{A})$ is called *R-cyclic* if $\kappa_r(a^{(l_1)}_{i_1 j_1}, \ldots, a^{(l_r)}_{i_r j_r}) = 0$ whenever it is not true that $j_1 = i_2$, $j_2 = i_3$, ..., $j_{r-1} = i_r$ and $j_r = i_1$.

Thus Proposition 2.4 ensures that $\{u, u^*\}$ is an R-cyclic family. Moreover, the cyclic free cumulants do not depend on the indices.

Since we want to look at powers of the generating matrix u, we need the following property:

Proposition 2.6 *([8, Theorem 4.3]). Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. Let d be a positive integer, and let $A_1, \ldots, A_s$ be an R-cyclic family of matrices in $\mathcal{M}_d(\mathcal{A})$. We denote by $\mathcal{D}$ the algebra of scalar diagonal matrices in $\mathcal{M}_d(\mathcal{A})$, and by $\mathcal{C}$ the subalgebra of $\mathcal{M}_d(\mathcal{A})$, which is generated by $\{A_1, \ldots, A_s\} \cup \mathcal{D}$. Then every finite family of matrices from $\mathcal{C}$ is R-cyclic.*

In particular every finite subset of $\{u^k\}_{k\geq 1} \cup \{(u^*)^k\}_{k\geq 1}$ is also an R-cyclic family of matrices.

3. Main result

Theorem 3.1. *The family of variables* $\left(\chi(u^p)\right)_{p\geq 1}$ *is asymptotically, as the dimension tends to infinity, a family of* $*$*-free circular variables of mean* 0 *and covariance* 1.

To prove this, let us use the R-cyclicity of u and u^*. First, let us note that $\chi(u^p)^e = \chi((u^e)^p)$ for any $p \geq 1$ and any $e \in \{\emptyset, *\}$. This means that we can see the calculation of $\kappa_s(\chi(u^{p_1})^{e_1}, \ldots, \chi(u^{p_s})^{e_s})$ in a more general framework and compute $\kappa_s(\chi(A_{l_1}^{p_1}), \ldots, \chi(A_{l_s}^{p_s}))$ for $\{A_l\}_{l\in I}$, an R-cyclic family of matrices.

Lemma 3.2. *Let* $\{A_l\}_{l\in I}$ *be an R-cyclic family of matrices such that the cumulants* $\kappa_\pi\left[a_{i_1j_1}^{(l_1)}, \ldots, a_{i_sj_s}^{(l_s)}\right]$ *depend only on the cyclicity of the indices. Let us denote by* $\kappa_\pi\left[a^{(l_1)}, \ldots, a^{(l_s)}\right]$ *the value of the cumulant with cyclic indices, i.e. such that* $j_1 = i_2, \ldots, j_s = i_1$*. Then*

$$\kappa_s(\chi(A_{l_1}^{p_1}), \ldots, \chi(A_{l_s}^{p_s})) = n^{p+2-s} \sum_{\substack{\pi\in NC(p)\\ \pi\vee\gamma_{\underline{p}}=1_p}} n^{-|\pi|}\kappa_\pi\left[a^{(l_1)}, \ldots, a^{(l_1)}, a^{(l_2)}, \ldots, a^{(l_s)}\right].$$

Note that this calculation is similar to the one in the last section of [6], and we will use similar arguments. Since $\{A_{l_1}^{p_1}, \ldots, A_{l_s}^{p_s}\}$ is also an R-cyclic family, by Proposition 2.6, if p is the sum of all the p_i's and $a_{ij}^{(l)}$ denotes the coefficient (i, j) of the matrix A_l,

$$\kappa_s(\chi(A_{l_1}^{p_1}), \ldots, \chi(A_{l_s}^{p_s})) = \sum_{\substack{1\leq i_1,\ldots,i_p\leq n\\ i_1=i_{p_1+1}=\ldots=i_{p-p_s+1}}} \kappa_s\left(a_{i_1i_2}^{(l_1)}\ldots a_{i_{p_1}i_{p_1+1}}^{(l_1)}, \ldots, a_{i_{p-p_s+1}i_{p-p_s+2}}^{(l_s)}\ldots a_{i_pi_1}^{(l_s)}\right).$$

By Proposition 2.3, we obtain

$$\kappa_s(\chi(A_{l_1}^{p_1}), \ldots, \chi(A_{l_s}^{p_s})) = \sum_{\substack{1\leq i_1,\ldots,i_p\leq n\\ i_1=i_{p_1+1}=\ldots=i_{p-p_s+1}}} \sum_{\substack{\pi\in NC(p)\\ \pi\vee\gamma_{\underline{p}}=1_p}} \kappa_\pi\left[a_{i_1i_2}^{(l_1)}, \ldots, a_{i_{p_1}i_{p_1+1}}^{(l_1)}, \ldots, a_{i_pi_1}^{(l_s)}\right].$$

By definition of κ_π, we can restrict ourselves to the study of a block $V = \{v_1 < \ldots < v_r\}$ of π, and look at $\kappa_r(a_{i_{v_1}i_{v_1+1}}^{(\lambda_{v_1})}, \ldots, a_{i_{v_r}i_{v_r+1}}^{(\lambda_{v_r})})$ where $\underline{\lambda} = (l_1, \ldots, l_1, l_2, \ldots, l_s) \in I^p$. In order to have a non-zero contribution, the indices have to be cyclic, i.e. to satisfy

$$\forall 1 \leq j \leq r-1, i_{v_j+1} = i_{v_{j+1}} \text{ and } i_{v_r+1} = i_{v_1}.$$

Let us denote by σ_π the permutation associated with the partition π, by considering the elements of a block of π in increasing order as a cycle of σ_π. Hence the conditions above can be written as $i_{\gamma(v_i)} = i_{\sigma_\pi(v_i)}$, where $\gamma = (1, 2, \ldots, p)$. Since this should be true for each block of π, it means that $i_j = i_{\gamma\circ\sigma_\pi^{-1}(j)}$ for all $1 \leq j \leq p$. Thus, we get

$$\kappa_s(\chi(A_{l_1}^{p_1}), \ldots, \chi(A_{l_s}^{p_s})) = \sum_{\substack{\pi\in NC(p)\\ \pi\vee\gamma_{\underline{p}}=1_p}} \sum_{\substack{1\leq i_1,\ldots,i_p\leq n\\ i_1=i_{p_1+1}=\ldots=i_{p-p_s+1}\\ i_j=i_{\gamma\circ\sigma_\pi^{-1}(j)}}} \kappa_\pi\left[a_{i_1i_2}^{(l_1)}, \ldots, a_{i_{p_1}i_{p_1+1}}^{(l_1)}, \ldots, a_{i_pi_1}^{(l_s)}\right].$$

Since, moreover $\kappa_\pi\left[a_{i_1i_2}^{(l_1)}, \ldots, a_{i_pi_1}^{(l_s)}\right]$ depends only on the cyclicity of the indices, and is denoted by $\kappa_\pi\left[a^{(l_1)}, \ldots, a^{(l_1)}, \ldots, a^{(l_s)}\right]$, we obtain

$$\kappa_s(\chi(A_{l_1}^{p_1}), \ldots, \chi(A_{l_s}^{p_s})) = \sum_{\substack{\pi\in NC(p)\\ \pi\vee\gamma_{\underline{p}}=1_p}} \kappa_\pi\left[a^{(l_1)}, \ldots, a^{(l_1)}, a^{(l_2)}, \ldots, a^{(l_s)}\right] c_\pi$$

where c_π denotes the quantity

$$\#\left\{\underline{i} \in \{1, \ldots, n\}^p,\ i_1 = i_{p_1+1} = \ldots = i_{p-p_s+1},\ \forall 1 \leq j \leq p, i_j = i_{\gamma\circ\sigma_\pi^{-1}(j)}\right\}.$$

Thanks to [7, Lemma 14], $\pi \vee \gamma_{\underline{p}} = 1_p$ if and only if $\sigma_\pi^{-1} \circ \gamma$ separates p_1, $p_1 + p_2$, ... and p, which is equivalent to the fact that 1, $p_1 + 1$, ... and $p - p_s + 1$ are all in different blocks of $\gamma \circ \sigma_\pi^{-1}$. Thus, $c_\pi = n^{\#(\gamma\circ\sigma_\pi^{-1})-s+1}$, where $\#\sigma$ denotes the number of cycles in the cycle decomposition of a permutation $\sigma \in \mathfrak{S}_p$. Notes that $\gamma \circ \sigma_\pi^{-1}$ corresponds to the Kreweras complement [9] of π, denoted $K(\pi)$, conjugated by γ. Hence

$$\#(\gamma \circ \sigma_\pi^{-1}) = |K(\pi)| = p + 1 - |\pi|$$

where $|\pi|$ is the number of blocks of π, and then $c_\pi = n^{p+2-s-|\pi|}$, which proves the lemma.

In particular, in the dual unitary group endowed with the free Haar trace, we have $I = \{\emptyset, *\}$ and Proposition 2.4 ensures that, if π is $\underline{\lambda}$-adapted with $\underline{\lambda} = (l_1, \dots, l_1, l_2, \dots, l_s)$,

$$\kappa_\pi \left[a^{(\lambda_1)}, \dots, a^{(\lambda_p)} \right] = n^{|\pi|-p} (-1)^{\frac{p}{2}-|\pi|} \prod_{V \in \pi} C_{\frac{\#V}{2}-1}$$

otherwise the cumulant vanishes. Here, $\pi \in \mathrm{NC}(p)$ is said to be $\underline{\lambda}$-adapted when the following conditions are true for each block $V = \{v_1 < \dots < v_l\}$ of the noncrossing partition π:

- $\#V := l \in 2\mathbb{N}$,
- $\forall 1 \leq i \leq l-1, \ \lambda_{v_i} \neq \lambda_{v_{i+1}}$.

Finally, we get, with $\underline{\epsilon} = (e_1, \dots, e_1, e_2, \dots, e_s)$,

$$\kappa_s \left(\chi(u^{p_1})^{e_1}, \dots, \chi(u^{p_s})^{e_s} \right) = n^{2-s} (-1)^{\frac{p}{2}} \sum_{\substack{\pi \in NC(p) \\ \pi \vee \gamma_{\underline{p}} = 1_p \\ \underline{\epsilon}-\text{adapted}}} (-1)^{|\pi|} \prod_{V \in \pi} C_{\frac{\#V}{2}-1} .$$

If $s > 2$, it is clear that the cumulants vanish asymptotically. If $s = 2$, the cumulant is non-zero if and only if $p_1 = p_2$ and $e_1 \neq e_2$; in this case, the cumulant equals 1. Moreover, if $s = 1$, there is no (e)-adapted partition, and the cumulant is zero. This is the description of the cumulants of a $*$-free family of circular random variables.

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